Deriving Geometry Theorems by Automated Tools

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Abstract

Derivation of geometry theorems belongs to mighty tools of automated geometry theorem proving. By elimination of suitable variables in the system of algebraic equations describing a geometric situation we get required formulas. The power of derivation is presented on computation of the area of planar polygons given by their lengths of sides and diagonals. This part we conclude with derivation of a formula of Robbins for the area of a cyclic pentagon given by its side lengths.
Searching for loci of points of given properties is a special case of derivation. This topic belongs to the most difficult parts of school mathematics all over the world. New technologies DGS and CAS enable to overcome this problem. We demonstrate it in a few examples from elementary geometry.

1 Introduction

In the paper we will be concerned with derivation of geometric theorems by automated tools.

First we introduce derivation as a part of the theory of automated geometry theorem proving. We continue with deriving formulas for the area of a quadrilateral and a pentagon in the plane given by their lengths of sides and diagonals. Then the formula of Brahmagupta for the area of a cyclic quadrilateral given by its side lengths is investigated. This part is concluded by derivation of the analogous formula for a cyclic pentagon. Whereas Brahmagupta formula comes from 6th century AD, it took almost 1400 years until 1994 American Robbins found it [11]. In this paper author's specific approach of finding this formula is shown, see also [7].
The second part of the paper is devoted to a special class of derivation — searching for loci of points of given properties. This topic belongs to the most difficult parts of school mathematics. New technologies such as DGS and CAS facilitate this problem. We will show how to search for loci using automated tools. First we use DGS to state a conjecture, then we apply CAS to find the searched locus exactly. The method is suitable for all school levels from elementary schools to universities.
During computations we will use dynamic geometry system GeoGebra, and computer algebra systems CoCoA[1] based on Gröbner bases (GB) computation and Epsilon[2] based on Wu-Ritt (WR) approach.

[1]Software CoCoA is freely distributed at the address http://cocoa.dima.unige.it
[2]Software Epsilon is freely distributed at http://www-calfor.lip6.fr/~wang/epsilon/

2 Automated derivation

Automatic derivation is a part of automatic discovery of theorems in geometry. Whereas in automatic discovery we search for complementary hypotheses for a geometric statement to become true, by automatic derivation of theorems we mean finding geometric formulas holding among prescribed geometric magnitudes which follow from the given assumptions [10]. Let us say it more precisely.

Denote by $K[x_1,\dots,x_n]$ the ring of polynomials of n indeterminates $x=(x_1,\dots,x_n)$ with coefficients in the field K, where K is a field of characteristic 0, for instance the field of rational numbers. Assume that polynomial equations $h_1(x_1,\dots,x_n)=0,\dots,h_r(x_1,\dots,x_n)$ $=0$ express geometric properties of some objects. Let $x_1,\dots,x_m$ be independent variables (parameters) and $x_{m+1},\dots,x_n$ dependent variables. Eliminating variables (dependent or independent) we get the elimination ideal which contains only polynomials in those variables we did not eliminate. Usually we eliminate independent variables $x_1,\dots,x_m$ or, if needed, some dependent variables $x_{m+1},\dots,x_p$, $m\le p\le n$ to obtain a geometric statement expressed by the equation $c(x_{p+1},\dots,x_n)=0$ which follows from the assumptions $h_1=0,\dots,h_r=0$. The theorem holds [7]:

Theorem: *Let $I=(h_1,\dots,h_r)\subset K[x_1,\dots,x_n]$ and $c\in I\cap K[x_{p+1},\dots,x_n]$, for $p\le n$. Then*

$$h_1(x_1,\dots,x_n)=0,\dots,h_r(x_1,\dots,x_n)=0\Rightarrow c(x_{p+1},\dots,x_n)=0. \tag{1}$$

In the next section we will show several examples on derivation of known or less known formulas from geometry of polygons.

3 Area of polygons

We will study the area of a planar polygon $A_1A_2\dots A_n$ which is given by its lengths of sides and diagonals. The (signed) area of a polygon is defined regardless of whether it intersects itself or not. The theorem holds [6]:

Theorem: *Let $d_{ij}=|A_iA_j|^2$ denote a square of the distance of the vertices A_i,A_j. Then the area p of an n-gon $A_1A_2\dots A_n$ is given by*

$$16p^2=\sum_{i,j=1}^{n}\begin{vmatrix} d_{i,j} & d_{i,j+1}\\ d_{i+1,j} & d_{i+1,j+1}\end{vmatrix}. \tag{2}$$

In the following we derive in automated way special cases of the theorem above — formulas for a quadrilateral and a pentagon given by their lengths of sides and diagonals.
For $n=3$ we get the well-known formula of Heron

$$16p^2=-a^4-b^4-c^4+2a^2b^2+2b^2c^2+2c^2a^2 \tag{3}$$

or

$$16p^2=(a+b+c)(-a+b+c)(a-b+c)(a+b-c). \tag{4}$$

As automated derivation of the Heron's formula is quite frequent in the literature, we omit it.
Let us derive a formula for the area of a quadrilateral:

Consider a planar quadrilateral $ABCD$ with lengths of sides a, b, c, d and diagonals e, f. We are to express the area p of $ABCD$ in terms of a, b, c, d, e, f.

