Solving Problems from *Sangaku* with Technology
- For Good Mathematics in Education -

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Abstract: *Wasan (traditional Japanese mathematics) as well as their important spin-offs the Sangaku (mathematics tablets) discussed in this article are a part of Japan's unique cultural heritage. They were written on Ema (wooden votive tablets) and presented as offerings at shrines and temples in order to thank the gods.*

In the present article, I first will introduce a brief history of Wasan, with strong emphasis on the development of Sangaku, which is followed by two examples of Sangaku problems from Kon'ou Shrine and their modern mathematical solutions. Then, the procedure for drawing geometric constructions in Sangaku is explained, with the use of a graphics calculator Casio fx-9860G II and a software Casio fx-9860G II Manager PLUS which runs on the PC. Finally, one attempt to apply this procedure to mathematics education is presented.

It is my sincere hope that the elegance of Wasan's way of looking at things and way of thinking, the very essence of mathematics in Edo culture, will be brought to life once again in modern mathematics education.

1. Introduction

Western mathematics was introduced in the Japanese school system at the beginning of the Meiji Period. In order to distinguish between the mathematics of the Edo Period and Western mathematics, the mathematics of the Edo Period is referred to as *Wasan*. With the introduction of Western mathematics in Japanese schools, the number of people learning *Wasan* steadily declined. At present, with *Wasan* researchers are at the forefront, and research on *Wasan* is being actively conducted, through writings and events, *Wasan* culture is being spread to the world. In 2008, events were held on a national scale in commemoration of the 300th anniversary of the death of Takakazu Seki, a man who is referred to as a "sage of mathematics." At the same time, *Wasan* is also increasingly being reevaluated. Furthermore, one increasingly sees reports of practical lesson trials or other research which attempts to make use of *Wasan* in mathematics education.

The custom of making *Sangaku offerings*, which is discussed in this article, is a part of Japan's unique mathematics culture which began around the middle of the Edo Period. It is thought that in Edo, the political, economic and cultural center of Japan, an enormous number of *Sangaku* were dedicated as offerings. It is written in *Sangaku Shinteiki* (Yoshitaka Muraki, 1681) that *Sangaku* were being dedicated as offerings throughout Edo at the time. It has also been noted elsewhere that *Sangaku* were dedicated as offerings at *Meguro Fudo* (Ryusenji), near our school. From the publication of *Shinpeki Sanpou* (Sadatsugu Fujita, 1789), a collection of *Sangaku*, and others, we can surmise that a great number of people from all walks of life held an active interest in *Sangaku*.

However, most of the *Sangaku* dedicated in the *Ema* halls of shrines and temples have been damaged by the weathering of time or destroyed by fires and wars; to the point that only about 1,000 *Sangaku* survive intact to this day in all of Japan. Especially in Tokyo, where such a large number of *Sangaku* had been presented as offerings, for reasons such as those listed above, the number of surviving *Sangaku* is extremely small. In Iwate Prefecture, Yamagata Prefecture and other parts of the Tohoku Region and in Nagano Prefecture, there are a large number of surviving *Sangaku* and, in turn, *Wasan* research in these areas is very active.

2. *Wasan* and *Idai Keishou* (The Passing on of Difficult Problems)

When introducing *Wasan*, it is first necessary to introduce the *Wasan* treatise, *Jinkouki*. Written and published by Mitsuyoshi Yoshida in 1627, *Jinkouki*, is a mathematical treatise well-known to many people. Mitsuyoshi Yoshida, a member of the Kyoto Suminokura Clan of merchants, had studied the Chinese mathematical treatise, *Sanpou Tousou*, under the tutelage of father and son, Ryoui Suminokura and Soan Suminokura. Soan Suminokura is the man best known for publishing the great reproduction of classical writings, *Sagabon*. Yoshida, using *Sanpou Tousou* as a model, created mathematical problems which were intimately related to the realities of day-to-day life in Japan at the time, publishing them as *Jinkouki*. From its illustrations to its overall style, *Jinkouki* is an exceptional book that draws from the literary legacy of the great work, *Sagabon*. *Jinkouki* was also a lifestyle manual explaining, in both minute and considerate detail, methods for using the calculation tool called the *Soroban* (abacus), which was necessary for people in their daily lives at that time. As the abacus was coming into common use across Japan, a slew of *Wasan* treatises imitating *Jinkouki* also began to appear. Also, because these imitations of *Jinkouki* continued to be published without relent, Yoshida revised *Jinkouki* several times. In Revised *Jinkouki*, published in 1641, Yoshida included 12 mathematics problems, without answers, at the back of the book, writing, There are those out there who are teaching mathematics at the level of *Jinkouki*. People studying mathematics probably have no way of knowing whether their teacher is competent or not, so I will teach you a method for judging the ability of your teacher. I shall write here twelve problems without answers. You may judge the ability of your teacher by whether or not he can solve these problems. These problems are called *Idai* (problems left behind) or *Konomi* (favorites).

