Solving Wasan Problems with an Inversion Method - Development of teaching materials for approaching complex geometry -

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Abstract

The idea of transformation is very important for engineers of computer science, computer graphics, etc. But, this is not emphasized in Japanese elementary and secondary education curricula. Therefore, most students think of "stretching", "shrinking" and "symmetry" as separate ideas. Students who major in engineering have a hard time learning various transformations. Even though complex geometry is related to complex analysis which is an important subject for them, there are few educational materials to help in understanding it. It has now become easy for students to visually understand transformation using dynamic geometry software (DGS). When we use DGS to express an inversion transformation, you can see the unexpected results. It is very interesting and useful for some intractable problems. We can also simply express inversion by using complex numbers. So, we expect to develop teaching materials to improve student understanding of complex geometry using an inversion approach. In this study, we show some teaching materials employing an inversion method. One of them is a Wasan problem. Wasan problems are found in very old Japanese mathematics books and had been solved by another method. We show how inversion is powerful enough to solve some of them and how students can effectively approach complex geometry.

1. Background

The idea of transformation is very important for electronic engineering, computer science, computer graphics, etc. In the new course of study announced in 2008 and which has been partially implemented, sixth grade students are expected to understand "stretching" and "shrinking" and "symmetry". In junior high school, the first grade students learn "translation" and "rotation", second grade students learn "congruence" and third grade students learn "similarity". Each topic is taught separately. The Japanese word "ido", which means "move from one position to another position" is used for transformation of figures. It is emphasized to students to learn the conditions for congruence triangles and for similar triangles. Therefore, most students compare the shape of two figures and only think about the positional relation between each side, angle, etc. So, they never think one figure is translated to the other. Students major in engineering have a hard time learning various transformations in the university.

Now, it has become easy for students to visually understand transformations using dynamic geometry software (DGS). An inversion especially causes unexpected results and unpredictable situations. It is a very interesting activity for investigation.

The objects of this study are to develop teaching materials to help promote an understanding of transformation focused on inversion, to show how inversion is powerful enough to solve problems and how students can use it to approach complex geometry. In our study, the inverse transformation is used to solve some Wasan problems found in very old Japanese mathematics books and which are usually solved by another method.

2. Learning transformation.

In the "Erlangen Program" Klein, Felix said geometric properties are not changed by the transformations of the principal group (see [1]). And, conversely, geometric properties are characterized by their remaining invariant under the transformations of the principal group. It means transformation is the way to find the invariant properties in a geometric figure. Therefore, we expect students to focus on the variant or invariant properties of a geometric figure by transformation. We would like to categorize translation, rotation, reflection, line symmetry, point symmetry, dilation and similar transformations which are taught in elementary and secondary schools as invariant properties. Inversion is not taught in elementary and secondary schools. But these days, it is easy to demonstrate it using DGS and there are many interesting topics related to inversion. When we draw a figure by inversion, we always think how to transfer each point on the figures. Therefore, when students learn transformation methods like translation, rotation, etc., they also have to think how each point is transferred on the figure and connect the relation between points on the figure and in this way they can learn the concept of transformation. The thought process will help them to learn advanced math.

3. What is Circle Inversion?

3.1 Definition of inversion

Figure 1 shows a point P and a circle R(T, t). In this paper, "circle R(T, t)" means a circle R which has center point T and radius t. When point P' exists on the line TP with $(TP)\cdot(TP')=t^2$, point P' is the inversion of point P with respect to the circle R (See Fig. 1). This circle R used for the definition is called "the reference circle". The equation $(OP)\cdot(OP')=t^2$ means the t is the geometrical average of length of TP and TP'. Fig. 1(a) shows point P is inside the reference circle and Fig. 1(b) shows point P is outside the reference circle. In Fig. 2, when point P moves on triangle ABC, the locus of P' which is inverted using the reference circle R is the funny shape inside the circle. The figure shows this kind of inversion keeps any big figure in the given circle R (the reference circle). Inversion becomes easy to draw even for some complex figures such as Steiner's rings (see Fig. 6).

Furthermore, inversion transforms any complex number z into $1/\bar{z}$, when the unit circle is used as the reference circle. Therefore, we can also use the expression of the complex number to express this transformation (see [2]).