Introduce a rectangular coordinate system such that $A = [0, 0]$, $B = [a, 0]$, $C = [u, v]$, $D = [w, z]$, Fig. 1. We express relations between a, b, c, d, e, f, p and coordinates a, u, v, w, z by

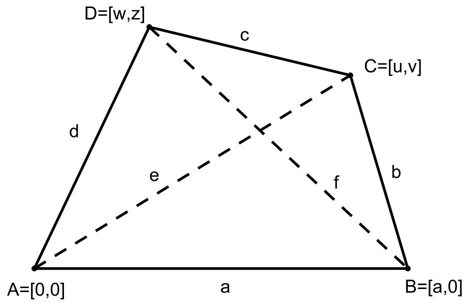

Figure 1:

the following system of algebraic equations:

$b = |BC| \Rightarrow h_1 := (u - a)^2 + v^2 - b^2 = 0,$

$c = |CD| \Rightarrow h_2 := (w - u)^2 + (z - v)^2 - c^2 = 0,$

$d = |DA| \Rightarrow h_3 := w^2 + z^2 - d^2 = 0,$

$e = |EF| \Rightarrow h_4 := u^2 + v^2 - e^2 = 0,$

$f = |EF| \Rightarrow h_5 := (w - a)^2 + z^2 - f^2 = 0,$

$p =$ area of $ABCD \ \Rightarrow h_6 := av - vw + uz - 1/2p = 0.$

Elimination of variables u, v, w, z from the system $h_1 = 0,\ h_2 = 0,\ \dots\ h_6 = 0$ gives the elimination ideal in variables a, b, c, d, e, f, p. In CoCoA we get

```
Use R::= Q[a,b,c,d,e,f,p,u,v,w,z];
I:=Ideal((u-a)^2+v^2-b^2,(w-u)^2+(z-v)^2-c^2,w^2+z^2-d^2,
u^2+v^2-e^2,(w-a)^2+z^2-f^2,av-vw+uz-1/2p);
Elim(u..z,I);
```

several polynomials. One of them leads to the equation

$$16p^2 = 4e^2 f^2 - (a^2 - b^2 + c^2 - d^2)^2 \tag{5}$$

which is the desired result. We can verify that (5) is in accordance with (2).
In the list of polynomials of the elimination ideal we get another polynomial

$M := -2(a^4c^2 - a^2b^2c^2 + a^2c^4 - a^2b^2d^2 + b^4d^2 - a^2c^2d^2 - b^2c^2d^2 + b^2d^4 + a^2b^2e^2 - a^2c^2e^2 - b^2d^2e^2 + c^2d^2e^2 - a^2c^2f^2 + b^2c^2f^2 + a^2d^2f^2 - b^2d^2f^2 - a^2e^2f^2 - b^2e^2f^2 - c^2e^2f^2 - d^2e^2f^2 + e^4f^2 + e^2f^4)$

which is equivalent to

$$M := \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & e^2 & d^2 \\ 1 & a^2 & 0 & b^2 & f^2 \\ 1 & e^2 & b^2 & 0 & c^2 \\ 1 & d^2 & f^2 & c^2 & 0 \end{vmatrix}. \tag{6}$$

M is the well-known Cayley–Menger determinant [1]. The condition

$$M = 0 \tag{7}$$

expresses a mutual dependence of all six distances between the vertices of a planar quadrilateral $ABCD$.

Similarly we can derive a special case of (2) for a planar pentagon $ABCDE$ with lengths of sides a, b, c, d, e and diagonals i_1, i_2, i_3, i_4, i_5 with the area p. If we denote $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DE|$, $e = |EA|$, and $i_1 = |CE|$, $i_2 = |AD|$, $i_3 = |BE|$, $i_4 = |AC|$, $i_5 = |BD|$, Fig. 2, then

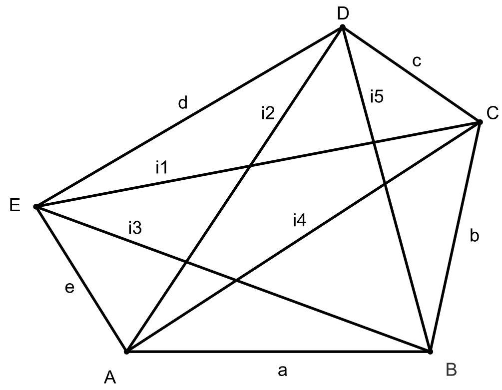

Figure 2: Area of a pentagon

$$\begin{aligned}16p^2 = &-(a^4 + b^4 + c^4 + d^4 + e^4) + 2(a^2b^2 + b^2c^2 + c^2d^2 + d^2e^2 + e^2a^2)\\ &+2(i_1^2i_2^2 + i_2^2i_3^2 + i_3^2i_4^2 + i_4^2i_5^2 + i_5^2i_1^2) - 2(a^2i_1^2 + b^2i_2^2 + c^2i_3^2 + d^2i_4^2\\ &+e^2i_5^2).\end{aligned} \tag{8}$$

4 Area of cyclic polygons

In this part we will study cyclic polygons, i.e., those whose vertices lie on a circle. We will derive area of cyclic polygons in terms of their side lengths. We start from well-known formulas for area of a triangle and a cyclic quadrilateral. This part we conclude with derivation of the formula of Robbins [11] for the area of a cyclic pentagon.

As any triangle is cyclic then the formula for the area of a triangle with side lengths a, b, c is the same as the formula of Heron (3), (4).