The solutions to Yoshida's *Idai* were revealed 12 years later, in 1653, in *Sanryouroku*, which was written by a young mathematician named Tomosumi Enami. In imitation of Yoshida, Enami also wrote his own set of eight *Idai* problems in the back of his book. Enami's publication stimulated other mathematicians to publish their works with solutions to the *Idai* of *Jinkouki* along their own sets of *Idai*. Examples include *Enpou Shikanki*, written by Jyushun Hatsusaka in 1657, *Sanpou Ketsugishou*, written by Yoshinori Isomura in 1659 and *Sanpou Kongenki*, written by Masaoki Satoh in 1669.

So began the mathematical question and answer "relay," in which one would publish a *Wasan* treatise with solutions to previous *Idai* while also including new *Idai* of one's own creation. This relay process is called *Idai keishou* (the passing on of difficult problems). It goes without saying that these *Idai* became progressively more difficult, while at the same time spurring the development of new kinds of mathematics operations.

It can rightly be said that *Idai keishou* contributed greatly to the development of *Wasan*.

3. The Offering of *Sangaku*

The term *Sangaku* refers to *Ema* (votive tablets) on which mathematical problems were written and which were dedicated at shrines and temples. It is said that the custom of offering *Sangaku* began in about 1660, in the middle of the Edo Period. Common people of the Edo Period dedicated *Sangaku* as expressions of gratitude to the gods for having been able to solve mathematical problems and as prayers to be able to apply themselves ever more to their studies. With shrines and temples, as centers of social discourse for common people at the time, also being places for showing one's achievements, there also appeared *Sangaku* which were dedicated with only *Idai*, without answers. About 1,000 *Sangaku* exist to this day. The custom of offering *Sangaku* is considered to be a completely unique aspect of Japanese culture for which nothing similar exists anywhere in the world.

As can also be understood from the background which produced *Sangaku*, the mathematical culture of the Edo Period was extremely advanced, had diverse contents ranging from mathematical games to fully-fledged mathematics and also offers us a wealth of content and topics which are applicable to the modern mathematics taught in schools. In the Meiji Period, Japan made the conversion to Western mathematics, so few people now know of *Wasan*. Amidst such a backdrop, however, *Wasan*, with its multitude of topics intimately related to daily life, is being reevaluated, as is the case with the application of *Wasan* in the mathematics taught in school. I would like, by all means, to apply *Sangaku* and *Wasan*, both invaluable cultural assets left to us by our ancestors, to the mathematics classroom.

4. *Sangaku* of Kon'ou Shrine

Kon'ou Shrine is located in one of the more serene areas of Shibuya and is also filled with many historical artifacts. Here, I shall introduce the existing *Sangaku* of Kon'ou Shrine, along with translations of the texts of their problems into modern Japanese and examples of methods for solving them for use in junior and senior high school classes. I shall also, as much as possible, touch upon how the problems were solved using *Wasan*.

4.1. How to Look at *Sangaku*

In general, *Sangaku* has a figure drawn on it and the figure itself is the problem.

(1) Problem text
(2) Figure
(3) Answer
(4) Explanation (*Jojutsu* formula)
(5) Date of dedication, name

The explanation/formula (4) lays out the method used to find the answer. These explanations often make use of *Jojutsu* (formulas) which are unique to *Wasan*, such as the Formula of the Pythagorean theorem, and contain parts of those formulas or their results.

4.2. Tomitarou Noguchi: The *Sangaku* of Minamoto-no-Sadanori (*Sangaku* 1)

The following *Sangaku* was dedicated with the name Minamoto-no-Sadanori in 1864. Its fan-like shape is different from the rectangular shape most commonly seen in *Sangaku*, making it a distinctive and rare example. The problem presents three circles—one large, one medium and one small—and seeks to find the diameter of the large circle when those of the small and medium circle are given.

