Does Technology Empower Students to do Research at the Secondary and Undergraduate Levels?

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Abstract. *Traditionally, most students at the undergraduate, not to mention at the secondary level, are rarely exposed to any research in mathematics. This is mostly due to their lack of mathematical background and the time involved in such activity, in an already crowded curriculum. However, we live in a time when information is becoming readily available, and when basic technology such as hand-held graphing technology (HHGT) allows bridging over cumbersome calculations, and facilitates the access to a variety of new and relevant topics at basic levels. It would seem reasonable to expect that the integration of these innovations is followed by an increased focus on conceptual understanding, applications, and exploration in the mathematics classroom. But, is it reasonable to expect that regular students at these levels will get involved in research and discovery? In this article, we will show some new mathematical results obtained by mostly secondary students empowered by technology. We will also look at some basic conditions that we can adopt in our teaching that seem to foster students' exploration and discovery in mathematics. In addition we will show two new results obtained using multiple representations illustrating the capability that the latest hand-held graphing technology provides for linking in every platform, a variable defined in one of them.*

Few students at the undergraduate and secondary levels seem to be exposed to basic exploration and research in mathematics or even to less traditional problems outside the textbook. One may wonder if regular young students, other than "wonder boys" such as Gauss and Pascal, are capable of doing some research and even finding new results in mathematics. Traditional wisdom points to the lack of students' mathematical background and an already extensive mathematics curriculum as the main reasons for why so little exposure to basic research is taking place. In recent secondary mathematics programs, students are guided to discover concepts and principles on their own. However, as Diane Resnek tell us

"Teachers in these programs must give students guidance when needed but must have enough faith in the process that they do not provide too much guidance. Giving too much guidance is akin to "telling," and students may not end up constructing their own meaning. If high school teachers have learned mathematics only through direct instruction, they have difficulty believing that people—their students in particular—can learn through guided discovery. Without a belief in guided discovery, teachers—often with the best of intentions—will sabotage new programs by teaching them the way that they have always learned."

Thus, one may also wonder if this is not a self perpetuating situation; the lack of exposure to investigation and discovery that many teachers at these levels have received in the traditional curriculum, contributes to their lack of interest or self-confidence that results in not exposing students to basic “unstructured” problems promoting investigation and inquiry.

In the nineties, I became interested and started collecting some discoveries by mostly secondary students empowered by technology. I also realized how attractive these discoveries were for teachers at every level, since, whenever I incidentally mentioned in a conference a discovery by a secondary student, many of the questions by the audience focused on the student’s result, rather than on the main topic I had covered. Finally, in 2001, I published an article that was later reprinted (Quesada, 2001a, b) on the fact that the amount of secondary students’ discoveries in the previous decade suggests that technology, in particular dynamic geometry software and HHGT, was empowering secondary students to find new mathematical results. The article included some interesting results found by students at this level. A synopsis of some of these results is included in Table 1.

Later on, I found a second common thread supporting these discoveries after thinking about these students’ results and talking with some of their teachers. As already mentioned the first thread is that these students used interactive geometry software and graphing calculators. Clearly, these technologies allow students to navigate cumbersome calculations, and, when properly used, facilitate an inquiry-based approach that promotes exploration and discovery. The second thread, not surprisingly, is that most of the students involved have been challenged by their teachers with problems beyond the traditional book exercises. Making problem solving a central part of our everyday mathematics class has always been fundamental. However, the need for covering a syllabus while facilitating the students’ learning of algorithmic processes has often focused most of the students’ work on solving exercises, rather than on true problem solving via inquiry in the Polya’s sense (Polya, 1957). In a time when information and computer algebra systems are readily accessible, the lack of sufficient exposure to problem solving that, in my estimation, a good number of our college seniors (including preservice teachers) and graduate students seem to have, can hardly be justified. In our work with preservice and inservice teachers, we share with them the following questions that we could be asking ourselves to be sure that we are challenging our students on a daily basis:

1. *Do we ask our students to try to generalize their solutions?*

Take, for instance, the simple problem of finding how many games need to be programmed for a round robin tournament involving 10 teams. Students find that 45 games are needed and stop. However, it is easy to extend the question simply by asking how many games would be needed if 12 teams participate? What if n teams participate?