The analogy of the formula of Heron for a cyclic convex quadrilateral with side lengths a, b, c, d and the area p is the following formula of Brahmagupta (Brahmagupta — Indian mathematician, 598–c. 665 A.D.)

$$16p^2 = (-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d). \tag{9}$$

Since that time no formula for a cyclic pentagon with given side lengths a, b, c, d, e and the area p, despite a great effort of mathematicians, appeared until 1994 when American

D. P. Robbins [11] discovered it. It took almost 1400 years than the formula for the area of a cyclic pentagon appeared. The reason why it lasted so long is a big complexity of such a formula.
Whereas Robbins combined several methods to discover the formula for a cyclic pentagon, we will demonstrate a method of deriving such a formula based on the theory of automated derivation.

First we will derive formula of Brahmagupta, then we show how to derive the formula of Robbins.

Problem (Brahmagupta): *Given a cyclic quadrilateral with side lengths a, b, c, d and the area p. Find a relation among a, b, c, d, p.*

We will solve the problem using coordinate-free approach. Consider a cyclic quadrilateral $ABCD$ with side lengths a, b, c, d with the area p. Denote by e, f its lengths of diagonals, Fig. 3.

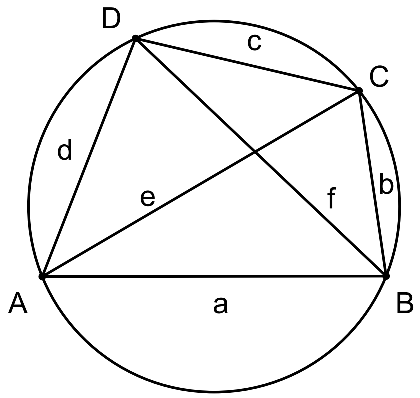

Figure 3: Cyclic quadrilateral $ABCD$

By well-known formulas of Ptolemy [2] we express that $ABCD$ is cyclic. It holds

$$ac + bd - ef = 0 \tag{10}$$

for a cyclic convex quadrilateral and

$$ac - bd + ef = 0 \quad \text{or} \quad -ac + bd + ef = 0 \tag{11}$$

for a cyclic non-convex quadrilateral.
First suppose that $ABCD$ is a convex cyclic quadrilateral. Consider the following hypotheses:

p is the area of $ABCD$ $\Rightarrow h_1 := 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2 - 16p^2 = 0$

by the formula (5), and

$ABCD$ is cyclic and convex $\Rightarrow h_2 := ac + bd - ef = 0.$

The elimination of variables e, f from the system $h_1 = 0$, $h_2 = 0$ gives

```
Use R ::= Q[a,b,c,d,e,f,p];
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```
I:=Ideal(4e^2f^2-(a^2-b^2+c^2-d^2)^2-16p^2,ac+bd-ef);
Elim(e..f,I);
```

the formula (9).
Now consider that a quadrilateral $ABCD$ is cyclic non-convex. Then

$ABCD$ is cyclic and non-convex $\Rightarrow$

$h_3 := ac - bd + ef = 0$ or $h_4 := -ac + bd + ef = 0.$

The elimination of e, f in the system $h_1 = 0$ and $h_3 = 0$ gives

```
Use R ::= Q[a,b,c,d,e,f,p];
I:=Ideal(4e^2f^2-(a^2-b^2+c^2-d^2)^2-16p^2,ac-bd+ef);
Elim(e..f,I);
```

the formula

$$16p^2 = (-a+b-c+d)(a-b-c+d)(a+b+c+d)(a+b-c-d). \tag{12}$$

The remaining relation $h_4 = 0$ gives the same result (12).
Thus for a cyclic quadrilateral with given side lengths a, b, c, d we get *two* formulas (9) and (12) which differ only in one term. Namely if we compute the products in (9) and (12) we get

$$16p^2 = -(a^4+b^4+c^4+d^4) + 2(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2) + 8abcd \tag{13}$$

in a convex case, and

$$16p^2 = -(a^4+b^4+c^4+d^4) + 2(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2) - 8abcd \tag{14}$$

in a non-convex case.

Note, that both polynomials in (13) and (14) on the right are symmetric polynomials, i.e., by any change of the order of variables a, b, c, d the formulas remain unchanged. Denote by k, l, m, n elementary symmetric functions in variables a^2, b^2, c^2, d^2, i.e.,

$k = a^2 + b^2 + c^2 + d^2,$

$l = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2,$

$m = a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2,$

$n = a^2b^2c^2d^2.$

and let $s = 16p^2$. Then both formulas (13) and (14) can be expressed by one formula

$$(k^2 - 4l + s)^2 - 64n = 0. \tag{15}$$

Similarly we can express the formula of Heron which reads

$$k^2 - 4l + s = 0, \tag{16}$$

where $k = a^2 + b^2 + c^2$, $l = a^2b^2 + a^2c^2 + b^2c^2$ and $s = 16p^2$.

Remark:

Note the similarity of the formulas (15) and (16). If we put for instance $d = 0$ then a quadrilateral becomes a triangle and the formula (15) becomes (16).

Now we are ready to derive the formula of Robbins.

Problem (Robbins): *Let $ABCDE$ be a cyclic pentagon with side lengths a, b, c, d, e and the area p. Find a relation among a, b, c, d, e, p.*

To solve the problem we will use coordinate-free approach. Consider a cyclic pentagon $ABCDE$ with sides $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DE|$, $e = |EA|$, and diagonals $i_1 = |AC|$, $i_2 = |AD|$, $i_3 = |BD|$, $i_4 = |BE|$, $i_5 = |CE|$, Fig. 4.

