# Integrating Certain Types of Rational Functions without Partial Fractions


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**Abstract:** In this paper, we will discuss a new method of integrating certain types of rational functions without using partial fractions. Our method mainly depends on writing the numerator of the integrand as a linear combination of the factors of the denominator. This method uses only basic algebra, and is generally more efficient than the traditional method. One weakness of the traditional method is that it usually requires solving large systems of equations, which is a tedious task. We have used certain features of the computer algebra system *Mathematica* to verify our results.


## 1. Introduction

In this section, we will briefly review the traditional method for integrating a rational function such as $1/[(x-a)(x-b)]$ where $a$ and $b$ are distinct real numbers. The traditional method is based on partial fractions as described below (see [3]):

First, set up the following identity, where $A$ and $B$ are constants to be determined:

$$\frac{1}{(x-a)(x-b)} \equiv \frac{A}{x-a} + \frac{B}{x-b} \tag{1.1}$$

One can reduce the above equation (1.1) to $1 \equiv A(x-b) + B(x-a)$. Then by substituting special values or by equating the corresponding coefficients, one can find the constants $A$ and $B$. In the latter method, one rewrites the last identity as $1 \equiv x(A+B) + (-bA - aB)$. Then equate the coefficients of $x$ in both sides to get the equation

$$A + B = 0 \tag{1.2}$$

Now equate the constant coefficients on both sides to obtain the following equation:

$$-bA - aB = 1 \tag{1.3}$$

The equations (1.2) and (1.3) define a system of two equations in two unknowns $A$ and $B$. One can solve this system to obtain that $A = 1/(a-b)$ and $B = -1/(a-b)$. Then we can compute the integral as follows, where $C$ is an arbitrary constant:

$$\int \frac{1}{(x-a)(x-b)}dx = \frac{1}{(a-b)}\int \frac{1}{x-a}dx - \frac{1}{(a-b)}\int \frac{1}{x-b}dx = \frac{1}{(a-b)}\ln\left|\frac{x-a}{x-b}\right| + C$$

In this particular case, the partial fraction method appears to be quite easy, but that is not the case when the degree of the denominator of the integrand is larger. In such cases, the partial fraction method might lead to solving large systems of equations, which is a tedious calculation in general. The paper proposes to find alternate methods for integration without using partial fractions. Our methods are largely based on rewriting the numerator of the integrand as a linear combination of the factors of the denominator, not necessarily with constant coefficients. In this paper we will use the term "linear combination" slightly different from the corresponding concept in linear algebra. Given two real polynomials $f(x)$ and $g(x)$, by a linear combination of $f(x)$ and $g(x)$ we mean an expression of the type $p(x)f(x)+q(x)g(x)$ where $p(x)$ and $q(x)$ are some real polynomials.

We can now illustrate the new method, for the integral just discussed above. We can easily write the numerator 1 as a linear combination of the factors $(x-a)$ and $(x-b)$. Observe that $a-b=(x-b)-(x-a)$, which yields the following equation:

$$1=\frac{1}{(a-b)}(x-b)-\frac{1}{(a-b)}(x-a) \tag{1.4}$$

This last equation (1.4) gives the numerator 1 as a linear combination of the factors $(x-a)$ and $(x-b)$. We can now perform the integration as follows, where $C$ is an arbitrary constant:

$$\begin{aligned}\int\frac{1}{(x-a)(x-b)}dx &= \frac{1}{(a-b)}\int\frac{(x-b)-(x-a)}{(x-a)(x-b)}dx = \frac{1}{(a-b)}\left[\int\frac{1}{x-a}dx - \int\frac{1}{x-b}dx\right] \\ &= \frac{1}{(a-b)}\ln\left|\frac{x-a}{x-b}\right| + C\end{aligned} \tag{1.5}$$

In the following sections, we will employ the method just illustrated to a wider class of rational functions. The new method seems quite surprising at first due to its creative aspect. Rather than contenting oneself with the textbook method of integrating using partial fractions, the curious reader will appreciate this new method. The impatient reader can perhaps skip section two momentarily and directly proceed to Examples 3.1 and 3.2 in section three of the paper, in order to see how the method works in the actual practice.

