Construction of common perpendicular in hyperbolic space

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Abstract: Construction problem is a good exercise for us to understand geometry deeply. Using dynamic geometry software, we can easily check whether our conjecture is right or not. In this paper, we introduce a construction of common perpendicular line to two lines in three-dimensional hyperbolic space. Inversion is the most essential operation for this procedure. In addition, we show that the angle of a pair of skew lines is simply measured with this construction.

1. Introduction

Our start point is a pair of geodesic lines in three-dimensional hyperbolic space $\mathbf{H}^3$. In this paper, assume that these two lines do not intersect but are in twisted position. There are two important models of hyperbolic space: the Poincare model and the half-space model. Both model are due to Poincare ([1] p.343). Our goal is to construct the common perpendicular line in these models. For this construction, inversion plays a very important role, because inversion in these models is an isometry which corresponds to reflection in Euclidean space. In Section 2, we recall the construction of the line perpendicular to a line through a point in two-dimensional hyperbolic plane $\mathbf{H}^2$. Inversion is effectively used in the construction. The construction of the perpendicular line in $\mathbf{H}^3$ is a natural extension of $\mathbf{H}^2$. In Section 3, we consider the construction of the common perpendicular in $\mathbf{H}^3$. The half-space model is fundamental for this construction. Key words are stereographic projection, Klein model ([1] p.327), pole and polar. In Section 4, we show how to measure the skew angle of two lines with the construction of common perpendicular. All pictures in this paper are made by dynamic geometry software *Cabri 3D*. With this type of software, we can intuitively understand that our constructions are reasonable.

2. Perpendicular line

In this section, let us recall the construction of perpendicular line in hyperbolic plane and space.

2.1. Perpendicular in the two-dimensional Poincare model

Geodesic line in the Poincare model is a minor arc perpendicular to the boundary circle $\mathbf{S}^1$ which is constructed in the following way (Figure 2.1):

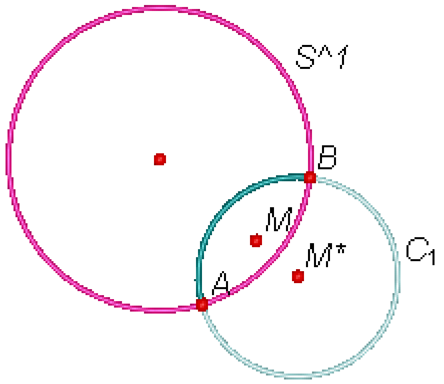

Figure 2.1 Geodesic line in the Poincare model.

Construction 1 (geodesic line).

0. (Input) Two points A and B on the boundary circle $\mathbf{S}^1$.
1. Point M, midpoint between A and B.
2. Point M*, inversion of M with respect to $\mathbf{S}^1$.
3. Circle C1, centered on M* and through A.
4. (Output) Minor arc AB on C1.

Note that Euclidean line AB is the polar of the pole M* with respect to $\mathbf{S}^1$ ([1] p.134).
For a geodesic line and a point, the perpendicular line is constructed as follows (Figure 2.2):

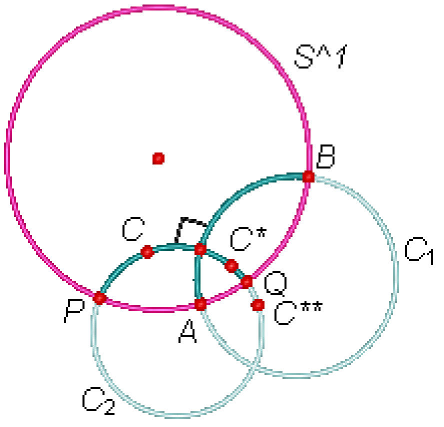

Figure 2.2 Perpendicular line in the Poincare model.

Construction 2 (perpendicular line in hyperbolic plane).