2. *Do we encourage our students to raise their own questions, and try to answer them?*

It has been said that assessment drives curriculum; similarly, for many students, rewards drive their interest. Hence, students’ encouragement goes hand–to-hand with the reward we give them for raising good questions, and further for conjecturing an answer. Having the students working in teams seems to have a positive result in this area.

3. *Do we create extensions—for example, more challenging questions—to the activities students do, and encourage them to do the same?*

In my experience, the more inquiry-based we model in our classes, the better. Raising the right questions is essential to solve problems, and to pose extensions. With some encouragement, and modeling continuously the “what if,” more students start to raise questions as part of what they do, first in their teams, and later in the whole classroom.

4. *Do we dare to ask our students truly challenging questions, questions for which we ourselves may not have an answer?*

I once read a comment attributed to Chesterton, saying that a book with no errors was not pedagogical, since it deprives students from learning to recognize and to correct them. The traditional teacher centered “master lesson” tends to create the wrong impression in many of our students about, among other things, how we do mathematics, and about the lack of open problems at most levels. It is interesting to see how the degree of attention of our students instantly grows the moment we make a mistake on the board. The way we navigate out of such situations, either by solving the problem underlining what created our error and possibly how to prevent it, or in some cases by simply saying : ”I don’t see the solution now, let me think about it,” may have a rather positive impact in our students. Students need to learn that it is normal to get stumped while exploring a problem; and that behind the few lines needed to cleanly state a mathematical theorem, there are many scratched papers and, often too, time of efforts and frustration before reaching a successful proof. Students must realize that there are many open problems at every level, and that if we keep asking the right questions, we are bound to run into one of them.

5. *How do we reward students who accept these challenges?*

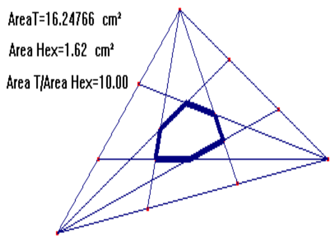
In addition to rewarding the students with the traditional point system, having students present their results to the class, or to prepare a poster to be displayed in the appropriate forum, further educates them and provides invaluable recognition and encouragement.

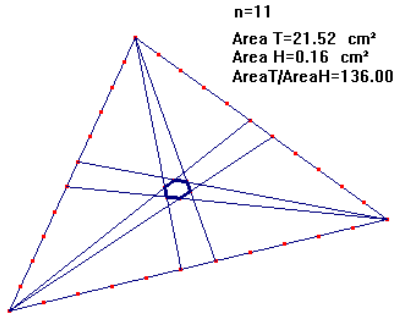
The way we answer these questions may help to foster students’ exploration and discovery in mathematics at any level. If we do not challenge our students or risk posing problems whose solutions we do not know, we will be hiding the true nature of mathematics, and we will miss the opportunity to be rewarded with their findings! (Quesada, 2009)

In the last few years, particularly while working with preservice teachers, I have used this approach, and although some students seem apprehensive about it, the use of teams to tackle unstructured problems that do not require an extensive mathematical background have helped to ease their lack of self-confidence. Students find extensions to some of the problems we proposed in class, and in many cases their explorations yield known results, although some of them are not that well known.

We show next two new results currently submitted for publication. Both were obtained using multiple representations illustrating the capability that the latest hand-held graphing technology provides for linking in every platform, a variable defined in one of them. The limitation on size of the proceedings’ articles prevents us from including the entire proofs. The first is a recent result found by one of my undergraduate students, Andrew Cooper.

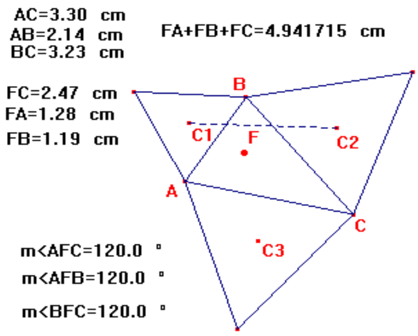
The jagged boundary-edge problem: Given an area of land bounded by two parallel lines and separated by a jagged-edge of arbitrary length into two properties, how can we find a straight segment from one line to the other that maintains the original areas of each property? Is there a solution segment of minimal length? Can the solution be extended to the general case of a jagged-edge between boundary lines that meet? (See figures 1 and 2)