## 2. Integrals of the Type $\int\frac{f(x)}{(x-a)(x-b)}\,dx$ where *a* and *b* are Distinct Real Numbers

In this section we will consider the integral of a rational function of the type $f(x)/[(x-a)(x-b)]$ where $f(x)$ is any polynomial and, $a$ and $b$ are distinct real numbers. Without loss of generality, one can assume that the degree of $f$ is less than two. This is so because, otherwise one can use the long division to divide the numerator $f(x)$ by the denominator $(x-a)(x-b)$ to cut down the degree of $f(x)$ (see [3]). This just means one can write $f(x)$ as a linear function, i.e. $f(x)=px+q$ where $p$ and $q$ are real numbers. Thus the problem reduces to calculating just the two integrals $\int x\,dx/[(x-a)(x-b)]$ and $\int dx/[(x-a)(x-b)]$. Since the latter was just discussed in the previous section, we only have to the consider the integral $\int x\,dx/[(x-a)(x-b)]$.

One can easily write the numerator $x$ as a linear combination of the factors $(x-a)$ and $(x-b)$. Since $(a-b)x = a(x-b) - b(x-a)$, we have the following:

$$x = \frac{a}{(a-b)}(x-b) - \frac{b}{(a-b)}(x-a) \tag{2.1}$$

Thus we can calculate the required integral as follows, where $C$ is an arbitrary constant:

$$\begin{aligned}\int \frac{x}{(x-a)(x-b)}dx &= \frac{1}{(a-b)}\int \frac{a(x-b)-b(x-a)}{(x-a)(x-b)}dx = \frac{a}{(a-b)}\int \frac{1}{(x-a)}dx - \frac{b}{(a-b)}\int \frac{1}{(x-b)} \\ &= \frac{a}{(a-b)}\ln|x-a| - \frac{b}{(a-b)}\ln|x-b| + C\end{aligned} \tag{2.2}$$

The required integral $\int \frac{px+q}{(x-a)(x-b)}dx$ is just a linear combination of the integrals (1.3) and (2.2), given by the following:

$$\int \frac{px+q}{(x-a)(x-b)}dx = \frac{(pa+q)}{(a-b)}\ln|x-a| - \frac{(pb+q)}{(a-b)}\ln|x-b| + C \tag{2.3}$$

## 3. Integrals of the Type $\int \frac{f(x)}{(x-a)(x^2+bx+c)}dx$ where *a*, *b*, and *c* are Real Numbers

In this section we will consider the integral of a rational function of the type $f(x)/[(x-a)(x^2+bx+c)]$ where $f(x)$ is any polynomial and, $a$, $b$, $c$ are real numbers. As described in the previous section, it is enough to consider the case where $f(x) = px^2+qx+r$ with $p$, $q$, and $r$ real numbers. Thus the problem reduces to calculating the following three integrals $I_1, I_2$ and $I_3$:

$$I_1 = \int \frac{1}{(x-a)(x^2+bx+c)}dx \tag{3.1}$$

$$I_2 = \int \frac{x}{(x-a)(x^2+bx+c)}dx \tag{3.2}$$

$$I_3 = \int \frac{x^2}{(x-a)(x^2+bx+c)}dx \tag{3.3}$$

We will first consider the integral $I_1$. The significance of the following two cases will be apparent in a short while.

