Tilings, Patterns and Technology

Ma. Louise Antonette N. De Las Peñas
mlp@math.admu.edu.ph
Department of Mathematics, Ateneo de Manila University
Philippines

Angela Fatima H. Guzon
afguzon@math.admu.edu.ph
Department of Mathematics, Ateneo de Manila University
Philippines

Abstract: In this paper we discuss various situations where tilings and patterns, with the aid of technology, facilitate the teaching of mathematics and serve as tools in understanding and developing new mathematical ideas. We also illustrate how technology makes possible cultural connections in the study of mathematics using Islamic tilings and patterns.

1. Introduction

The theory of tilings and patterns has been an interesting field of study by mathematicians for more than 100 years. It includes ideas from various areas of mathematics such as geometry, algebra, topology and number theory. Tilings and patterns are very rich resources both in the teaching and discovery of mathematical ideas. It has facilitated the link of mathematics to other disciplines such as physics, chemistry, crystallography, art and architecture. The emergence and development of technological tools the past years has facilitated new and more exciting dimensions in this particular branch of mathematics. In this paper, we expound on the following various aspects of the theory of tilings and patterns.

First, we show how tilings and patterns, with the aid of technology, can be used as tools in teaching and learning algebraic and geometric concepts, and in the development of critical thinking. We give varied examples we have used in our teaching involving the use of dynamic geometry software, online interactive software and animated demonstrations.

In the past years we have seen new development in tiling theory such as the emergence of non-euclidean tilings, and tessellations in higher dimensions. We present briefly the roles of technological tools in understanding the deeper mathematical ideas presented by these tilings and in facilitating the solution and proof of challenging mathematical problems suggested by tilings.

In the third part of the paper, we focus on Islamic patterns and tilings. We give particular situations how technology has facilitated the use of these designs in our teaching. Lastly, we highlight the role of technology in describing the cultural connections of Islamic tilings and patterns to the mathematics being discussed.

2. As a Teaching Tool

A *tiling* has been defined mathematically as a countable family of closed sets called *tiles* that cover the plane without gaps or overlaps [14]. Due to its inherent structure, tilings have been often used to represent and exhibit geometric concepts such as when teaching students basic ideas on symmetries and geometric transformations.

One approach is to have students experience the meaning of these abstract ideas by construction, through the use of dynamic geometry software. By creating and replicating geometric designs using various symmetries applied to a basic motif, the students can visualize the effect of each transformation. This also allows the students to recognize the different symmetries present in the tiling. For instance, consider the 3·6·3·6 tiling consisting of regular hexagons and equilateral triangles shown in Figure 1. One possibility to obtain the tiling is to start with a regular hexagon and apply 180° rotational symmetries about its vertices to obtain copies of the hexagon. Then the triangles may be drawn by connecting the vertices of the hexagons (See Figure 2).

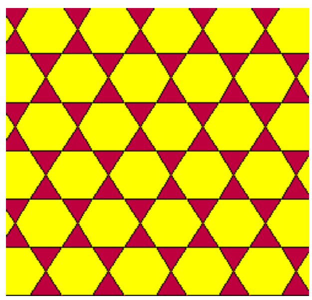

Figure 1: The 3·6·3·6 tiling

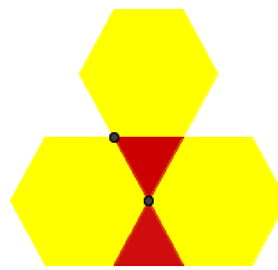

Figure 2: A part of the tiling with centers of 180° rotation

This teaching strategy develops students' critical thinking skills through the processes they undergo in constructing the tiling using the basic motifs (hexagon and triangle, in this case) with the aid of technology that serves as a scaffold that facilitates the construction and reconstruction process. Recognizing the different symmetries in the tiling requires an understanding of the connections among the properties of the geometric figures involved. Reflective thinking is developed when students make adjustments (when deemed necessary) to come up with the tiling, seeing the symmetries and using correct transformations. They must also have a disposition of openness to see different possible constructions that can be made based on different possible interpretations of the motifs. Note that the example above showed only one way of constructing the tiling. In addition, allowing students to present different ways of constructing the tiling coupled with clear explanations of their analysis of how each tiling was produced will further enhance their critical thinking skills.

In the past years we have seen the emergence of interactive software which generates tilings and patterns designed to facilitate the understanding of geometric transformations, hand-in-hand with concepts in group theory. Using this software when teaching students higher algebra for example, eases the abstraction of the course, and provides the platform for a less intimidating learning environment.