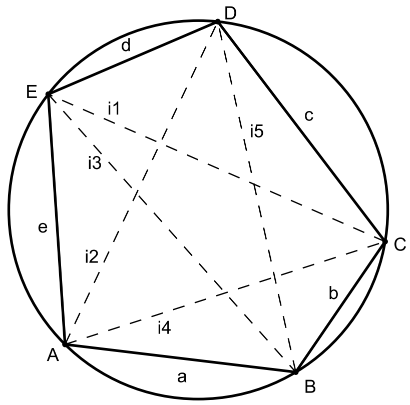

Figure 4: Cyclic pentagon $ABCDE$

First suppose that $ABCDE$ is a convex cyclic pentagon. We will use the formula (8) to express the area p of $ABCDE$ in terms of its lengths of sides a, b, c, d, e and diagonals i_1, i_2, i_3, i_4, i_5. Now we need conditions for a pentagon $ABCDE$ to be cyclic. Using the Ptolemy theorem on cyclic convex quadrilaterals $ABCD$, $BCDE$, $CDEA$, $DEAB$ and $EABC$ we get

$h_1 := ce + di_4 - i_1 i_2 = 0,$

$h_2 := da + ei_5 - i_2 i_3 = 0,$

$h_3 := eb + ai_1 - i_3 i_4 = 0,$

$h_4 := ac + bi_2 - i_4 i_5 = 0,$

$h_5 := bd + ci_3 - i_5 i_1 = 0.$

Applying the Ptolemy conditions $h_1 = 0$, $h_2 = 0$, $h_3 = 0$, $h_4 = 0$, $h_5 = 0$, to the formula (8) we get the important relation

$$k^2 - 4l + s = 4(eabi_1 + abci_2 + bcdi_3 + cdei_4 + deai_5), \tag{17}$$

where

$k = a^2 + b^2 + c^2 + d^2 + e^2,$

$l = a^2b^2 + a^2c^2 + a^2d^2 + a^2e^2 + b^2c^2 + b^2d^2 + b^2e^2 + c^2d^2 + c^2e^2 + d^2e^2,$

$m = a^2b^2c^2 + a^2b^2d^2 + a^2b^2e^2 + a^2c^2d^2 + a^2c^2e^2 + a^2d^2e^2 + b^2c^2d^2 + b^2c^2e^2 + b^2d^2e^2 + c^2d^2e^2,$

$n = a^2b^2c^2d^2 + a^2b^2c^2e^2 + a^2b^2d^2e^2 + a^2c^2d^2e^2 + b^2c^2d^2e^2,$

$o = a^2b^2c^2d^2e^2$

are elementary symmetric functions and $s = 16p^2$.

Now we need "to get rid" of variables i_1, i_2, i_3, i_4, i_5 in (17) to obtain a formula in a, b, c, d, e and p. To ensure the planarity of $ABCDE$ it suffices to ensure planarity of quadrilaterals $ABCD$, $BCDE$, $CDEA$, $DEAB$ and $EABC$. We remind the relation (7) which is a necessary and sufficient condition for a quadrilateral to be planar. In this case elimination of i_1, i_2, i_3, i_4, i_5 is very complex. Therefore we introduce the following simplification.
If a quadrilateral is cyclic then (7) can be simplified by the following formula [12], [8]:

$$S^2 = PV - 1/2M, \tag{18}$$

where $P = ac + bd - ef$, $V = ac(-a^2 - c^2 + b^2 + d^2 + e^2 + f^2) + bd(a^2 + c^2 - b^2 - d^2 + e^2 + f^2) - ef(a^2 + c^2 + b^2 + d^2 - e^2 - f^2)$ and $S = e(ab + cd) - f(bc + ad)$. Suppose that $P = 0$. Then (18) implies that instead of the condition $M = 0$ we can take $S = 0$. This gives five conditions for quadrilaterals $ABCD$, $BCDE$, $CDEA$, $DEAB$, $EABC$ to be planar:

$h_6 := i_4(ab + ci_2) - i_5(bc + ai_2) = 0,$

$h_7 := i_5(bc + di_3) - i_1(cd + bi_3) = 0,$

$h_8 := i_1(cd + ei_4) - i_2(de + ci_4) = 0,$

$h_9 := i_2(de + ai_5) - i_3(ea + di_5) = 0,$

$h_{10} := i_3(ea + bi_1) - i_4(ab + ei_1) = 0.$

Now we are ready to derive the formula of Robbins. Let us express the right side of (17) in terms of a, b, c, d, e. Denote

$h_{11} := 4(eabi_1 + abci_2 + bcdi_3 + cdei_4 + deai_5) - t = 0,$

where t is a slack variable. Elimination of variables i_1, i_2, i_3, i_4, i_5 in the ideal I which is generated by the polynomials $h_1, h_2, \ldots, h_{11}$ gives

```
Use R::=Q[a,b,c,d,e,i[1..5],t];
I:=Ideal(ce+di[4]-i[1]i[2],da+ei[5]-i[2]i[3],eb+ai[1]-i[3]i[4],
ac+bi[2]-i[4]i[5],bd+ci[3]-i[5]i[1],i[4](ab+ci[2])-i[5](bc+ai[2]),
i[5](bc+di[3])-i[1](cd+bi[3]),i[1](cd+ei[4])-i[2](de+ci[4]),
i[2](de+ai[5])-i[3](ea+di[5]),i[3](ea+bi[1])-i[4](ab+ei[1]),
t-4(eabi[1]+abci[2]+bcdi[3]+cdei[4]+deai[5]));
Elim(i[1]..i[5],I);
```

in 1m7s a polynomial in a, b, c, d, e, t with 827 terms. Substitution of elementary symmetric functions k, l, m, n, o and elimination of a, b, c, d, e gives a polynomial equation $Q = 0$ with 37 terms, where