**(a) Case 1**: $a^2 + ab + c \neq 0$

Our first task is to write the numerator 1 of the integral $I_1$ as a linear combination of the factors $(x-a)$ and $(x^2+bx+c)$. It is easy to verify the following two equations:

$$x(x-a)-(x^2+bx+c) = x(-a-b)-c \tag{3.4}$$

$$(x+a)(x-a)-(x^2+bx+c) = -bx-(a^2+c) \tag{3.5}$$

Multiply the equation (3.4) by *b*, equation (3.5) by $(-a-b)$, and add the two results to obtain the new equation $a^3+a^2b+ac=(x-a)(-ax-a^2-ab)+a(x^2+bx+c)$. If $a \neq 0$, one can divide both sides by *a* to obtain the equation $a^2+ab+c=(x-a)(-x-a-b)+(x^2+bx+c)$. However, this last equation is true even if $a=0$, as it can be verified directly. Since we are assuming that $a^2+ab+c \neq 0$, one can divide both sides of the last equation by $a^2+ab+c$ to obtain 1 as a linear combination of the factors $(x-a)$ and $(x^2+bx+c)$:

$$1 = \frac{-(x+a+b)}{(a^2+ab+c)}(x-a) + \frac{1}{(a^2+ab+c)}(x^2+bx+c) \tag{3.6}$$

Thus, we are now in a position to calculate the first integral $I_1$ :

$$\begin{aligned}(a^2+ab+c)I_1 &= \int \frac{-(x+a+b)(x-a)+(x^2+bx+c)}{(x-a)(x^2+bx+c)}dx = \int \frac{-(x+a+b)}{(x^2+bx+c)}dx + \int \frac{1}{(x-a)}dx \\ &= \ln|x-a| - \int \frac{(1/2)(2x+b)+(a+b/2)}{x^2+bx+c}dx \\ &= \ln|x-a| - \frac{1}{2}\ln|x^2+bx+c| - \left(a+\frac{b}{2}\right)\int \frac{1}{x^2+bx+c}dx\end{aligned} \tag{3.7}$$

Thus, we have the following relationship where $J = \int dx/(x^2+bx+c)$:

$$I_1 = \frac{1}{(a^2+ab+c)}\ln|x-a| - \frac{1}{2(a^2+ab+c)}\ln|x^2+bx+c| - \frac{(2a+b)}{2(a^2+ab+c)}J \tag{3.8}$$

In order to evaluate the integral $J$ on the right-hand side of (3.8), we need to consider two cases. This is so because depending on *b* and *c*, the quadratic expression $x^2+bx+c$ may or may not have real factors. Recall from elementary algebra that $x^2+bx+c$ has real factors if and only if the discriminant $b^2-4c \geq 0$ (see [1]).

*Sub case* 1: $b^2-4c \geq 0$

In this case, one can write $x^2+bx+c=(x-u)(x-v)$ for some real numbers *u* and *v*, so that the integral $J$ can be calculated using the method described in the last half of the section one of the paper.

*Sub case* 2: $b^2 - 4c < 0$

In this case, $x^2 + bx + c$ does not have real factors, and one has to use the method of completing the square (see [3]):

$$J = \int \frac{1}{x^2 + bx + c} dx = \int \frac{1}{(x + b/2)^2 + (4c - b^2)/(4)} dx = \frac{2}{\sqrt{4c - b^2}} Arc\tan\left( \frac{2x + b}{\sqrt{4c - b^2}} \right) + C$$

Therefore, in either sub case, the integral $I_1$ can be calculated via equation (3.8), under the first case, where we have assumed that $a^2 + ab + c \neq 0$. We still have to calculate $I_1$ for the case where $a^2 + ab + c = 0$.

**(b) Case 2**: $a^2 + ab + c = 0$

This case implies that $x^2 + bx + c = x^2 + bx - a^2 - ab = (x - a)(x + a + b)$. Therefore, in this case, the denominator of the integral $I_1$ breaks into three linear factors. However, we don't need the partial fractions to calculate $I_1$, as illustrated below.

