A PROBLEM SOLVING METASTRATEGY

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Abstract: *The introduction of calculators (or computers) with built-in Computer Algebra Systems into senior secondary mathematics classrooms has exacerbated a problem caused by the introduction of graphical calculators. Algebraic calculators provide students with a greatly increased range of strategies and procedures that can be used in problem solving. Many problems can now be solved using numerical, graphical, algebraic or geometrical procedures. Some of these procedures provide exact solutions, while others provide only approximate solutions. Students need some strategies for deciding which procedure to employ in a particular problem. It is suggested that, especially when using mathematical modelling to solve real-world problems, one such strategy is to consider whether or not an exact solution is possible, justified and necessary, given the context of the problem.*

Introduction

Prior to the introduction of graphical calculators, many of the problems presented to secondary school students could be solved using either exact algebraic techniques or approximate graphical techniques. The accuracy of the graphical approaches was limited by the students' ability to draw accurate graphs.

Consider the problem:

> "Squaring a number gives the same result as adding five to four times the number. What is the number?"

The problem can be solved in a multitude of ways.

Many (possibly most) numerate readers would immediately think: *"this can be solved by writing an equation and using algebra to solve the equation and hence the problem.*"

e.g. let the number be x.

then the square of the number is x^2 so the problem can be expressed as $x^2 = 4x + 5$, and solving:

$$\begin{aligned} x^2 &= 4x+5 \\ x^2 - 4x - 5 &= 0 \\ (x-5)(x+1) &= 0 \\ x &= 5 \text{ or } {}^{-}1 \end{aligned}$$

the number was either 5 or ⁻1.

This problem (and related problems) can be solved using numerical, graphical, or geometric techniques as well as this, and other, algebraic techniques. Some of these methods for solving the word problem above and similar problems are classified in table 1.

A computer algebra system (CAS) on either a computer or calculator allows a year 11 student to solve the equation and hence the problem in all of these ways (except the diagrammatic). Throughout this paper, screen dumps from a Texas Instruments TI-89 have been used to illustrate, in outline form, how the procedure would be used on an algebraic calculator. Similar procedures are possible on other algebraic calculators such as the TI-nspire, Casio Classpad or Hewlett Packard HP39.

	Strategy			
	Numerical	**Graphical**	**Algebraic**	**Geometric**
Approach	Guess-and-check	Trace	Null factor theorem	Diagram
	Table of values	Intersection	Quadratic formula	
	Newtonian approximation	Intercepts	Complete the square	
			Factorise both sides	

Table 1 Classification of problem solving strategies and approaches

Numerical techniques

The simplest, guess-and-check method, involves entering the equation, guessing a value for *x*, substituting into the equation and checking if the answer is true.

$$\blacksquare\ x^2 = 4\cdot x + 5 \mid x = 4 \qquad \texttt{false}$$

```
x^2=4x+5|x=4
```

Figure 1 Solution by trial and error

The problem, for the student, with this method is that they have no feedback about how close their guess is, if it is false. (If a student used this strategy and pen-on-paper techniques, they would have some feedback because they would know the values of expressions on the left- and right-hand sides of the equation.)

$$\blacksquare\ x^2 = 4\cdot x + 5 \mid x = 5 \qquad \texttt{true}$$

```
x^2=4x+5|x=5
```

Figure 2 Solution by guess and check

If, as in this example, the solution is an integer, a student may well be able to correctly guess after a few trials without any feedback. But if the solution is irrational or a terminating decimal, the likelihood of success is small. The student could go on guessing indefinitely without ever getting the answer “true”.

Creating a table of values showing x, x^2, and $4x + 5$ requires no algebraic manipulation. The only algebraic concept involved is that of substituting a value for an unknown (variable) into an expression.

Entering $y1 = x^2$ and $y2 = 4x + 5$, then setting up a table with table start = 0 and increment = 1, generates the table:

Scrolling down, one of the two solutions becomes apparent.

✓y1=x^2
✓y2=$4 \cdot x + 5$

x	y1	y2
2.	4.	13.
3.	9.	17.
4.	16.	21.
5.	25.	25.
6.	36.	29.