For instance in Figure 3 we present screen dumps from the activities we have carried out with *Kaleidomania* [25], in the teaching of abstract algebra. These activities highlight the connection of tilings and patterns with group structures, via their symmetry groups. The set of distance preserving transformations of the plane that sends a tiling/pattern to itself forms a group under composition of functions, and is called the *symmetry group* of the tiling/pattern. The pattern on the left has 11-fold rotational symmetry about the center of the pattern, and this symmetry generates the symmetry group of the pattern, the cyclic group of order eleven, C_{11}. On the other hand, the pattern on the right has 6-fold rotational symmetry about the center of the pattern, as well as mirror reflections with axes shown. These symmetries make up the dihedral group of 12 elements, D_6, which is the symmetry group of the pattern. The 6-fold rotation together with one of the reflections, form a set of generators for the symmetry group. Observe that each pattern has been generated from the corresponding tiling to its right, by specifying a fundamental triangle, to which the respective symmetries are applied. These activities were designed primarily to allow students to visualize characteristics of cyclic groups and dihedral groups, including that of their generators and relations.

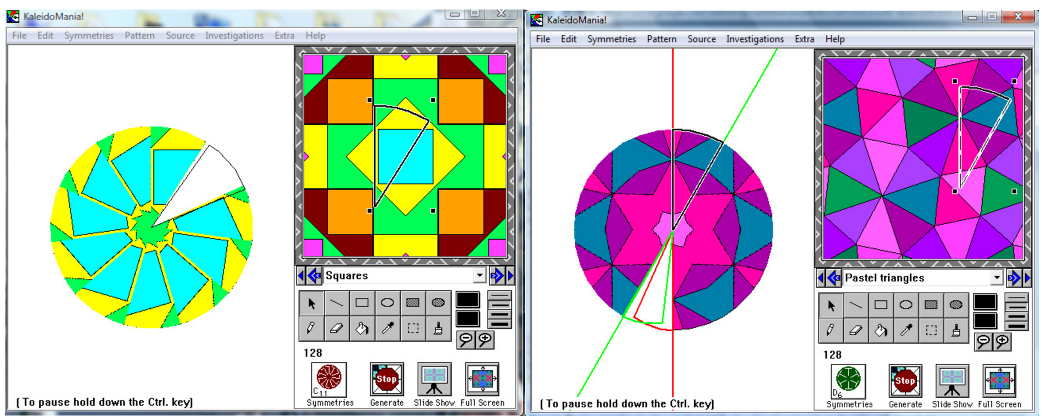

Figure 3: Patterns with symmetry groups a cyclic and dihedral group
(Outputs produced using *Kaleidomania* [26])

The use of technology in these activities to present a visualization of the abstract concepts develops students' appreciation and greater interest for the subject matter. This multi-representation of the concept allows students to make connections with geometric topics such as transformations and symmetry, and leads them to a deeper understanding of these previously-learned concepts. Being able to recognize the connection of cyclic and dihedral groups to particular patterns, and understanding the generators and relations of these groups through the visual representation, brings about a deeper understanding of these concepts, which can prepare students in learning more advanced concepts using higher levels of thinking. For instance, students would have to be able to read and comprehend proofs, as well as to construct them in the language of abstract algebra.

Escher Web Sketch [20] is an online interactive software that generates tilings based on an infinite symmetry group and can be used as a tool in teaching both abstract algebra and modern geometry. The software allows for a tiling to be generated by applying a motif on a unit cell, whose images under a specified symmetry group cover the plane (see Figure 4 for a sample output). By varying the symmetry groups and analyzing the tilings that result, students are able to understand more clearly the symmetries present in each of the 17 plane symmetry groups and distinguish one group from another. Experimenting on the symmetry groups also reveals that the possible n-fold rotational symmetries occur only when n = 1, 2, 3, 4 and 6 (commonly called the crystallographic restriction); n = 5, 8 or 12, for instance, are called forbidden symmetries for Euclidean patterns.

These investigations develop students' abilities to interpret the tiling in relation to the symmetry group and to analyze the symmetries and transformations occurring on the tiling. These will help the students develop the capability to classify planar patterns according to their symmetry group. Moreover, the explorations give the students opportunities to make conjectures on the possible symmetries that may occur and give explanations to prove or disprove such conjectures. These critical thinking skills can be developed and practiced through such activities to prepare students for a more advanced study of group theory. Here, technology is used as an aid in the exploration of the mathematical concepts.

The software may also be used to supplement the discussion on the derivation of the 17 groups as the only two dimensional symmetry groups in the Euclidean plane.