*Sub case* 1: $-2a - b \neq 0$

$$\begin{aligned} I_1 &= \int \frac{1}{(x-a)(x^2 + bx + c)} dx = \int \frac{1}{(x-a)^2 (x + a + b)} dx = \frac{-1}{(2a+b)} \int \frac{(x-a) - (x + a + b)}{(x-a)^2 (x + a + b)} dx \\ &= \frac{-1}{(2a+b)} \left[ \int \frac{1}{(x-a)(x+a+b)} dx - \int \frac{1}{(x-a)^2} dx \right] \\ &= \frac{-1}{(2a+b)(x-a)} + \frac{1}{(2a+b)^2} \int \frac{(x-a) - (x+a+b)}{(x-a)(x+a+b)} dx \\ &= \frac{-1}{(2a+b)(x-a)} + \frac{1}{(2a+b)^2} \ln\left| \frac{x+a+b}{x-a} \right| + C \end{aligned}$$

*Sub case* 1: $-2a - b = 0$

Under this case $x^2 + bx + c = (x-a)(x+a+b) = (x-a)(x+a-2a) = (x-a)^2$. Thus, we calculate:

$$I_1 = \int \frac{1}{(x-a)(x^2 + bx + c)} dx = \int \frac{1}{(x-a)^3} dx = - \frac{1}{2(x-a)^2} + C$$

The above discussion calculates the integral $I_1$ in all possible cases.

We must now consider the integral $I_2$ given by equation (3.2). Here, we will write the numerator $x$ as $(x-a) + a$, this time not necessarily as a linear combination of factors in the denominator, and then decompose $I_2$ into two simpler integrals:

$$I_2 = \int \frac{(x-a) + a}{(x-a)(x^2 + bx + c)} dx = \int \frac{1}{x^2 + bx + c} dx + a \int \frac{1}{(x-a)(x^2 + bx + c)} dx = J + I_1 \quad (3.9)$$

Since we have calculated $J$ and $I_1$ previously, the above equation (3.9) calculates the integral $I_2$.

We employ a similar technique to calculate the integral $I_3$ given by equation (3.3):

$$\begin{aligned} I_3 &= \int \frac{(x^2-a^2)+a^2}{(x-a)(x^2+bx+c)}dx = \int \frac{x+a}{x^2+bx+c}dx + a^2\int \frac{1}{(x-a)(x^2+bx+c)} \\ &= \int \frac{(2x+b)/2+(a-b/2)}{x^2+bx+c}dx + a^2 I_1 \qquad (3.10) \\ &= \frac{1}{2}\ln|x^2+bx+c| + \left(a-\frac{b}{2}\right)J + a^2 I_1 \end{aligned}$$

Since we have calculated $J$ and $I_1$ previously, the above equation (3.10) calculates the integral $I_3$. Finally the required integral $\int (px^2+qx+r)\,dx/[(x-a)(x^2+bx+c)]$ can be calculated as a combination of the integrals $I_1, I_2$ and $I_3$.

**Example 3.1** Calculate the integral $\int \frac{2x+3}{(x-1)(x^2+3x+4)}dx$.

Observe that the quadratic $x^2+3x+4$ does not have any real factors, so the traditional partial fraction method is even more tedious. However, we are now in a position to calculate this integral a bit faster!

$$\begin{aligned} \int \frac{2x+3}{(x-1)(x^2+3x+4)}dx &= \int \frac{2(x-1)+5}{(x-1)(x^2+3x+4)}dx = 2\int \frac{1}{x^2+3x+4}dx + 5\int \frac{1}{(x-1)(x^2+3x+4)}dx \\ &= 2\int \frac{1}{x^2+3x+4}dx + \frac{5}{8}\int \frac{-(x+4)(x-1)+(x^2+3x+4)}{(x-1)(x^2+3x+4)}dx \\ &= 2\int \frac{1}{x^2+3x+4}dx - \frac{5}{8}\int \frac{x+4}{x^2+3x+4}dx + \frac{5}{8}\ln|x-1| \\ &= 2\int \frac{1}{x^2+3x+4}dx - \frac{5}{8}\int \frac{(2x+3)/2+5/2}{x^2+3x+4}dx + \frac{5}{8}\ln|x-1| \\ &= \frac{7}{16}\int \frac{1}{(x+3/2)^2+7/4}dx - \frac{5}{16}\ln|x^2+3x+4| + \frac{5}{8}\ln|x-1| \\ &= \frac{\sqrt{7}}{8} Arc\tan\left(\frac{2x+3}{\sqrt{7}}\right) - \frac{5}{16}\ln x^2+3x+4| + \frac{5}{8}\ln|x-1| + C \end{aligned}$$