Figure 4.1 The Photograph of *Sangaku* 1, 1864

4.1.1. *Sangaku* Problem Text

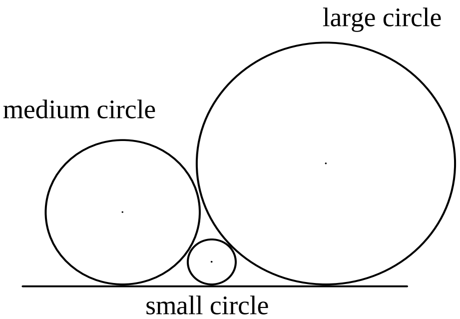


Figure 4.2 The Figure of *Sangaku* 1

如図
中圓径九寸
小圓径四寸
大圓径幾何問
答 三十六寸
術曰置中圓径除小圓径
開平方内減一箇自之以
除中圓径得大圓径合問
關流 水野興七郎門人
野口冨太郎
源 貞則
元治元 甲子年十一月吉日

Figure 4.3. The Text of *Sangaku* 1 in Japanese

As shown in the figure, if the diameter of the medium circle is 9 *sun* and the diameter of the small circle is 4 *sun*, what is the diameter of the large circle?

Answer: 36 *sun*.

Explanation (formula): first divide the segment of the medium circle by that of the small circle and take the square root of that number. Then, subtract 1 from that number and square the result. By dividing that number by the segment of the medium circle, we can find the segment of the large circle, and simplified,

$$\text{Diameter of large circle} = \frac{\text{medium}}{\left(\sqrt{\dfrac{\text{medium}}{\text{small}}}-1\right)^2}$$

The resulting number is equal to the diameter of the large circle.

Moreover, the *Sangaku* expresses the contact relationship between the large, medium and small spheres. That is, all three spheres are centered on a level plane.

4.2. The *Sangaku* of Takataka Youzaburou Yamamoto (*Sangaku 2*)

The following *Sangaku* was dedicated in 1859 by Takataka Youzaburou Yamamoto of the Saijou Domain of Iyo Province in Shikoku. It is thought that it was dedicated at Kon'ou Shrine because of the proximity of the shrine to the Saijou Domain's official residence in Edo. Also, as can also be seen from the photograph, this *Sangaku* has three problems.

Figure 4.4. The Photograph of *Sangaku 2*, 1859

4.2.3. Problem Text in *Sangaku* 1

Figure 4.5. The Problem of Left Side

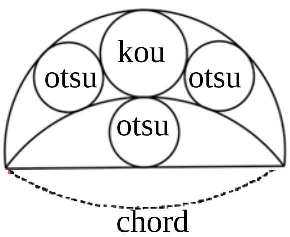


Figure 4.6. The Figure of *Sangaku* 2

（第三問）
今有如図員鋏内隔弧背容
甲員一個乙員三個甲員径五寸
乙員径四寸問弦幾何
答曰弦一十六寸
術曰以乙径除甲径内減一個
餘平方開之以除乙径倍之得弦合問
關流宗統六傳
御粥安本門人
西條藩 山本庸三郎貴隆撰
安政六年己未四月

Figure 4.7. The Text of *Sangaku* 2 in Japanese

As shown in the figure, there is a chord and four circles—one labeled *kou* and three labeled *otsu*, within a larger circle. If the diameter of *kou* is 5 *sun* and the diameter of *otsu* is 4 *sun*, then what is the length of the chord?

Answer: The length of the chord is 16 *sun*.

Explanation (formula) :first divide the diameter of *kou* circle by that of *otsu* circle and subtract 1 from that number. Then take the square root of that number. Second divide the double of the diameter of *otsu* circle by that number, and simplified,

$$\text{chord} = \frac{2otsu}{\sqrt{\dfrac{otsu}{kou} - 1}}$$

5. Explore *Sangaku* with a Graphing Calculator -Using fx-9860G II-

In many problems from *Sangaku* which include the contact relations about a circle and a line, a circle and a circle. Of course, we can draw geometric construction to the figure in *Sangaku* with a compass and a ruler, but it is difficult to draw the figure mathematically. I used Cabri Geometry for drawing the figures in this paper. Some figures are drawn approximately, it is not good from the perspective of mathematics education. Many mathematics teachers maybe think so. Therefore, many students are stuck at the entrance of the problem, and it is not easy to consider the problem going ahead.