Figure 3 Solution using a table of values

This method has the advantage of providing feedback regarding the closeness of each estimate.

If the exact result is not evident then an examination of the table may reveal the interval in which it lies. For example, the table of values for the quadratic equation $12x^2 - 79x + 117 = 0$ is shown in Figure 4 Since the sign changes twice, there must be solutions in the intervals $2 < x < 3$ and $4 < x < 5$

Adjusting the increment in x to 0.1 narrows the intervals in which the exact solution can be located to $2.2 < x < 2.3$ and $4.3 < x < 4.4$.

x	y1
2.	7.
3.	-12.
4.	-7.
5.	22.

x	y1
2.1	4.02
2.2	1.28
2.3	-1.22
2.4	-3.48

Figure 4 Solution by successive approximation

If the solution is irrational, the approximate solution, correct to 2 decimal places, should be found with three or four iterations.

Graphical techniques

The problem can be solved graphically in several different ways. The simplest (involving the least algebra) is to graph the two functions.

Using only basic graphing skills, the solution can be found using the trace function. This approach, however, provides only an approximate solution in most cases.

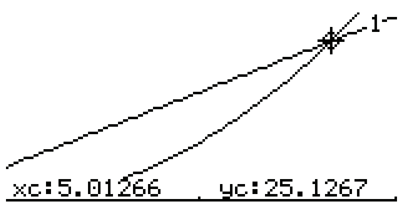

Figure 5 Solution using the trace facility

The exact solution (i.e. coordinates of the point of intersection) can be found using the CAS capability of the calculator or computer. (A graphical calculator will also give a close approximation using an iterative numerical technique.)

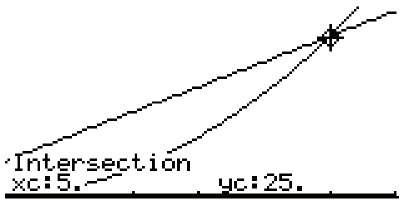

Figure 6 Solution using the intersection

Using a minimum of algebra, the equation can be expressed in the standard form for a quadratic as $x^2 - 4x - 5 = 0$. If the function $y = x^2 - 4x - 5$ is graphed, the zeros of the function will be the solution of the required equation.

With all of the approaches discussed above, there is a likelihood that students will not identify the

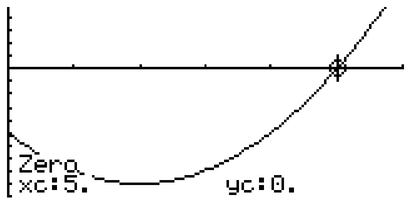

Figure 7 Solution using the zeros

second solution. The exception is the zeros method, where, in this example at least, students are likely to realise that the graph has a second *x*-intercept and change the domain and range so as to reveal the second solution.

All of these approaches are equally amenable to being performed on either a graphic or algebraic

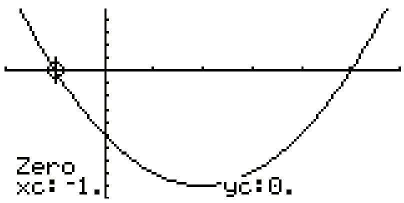

Figure 8 Complete solution using the zeros

calculator, or in the case of trial and error, on a four function or scientific calculator.

Algebraic techniques

The algebraic (CAS enabled) calculator can also perform symbolic manipulation. At its simplest, it can solve equations, as shown in Figure 9:

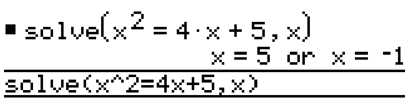

Figure 9 Using the solve facility

However, any of the usual pen-on-paper techniques employed for solving quadratic equations can be employed. For example,

factorisation:

```
■ factor(x^2 - 4·x - 5)
                    (x - 5)·(x + 1)
factor(x^2-4*x-5)
```

Figure 10 Factorising an expression

quadratic formula:

```
■ x | a = 1 and b = -4 and c =▶
                                  5
x|a=1 and b=-4 and c=-5
```