0. (Input) Geodesic line AB and point C.
1. Circle C1 which includes minor arc AB.
2. Point C*, inversion of C with respect to C1.
3. Point C**, inversion of C* with respect to $\mathbf{S}^1$.
4. Circle C2, through C, C* and C**.
5. Points P and Q, intersections of C2 and $\mathbf{S}^1$.
6. (Output) Geodesic line PQ (by Construction 1).

In hyperbolic geometry $\mathbf{H}^2$, C* corresponds to the reflection of C with respect to geodesic line AB.

2.2. Perpendicular in the half-plane model

Geodesic line in the half-plane model is a half circle perpendicular to the boundary line $\mathbf{R}^1$. The construction of perpendicular is almost the same as Construction 2 (Figure 2.3):

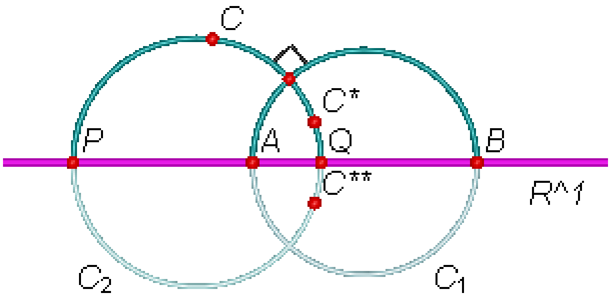

Figure 2.3 Perpendicular line in the half-plane model.

Construction 3 (perpendicular line in hyperbolic plane).

0. (Input) Geodesic line AB and point C.
1. Circle C1 which includes half circle AB.
2. Point C*, inversion of C with respect to C1.
3. Point C**, reflection of C* with respect to $\mathbf{R}^1$.
4. Circle C2, through C, C* and C**.
5. Points P and Q, intersections of C2 and $\mathbf{R}^1$.
6. (Output) Half circle PQ.

2.3. Perpendicular in the half-space model

Geodesic line in the half-space model is a half circle perpendicular to the boundary plane $\mathbf{R}^2$. The construction of perpendicular is realized by a combination of inversion and reflection (Figure 2.4):

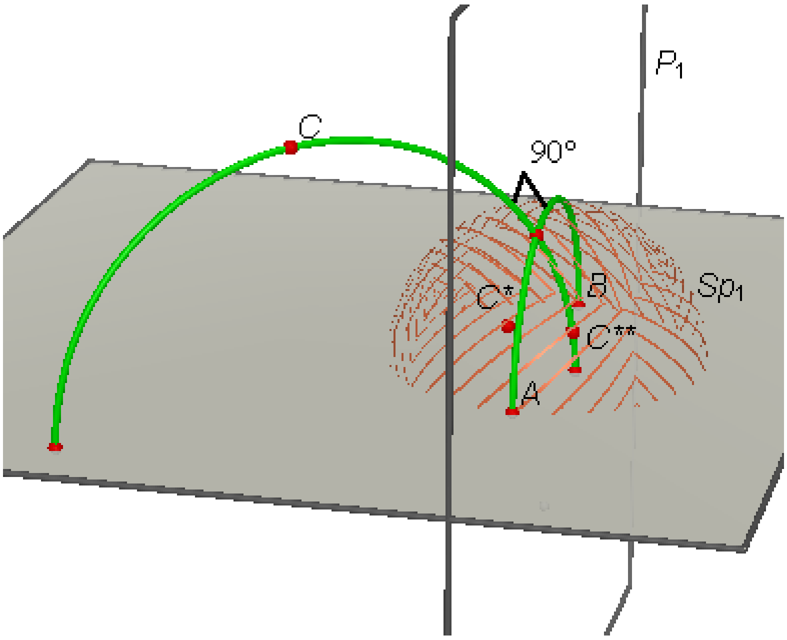

Figure 2.4 Perpendicular line in the half-space model.

Construction 4 (perpendicular line in hyperbolic space).