I have been making development of mathematics teaching materials using technology in the field of algebra and analysis, and I have wanted a good technology in the field of geometry. Now I found a Casio fx-9860G II graphics calculator.

The advantage in using this calculator is that you can easily draw geometric constructions. Speaking specifically, if you want to draw on a fixed radius of a circle and its' tangent.

(1) First, draw a circle and a line loosely.
(2) Set the radius of the circle.
(3) Set to touch the circles and the line.

Drawing a circle and its' tangent requires only this operation.

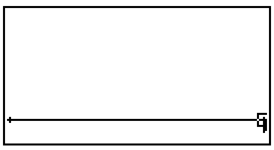
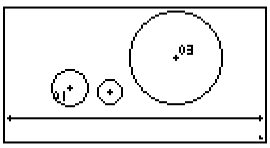
The figures below are presented in *Sangaku* 1 problem and *Sangaku* 2 problem, which are drawn by fx-9860G Ⅱ Manager PLUS.

5.1. Apply for Problem of *Sangaku* 1

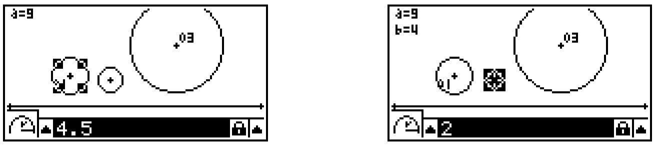
The procedure to draw the figure in *Sangaku* 1 problem with using fx-9860G II, is as follows.

<Process>

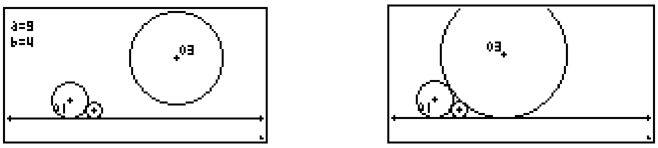
(1) Draw a line (tangent). Draw three circles on a tangent, medium one and small one and large one.

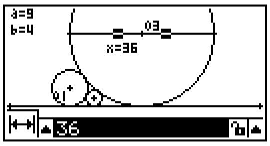
(2) Set the length, diameter of medium circle $a = 9$. Set the length, diameter of small circle $b = 4$.



(3) Set three circles contact onto tangent, and set three circles to contact each other.



(4) fx-9860 shows the length, diameter of large circle $x = 36$. Completed.

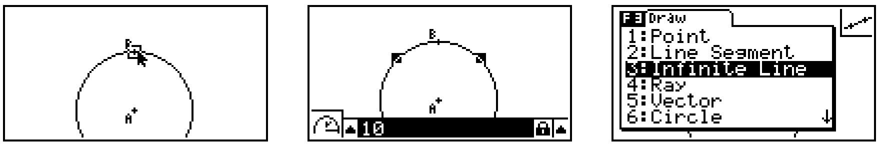


5.2. Apply for Problem of *Sangaku* 2

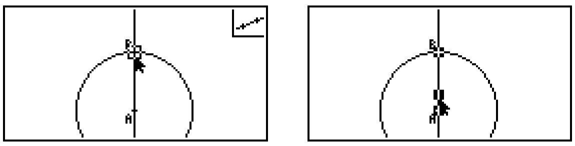
The procedure to draw the figure in *Sangaku* 2 problem with using fx-9860G II, is as follows.

<Process>

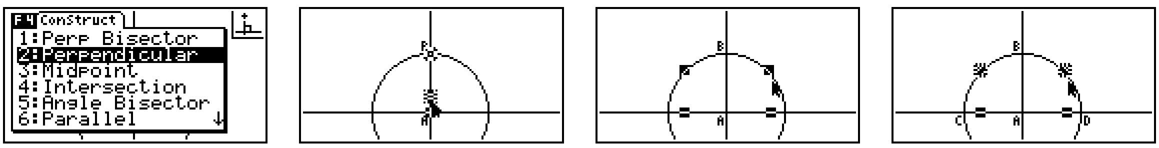
(1) Draw circle A whose center is a point A and radius is about 10.



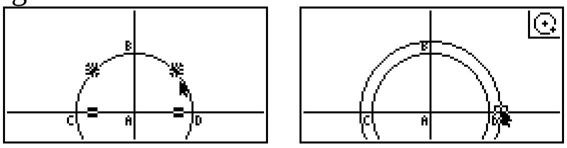
(2) Draw a line through from a point A to a point B.