Figure 11 Using the quadratic formula

completing the square

```
■ x^2 = 4·x + 5        x^2 = 4·x + 5
■ (x^2 = 4·x + 5) - 4·x
                       x^2 - 4·x = 5
ans(1)-4x
```

```
■ √(x^2 - 4·x + 4 = 9)     |x - 2| = 3
√(ans(1))
```

Figure 12 Completing the square

A little known algebraic method involves factorising both sides of a quadratic equation, without first rearranging it into the standard form $ax^2 + bx + c = 0$. For example, to "solve" the equation $x^2 + 5x + 6 = 132$, proceed as shown opposite:

$$x^2 + 5x + 6 = 132$$
$$(x+2)(x+3) = 11 \times 12 \text{ or } {}^{-}12 \times {}^{-}11$$
$$x+2 = 11 \text{ or } {}^{-}12$$
$$x = 9 \text{ or } {}^{-}14$$

This method works if the difference between the factors on the right-hand side is equal to the difference between the factors on the left-hand side.

The TI-89 is not particularly helpful using this algorithm. Factorising the equation does not clearly identify how the right-hand-side of the equation should be factorised.

```
■ factor(x^2 + 5·x + 6 = 132)
          (x + 2)·(x + 3) = 2^2·3·11
factor(x^2+5x+6=132)
```

Figure 13 Factorising an equation

Thus the introduction of the graphical calculator has provided the student with a greater range of strategies to provide an exact or close approximation to the answer to this relatively simple problem. The algebraic calculator has further increased this range of strategy options and transformed some of the approximate methods to exact methods.

Geometric techniques

Prior to the introduction of algebra, the Greeks would have solved this by logically arguing that the two shapes shown must have the same area:

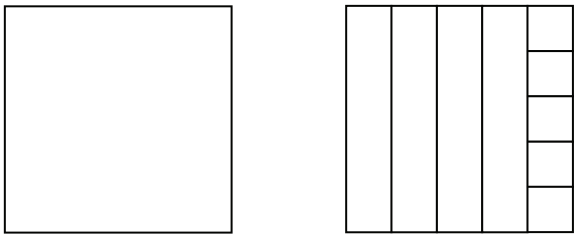

Figure 14 Diagrammatic technique

If the width of the rectangle is equal to the side of the small squares and the length of the rectangle is equal to the length of the side of the large square, then the total area of five small squares and four rectangles will equal the area of the large square, so the length of the large square is five times the length of the small squares, i.e. the number is 5.

As with some of the graphical techniques, this method does not lead to the alternative negative solution. If however the problem has two positive solutions, both will be “discovered.

Some solutions

Given the possibility of solving a simple problem in so many ways, should we teach students all of the possible methods or only selected ones? To what extent does our personal preference for some approaches to solving problems influence what we teach, the way we teach the methods we encourage students to use or the value we place (in assessment situations) on the various strategies?

Margaret Kendall (2001) uses the notion of teacher privileging to describe the tendency of teachers to favour some approaches over others. The recent introduction of graphical calculators, has exposed the tendency of some teachers to persist with algebraic approaches to certain tasks when a graphical approach could have been used. If a graphical approach is more suited to some students, e.g. because they are visual rather than symbolic thinkers, then these students have been disadvantaged by teacher privileging.

The term privileging was devised by Wertsch to explain how different forms of mental functioning dominate in different socio-cultural contexts. Privileging describes a teacher’s individual way of teaching and includes decisions about what is taught and how it is taught. (Wertsch, 1990)

A calculus problem

On another level, how do we encourage students to determine the nature of a stationary point? If they have access to a CAS which enables them to: evaluate an algebraic expression; create a table of values, graph a function; calculate the value of the derivative function for any value of the independent variable; and find the second derivative and determine its sign for any value for the independent variable.

Which method(s) should we teach students and which method(s) should we encourage students to use?

Consider this task, for example.