0. (Input) Geodesic line AB and point C.
1. Sphere Sp1, through A and B as antipodal points.
2. Plane P1 which includes half circle AB.
3. Point C*, inversion of C with respect to Sp1.
4. Point C**, reflection of C* with respect to P1.
5. (Output) Half circle through C and C** (by Construction 3).

In hyperbolic geometry $\mathbf{H}^3$, Sp1 and P1 are two geodesic planes, perpendicular to each other. Hence C* corresponds to the half turn of C with respect to geodesic line AB.

3. Common perpendicular line

In this section, let us see the construction of common perpendicular line in hyperbolic space.

3.1. Orthogonal lines in geodesic plane

Let us consider the inverse of the stereographic projection ([1] p.261) of the Poincare model ($\mathbf{H}^2$) onto the upper hemisphere with the south pole S as in the left of Figure 3.1. Let two geodesic lines AB and CD be perpendicular to each other in the Poincare model. Point P (resp. Q) is the center of the circle which includes minor arc AB (resp. CD). Note that Euclidean line AB (resp. CD) is the polar of the pole P (resp. Q) with respect to the equator $\mathbf{S}^1$. The inverse images of these two minor arcs are also two half circles in the upper hemisphere, because the stereographic projection transforms circles in the sphere into circles in the plane. In addition it preserves angles, therefore, two half circles in the upper hemisphere are also perpendicular to each other. These two half circles correspond to two orthogonal geodesic lines on the same geodesic plane in the half-space model. On the other hand, the top view (the right of Figure 3.1) looks like the Klein model ([1] pp. 326-331). We can see that half circle CD is perpendicular to half circle AB if and only if the pole Q of Euclidean line CD is on Euclidean line AB. This fact makes us possible to construct orthogonal geodesic lines on the geodesic plane in $\mathbf{H}^3$.

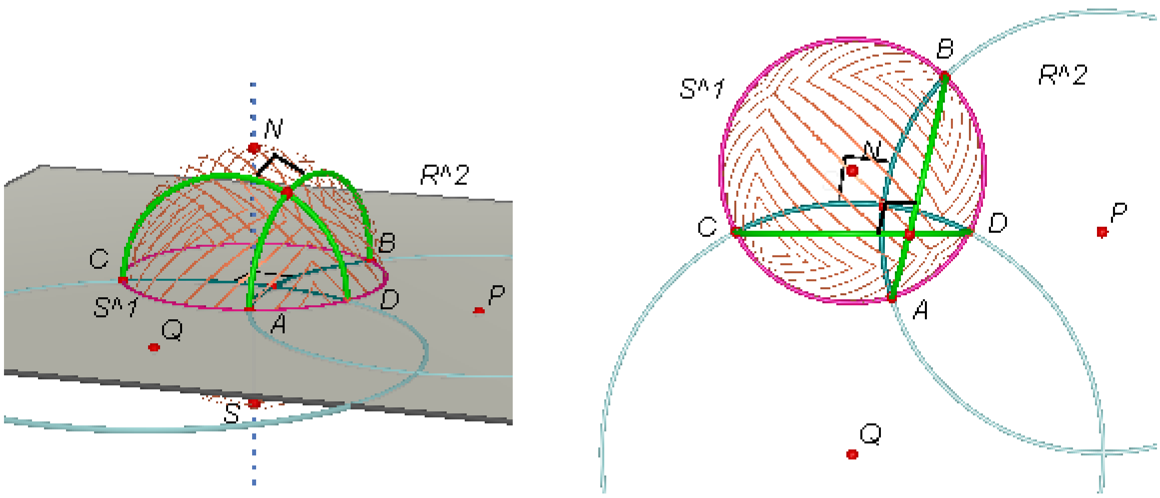

Figure 3.1 Inverse image of the stereographic projection (left) and its top view (right).