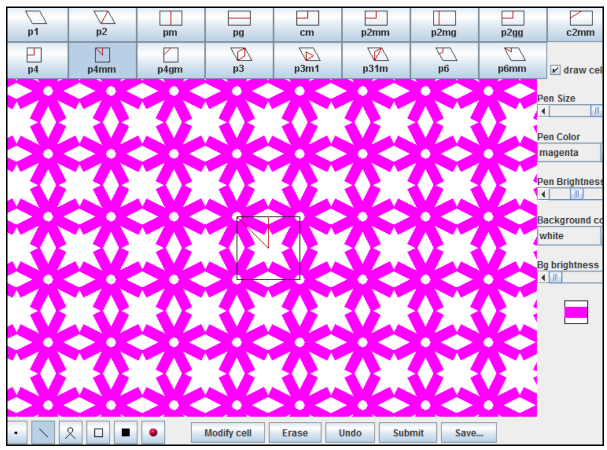

Figure 4: Tiling with symmetry group *p*4*m*, with the unit cell shown (Output produced using *Escher Web Sketch* [20])

The Wolfram Demonstrations project site [34] has a huge collection of animated demonstrations involving a wide range of mathematical topics. There are interesting examples for use in teaching abstract algebra, one of which involves the use of colored patterns to illustrate the concept of group actions, orbits and stabilizers. The idea is to click any of the 16 squares to change its color. For a particular coloring, the images under the group action on the coloring are displayed, as are the orbits of colors and stabilizer. The group can be a choice of either a dihedral group of order 8 or a cyclic group of order 4, acting on either two, three, or four colors. A still image of the animation is presented in Figure 5.

By analyzing the different colored patterns that arise after varying the groups and the color assignments to the squares, the concepts on orbits and group actions are better understood. The use of technology in such activity is two-fold: it allows a geometric representation of the abstract concepts through the use of colored patterns and it facilitates the exploration process that leads to a deeper understanding of the concept. Each colored pattern represents the resulting image of the action under a group element. The fact that technology can generate these images with ease allows the students to observe group properties through the colored patterns that result from the group action. By also giving representations to the colors (e.g. a color represents an atom or molecule in a given crystal) the students will appreciate the connections of the concepts to real world situations. In future studies this will help the students integrate such mathematical concepts with other fields of study.

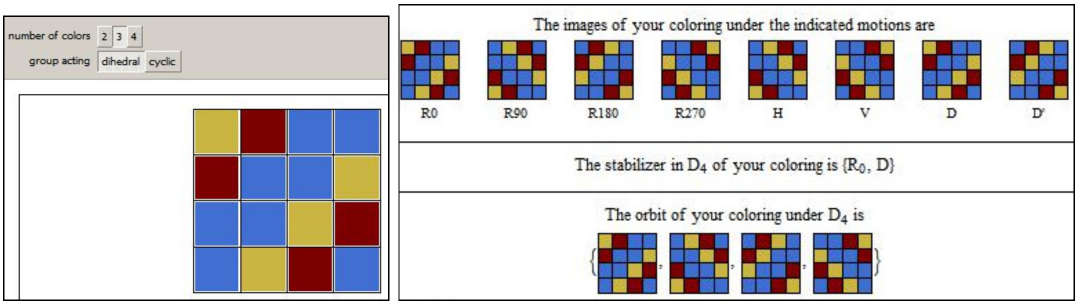

Figure 5: Brodie's *Orbits and Stabilizers of Groups Acting on Colorings of* 4x4 *Checkerboards* (Still image from Wolfram Demonstration's Project [28])

3. As an aid in Research

The mathematical theory behind tilings is very rich and to this day, still involves challenging and unsolved problems. New developments on non-Euclidean tilings and tessellations in higher dimensions for instance, have posed interesting questions on the mathematical structures corresponding to these tilings such as symmetry groups and color groups. Technology has found its mark in the quest to find answers to these questions either through computational and approximation methods, and exploratory techniques to visualize the different aspects of the problems. In this part of the paper, we give some examples on the different technological approaches that have facilitated our research involving tilings.

Figure 6: The 5·3·5 ·3 ·5·3 tiling superimposed on a modification of Escher's *Circle Limit III* [12]

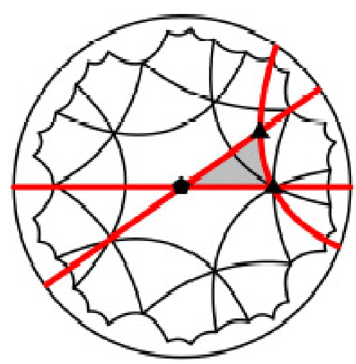

Figure 7: Finding symmetries of the 5·3 ·5·3 · 5 ·3 tiling on a fundamental triangle [7]