> The function $y = 6x^5 + 15x^4 - 130x^3 - 210x^2 + 720x$ has a stationary point at (1, 401). Determine the nature of the stationary point.

Numerically, the value of the dependent variable, y, is less than 401 if the value of the independent variable, x, is slightly smaller or larger than 1.

```
■ 6·x^5 + 15·x^4 − 130·x^3 − 21▶
                          396.514
…130x^3-210x^2+720x|x=0.9

■ 6·x^5 + 15·x^4 − 130·x^3 − 21▶
                          396.495
…130x^3-210x^2+720x|x=1.1
```

Figure 15 Characterisation using the value of the function

Similarly a table of values for the function shows that the function has a local maximum value at $x = 1$.

x	y1
.8	383.15
.9	396.51
1.	401.
1.1	396.49

Figure 16 Characterisation using a table of values

Graphically, the function has a maximum at $x = 1$

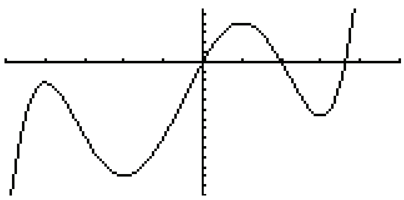

Figure 17 Characterisation using a graph

Using differential calculus: $y' = 30x^4 + 60x^3 - 390x^2 - 420x + 720$, then using the sign test: $y'(0) = 720 > 0$ and $y'(2) = {}^{-}720 < 0$

$\therefore$ the turning point is a maximum.

```
■ d/dx(6·x^5 + 15·x^4 − 130·x^3 −▶
  30·x^4 + 60·x^3 − 390·x^2 − 42▶
…30*x^3-210*x^2+720*x,x)

■ ◀·390·x^2 − 420·x + 720 | x = 0
                              720
■ ◀·390·x^2 − 420·x + 720 | x = 2
                             ⁻720
…^3-390*x^2-420*x+720|x=2
```

Figure 18 Characterisation using the first derivative or using the second derivative test: $y'' = 120x^3 + 180x^2 - 780x - 420$; $y''(1) = {}^{-}900 < 0$

$\therefore$ the turning point is a maximum

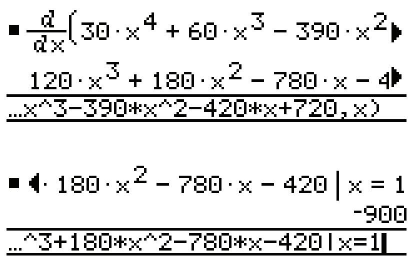

Figure 19 Characterisation using the second derivative

So again, students have numerous methods available for solving the problem.

Discussion

Should we teach our students all of the possible methods? If so, on what basis do they select a strategy to use for a particular problem? Herget, Heugl, Kutzler and Lehman (2000) discuss this and emphasise that the use of a CAS highlights the *distinction between the goals "perform an operation" (to some extent this can be delegated to a calculator) and "choose a strategy" (this cannot be done by the calculator).* They conclude that algebraic skills can be divided into at least three categories: those which should be taught and which students should be able to perform on paper; those which students should be exposed to, but not necessarily be able to perform on paper; and those which students should be able to perform using appropriate technology.

Given that the calculator will "do the algebra" for a student, the class-time made available should be devoted to the setting up of the model which will enable the problem to be solved. (See Kutzler, 2000). Kaye Stacey uses the phrase "gobbling up the algebra" to describe this phenomenon and the opportunities it provides teachers.

One possible approach is for students to consider whether an exact or approximate answer is acceptable, possible, and justifiable. Some equations cannot be solved using available algebraic procedures. In such cases, one of the numerical or graphical approaches will be required. But the student still has the task of selecting which strategy to use. If the equation to be solved is based on real-life data, then the parameters or coefficients in the equation are probably not exact, so an exact method cannot be justified. (Although a pseudo exact answer may be obtained then expressed as an approximate answer.) Depending on the context of the problem, it may be that an approximate answer will be sufficient even though an exact answer is justifiable and can be found.