$Q := t^7 + t^6l + t^5km + t^4k^2n + t^3k^3o + t^4m^2 - 12t^5n - 12t^4ln - 8t^3kmn - 8t^2k^2n^2 - 36t^4ko - 36t^3klo - 30t^2k^2mo - 36tk^3no - 27k^4o^2 - 8t^2m^2n + 48t^3n^2 + 48t^2ln^2 + 16tkmn^2 + 16k^2n^3 -$

$72t^3mo - 72t^2lmo - 96tkm^2o + 144t^2kno + 144tklno - 72k^2mno + 216tk^2o^2 + 216k^2lo^2 + 16m^2n^2 - 64tn^3 - 64ln^3 - 64m^3o + 288tmno + 288lmno - 432t^2o^2 - 864tlo^2 - 432l^2o^2$.

Next substitution $(k^2-4l+s)-4t = 0$, $(k^2-4l+s)^2-64n-u = 0$, $k(k^2-4l+s)+8m-v = 0$, $128o - w = 0$ together with elimination of k, l, m, n, o, t in the ideal L

```
Use R::=Q[u,v,w,k,l,m,n,o,t,s];
L:=Ideal(Q,k^2-4l+s-4t,(k^2-4l+s)^2-64n-u,k(k^2-4l+s)+8m-v,128o-w);
Elim(k..t,L);
```

gives the final result

$$u^3s + u^2v^2 - 16v^3w - 18uvws - 27w^2s^2 = 0 \tag{19}$$

which is the formula of Robbins.

Similarly we proceed in the case of a non-convex cyclic pentagon. Also in this case we get the same result (19) [11].

Remark:

1. Notice that the formula (19) is of the 7th degree in $s = 16p^2$, where p is the area of a pentagon. This means that there exist at most *seven* cyclic pentagons with given side lengths a, b, c, d, e and different radii.

2. If we put for instance $e = 0$ in the pentagon $ABCDE$ then it becomes a quadrilateral and the formula (19) transforms into the formula (15) for a quadrilateral.

3. As far as I know, the explicit formula for the area of a cyclic n-gon exists for $n = 3, 4, 5, 6, 7, 8$, see [5].

5 Derivation of locus equations

The method of derivation can be also used to determine the locus equations of a motion whose geometric description is given, see [14], [15].
Searching for loci of points forms one part of the geometry seminar which I lead for several years at the University of South Bohemia. The reason why this topic is included into the seminar is natural. In practice we often meet situations in which we are to determine a trajectory of a point by a given motion. Another reason is that searching for loci belongs to the most difficult parts of a school curricula. To describe derivation of locus equations orderly, we start with the following problem:

Let ABC be a triangle with the given base AB and the vertex C on a line k. Find the locus l of the orthocenter G of ABC when C moves on the line k.

First we will use dynamic geometry system GeoGebra

- to demonstrate the problem,
- to find the locus.

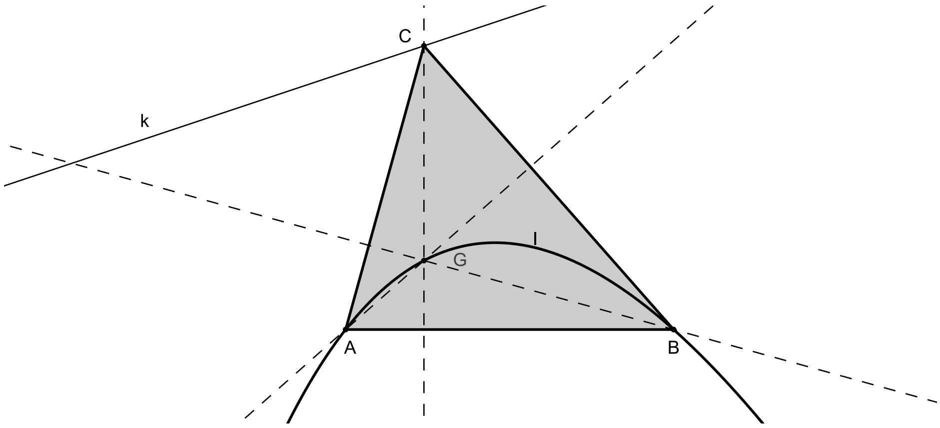

Figure 5: When C moves on k then G moves on a curve similar to parabola

Demonstration in GeoGebra supports students' activity and helps students familiarize with the problem. When we interactively move the vertex C along the line k we see that the orthocenter G moves along the curve which is similar to parabola, Fig. 5. To verify this we use the icon Locus to form the locus of C. Another position of the line k gives a curve which is similar to hyperbola, Fig. 6.