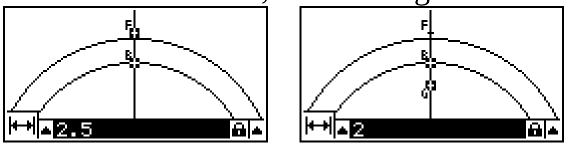
(3) Draw a perpendicular line to AB through a point A, and intersect at circle A for label C and D.



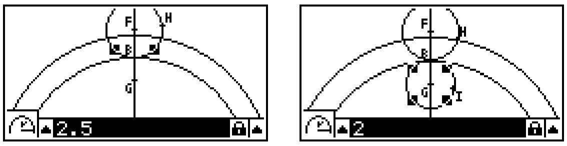
(4) Draw a concentric circle at point A, and draw two circles whose centers are on the line AB as the below figure shows.



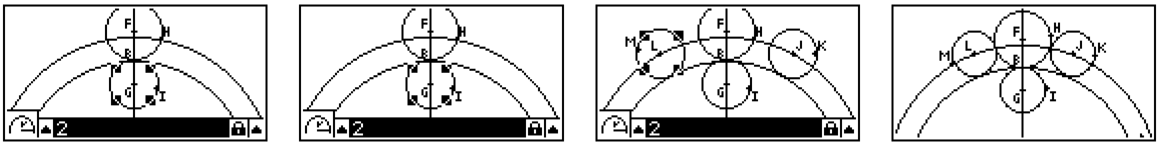
(5) Set two points F and G on the line AB, and set lengths as BF=2.5, BG=2.



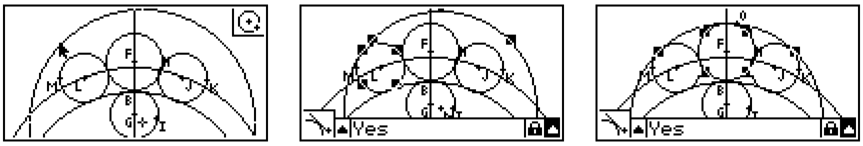
(6) Draw two circles whose center are a point of F and G on the line AB, and set circle F as radius 2.5, and circle G as radius 2.



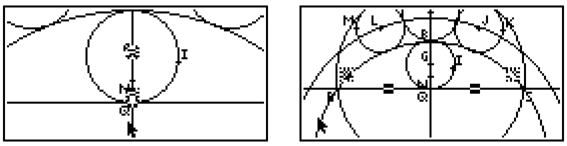
(7) Draw three circles whose centers are on a concentric circle as radius 2. And set circle L and circle J contact circle F.



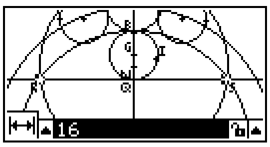
(8) Draw circle N whose center on the line AB, and set three circles F, J and L contact to circle N.



(9) Draw a perpendicular line to the line AB, and set circle G contact to it. Make intersection points circle N and it as R and S.



(10) fx-9860G II shows the distance from R to S for 16. Of course, the value fits. Completed.



5.3. The Knowledge with Using fx-9860G II

Using fx-9860G II to draw those *Sangaku* figures has such great effects on mathematics education as follows:

(1) In the solution of geometric problems, it will be very important to try to draw a figure first. By thinking about the method of the figure, students can approach the solution by the drawing requirements.

(2) Therefore, by using fx-9860G II which supports many students in their mathematical activities, it is expected to help in the following in mathematics education.

(3) It is able to visualize the geometric construction.

(4) It is able to show the geometric construction steps.

(5) Using this graphics calculator, it is possible to avoid the difficulty of problems.

(6) Students will be able to understand clearly what is required in the problem in order for them to operate this graphics calculator. And this experience will extend mathematics thinking.

(7) It will be possible for students to read the geometric construction in the guidance.

(8) It is necessary some conditions for drawing the geometric construction and it will be linked to the solving of algebraic equations.

6. In Closing

I myself, having thought for quite some time that the content of *Wasan* may become a breakthrough, ushering in a new era of mathematics education, and have conducted demonstrative research into *Wasan,* including the usage of *Wasan* geometry teaching materials, the making of *Sangaku* and so on. I have also published my results through various organs, including this school's Research for Super Science High School, and the Japan Society for Mathematical Education.