It is important to verify the accuracy of the new method by some other means. One option is to redo the calculation using partial fractions. However, since this is a time consuming task, a computer algebra system such as *Mathematica* can come to our rescue (see [2] and [4]). We have used *Mathematica* version 5.0 on a Windows XP platform. For example, the "**Integrate**" command of *Mathematica* can be used to verify the above integral:

**Input:** **Integrate[(2x+3)/((x-1)(x^2+3x+4)),x]**

To execute the command, press "**Shift-Enter**" at the end of the command line. The output was identical to our answer. ■

**Example 3.2** Calculate the integral $\int \frac{2x^2+3x+1}{x^3+1}dx$.

Note that $x^3+1=(x+1)(x^2-x+1)$ and use the factors $(x+1)$ and $(x^2-x+1)$ to rewrite the numerator at the *right time*.

$$\begin{aligned}
\int \frac{2x^2+3x+1}{x^3+1}dx &= 2\int \frac{x^2}{x^3+1}dx + \int \frac{3(x+1)-2}{x^3+1}dx \\
&= \frac{2}{3}\ln|x^3+1| + 3\int \frac{1}{x^2-x+1}dx - \int \frac{(x+1)+(x^2-x+1)-x^2}{x^3+1}dx \\
&= \frac{2}{3}\ln|x^3+1| + 3\int \frac{1}{x^2-x+1}dx - \int \frac{1}{x^2-x+1}dx - \ln|x+1| + \int \frac{x^2}{x^3+1}dx \\
&= \frac{2}{3}\ln|x^3+1| + 2\int \frac{1}{(x-1/2)^2+3/4}dx - \ln|x+1| + \frac{1}{3}\ln|x^3+1| \\
&= \ln|x^3+1| - \ln|x+1| + \frac{4}{\sqrt{3}}Arc\tan\left(\frac{2x-1}{\sqrt{3}}\right) + C \\
&= \ln|x^2-x+1| + \frac{4}{\sqrt{3}}Arc\tan\left(\frac{2x-1}{\sqrt{3}}\right) + C
\end{aligned}$$

Let us check the above answer using *Mathematica*:

**Input: Integrate [(2x^2+3x+1)/(x^3+1), x]**

Press "**Shift-Enter**" to execute the above command. We found the output to be identical to the above answer, verifying the accuracy of the method. ■

Using a method very similar to Example 3.2, one can calculate the integral $\int \frac{px^2+qx+r}{x^3+a^3}dx$, in general terms, where $p$, $q$, $r$, and $a$ are arbitrary real numbers. The reader is encouraged to go through the calculation in order to properly understand the linear combination method presented in this paper.

In the next section, we will consider the integration of yet a different class of rational functions without partial fractions.

## 4. Integrals of the Type $\int \frac{f(x)}{x^4+a^4}\,dx$ where *a* is a Nonzero Real Number

As you can see, we are gradually increasing the degree of the denominator, this time to four. The denominator is not the general fourth degree polynomial, but for the time being, we will content ourselves with this type. Like before, there is no harm in assuming that $f(x)$ is of the type $px^3+qx^2+rx+s$, where *p*, *q*, *r*, and *s* are real numbers.