3.2. Common perpendicular in the half-space model (special case)

Let us consider a special case in the half-space model that one of two geodesic lines is Euclidean line as in Figure 3.2. General case will be considered in the next subsection. In this special case, the construction of common perpendicular is as follows:

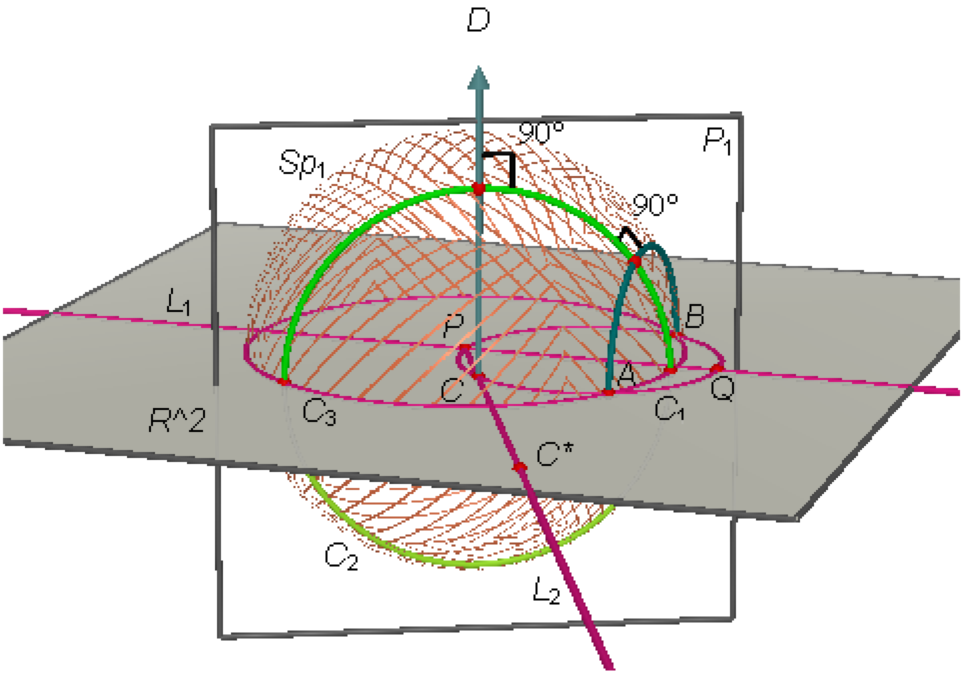

Figure 3.2 Common perpendicular in the half-space model (special case).

Construction 5 (common perpendicular).

0. (Input) Geodesic lines AB and CD(∞).
1. Circle C1, through A, B and C on $\mathbf{R}^2$.
2. Line L1, perpendicular bisector between A and B on $\mathbf{R}^2$.
3. Points P and Q, intersections of C1 and L1 (P and C are in the same side with respect to the plane including half circle AB).
4. Sphere Sp1, centered on P and through A.
5. Line L2, through P and C.
6. Plane P1, perpendicular to L2 and through C.
7. Circle C2, intersection of P1 and Sp1.
8. (Output) Half circle in C2.

Let circle C3 be the intersection of Sp1 and $\mathbf{R}^2$, then the center of C3 is P as in Figure 3.2. Since P is on C1, notice that the inversion of C1 with respect to C3 is Euclidean line AB, especially, point C*, inversion of C with respect to C3 is on Euclidean line AB. In addition, since $\angle PCQ=90°$, C* is the pole of Euclidean line CQ with respect to C3. That is why C2 intersects orthogonally with half circle AB. Since the center of C2 is C, it is trivial that C2 also intersects orthogonally with ray CD.

3.3. Common perpendicular in the half-space model (general case)

General case in the half-space model is simply reduced to special case by inversion. For two geodesic lines AB and CD, let us take the inversion with respect to the sphere centered on D and through C as in Figure 3.3. Then, we can use Construction 5 in the previous subsection.