Actually observing genuine *Sangaku* alongside my students while doing field-work, I get an impression of how wonderful *Wasan* is, and, with that impression, a strong sense of interest in the fact that such mathematical research was carried out during the Edo Period. Furthermore, I felt that my students' experience of making *Sangaku*, that is, having my students make *Sangaku*, added a new layer of depth to the content of my normal classes.

Acknowledgements The author would like to thank his colleagues for their valuable comments and input. In addition, I have received appropriate advice from Dr. Kimio Watanabe who is a professor of Waseda University.

References

[1] Mitsuyoshi Yoshida, (1641). *Revised Jinkouki* (in Japanese), from the archives of the Tohoku University Library.

[2] Shinichi Ooya, (1987). *A Guide to Wasan* (in Japanese), Nippon-Hyoron-Sha.

[3] Hidetoshi Fukugawa, (1998). *Japanese Mathematics and Sangaku* (in Japanese), Morikita Publishing.

[4] Hideyo Makishita, (2000). *Coexistence of Mathematical Thinking with Technology* (in Japanese), The 9th International Congress on Mathematical Education

[5] Hideyo Makishita, (2001). *Practical Lessons Incorporating the History of Mathematics—Creating Teaching Materials and General Studies Using Sangaku* (in Japanese), Junior and Senior High School at Komaba, University of Tsukuba, Article Collection No. 40; p.145-171.

[6] Hideyo Makishita, (2002). *Sangaku Doujo* (in Japanese), Kenseisha.

[7] Hideyo Makishita (2003). *Enhancing General Studies through Mathematics Education, with Practical Research—from Practical Application of Thematic Assignments and Related Field Work Using Sangaku* (in Japanese), Junior and Senior High School at Komaba, University of Tsukuba, Article Collection No. 40; p.193-221.

[8] Hiroshi Kotera, (2007). *Enjoying Sangaku in Edo period* (in Japanese), Kenseisha.

[9] Hidetoshi Fukagawa, (2008) *Sacred Mathematics: Japanese Temple Geometry* (in English), Princeton University Press.

Internet Sources of Useful Information about *Sangaku* and *Wasan*

[1] R. Nolla & R. Masip, *Sangaku Recursos de Geometria*. http://www.xtec.cat/~rnolla/Sangaku/SangWEB/PDF/Sangak_4.pdf

[2] Contemplació i raó, R. Nolla, *SANGAKUS*, http://www.xtec.es/~rnolla/Sangaku/Sangakus3b.pdf

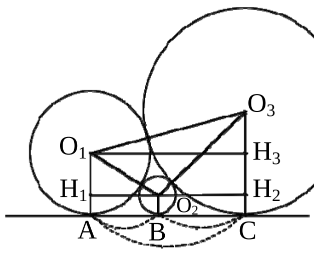
[3] *Sangaku*, http://www.sangaku.info/

[4] *Japanese Temple Geometry Problem* http://www.wasan.jp/

Apenddix

A Modern Mathematical Solving the Problem from *Sangaku 1*

Let a be the diameter of medium circle O_1, b be the diameter of small circle O_2 and x be the value of the diameter of large circle O_3.



The Figure of the problem from *Sangaku 1*

Let A, B and C be the points of contact between the shared plane and the medium circle, small circle and large circle, respectively. Then: AB＋BC＝AC

So, by the Pythagorean theorem:

$$AB = O_2H_1 = \sqrt{(a+b)^2-(a-b)^2} = 2\sqrt{ab}$$

$$BC = O_2H_2 = \sqrt{(x+b)^2-(x-b)^2} = 2\sqrt{bx}$$

$$AC = O_1H_3 = \sqrt{(x+a)^2-(x-a)^2} = 2\sqrt{ax}$$

Thus:

$$2\sqrt{ab} + 2\sqrt{bx} = 2\sqrt{ax} \ ,$$

Thus, we get:

$$(\sqrt{a}-\sqrt{b})\sqrt{x}=\sqrt{ab}$$

$$\sqrt{x}=\frac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}}$$

$$x=\left(\frac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}}\right)^2 \qquad (*)$$

In short, $a=9$ (sun), $b=4$ (sun) gives us:

$$x=\left(\frac{\sqrt{9\times 4}}{\sqrt{9}-\sqrt{4}}\right)^2=\left(\frac{6}{3-2}\right)^2=36\ (sun)$$

Moreover, the equation used (*) is:

$$x=\left(\frac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}}\right)^2=\frac{a}{\left(\sqrt{\frac{a}{b}}-1\right)^2}$$

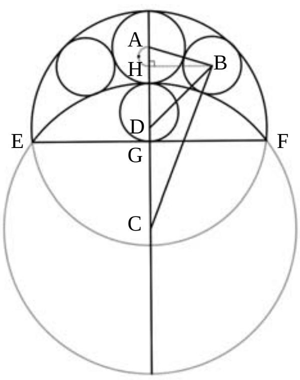
This equation is exactly the same as what is described in the explanation written on the *Sangaku*.