In [7,9,10], we have addressed the problem of characterizing the symmetry groups of semi-regular tilings on the hyperbolic plane. By a *semi-regular* $p_1 \cdot p_2 \cdot \ldots \cdot p_q$ *tiling* we mean an edge to edge tiling having regular polygons as its tiles, with a p_1-gon, a p_2-gon,…, and a p_q-gon surrounding each vertex in cyclic order, and satisfying the additional property that the symmetries of the tiling act transitively on its vertices. The tiling presented in Figure 1 is an example of a semi-regular tiling. In our study, we used computer generated hyperbolic patterns obtained from semi-regular tilings, to provide insights on the interesting properties of their corresponding symmetry groups. Dunham's beautiful computer modification of the Dutch artist Escher's *Circle Limit III* for example, shown in Figure 6, depicts the semi-regular 5·3·5 ·3 ·5·3 tiling exhibited in the Poincare model of hyperbolic geometry [12]. The group of symmetries for this tiling is of type *533, characterized by its generators which are the reflections with axes passing through the sides of a fundamental triangle, a 5-fold rotation, and two 3- fold rotations with centers on vertices of a fundamental triangle as highlighted in Figure 7. In the colored pattern, the 5-fold and 3-fold rotational symmetries have centers where the fins and the mouths of the fishes meet, respectively.

Among the challenges encountered in answering the question of determining the symmetry groups of hyperbolic semi-regular tilings was the fact that the hyperbolic symmetry groups are infinite groups consisting of an immense variety of isometries and an infinite number of such groups exist. It was important to experiment on as many tilings as possible to be able to come up with generalizations on the symmetry groups given a class of semi-regular tilings. The software *Mathematica* with its *L2Primitive* and *Tess* add-on packages [31] provided access to different classes of semi-regular tilings with symmetry groups of varying properties and facilitated exploration of tilings on the Poincare' model before the formulation of the proof on the symmetry group of the uncolored tiling. The result on the symmetry group for the 3·4 ·3 ·4· 3 ·4 tiling for instance, generalizes to that of the $p \cdot 3 \cdot p \cdot 3 \cdot p \cdot 3$ family of tilings, and is given to be *p33. The methodology of arriving at the proof with the aid of *Mathematica* was discussed extensively in [9]; see Figure 8 for a sample semi-regular tiling generated in the process. Alternatively, investigations on these tilings can also be carried out via applets readily available from the web which generates hyperbolic regular and semi-regular tilings, such as Don Hatch's *Hyperbolic Tessellations* [23], and Jeff Weeks' *Kaleidotile* [26]. A sample output of the latter is given in Figure 9.

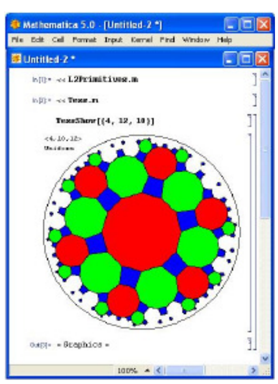

Figure 8: An output of a 4 · 12 · 10 tiling (Produced using Mathematica [31])

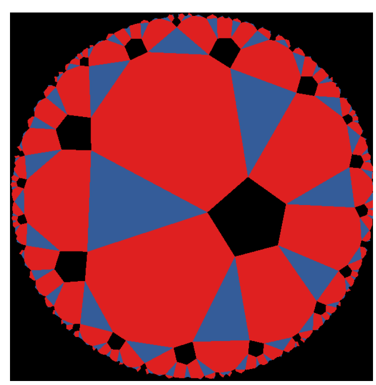

Figure 9: Week's *Kaleidotile* tesselations applet [26]

In the three-dimensional case, instead of tilings by polygons there exist tilings by polyhedra, called *honeycombs*, where the polyhedra are fitted together to fill space, so that every face of each polyhedron belongs to exactly one other polyhedron. Geometric properties, symmetry groups and subgroup structures of these tilings in hyperbolic space, as well as Euclidean and spherical counterparts, have also been studied hand-in-hand with technological tools.

In [8], we worked on the open problem of finding the low index subgroups of three dimensional hyperbolic groups. A framework was presented in obtaining subgroups of hyperbolic groups, and the approach was geometric in nature, by considering these groups as symmetries of three dimensional honeycombs and tilings in hyperbolic space, and applying concepts in color symmetry theory. The first step was to assume a fundamental polyhedron, (hyperbolic tetrahedron), and to consider the group generated by reflections about the faces of the given tetrahedron, called the tetrahedron group G. Specifically, in determining the subgroups of G, we used the correspondence between the index n subgroups of G and n-colorings of the tilings where the elements of G effect a permutation of the n colors and G is transitive on the set of n colors. When the tetrahedron is an orthoscheme, that is, a tetrahedron whose faces may be colored so that two that are not consecutive are orthogonal, the group G is the symmetry group of a regular honeycomb in hyperbolic three-space.