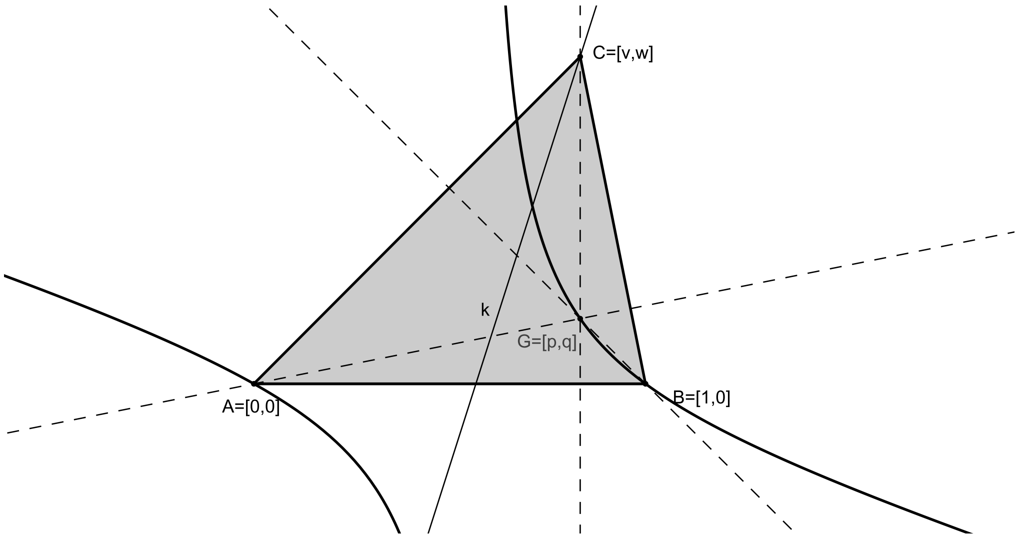

Figure 6: When C moves on k then G moves on a curve similar to hyperbola

We can conclude that the locus is probably hyperbola or parabola. To decide this we will use computer algebra system.
Let us place a rectangular coordinate system so that $A = [0, 0]$, $B = [1, 0]$, $C = [v, w]$, $G = [p, q]$ and let k be an arbitrary line with the equation $k : ax + by + c = 0$, Fig. 6. We translate the geometry situation into the following set of polynomial equations:

For the intersection $G = [p, q]$ of heights h_{AB} and h_{BC} it holds:

$G \in h_{AB} \Rightarrow h_1 : p - v = 0,$

$G \in h_{BC} \Rightarrow h_2 : (v-1)p + wq = 0.$

Further

$C \in k \Rightarrow h_3 : av + bw + c = 0.$

We get the system of three equations $h_1 = 0$, $h_2 = 0$, $h_3 = 0$ in variables v, w, p, q, a, b, c, where a, b, c are independent variables, whereas v, w, p, q are dependent variables. To find the locus of $G = [p, q]$ we have to eliminate variables v, w in the ideal $I = (h_1, h_2, h_3)$ to get a relation in p, q which depends on a, b, c. In CoCoA we enter

```
Use R::=Q[abcvwpq];
I:=Ideal(av+bw+c,p-v,(v-1)p+wq);
Elim(v..w,I);
```

and get a polynomial C which leads to the equation

$$C(p,q) := bp^2 - apq - bp - cq = 0. \tag{20}$$

We can suppose that $(a, b) \neq (0, 0)$ since in this case the line k is not defined. Then (20) is the equation of a conic $C(p, q) = 0$.
The cases $k = h_{AB}$, $k = AC$, or $k = BC$ lead to singular conics which consist of two intersecting lines which we will not depict.

Considering regular conics we get *two* cases:

If $k \parallel AB$ the locus $C(p, q) = 0$ is a parabola with the vertex $[1/2, -b/(4c)]$ and a parameter $|c/(2b)|$, Fig. 7.

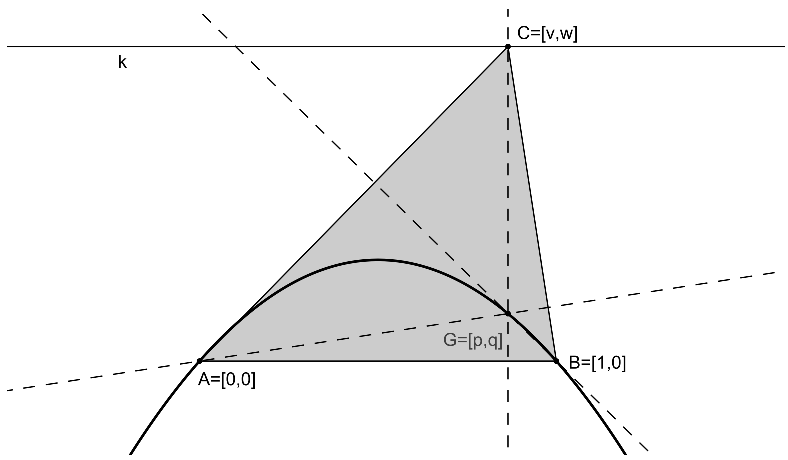

Figure 7: If C moves on $k \parallel AB$ then G lies on parabola

If $k \nparallel AB$ we obtain a hyperbola centered at $[-c/a, -b(a + 2c)/a^2]$ with one asymptote perpendicular to AB and the second asymptote perpendicular to the line k, Fig. 6.

In the given example we see that the use of DGS does not suffice to determine a curve exactly and that CAS was needed.

Remark:

The loci above were found by algebraic and computer tools. It would be interesting to find a classical geometric proof.

The next example shows that the locus can be an algebraic curve of higher degree.