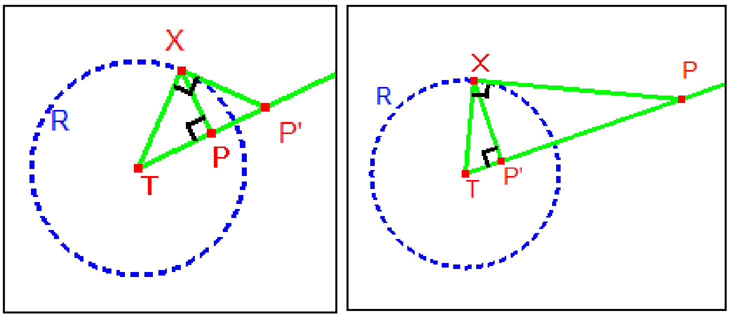


Fig. 1(a) and 1(b) *Point P is inside or outside the circle*

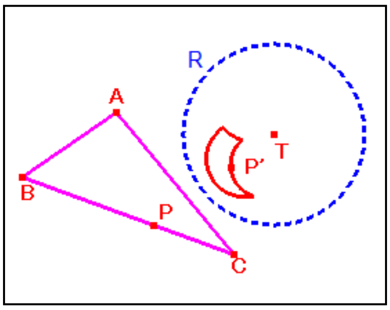


Fig. 2 *Inversion of triangle ABC*

3.2 Proposition

Here is a proposition to prove a problem with transformation by inversion. It shows the relation between the radiuses of an original circle, the radiuses of a transformed circle and the length of tangent.

Proposition 1: When circle C(O, r) is transformed into circle C'(O', r') by inversion with the reference circle R(T, t), there is the following relationship exists (see Fig. 3).

$$\frac{r}{r'} = \frac{t^2}{d^2}$$

d: the length of tangent from T to point of contact F on circle C'.

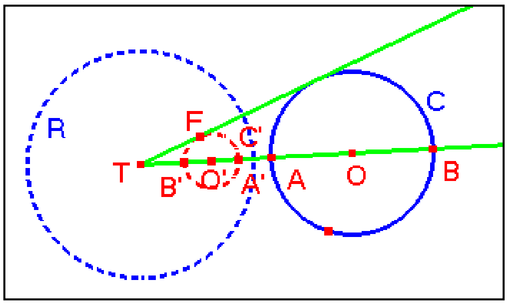


Fig. 3 *Proposition*

[Proof]

From the definition of inversion

$$TA = \frac{t^2}{TA'} , \quad TB = \frac{t^2}{TB'}$$

$$2r = TA - TB = \frac{(TB' - TA') \cdot t^2}{TA' \cdot TB'} = \frac{2r' \cdot t^2}{d^2}$$

i.e. $\frac{r}{r'} = \frac{t^2}{d^2}$ end

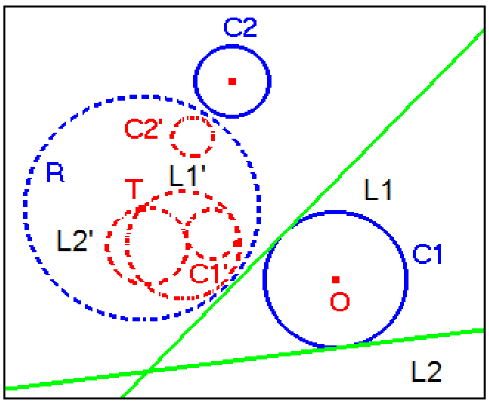
In the next section we would like to show some teaching materials which assist students to learn inversion. Students will learn how to draw these beautiful figures from easy simple figures. They do this through activities which determine the relationship between a base figure and the transformed inversion figure.

4. Examples of an inversion teaching material

Students find rules for the relationship between a base figure and the transformed figure by inversion through activities 4.1 to 4.3. They then try to draw Steiner's rings and Wasan problems to get hints to prove 4.4 to 4.5.

4.1 How a line and a circle which are outside the reference circle R are transformed by inversion. (Fig. 4(a))

(a) There are lines and circles outside of the reference circle R.
(b) Line L1 is transformed into the circle L1' and the line L2 into the circle L2'
(c) Circle C1 which is tangent to line L1 and L2 is transformed into the circle C1', circle C2 into the circle C2'.
(d) Are there any differences between the circle C1' and C2' or L1' and L2'?



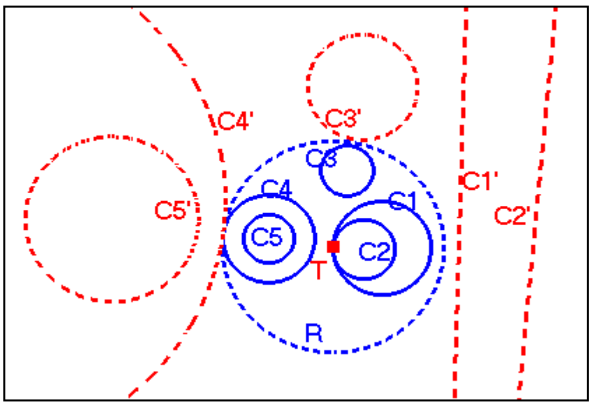


Fig. 4(a) and 4(b) *Figures outside of the reference circle R or inside it*

4.2 How a line and a circle which are inside the reference circle R are transformed by inversion. (Fig. 4(b))

(a) Circles C1 and C2 pass through point T (the center of the reference circle R). They are transformed into the line C1', C2'.