During the initial phase of the work, in attempting to arrive at an appropriate approach in determining subgroups of three dimensional hyperbolic groups - regular honeycombs and their symmetry groups were studied in detail. It was important to visualize the connection between the structure of the groups and the corresponding three dimensional tilings. A useful tool in the study of the symmetry properties of three-dimensional honeycombs includes Jeff Weeks' *Curved Spaces* [19] (see Figure 10). Curved Spaces is a flight simulator for multi-connected universes. This program allows the user to "fly through" spaces such as the three-dimensional spherical, Euclidean, and hyperbolic space filled with certain polyhedra, allowing the study of various properties of honeycombs. The approach is looking at the tiling from inside the space. This allows a clear visualization of the local structure of the tiling. Other ways to visualize hyperbolic 3-space are through the videos *Not Knot* [36], which is a guided tour through hyperbolic space (see Figure 11 for a still image) and *Hyperbolic Space Tessellated Through Dodecahedrons* [35], among others.

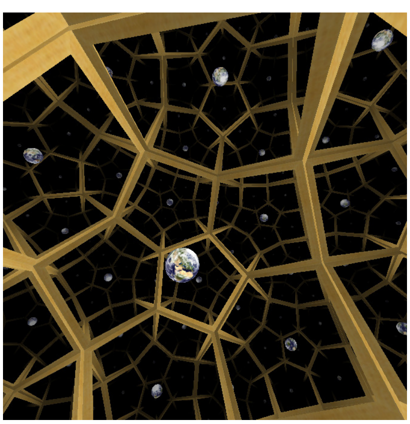

Figure 10: Weeks' *Curved Spaces* [19]

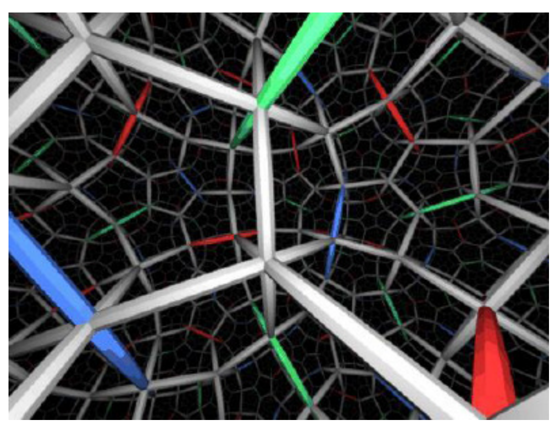

Figure 11 *Not Knot* [36]

Other computer generated renderings of tilings of hyperbolic space useful to our study on three dimensional hyperbolic groups have been provided by other mathematicians, such as the work of Bulatov [3]. In Figure 12 we see a regular dodecahedron, which serves as a fundamental polyhedron for a subgroup of a tetrahedron group. There are 12 generators for this subgroup, which consists of the reflections about the faces of the polyhedron. Reflecting the dodecahedron about its sides will result in a tiling in hyperbolic 3-space as shown in Figure 13 for the first and second iteration of reflections.

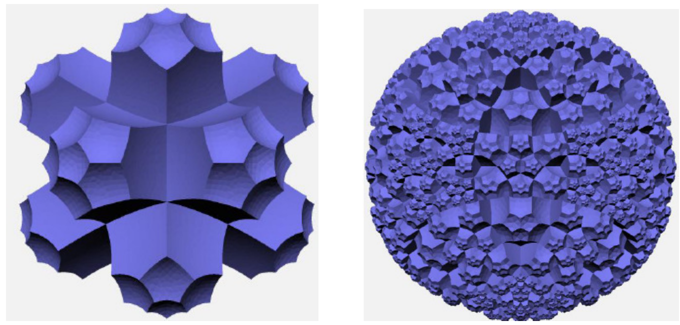

Figure 12: A regular right angled dodecahedron [3]

Figure 13: Tiling of hyperbolic 3-space by a regular right angled dodecahedron [3]

Finally, the still picture shown in Figure 14 comes from a demonstration [22] from the Wolfram Projects Demonstration site. The application provides a visual image of the horosphere (sphere of infinite radius) packings of the {3,3,6} honeycomb. The {3,3,6} honeycomb is a tiling of hyperbolic 3-space by regular asymptotic tetrahedra. The outermost sphere is the Cayley-Klein model of hyperbolic 3-space; its points represent points at infinity. Horosphere packings are given where the centers of the balls are at the lattice points of the honeycomb. The demonstration illustrates one tetrahedral cell of the tiling with balls around its vertices.