Let ABC be a triangle with the given side AB and the vertex C on a circle k. Find the locus l of the orthocenter G of ABC when C moves on k.

First we construct the triangle ABC with the point C at the circle k in GeoGebra. Using a window Locus we construct the locus of the orthocenter G when C moves along k.

In the second step we derive the locus equation by CAS. We will use the same notation as in the previous case. Additionally let k be a circle centered at $B = [a, 0]$ with the radius $r = a$, Fig. 8. The situation is described by the following system of equations:

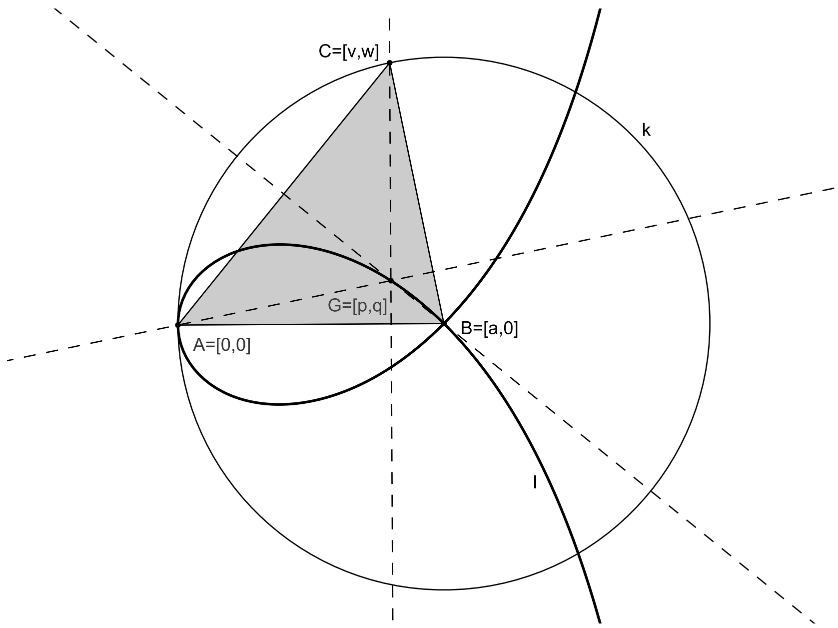

Figure 8: When C moves on k then G moves on the strophoid

$G \in h_{AB} \Rightarrow h_1 : p - v = 0,$

$G \in h_{BC} \Rightarrow h_2 : (v - a)p + wq = 0.$

The equation of the circle k is $(x - a)^2 + y^2 - a^2 = 0$ then

$C \in k \Rightarrow h_3 : (v - a)^2 + w^2 - a^2 = 0.$

Elimination of v, w in the system $h_1 = 0$, $h_2 = 0$, $h_3 = 0$ gives in the program Epsilon

```
with(epsilon);
U:=[p-v,(v-a)*p+w*q,(v-a)^2+w^2-a^2]:
X:=[p,q,v,w]:
CharSet(U,X);
```

the equation

$$p^3 + pq^2 - 2p^2a - 2q^2a + pa^2 = 0 \tag{21}$$

which is the equation of a cubic curve called *strophoid* [13], Fig. 8. (21) is usually written as

$$(p^2 + q^2)(p - 2a) + pa^2 = 0.$$

The strophoid or more exactly the right strophoid has many interesting properties [3], [13], [4]. One of them is as follows:

Let k be a circle which is tangent to a given line r at the point B. Denote by s a line which is perpendicular to the tangent r at B and let u be a line through a given point A on r and the center S of k. Determine the locus of the intersections P of the circle k and the line u when S moves along the line s.

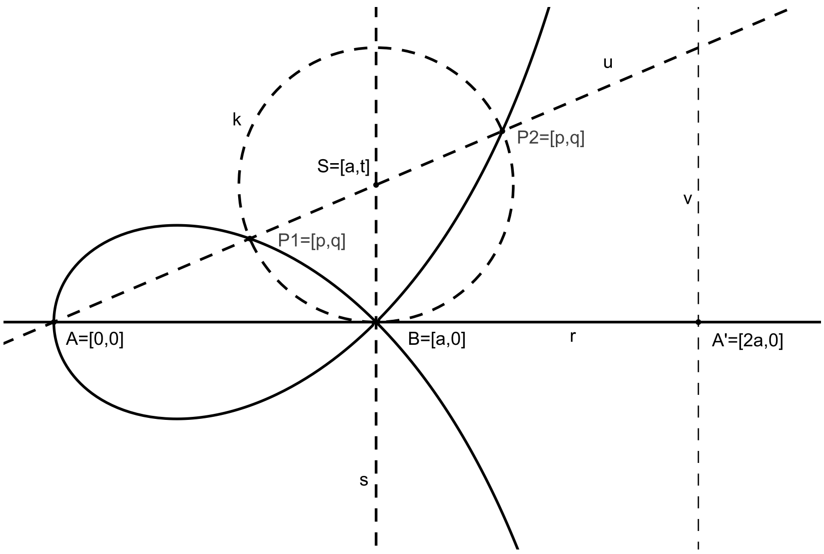

Figure 9: Strophoid

We will derive the locus by elimination of suitable variables in a coordinate system. Adopt a rectangular coordinate system so that $A = [0, 0]$, $B = [a, 0]$, $S = [a, t]$, $P = [p, q]$, Fig. 9. Algebraic description is as follows:

$P \in k \ \Rightarrow h_1 := (p - a)^2 + (q - t)^2 - t^2 = 0,$

P, S, A are collinear $\Rightarrow h_2 := \begin{vmatrix} p & q & 1 \\ a & t & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.$

Elimination of t in the system $h_1 = 0$, $h_2 = 0$ in Epsilon gives

```
with(epsilon);
U:=[(p-a)^2+(q-t)^2-t^2,p*t-q*a]:
X:=[p,q,a,t]:
CharSet(U,X);
```

the equation

$$p^3 + pq^2 - 2p^2a - 2q^2a + pa^2 = 0 \tag{22}$$

which is (21). We get the same locus as in the previous case.