Even the elementary algebra student is familiar with the real factors of $x^4-a^4$, but the real factors of $x^4+a^4$ are less known. However, using a simple, but an elegant trick, one can produce the real factors of $x^4+a^4$:

$$x^4+a^4=\left(x^2+a^2\right)^2-2a^2x^2=\left(x^2+a^2\right)^2-\left(ax\sqrt{2}\right)^2=\left(x^2+ax\sqrt{2}+a^2\right)\left(x^2-ax\sqrt{2}+a^2\right) \quad (4.1)$$

The factors given in (4.1) will be very useful to us throughout this section. Since our task is to consider the integrals of the type $\int (px^3+qx^2+rx+s)\,dx/(x^4+a^4)$, it is sufficient to analyze the following four simple types of integrals $I_n$, where $n=0,1,2,$ and 3:

$$I_n = \int \frac{x^n}{x^4+a^4}\,dx \quad (4.2)$$

First observe that the two integrals $I_1$ and $I_3$ are quite trivial. The *u*-substitution $u=x^2$ reveals that $I_1=(1/(2a^2))Arc\tan(x^2/a^2)+C$, and it is also easy to see that $I_3=(1/4)\ln|x^4+a^4|+C$. Thus, we only have to analyze the other two integrals $I_0$ and $I_1$. Our method of bypassing partial fractions is based on the following two identities:

$$(x^2+ax\sqrt{2}+a^2)+(x^2-ax\sqrt{2}+a^2)=2x^2+2a^2 \quad (4.3)$$

$$x(x^2+ax\sqrt{2}+a^2)-x(x^2-ax\sqrt{2}+a^2)=2a\sqrt{2}\,x^2 \quad (4.4)$$

In order to write the numerator of the integrals $I_0$ and $I_2$ as a linear combination of the factors $(x^2+ax\sqrt{2}+a^2)$ and $(x^2-ax\sqrt{2}+a^2)$ of the denominator, we will actively employ the equations (4.3) and (4.4).

Here is the calculation for $I_2$, which uses equation (4.4):

$$I_2 = \int \frac{x^2}{x^4 + a^4} dx = \frac{1}{2a\sqrt{2}} \int \frac{x(x^2 + ax\sqrt{2} + a^2) - x(x^2 - ax\sqrt{2} + a^2)}{(x^2 + ax\sqrt{2} + a^2)(x^2 - ax\sqrt{2} + a^2)} dx$$

$$= \frac{1}{2a\sqrt{2}} \int \frac{x}{(x^2 - ax\sqrt{2} + a^2)} dx \;-\; \frac{1}{2a\sqrt{2}} \int \frac{x}{(x^2 + ax\sqrt{2} + a^2)} dx$$

$$= \frac{1}{2a\sqrt{2}} \int \frac{(2x - a\sqrt{2})/2 + a/\sqrt{2}}{(x^2 - ax\sqrt{2} + a^2)} dx \;-\; \frac{1}{2a\sqrt{2}} \int \frac{(2x + a\sqrt{2})/2 - a/\sqrt{2}}{(x^2 + ax\sqrt{2} + a^2)} dx$$

$$= \frac{1}{4a\sqrt{2}} \ln\left|\frac{x^2 - ax\sqrt{2} + a^2}{x^2 + ax\sqrt{2} + a^2}\right| \;+\; \frac{1}{4}\int \frac{1}{(x - a/\sqrt{2})^2 + a^2/2} dx \;+\; \frac{1}{4}\int \frac{1}{(x + a/\sqrt{2})^2 + a^2/2} dx$$

$$= \frac{1}{4a\sqrt{2}} \ln\left|\frac{x^2 - ax\sqrt{2} + a^2}{x^2 + ax\sqrt{2} + a^2}\right| + \frac{\sqrt{2}}{4a} Arc\tan\left(\frac{x\sqrt{2} - a}{a}\right) + \frac{\sqrt{2}}{4a} Arc\tan\left(\frac{x\sqrt{2} + a}{a}\right) + C$$

The above indicated method of calculating $I_2$ based on the equation (4.4) has a definite advantage over the traditional partial fraction method. The traditional method would have required solving a tedious 4X4 system of equations.