(b) Circle C3 and C5 which have the same size radius are located in different places. They are transformed into the circle C3' and C5'.

(c) Circle C4 and C5 have the same center point and different radius. They are transformed into the circle C4' and C5'.

(d) Are there any differences between the cirlces C1' and C2', C3' and C5', C4' and C5'?

4.3 When is the transformed figure equal to the original one?

In Fig. 5, you can see many transformed figures (dotted line) when the center point of original circle C(O, r) moves from P1 to P2. Those inverted figures become a line or a circle depending on the position of circle C(O, r). The figure on the right shows point O moves farther from P2. We notice there is a chance that the transformed figure is equal to the original one.

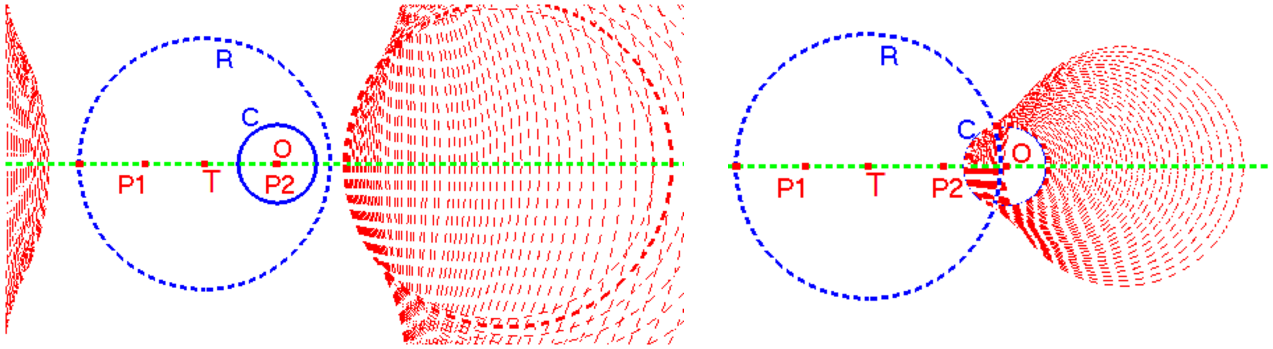


Fig. 5 *Transformed figure (dotted line) depends on the original figure*

4.4 Draw Steiner's rings

Fig.6 shows Steiner's rings drawn by inversion. The parallel lines contain circles that are almost the same size as the original figures, but we can also see various transformed figures. The transformed figures show an important relation of size and location between the original figure and the reference circle. Moreover, they show that curves transformed by inversion keep touching each other when the original curves touch each other.

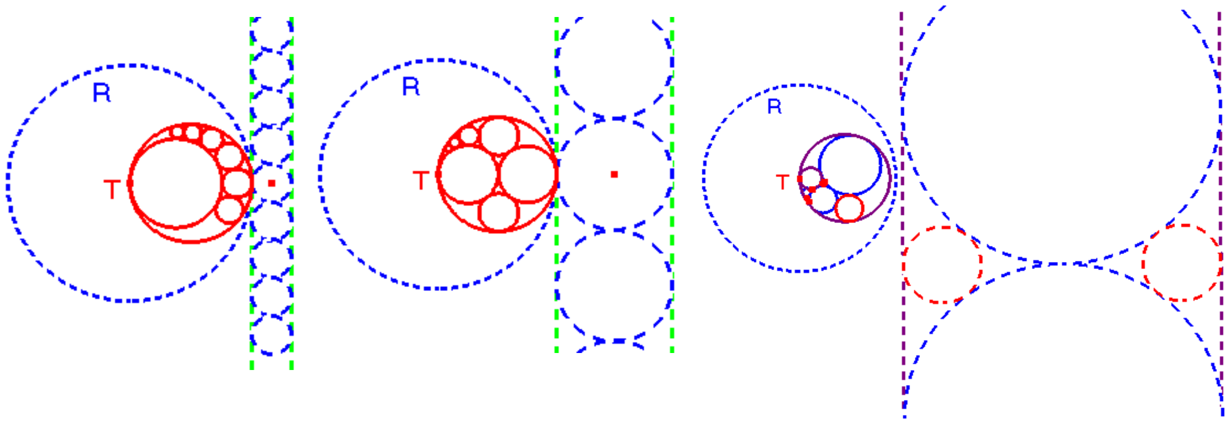


Fig. 6 *Examples of the Steiner's rings drawn by inversion.*

4.5 Examples of Inversion to solve Wasan problems

Wasan problems are found in very old Japanese mathematics books and they were traditionally solved by other methods such as the Patagonian theorem. When they were solved, their solutions were dedicated at a Shinto shrine (see [3] & [4]).