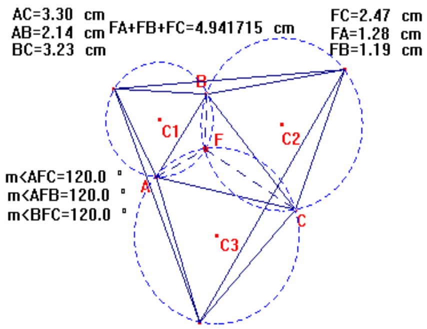




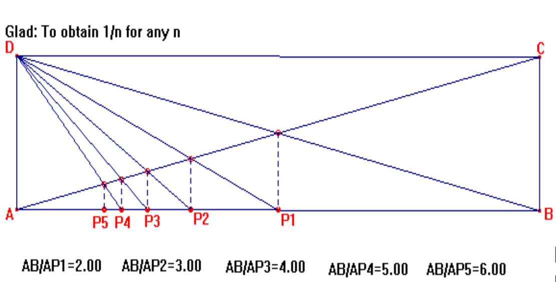
Ryan Morgan's result: For *n* odd, if the central *n*-section points of the sides of any triangle are connected to the opposite vertices, the ratio of the area of the original triangle to the area of the resulting hexagon is $(9n^2 - 1)/8$ to 1.

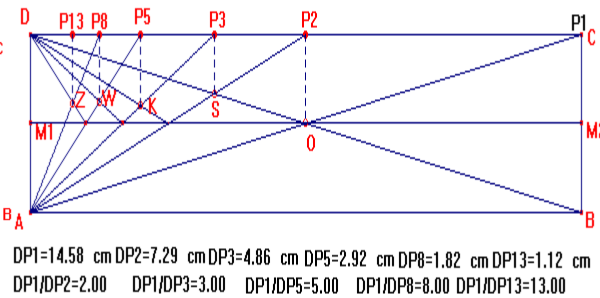
1. Marion Walter's theorem and the extension obtained by Ryan Morgan.



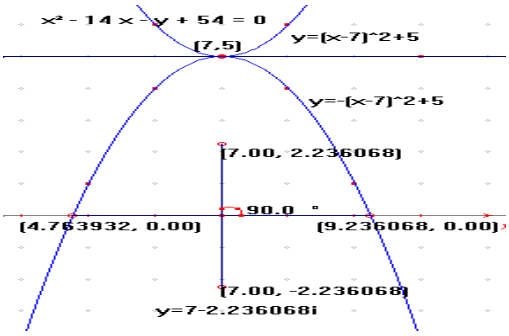


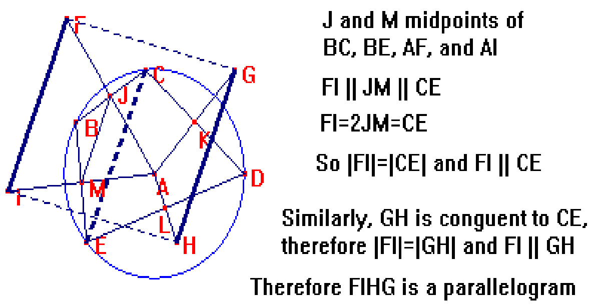
2. A different way of obtaining Fermat's point as a reflection of a vertex on the line connecting the external equilateral triangles centroids.





3. The GLaD construction to subdivide a segment in *n* parts without a compass, and the Fibonacci sequence obtained by D. Goldenheim & D. Litchfield



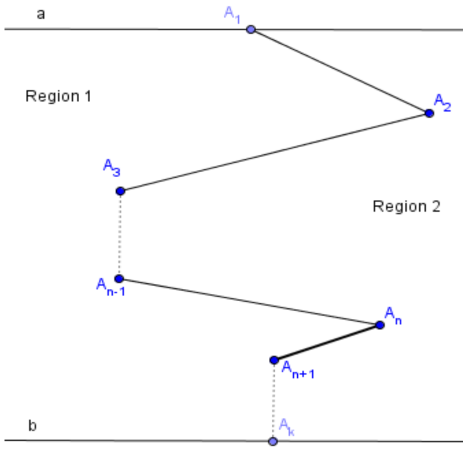


4. A) Shaun Pieper's graphical representation of the imaginary solutions of a quadratic.

B) Solution by Lori Sommar to a geometric problem.