We will now calculate $I_0$, using equation (4.3):

$$I_0 = \int \frac{1}{x^4 + a^4} dx = \frac{1}{2a^2} \int \frac{(x^2 + ax\sqrt{2} + a^2) + (x^2 - ax\sqrt{2} + a^2) - 2x^2}{(x^2 + ax\sqrt{2} + a^2)(x^2 - ax\sqrt{2} + a^2)} dx$$

$$= \frac{1}{2a^2} \int \frac{1}{(x^2 - ax\sqrt{2} + a^2)} dx \;+\; \frac{1}{2a^2} \int \frac{1}{(x^2 + ax\sqrt{2} + a^2)} dx \;-\; \frac{1}{a^2} \int \frac{x^2}{x^4 + a^4} dx$$

$$= \frac{1}{2a^2} \int \frac{1}{(x - a/\sqrt{2})^2 + a^2/2} dx \;+\; \frac{1}{2a^2} \int \frac{1}{(x + a/\sqrt{2})^2 + a^2/2} dx \;-\; \frac{1}{a^2} I_2$$

$$= \frac{1}{a^3\sqrt{2}} ArcTan\left(\frac{x\sqrt{2} - a}{a}\right) \;+\; \frac{1}{a^3\sqrt{2}} ArcTan\left(\frac{x\sqrt{2} + a}{a}\right) \;-\; \frac{1}{a^2} I_2$$

$$= \frac{\sqrt{2}}{4a^3} ArcTan\left(\frac{x\sqrt{2} - a}{a}\right) \;+\; \frac{\sqrt{2}}{4a^3} ArcTan\left(\frac{x\sqrt{2} + a}{a}\right) \;-\; \frac{1}{4a^3\sqrt{2}} \ln\left|\frac{x^2 - ax\sqrt{2} + a^2}{x^2 + ax\sqrt{2} + a^2}\right| + C$$

We have now calculated all four integrals $I_0, I_1, I_2,$ and $I_3$. The required integral $\int (px^3 + qx^2 + rx + s)\, dx/(x^4 + a^4)$ is simply equal to $pI_3 + qI_2 + rI_1 + sI_0$, and thus can be calculated easily. We will illustrate the foregoing procedure by a concrete example:

**Example 4.1** Calculate the integral $\int \frac{2x^3+3x^2+x+1}{x^4+1}dx$.

First observe that $x^4+1=(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1)$. These factors are used in the calculation below:

$$\int \frac{2x^3+3x^2+x+1}{x^4+1}dx = 2\int \frac{x^3}{x^4+1}dx + \int \frac{x}{x^4+1}dx + 3\int \frac{x^2}{x^4+1}dx + \int \frac{1}{x^4+1}dx$$

$$= \frac{1}{2}\ln|x^4+1| + \frac{1}{2}ArcTan(x^2) + 3\int \frac{x^2}{x^4+1}dx + \frac{1}{2}\int \frac{(x^2+x\sqrt{2}+1)+(x^2-x\sqrt{2}+1)-2x^2}{x^4+1}$$

$$= \frac{1}{2}\ln|x^4+1| + \frac{1}{2}ArcTan(x^2) + \frac{1}{2}\int \frac{1}{(x^2-x\sqrt{2}+1)}dx + \frac{1}{2}\int \frac{1}{(x^2+x\sqrt{2}+1)}dx + 2\int \frac{x^2}{x^4+1}dx$$

$$= \frac{1}{2}\ln|x^4+1| + \frac{1}{2}ArcTan(x^2) + \frac{1}{2}\int \frac{1}{(x^2-x\sqrt{2}+1)}dx + \frac{1}{2}\int \frac{1}{(x^2+x\sqrt{2}+1)}dx$$

$$+ \frac{1}{\sqrt{2}}\int \frac{x(x^2+x\sqrt{2}+1)-x(x^2-x\sqrt{2}+1)}{x^4+1}dx$$

$$= \frac{1}{2}\ln|x^4+1| + \frac{1}{2}ArcTan(x^2) + \frac{1}{2}\int \frac{1}{(x^2-x\sqrt{2}+1)}dx + \frac{1}{2}\int \frac{1}{(x^2+x\sqrt{2}+1)}dx$$