(1) Wasan Problem 1 (see [3] p. 28)

The circles C0(O0, r), C1, C2, C3, and C4 are inscribed within the half circle C(O, 2r). All circles are tangent to each other (Fig. 7).

If the radius of circle C0 is r, find the length of each radius r_n (n=1,2,3・・・)

<Answer>

$$r_n = \frac{2r}{2+\{\sqrt{2}+(n-1)\}^2}$$

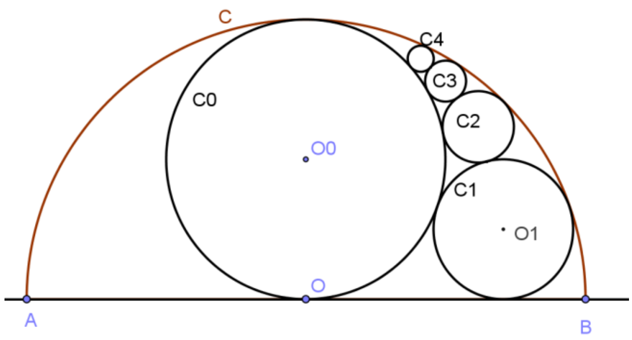


Fig. 7 *Wasan problem 1*

<u>The original figure (Fig. 7 or dotted lines in Fig. 8)</u>

(a) In half circle C(O, 2r) with the diameter AB, circle C0(O0, r) is inscribed within circle C and is tangent to the middle point O of AB.

(b) Circles Ci (i=1,2,3…) are inscribed within circle C and circumscribed with circle C0.

(c) Each circle should be connected with one point.

(d) The circle C1(O1, r1) should be tangent with diameter AB of circle C.

<u>Find the reference circle and invert it.</u>

It's important to define the reference circle. We got the idea of this reference circle from the Steiner's rings. Circle R(T, 2r) was used. Point T is the contact point of circle C and circle C0, and TO is its radius. Results of the inversion (see Fig. 8) are as follows,

<u>The inverted figure (solid lines in Fig. 8)</u>

(a) The circumscribed circles C and C0 are transformed into parallel lines C' and C0'.

(b) Circle C1 is transformed to C1'(O1', r1') which is the same as C1(O1, r1) (see Fig. 5).

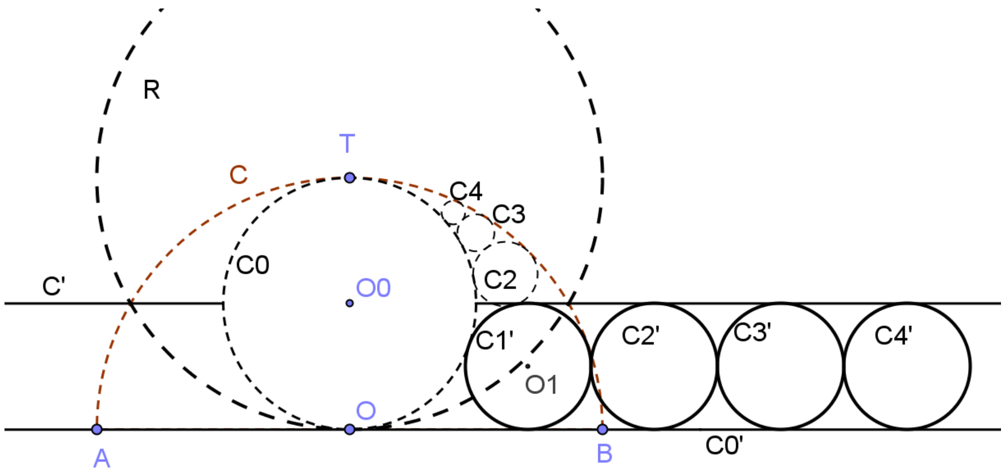


Fig. 8 Inverted figure of Wasan problem 1 (solid line)

(c) Circles C2, C3 and C4 are inverted to C2'(O2', r2'), C3' (O3', r3') and C4' (O4', r4') between parallel lines C' and C0'.

We get the transformed figure, and the figure which makes it easy to solve this problem. The details of this answer are described in Reference Materials [1].

(2) Wasan Problem 2 (see [3] p. 27)

The circle C0(O0, r), C1, C3 and C4 are inscribed within the half circle C(O, 2r). Circle C2 is tangent circle C0, C1, C3 and circle C4. All circles are tangent to each other (see fig. 9).