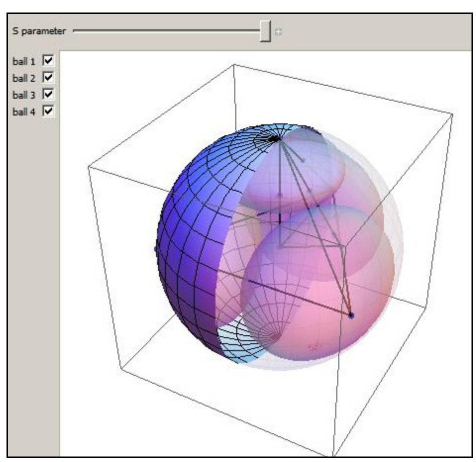

Figure 14: *Horosphere Packings of the* {3,3,6} *Coxeter Honeycomb in Three Dimensional Hyperbolic Space* (Still image from Wolfram's Demonstration Project [22])

4. Islamic Tilings and Patterns

In this part of the paper, our focus of attention will be Islamic tilings and patterns. These geometric objects are reflective of highly symmetric Islamic art and adorn buildings such as mosques, mausoleums and tombs in regions throughout the Islamic world. Geometric and group theoretic concepts combined with computer software are tools that allow for creating modern day variations of these historical tilings, making these valuable resources of instruction and research in mathematics. For example, Abas and Salman, have presented methods for the computer generation of Islamic patterns [1], Kaplan has designed a program to draw Islamic star patterns [15, 29] (Figure 18), and Dunham has rendered hyperbolic versions of these patterns [11](Figure 19).

Reforms in mathematics education have advocated realistic and contextual avenues in the teaching and learning of mathematics. In recent years, mathematics educators have investigated teaching methodologies for introducing mathematics with the aid of cultural connections. Islamic tilings and patterns are very rich resources when adding a cultural dimension to the mathematics being discussed. These provide a springboard for developing theoretical ideas of construction, transformation geometry and group theory. Such connections spark students' interest and participation in the learning process. Thus, teachers and educators are able to raise students' level of thinking and analysis as well as their openness to discover new ideas and their creativity towards new constructions and connections.

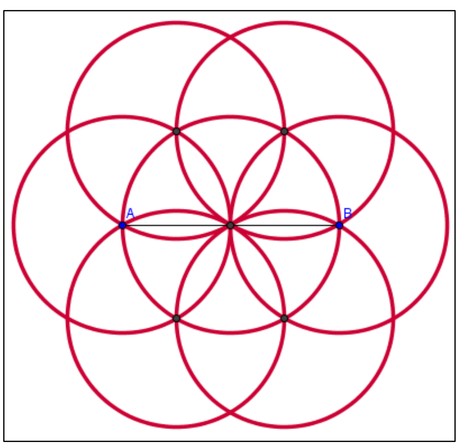

Figure 15: Construction of 6-point rosette (Plot produced using Geogebra [16])

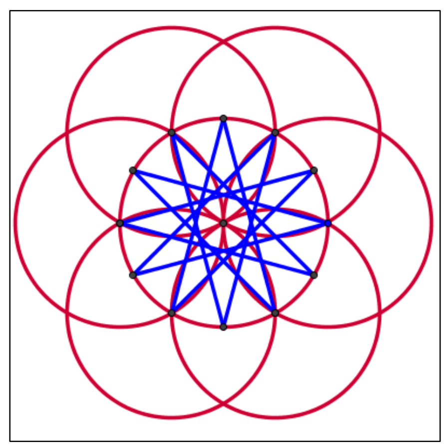

Figure 16: Construction of 12-points star (Plot produced using Geogebra [17])

Repetitive interlaced patterns, rosettes and stars are common features of Islamic designs, and serve as interesting examples of geometric construction for use in the classroom. For example, in Figures 15 and 16, we present plots from the activities ***creating_rosette.html*** [16] and ***creating_star.html*** [17] respectively, where students explore the construction of a rosette and a 12-point star. The starting point would be to construct the segment *AB*, and then using the points *A*, *B* and the midpoint *M* of *AB*, students construct 7 intersecting circles of equal radius. In the case of the rosette, the centers of the 7 circles will serve as the tips of its petals. For the 12-point star, the students will have to construct the remaining points on the inner circle, to obtain 12 equally spaced points on the circle. Then explorations are carried out on how the star is constructed. For instance, the 12-point star can be obtained by connecting every fifth point on the circle. Further investigations will allow students to make generalizations on the relationship between the number of points on the star, and the *k*th-point being connected to form the star. This investigative approach provides the students with a stronger foundation of the basic concepts, at the same time exposes them to advanced ideas present in higher mathematics.