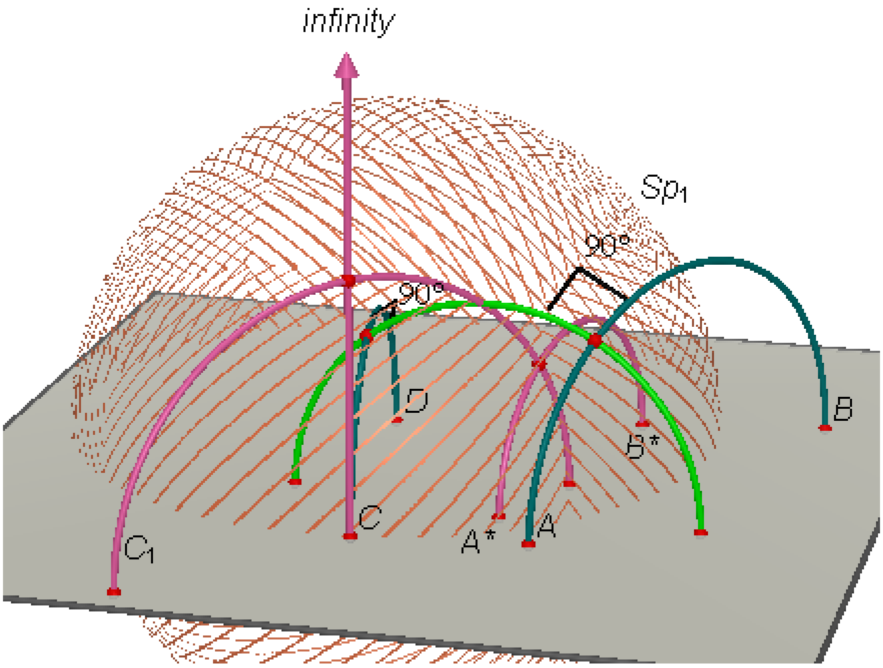

Figure 3.3 Common perpendicular in the half-space model (general case).

Construction 6 (common perpendicular).

0. (Input) Geodesic lines AB and CD.
1. Sphere Sp1, centered on D and through C.
2. Points A* and B*, inversions of A and B with respect to Sp1.
3. Half circle C1, common perpendicular to lines A*B* and $C\infty$. (by Construction 5).
4. (Output) Half circle, inversion of C1 with respect to Sp1.

3.4. Common perpendicular in the Poincare model

The construction of the common perpendicular in the Poincare model is simply reduced to special case by stereographic projection. For two geodesic lines AB and CD in the Poincare model, let us take the stereographic projection of boundary sphere onto a plane with D as the north pole as in Figure 3.4. Then, we can use Construction 5 again.