If the radius of circle C0 is r, find the length of each radius r_n (n=1,2,3,4)

<Answer>

$$r_1=\frac{r}{2},\quad r_2=\frac{2r}{15},\quad r_3=\frac{r}{6},\quad r_4=\frac{r}{5}$$

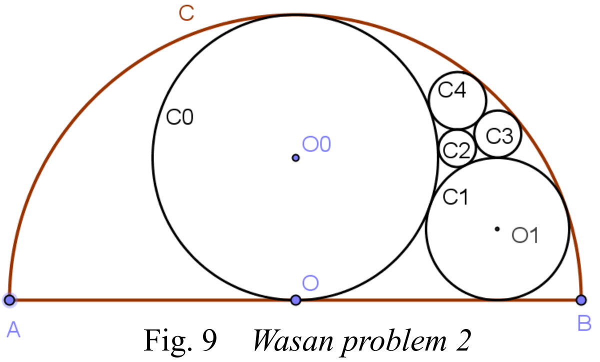


Fig. 9 *Wasan problem 2*

We can define the reference circle R(T, t) based on problem 1. The circle R(T, 2r) was used. Point T is the contact point of circle C and circle C0, and TO is its radius. Results of the inversion (see Fig. 10) are as follows:

(a) The circumscribed circles C and C0 are transformed into parallel lines C' and C0'.

(b) Circle C1 is transformed to C1'(O1', r1') which is the same as C1(O1, r1) (see Fig. 5).

(c) Circles C2, C3 and C4 are inverted to C2'(O2', r2'), C3' (O3', r3') and C4' (O4', r4') between parallel lines C' and C0'.

Once again, we get the transformed figure which makes it easy to solve this problem. The details of this answer are described in Reference Materials [2].

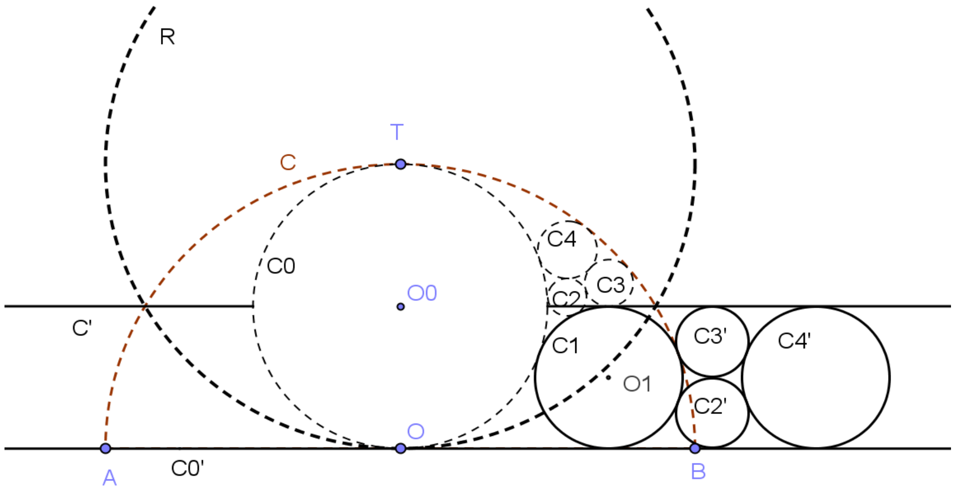


Fig. 10 *Inverted figure of Wasan problem 2 (solid line)*

5. Discussion

We showed that we could solve two Wasan problems by inverted circle figures using GeoGebra software. The software facilitates drawing the transformed figures and makes it easy to solve a problem. Next, we show how the inversion relates to the transformation of a complex number.

Johann Carl Friedrich Gauss connected “complex number” with “complex plane”. Although, complex numbers had been thought of as only a symbol to express a number formally, they were later connected to geometry and this idea has expanded to “complex analysis”, electromagnetism and other applications.

Therefore, we can apply this geometrical relation to the expression of a complex number. Circle inversion is an important and powerful geometric method. When we express this transformation by complex numbers, it’s described very simply.

Now, let’s examine a case in which the unit circle’s center is the origin. The unit circle is used as the reference circle. Geometrical inversion has a relation OP’=1/OP (see Figs. 1(a)). In function theory, inversion means that any complex number z is transformed $1/\bar{z}$. We will show this linear fractional transformation of complex numbers concept visually by using GeoGebra software (see Fig. 11). In Fig. 11, an inversion of point E using the unit circle (its center is the origin) is made by two methods.

(1) E’: an inverted point using a tool (geometrically).

(2) z= $1/\bar{E}$: an inverted point which is calculated using a complex number.