The group theoretic basis for the construction of stars illustrates the high level of mathematics involved in Islamic art. In ***cyclic.nb*** [18] (see Figure 17), we give outputs of the *Mathematica* package, which we have used in the teaching of abstract algebra, particularly to illustrate properties of the cyclic group of order n, Z_n. Using *Mathematica*, the elements of Z_n are presented as numbers on an n hour clock where the nth element is viewed as being equivalent to zero. Addition of two numbers under $+_n$ in Z_n will be treated like adding numbers on the clock. The students explore the generators of Z_n - these generators are elements a^k such that $(k,n) = 1$. Moreover, students can also visualize the construction of a star of n sides and points from a generator a^k of Z_n where $\gcd(n,k) =1$ and $1< k < n-1$. The order in which the elements of Z_n will be generated is by multiples of k, namely k, $2k$, $3k$,… If we connect every kth point as we go through the elements of the group, a star of n sides will be formed. The table in Figure 17 for instance shows the order of all elements of Z_8 with the generators 1, 3, 5 and 7. The generators 3 and 5 give rise to a star of 8 sides and 8 points. The students can also explore that k and $n-k$ gives rise to the same star. Note that this concept also relates directly to the construction of the star in Figure 16 where connecting every 5th point gives rise to a 12 sided and 12 point star. Such relationship between geometry and abstract algebra allows students to appreciate the connections between and among different areas of mathematics. The visual representation of the cyclic group Z_n given by the construction of the n-point star is an interesting motivational tool that can be used in the teaching of such abstract concepts. This activity also suggests a profound appreciation of the mathematics involved in the construction of Islamic star patterns.

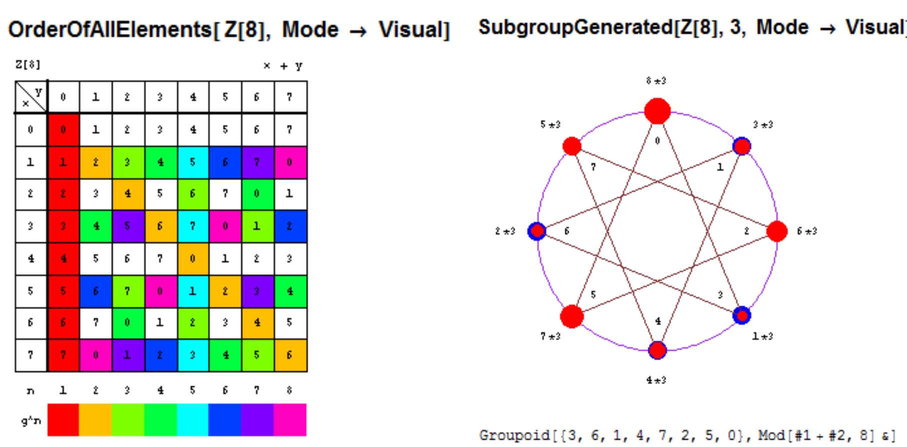

Figure 17: Order of elements of Z_8 and the construction of an 8-pointed star
(Output produced usinng *Mathematica* [18])

Another technological tool that can facilitate the design and rendition of Islamic tilings and patterns is the online interactive software called *Taprats* [29]. Using rosettes and stars as motifs, tilings may be rendered through a technique which uses regular and semi-regular tilings of the plane as basis (see Figure 18).

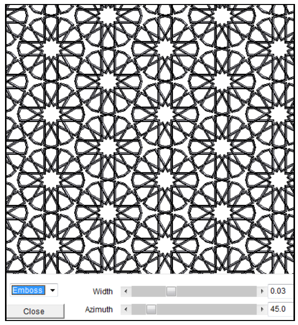

Figure 18: Islamic pattern from 3·12·12 tiling (Output produced using *Taprats* [29])

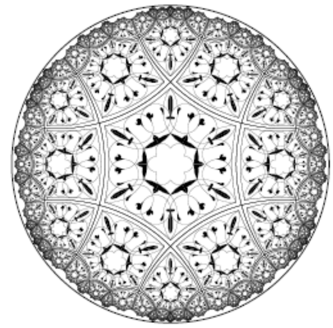

Figure 19: Dunham's hyperbolic islamic pattern [11]

The *Pattern in Islamic Art* website [32] is a rich library of resources for the teaching of symmetry alongside Islamic art and ethnography. It makes available over 4000 images of patterns and tilings obtained from regions in the Islamic world.