Table 1

With some guidance, Andrew solved this problem geometrically. He also found a recursive algebraic solution based on the coordinates of the points that define the jagged edge. A sketch of his approach follows.



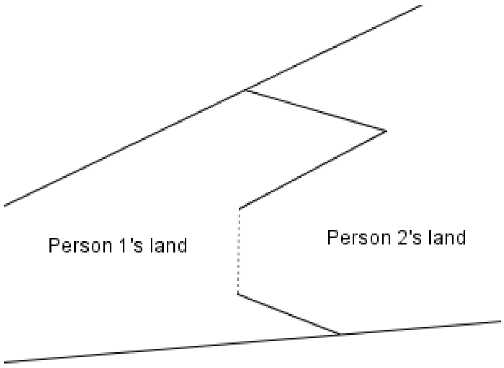
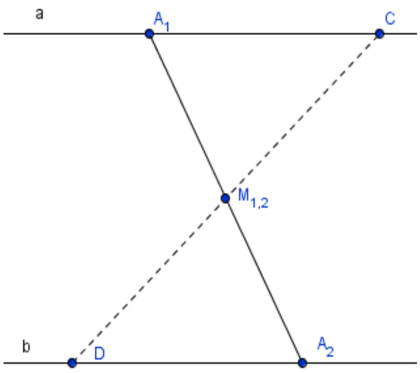
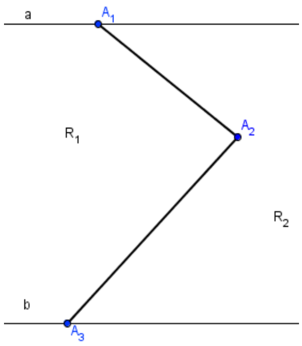


Figure 1 Figure 2

The first lemma (Figure 3) considers a region bounded by two parallel lines *a* and *b* and let $\overline{\boldsymbol{A}_1\boldsymbol{A}_2}$ be a line segment from one parallel to the other that splits the given region into two sub regions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$. It then shows that any other segment from one parallel to the other, that passes through the midpoint $\boldsymbol{M}_{12}$ of the given segment, will preserve the areas of the two sub regions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$.





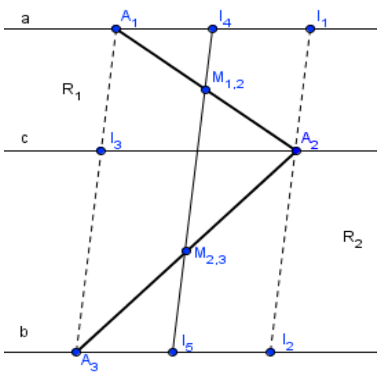


Figure 3 Figure 4 Figure 5

Figures 4 and 5 show how the lemma can be extended to the case of two regions separated by a two-segment boundary, $\overline{\boldsymbol{A}_1\boldsymbol{A}_2}$ and $\overline{\boldsymbol{A}_2\boldsymbol{A}_3}$, by connecting the midpoints $\boldsymbol{M}_{12}$ and $\boldsymbol{M}_{23}$ of these segments. In Figures 6 and 7 we see how the result can be generalized using induction. We have seen a geometrical solution to the jagged-edge problem finding the existence of a "*center point*" *M* such that any segment between the parallel boundaries passing through *M* is a solution. In particular, the minimal solution is the perpendicular to the boundaries through *M*.