You can see point E’ and point z are the same. Then, we can express inversion geometrically by simply using a complex number.. We know this transformation is one of the linear fractional transformations of a complex number

In Japanese secondary schools, students learn translation, rotation and similarity in Euclidean geometry. They learn these topics separately. But when they use Cabri Geometry or GeoGebra software, they are expressing these topics visually and dynamically. Students also show visually figures transformed by inversion by means of those software. Through these activities, students can identify and connect these inversion relationships to each other. This time, we treated inversion only geometrically because this software cannot draw figures on a complex plane. If we handle inversion

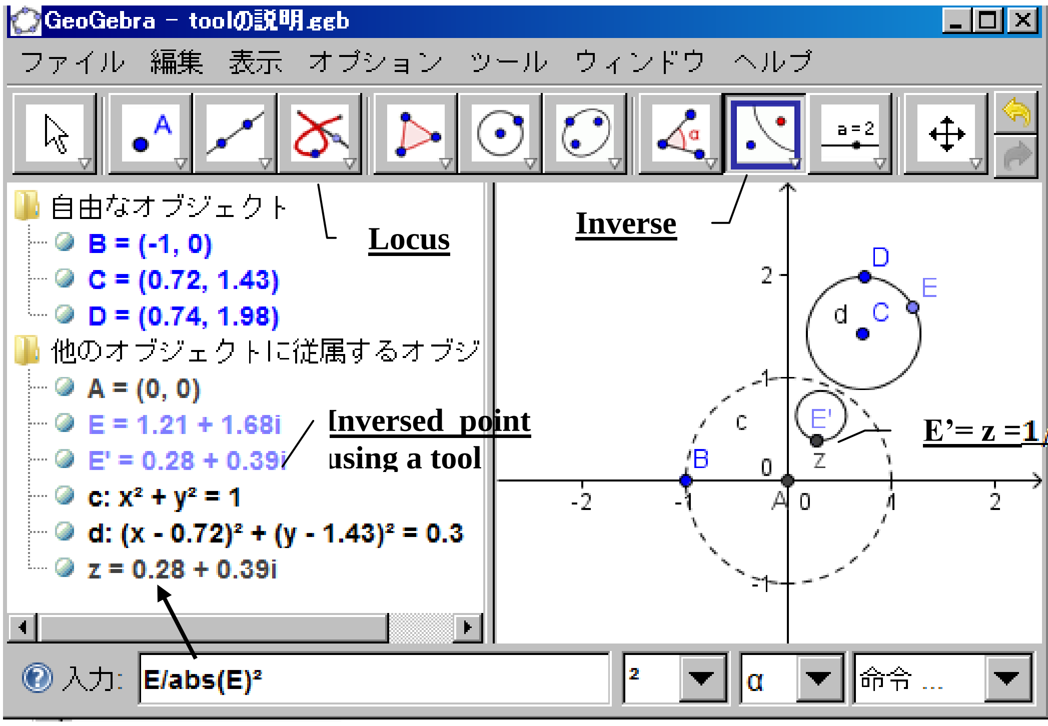


Fig. 11 *An inverted figure by GeoGebra*

using complex numbers, inversion can be treated as a linear fractional transformation.

6. Conclusion

(1) Many difficult problems in geometry become much more tractable when an inversion is applied. We can solve Wasan problems by using inversion. The inversion of the problem can be drawn by software such as Cabri Geometry or GeoGebra which facilitate solving the problem.
(2) The concept of an inversion can be expanded to complex transformation and complex analysis which are basic knowledge for students majoring in electronic engineering, computer science, computer graphics and so on. But, these topics are difficult to learn. Therefore, effective inversion teaching materials which simplify learning transformation and complex transformation have to be developed.
(3) We also need to investigate how students connect these activities in trying to understand complex geometry and complex analysis.

Software for Education

[a] Cabri Geometry II Plus. Dynamic Geometry Software, Product of Cabrilog. http://www.cabri.com/
[b] GeoGebra. Developed by Markus Hohenwarter, It's a multi-platform dynamic mathematics software for all levels of education that combines arithmetic, geometry, algebra and calculus. Recently includes spreadsheets. Free download from following website. http://www.geogebra.org/

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[3] Fukazawa Hidetosi & Dan Sokolowsky. (1991). *Japanese Geometry –How many can you prove?*. Tokyo, Japan. Morikita Syuppan.

[4] Fukazawa Hidetosi and Dan Sokolowsky (1994). *Japanese Geometry –How many can you prove? Vol. 2: Geometric Problems (Triangles, Circles and ellipses)*. Tokyo, Japan. Morikita Syuppan.

[5] Iwata Shikou ed. (1971). *Comprehensive dictionary in geometry*. Tokyo, Japan. Maki-shoten,

Reference Materials

[1] Detailed answer: Wasan problem 1

Draw a tangent line of circle C_i' from point T which has a contact point P_i' (i=0,1,2...) (see Fig. 12).