Remark:

We observe that the strophoid is bounded from the right by the asymptote v which is perpendicular to r and goes through the point $A' = [2a, 0]$, Fig. 9.

The fact that we obtained the same curve as the locus of *two* motions means that the strophoid has two following properties — it is the locus $\mathcal{L}$ of the orthocenter of ABC when C moves on a given circle k_1, and it is also the locus $\mathcal{M}$ of the intersection P of the line AS and the circle k_2. Let us prove it classically! Fig. 10.

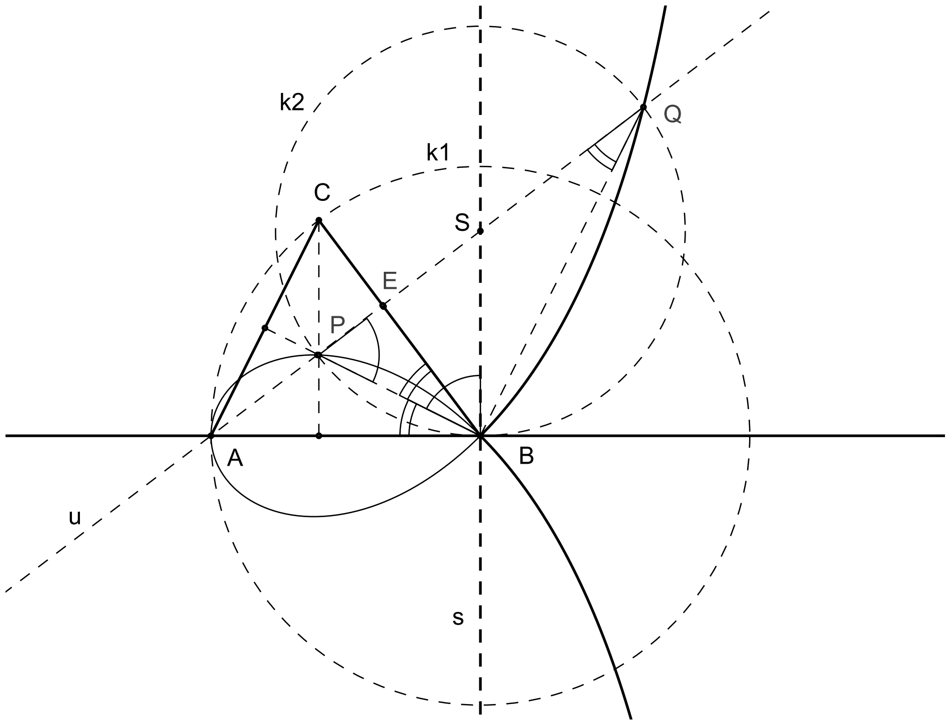

Figure 10: Strophoid as two loci

We will prove the following theorem:

Theorem: *Let $\mathcal{L}$ and $\mathcal{M}$ are as above. Then*

$$\mathcal{L} \;=\; \mathcal{M}. \tag{23}$$

Proof: First we prove that $\mathcal{M} \subseteq \mathcal{L}$. Let P be an arbitrary point of a strophoid which is the intersection of the line AS and the circle k_2. Construct the triangle ABC with the orthocenter P. We are to show that $C \in k_1$, Fig. 10. By the theorem of Thales the triangle PBQ is right from which

$$|\angle BPQ| + |\angle PQB| = 90^\circ \tag{24}$$

follows. Triangles PBE and PQB are similar which implies $|\angle PQB| = |\angle PBE|$. Since the triangle PBS is isosceles with $|PS| = |BS|$ we get $|\angle BPQ| = |\angle PBS|$. As

$$|\angle ABP| = 90^\circ - |\angle PBS| \tag{25}$$

then from (24) we get

$$|\angle ABP| = |\angle PBE| \tag{26}$$

and the triangle ABC is isosceles with $|AB| = |BC|$. Hence $C \in k_1$.

Conversely suppose that $P \in \mathcal{L}$. We are to prove that $P \in \mathcal{M}$. Let P be the orthocenter of ABC with C on k_1. Denote by s the line which is perpendicular to AB through B and let S be the intersection of s and AP, Fig. 10. As A, C lie on k_1 then ABC is isosceles with $|AB| = |BC|$. This implies $|\angle ABP| = |\angle PBE|$. From the right triangle PBE we get the equality

$$|\angle BPE| + |\angle PBE| = 90^\circ. \tag{27}$$

Since $|\angle ABS| = 90^\circ$ then from (27) the relation $|\angle PBS| = |\angle BPE|$ follows and PBS is isosceles with $|SP| = |SB|$. Similarly we prove that the triangle BQS is isosceles. This implies the equality $|SB| = |SQ|$. Hence $|SP| = |SB| = |SQ|$ and P, B, Q lie on a circle k_2. The theorem is proved.

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