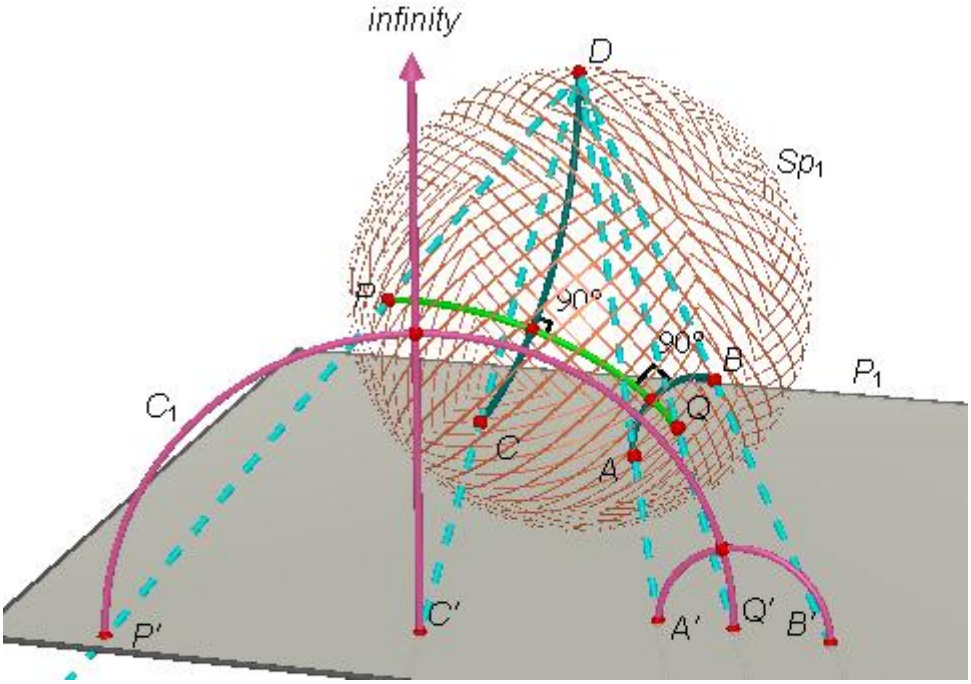

Figure 3.4 Common perpendicular in the Poincare model.

Construction 7 (common perpendicular).

0. (Input) Geodesic lines AB and CD inside of sphere Sp1.
1. Points A', B' and C', stereographic projections of A, B and C onto plane P1 with D as the north pole.
2. Half circle C1, common perpendicular to lines A'B' and C'∞. (by Construction 5).
3. Points P' and Q', intersections of C1 and P1.
4. Points P and Q, inverse stereographic projections of P' and Q'.
5. (Output) Geodesic line PQ inside of Sp1.

4. Skew angle

With the construction of common perpendicular, we can easily measure the skew angle of two geodesic lines as in Figure 4.1. Let Sp1 (resp. Sp2) be the half-sphere including AB (resp. CD) and the common perpendicular. These spheres correspond to geodesic planes in $\mathbf{H}^3$, and the skew angle of the two geodesics is the angle between Sp1 and Sp2.

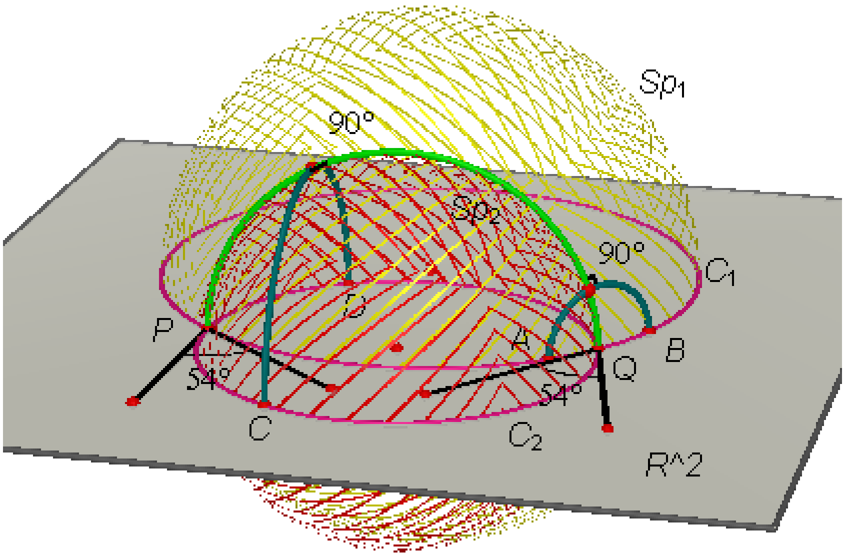

Figure 4.1 Skew angle of two geodesics.

Construction 8 (skew angle).

0. (Input) Geodesic lines AB and CD.
1. Geodesic PQ, common perpendicular to lines AB and CD. (by Construction 6).
2. Circle C1, through A, B, P and Q.
3. Circle C2, through C, D, P and Q.
4. (Output) Angle between C1 and C2.

For two geodesics AB and CD in Figure 4.1, the skew angle measures $54°$.

References

[1] Berger, M. (1987). *Geometry II.* Berlin Heidelberg, Germany: Springer-Verlag.