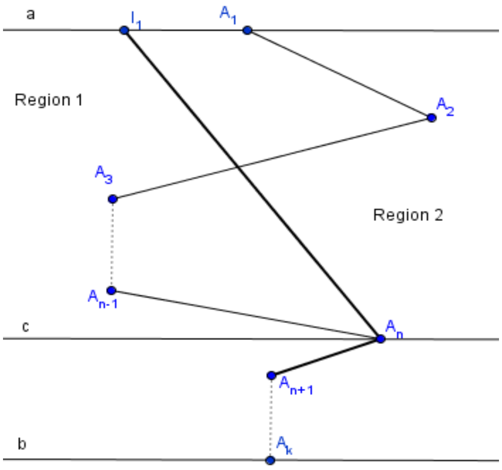


Figure 6

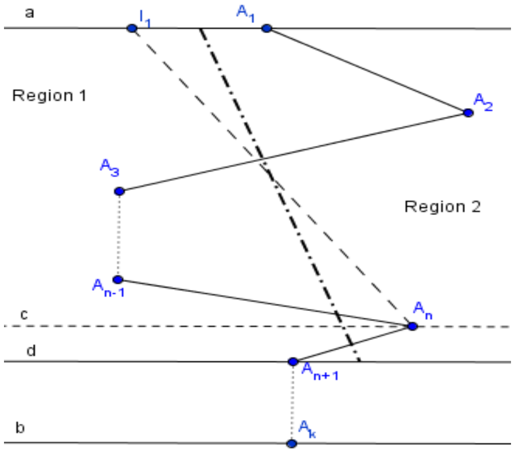


Figure 7

A recursive analytical solution was also found: For any jagged edge $A_1A_2\ldots A_k$ with coordinates (x_i, y_i) between two parallel boundary lines *a* and *b*, the coordinates of the *center point* $M_{12\ldots k}$ are given by:

$$\left(x_{M_{12\ldots k}}, y_{M_{12\ldots k}}\right) = \left(x_{M_{k-1,k}} + (y_{M_{12\ldots k}} - y_{M_{k-1,k}}) \frac{x_{M_{12\ldots k-1}} - x_{M_{k-1,k}}}{y_{M_{12\ldots k-1}} - y_{M_{k-1,k}}}, \frac{a+b}{2} \right).$$

The solution of this problem was extended to the case of the jagged-edge between non-parallel boundaries, which required more involved preparation.

Our last result is a generalization of the GLAD algorithm, initially found by using a TI-Nspire calculator to analyze numerically and graphically the changes that take place in figure 8, drawn using dynamically geometry software .

Theorem. Let $y < h$, and $y, h \in \mathbf{R}^+$. Given a segment $\overline{AD}$, the sequence of points $P_1 = D, P_2, P_3, \ldots, P_n, \ldots$, on $\overline{AD}$ is obtained satisfying that the ratio of the length of each segment of the sequence of segments $\overline{AP_i}$, $i \geq 2$ to AD is given by

$$\frac{AP_i}{AP_1} = \frac{y}{(i-2)y + h}, i \geq 2.$$

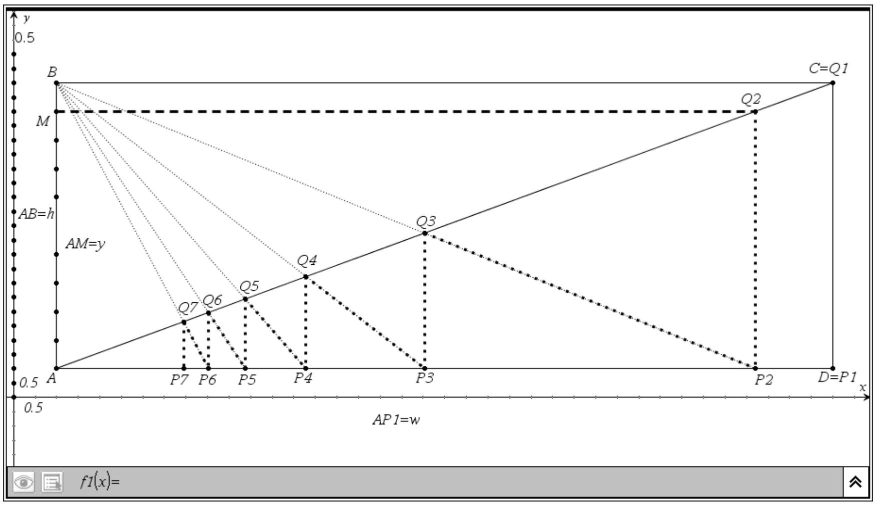


Figure 8

It is easy to check that if *M* is the middle point of *AB*, the result coincides with the GLaD algorithm, that is, $AP_i / AP_1 = 1/n, i \geq 2$.

Conclusion. Currently, technology allows bypassing cumbersome calculations while making information and data readily available. This facilitates the increased use of exploration and discovery in teaching and learning. At the same time, we are called to increase the mathematical knowledge of all students while attracting more students to the areas of science, technology, engineering, and mathematics. Nothing conveys both the beauty and usefulness of mathematics like solving problems. It is worthwhile to increase our students exposure to both structured (guided) and unstructured explorations while showing them why we love our subject.

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