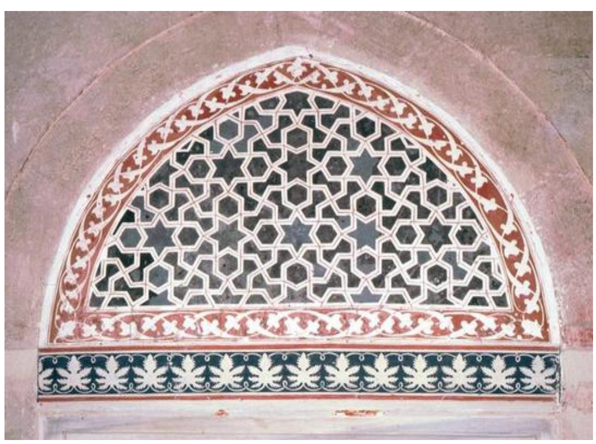

Figure 20: Islamic pattern from a mosque in Turkey [32]

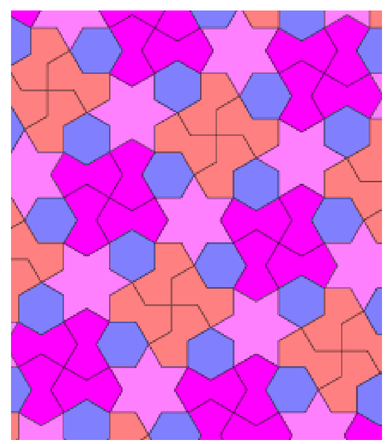

Figure 21: computer generated rendition of the pattern [33]

In Figure 20 we present an image from the website, which captures a decorative panel above a doorway pictured from the Selimye Mosque in Edirne, Turkey, dating from the Ottoman Dynasty (1574). For example, when teaching the concept on symmetries in a geometry class, the students are asked to discover the various symmetries present in the given Islamic pattern by construction, using dynamic geometry software. The students are asked to analyze the various polygons evident from the pattern, in this case, a regular hexagon, a regular 6-pointed star polygon, together with an irregular 8-sided polygon (all with internal angles a multiple of 30°) which can serve as starting tiles or motifs in replicating the Islamic pattern. By trying to understand the construction of the tiling using the dynamic geometry software, the student can develop an appreciation of how the artists have arrived at the tiling at a period such as the Ottoman dynasty. Moreover, in understanding the intrinsic symmetries of the pattern, a student may be introduced to concepts of group theory. This is a useful endeavor for students regardless of whether or not they plan to take up a more rigorous study of abstract algebra.

An interesting feature of the website is that it links up to a tiling database [33] which gives a collection of tilings and patterns together with their geometric and group theoretic properties. The version of the actual Islamic pattern in Figure 20 has a counterpart from the tiling database as an edge-to-edge tiling of four colors, shown in Figure 21. The students can make good use of this version for their constructions. This approach is also an efficient way for making students visualize concepts on group theory in an abstract algebra class. In this case the symmetry group of the tiling is of type 4*2, generated by a 90° rotation together with a 180° rotation; the latter has center lying on an axis of reflection. The group also consists of glide reflections.

The Alhambra Palace, the 15th century Moorish architectural wonder in Granada, Spain also contains many excellent examples of Islamic constructions which can be used in teaching mathematics. These constructions are very rich in history and can provide the motivation for the students' interest in the subject matter. Are the 17 plane symmetry groups represented in the tilings of the Spanish palace in Alhambra? This question has not been completely settled at this point, and various interesting views have been expressed on the subject [See for instance 2,13]. Questions such as these can jumpstart discussions in class, and capture the interest of students. Studying symmetries of the tilings and patterns present in the Alhambra form the basis of a wide range of instructional materials that can be prepared for different audiences studying symmetry. The software *Escher Sketch* [20], *Kaleidomania* [25] and *Tess* [30], among others, are technological tools which can aid students in the recognition of symmetries and identification of the symmetry group of a given planar pattern.

5. Conclusion

Mathematics teachers and educators have always been formulating strategies to make the study of mathematics more interesting and meaningful to their students. In this work we discussed tilings and patterns as rich resources in the teaching, learning and discovery of mathematical concepts. Through the properties displayed in such resources, they provide avenues for the teaching and learning of geometric and algebraic concepts. Recent advances in technology have further enhanced the use of tilings and patterns in studying mathematical ideas. Technological tools have provided visual representations of the abstract mathematical concepts that allow for learning existing mathematical concepts and discovery of new mathematics. The interactive feature of technology also provides a constructive learning environment through exploration and observation. It brings about greater interest in the study of mathematics, an appreciation of the connections between mathematics and other fields of study, and the development of higher-level thinking skills.

In this paper, we have illustrated some teaching, learning and research activities using tilings and patterns with the aid of available technological tools. In addition, we focused on Islamic tilings, as a means to link the study of abstract mathematical concepts with art, history and culture. We hope we have sparked interest and opened new directions in the endeavor of integrating mathematics with other fields of study. Different histories and cultures throughout the world contain vast resources for use in the study of mathematics and other related fields. Technological advances have and will continue to provide means to facilitate the link. The challenge is to discover such links and integrate these discoveries in teaching, learning and research activities.

Acknowledgement. Ma. Louise De Las Peñas acknowledges the Ateneo de Manila University Office of the Vice President for funding support under the Special Conference Assistance Grant.

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