From the definition of inversion $\quad t^2 = (2r)^2 \cdots\cdots (1)$

$$(NO_1)^2 = (O_0O_1)^2 - (NO_0)^2 = (\frac{3}{2}r)^2 - (\frac{1}{2}r)^2 = (\sqrt{2}r)^2 \qquad NO_1 = \sqrt{2}r$$

$$(TO_n')^2 = (NO_n')^2 + (TN)^2 = \{\sqrt{2}r + (n-1)r\}^2 + (\frac{3}{2}r)^2 = [\{\sqrt{2} + (n-1)\}^2 + (\frac{3}{2})^2]r^2$$

$$(TP_n')^2 = (TO_n')^2 - (O_nP_n')^2 = [\{\sqrt{2}+(n-1)\}^2 + (\frac{3}{2})^2]r^2 - (\frac{r}{2})^2 = \left[2 + \{\sqrt{2}+(n-1)\}^2\right] r^2 \cdots (2)$$

From Proposition (see Fig. 3), (1) and $r_n' = \frac{r}{2}$ $\quad (TP_n')^2 = \frac{r_n'}{r_n}t^2 = \frac{r}{2r_n}t^2 = \frac{2r^3}{r_n}\cdots\cdots(3)$

(2) = (3) $\quad 2+\{\sqrt{2}+(n-1)\}^2 = \frac{2r}{r_n} \qquad r_n = \frac{2r}{2+\{\sqrt{2}+(n-1)\}^2}$

end

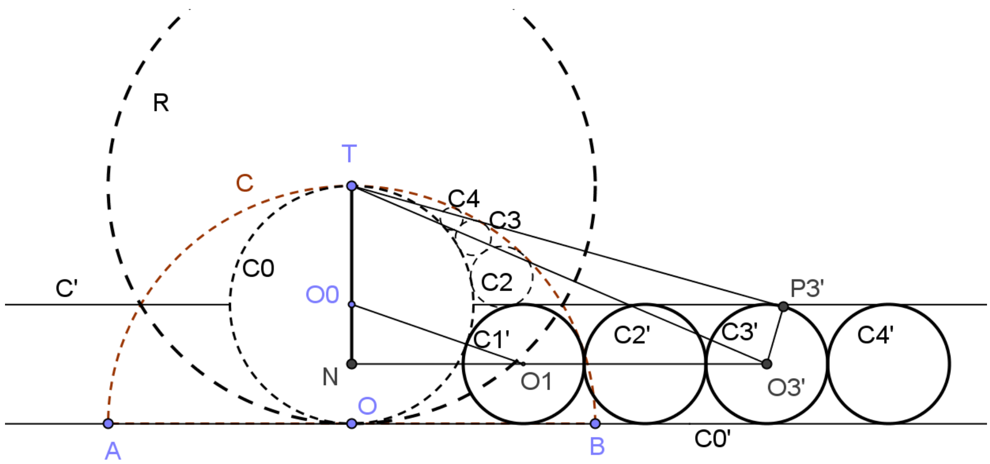

Fig. 12 *Inverted figure with some additional lines*

[2] Detailed answer: Wasan problem 2

Draw a tangent line of circle C_i from point T which has a contact point P_i' (i=0,1,2….) (see Fig. 13).
From the definition of inversion $\quad t^2 = (2r)^2 \cdots\cdots (1)$

From $\quad TO = \frac{t^2}{2r} \qquad r_0' = \frac{t^2}{4r}, \qquad r_1' = r_4' = \frac{t^2}{8r}, \qquad r_2' = r_3' = \frac{t^2}{16r}$

$$(TP_1')^2 = (TO_1')^2 - (r_1')^2 = (TN_1)^2 + (N_1O_1')^2 - (r_1')^2$$

$$= (\frac{3t^2}{8r})^2 + (2\sqrt{\frac{t^2}{4r}}\sqrt{\frac{t^2}{8r}})^2 - (\frac{t^2}{8r})^2 = \frac{t^4}{(2r)^2}$$

(see Reference Materials [3])

from the proposition (see Fig. 3) $\quad \frac{r_1}{r_1'} = \frac{t^2}{(TP_1')^2} \quad$ i.e. $\quad r_1 = \frac{r}{2}$

$$(TP_2')^2 = (TO_2')^2 - (r_2')^2 = \left(\frac{7t^2}{16r}\right)^2 + \left(2\sqrt{r \cdot r_1'} + 2\sqrt{r_1'r_2'}\right)^2 - (r_2')^2 = \frac{15}{8} \times \frac{t^4}{(2r)^2}$$

$$\frac{r_2}{r_2'} = \frac{t^2}{(TP_2')^2} \quad \text{and} \quad r_2' = \frac{t^2}{16r} \quad \text{i.e.} \quad r_2 = \frac{2r}{15}$$