A Modern Mathematical Solving the Problem from *Sangaku 2*

Let A be the center point of circle *kou* and B be the center point of the circle *otsu* (on the right).

Also, as shown in the illustration, let C be the center point of the third circle, D be the center point of the outer circle, H be a segment of a perpendicular line drawn from B to the diameter of the outer circle, G be the point of contact between the central *otsu* circle and E and F be the start and end points of the chord whose length we are seeking.

The diameters of the *kou* circle, the *otsu* circle, the third circle and the outer circle are expressed as $2R$, $2r$, $2a$ and $2b$, respectively. Also, the length of the chord is expressed as x and the length of AH is expressed as y.



By the similarity of triangle:

$$x^2=4\cdot 2r(2a-2r) \qquad (1)$$

Because the power of point of G is equal for both the outer circle and the third circle:

$$(2R+2r)(2b-2R-2r)=2r(2a-2r) \qquad (2)$$

In short,

$$(2R+2r)\cdot 2b-\left\{(2R)^2+2\cdot 2R\cdot 2r+2r\cdot 2a\right\}=0 \qquad (3)$$

By applying the *Wasan*’s Formula of like cosine law to triangle △ABD, we get:

$$(2b-2r)^2=(2R+2r)^2+(2b-2R)^2-4(2b-2R)y$$

$$(2R)^2+2R\cdot 2r-2b\cdot 2R+2b\cdot 2r-2(2b-2R)y=0 \qquad (4)$$

By applying the *Wasan*’s Formula of like cosine law to triangle △ABC, we get:

$$(2a+2r)^2=(2R+2r)^2+(2a+2R)^2-4(2a+2R)y$$

$$(2R)^2 + 2R\cdot 2r + 2a\cdot 2R - 2a\cdot 2r - 2(2a+2R)y = 0 \qquad (5)$$

If we eliminate y from (4) and (5), we get:

$$\left\{(2R)^2 + 2R\cdot 2r - 2b\cdot 2R + 2b\cdot 2r\right\}(2a+2R) = \left\{(2R)^2 + 2R\cdot 2r + 2a\cdot 2R - 2a\cdot 2r\right\}(2b-2r)$$

Putting these into order gives us:

$$\left\{(2R)^2 + 2a\cdot 2R - 2a\cdot 2r\right\}\cdot 2b - \left\{(2R)^3 + 2r\cdot(2R)^2 + 2a\cdot(2R)^2\right\} = 0 \qquad (6)$$

If we eliminate the outer parts of the problem by applying the ratio from (3) and (6), we get:

$$(2R+2r)\left\{(2R)^3 + 2r\cdot(2R)^2 + 2a\cdot(2R)^2\right\} = \left\{(2R)^2 + 2\cdot 2R\cdot 2r + 2r\cdot 2a\right\}\left\{(2R)^2 + 2a\cdot 2R - 2a\cdot 2r\right\}$$

$$(2R-2r)\cdot(2a)^2 + \left\{(2R)^2 - 2\cdot 2R\cdot 2r\right\}\cdot 2a - (2R)^2\cdot 2r = 0$$

Factoring gives us:

$$\left\{2(2R-2r)a - 2R\cdot 2r\right\}(2a+2R) = 0 \qquad (7)$$

Thus,

$$2a = \frac{2R\cdot 2r}{2R-2r}$$

Substituting this into (1) gives us:

$$x^2 = 4\cdot 2r\left(\frac{2R\cdot 2r}{2R-2r} - 2r\right) = \frac{4(2r)^3}{2R-2r}$$

Thus,

$$x = \frac{2\cdot 2r}{\sqrt{\dfrac{2R}{2r} - 1}}$$

Input the value: $2R = 5$ *sun*, $2r = 4$ *sun*, so:

$$x = \frac{2\times 4}{\sqrt{\dfrac{5}{4} - 1}} = 16 \quad (sun)$$

Fit it the answer.