$$+ \frac{1}{\sqrt{2}}\int \frac{x}{(x^2-x\sqrt{2}+1)}dx - \frac{1}{\sqrt{2}}\int \frac{x}{(x^2+x\sqrt{2}+1)}dx$$

$$= \frac{1}{2}\ln|x^4+1| + \frac{1}{2}ArcTan(x^2) + \frac{1}{2}\int \frac{1}{(x^2-x\sqrt{2}+1)}dx + \frac{1}{2}\int \frac{1}{(x^2+x\sqrt{2}+1)}dx$$

$$+ \frac{1}{2\sqrt{2}}\int \frac{(2x-\sqrt{2})/2+1/\sqrt{2}}{(x^2-x\sqrt{2}+1)}dx - \frac{1}{2\sqrt{2}}\int \frac{(2x+\sqrt{2})/2-1/\sqrt{2}}{(x^2+x\sqrt{2}+1)}dx$$

$$= \frac{1}{2}\ln|x^4+1| + \frac{1}{2}ArcTan(x^2) + \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right| + \int \frac{1}{(x^2-x\sqrt{2}+1)}dx + \int \frac{1}{(x^2+x\sqrt{2}+1)}dx$$

$$= \frac{1}{2}\ln|x^4+1| + \frac{1}{2}ArcTan(x^2) + \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right| + \int \frac{1}{(x-1/\sqrt{2})^2+1/2}dx$$

$$+ \int \frac{1}{(x+1/\sqrt{2})^2+1/2}dx$$

$$= \frac{1}{2}\ln|x^4+1| + \frac{1}{2}ArcTan(x^2) + \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right| + \sqrt{2}ArcTan(x\sqrt{2}-1)$$

$$+ \sqrt{2}ArcTan(x\sqrt{2}+1) + C$$

We were quite anxious to check the above result, especially since it was a long computation. Using *Mathematica*, we were able to verify the accuracy of the above. ■

In this section we showed how to calculate an integral of the type $\int (px^3 + qx^2 + rx + s)\,dx/(x^4 + a^4)$ without using partial fractions. Earlier we commented on the fact that the denominator of our integrand is not the general fourth degree polynomial. Note that any fourth degree polynomial with real coefficients can be written as a product of two real quadratic factors. Therefore, a more general problem is to calculate an integral of the type $\int (px^3 + qx^2 + rx + s)\,dx/[(x^2 + ax + b)(x^2 + cx + d)]$ without using partial fractions. Under special cases, such as $a = c$, or $b = d$, one can use a method very similar to this section to perform the calculation. However, the general problem seems to be a bit more challenging and the reader is invited to try it.

## 5. Integrals of Other Types of Rational Functions

There are many other types of integrals that can be computed using the methods presented in this paper. Here is a partial list, where $f(x)$ is any real polynomial, and $a$ is any nonzero real number.

(a) $\int \frac{f(x)}{x^4 - a^4}\,dx$ (b) $\int \frac{f(x)}{x^6 + a^6}\,dx$ (c) $\int \frac{f(x)}{x^8 + a^8}\,dx$

The author has devised methods to solve each one of the above types without using partial fractions. Due to the space limitations we are not able to present those solutions here, but the reader is invited to attempt them.

## Conclusion

In this paper, we showed how to integrate several types of rational functions without using partial fractions. The traditional integration method using partial fractions usually involves solving tedious systems of equations. However, our method seems to be quite efficient, and as evidenced by some of the examples given in the paper, has a definite advantage over the traditional method. Moreover, our method is quite simple as it only relies on basic algebra, and is even accessible to the first-year calculus student. One main difference between the two methods is that the partial fractions method is very methodical, and doesn't require any creativity on the student's part. On the other hand, our method based on linear combinations requires some kind of creativity, and it is an ideal method for the thinking student. The method we have presented is not given in any standard calculus text, so we hope that the reader will appreciate the creative aspect behind it. In order to become more familiar with the method, the reader is at the least encouraged to attempt the three types of integrals given in the last section of the paper.

**References**


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