(see Reference Materials [3])

$$(TP_3')^2 = (TO_3')^2 - (r_3')^2 = (TO_0 + r_4')^2 + \left(2\sqrt{r \cdot r_1'} + 2\sqrt{r_1' r_2'}\right)^2 - (r_3')^2$$

$$= \left(\frac{t^2}{4r} + \frac{t^2}{16r}\right)^2 + 4\left(\sqrt{\frac{t^2}{4r}}\sqrt{\frac{t^2}{8r}} + \sqrt{\frac{t^2}{8r}}\sqrt{\frac{t^2}{16r}}\right)^2 - \left(\frac{t^2}{16r}\right)^2 = \frac{3}{2} \cdot \frac{t^4}{(2r)^2}$$

$$\frac{r_3}{r_3'} = \frac{t^2}{(TP_3')^2} \quad \text{and} \quad r_3' = \frac{t^2}{16r} \quad \text{i.e.} \quad r_3 = \frac{r}{6}$$

(see Reference Materials [3])

$$(TP_4')^2 = (TO_4)'^2 - (r_4')^2 = \left(\frac{3t^2}{8r}\right)^2 + \left(2\sqrt{r \cdot r_1'} + 4\sqrt{r_1' r_2'}\right)^2 - (r_4')^2 = \frac{5}{2} \times \frac{t^4}{(2r)^2}$$

$$\frac{r_4}{r_4'} = \frac{t^2}{(TP_4')^2} \quad \text{and} \quad r_4' = \frac{t^2}{8r} \quad \text{i.e.} \quad r_4 = \frac{r}{5}$$

(see Reference Materials [3]) end

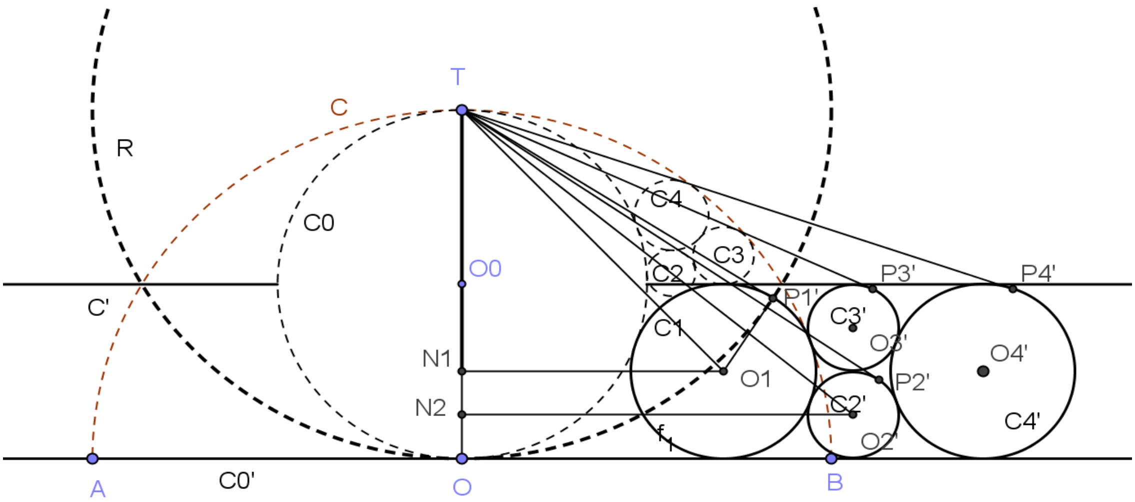

Fig. 13 *Inverted figure with some additional lines*

[3] Lemma: Distance between 2 tangent points.

Circle $C_1(O_1, r_1)$ is circumscribed to circle $C_2(O_2, r_2)$, and these two circles are tangent to common external tangent L. Distance between 2 tangent points T_1T_2 is as follows.

$$(T_1T_2)^2 = (O_1O_2)^2 - (O_1H)^2$$
$$= (r_1 + r_2)^2 - (r_1 - r_2)^2 = 4r_1r_2$$

i.e. $T_1T_2 = 2\sqrt{r_1r_2}$

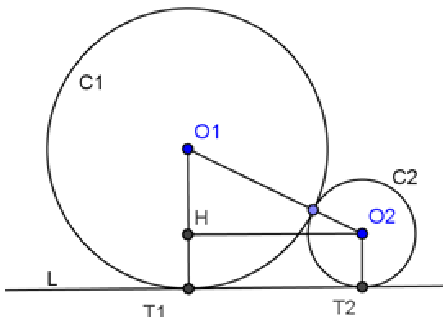

Fig. 14 *Lennma*