When locating the turning points of a function, the choice of either an exact method (differential calculus) or approximate method (graphical) can again be made on the basis of whether or not an exact method is acceptable, possible, and justifiable.

The other alternative is that we only teach some methods. This may be due to time constraints or a desire to provide the student with a manageable range of solution techniques. One suggestion would be that we omit the quadratic formula. Students in junior secondary school have difficulty comprehending why it works, even if they can use it. Students who do not proceed to formal mathematics at the senior secondary level are unlikely to need to find the exact (surdic) solution to a quadratic equation. Students who do senior mathematics will have other exact methods available (if they have access to a CAS) or sufficiently accurate approximations if they have access to a graphic calculator.

Herget, et. al. commented:

> we eliminate the formula for the solution of a quadratic equation from the list of indispensable manual skills. … The traditional; approach of solving quadratic equations with a procedure (by either applying the formula or performing the method of completion of a square) will become extinct. (Herget, et.al. 2000.)

In fact, completing the square has already all but disappeared from senior mathematics curricula in Queensland, a state of Australia with its own education system. It should be noted that in Queensland, teachers in schools have considerable influence over the selection of the content of their mathematics curricula.

When deciding whether or not to remove particular skills or procedures from the curriculum, the aim is **not** to make room for other skills, procedures or algorithms. Rather, in line with Berry (2002), the aim is to allow students greater opportunity to develop conceptual understanding and to have opportunity to apply their knowledge through modelling and problem solving activities.

In Queensland, students are assessed against standards descriptors for three criteria – Knowledge and procedures, Modelling and problem solving, and Communication and justification. These three criteria are equally weighted. (Queensland Board of Senior Secondary School Studies, 2000.)

When modelling and problem solving, it is generally true, that if the model or problem is purely mathematical (or theoretical) then an exact method is justifiable and should be used if possible. If the model or problem is based on a real-life situation, then an exact solution cannot be justified even if it is possible. In either case, an approximate solution will often be adequate depending on the use to be made of the solution

A problem solving metastrategy

The syllabus for Mathematics B in Queensland (a course for university bound year 11 and 12 students) includes the objectives of "selecting appropriate mathematical procedures required to solve a problem" and "selecting and using problem-solving strategies to test and validate any conjectures of theories". Part of the teacher's role, therefore, is to provide students with criterion on which to base their selection of procedures and strategies.

This criteria on which to base a decision for selecting a strategy could be considered a metastrategy - a strategy for choosing a strategy to solve the problem, comparable to a metacognition or metalanguage.

It would seem that in both of the contexts described in this paper, a valid criterion for selecting a strategy for solving the problem (i.e. a useful metastrategy or strategy selection strategy) is:

> "Is an exact (algebraic) or approximate (numerical / graphical) strategy / solution valid, justified, required?" If an exact solution is both valid and justified or required, then use an algebraic technique. Otherwise use a numerical or graphical strategy

When confronted with a problem, students who have been "trained" in the use of this metastrategy will have some basis to guide their thinking about how to approach the problem.

Subsidiary questions, which may help students select their strategy are:

"Is the data, equation, function being used exact (i.e. purely mathematical) or approximate (based on real-world data)?" Since real world data almost invariably involves measurement, which is rarely exact, the exact solution provided by an algebraic technique cannot by justified. This is not to say that a pseudo exact solution cannot be found and then expressed to a degree of accuracy which can be justified by the data, but this is a different topic which the author hopes to address in a subsequent paper.

“Is there an algebraic method available?” Although it may be difficult to determine this at the outset of a solution to a problem, if it is anticipated that an algebraic solution does not exist (or is unknown to the student at the current stage of their mathematical development), then an approximate graphical or numerical approach may be chosen before a fruitless algebraic strategy is undertaken.

“Do I need an exact answer, or will an approximate answer suffice?” If an exact solution is not required, then a "quick and easy" numerical or graphical solution may be quite acceptable.

Conclusion

To conclude, algebraic calculators, and to a lesser extent graphical calculators, have created opportunities for students to save time performing a variety of algebraic manipulation tasks. However, they have also created a difficulty by increasing the range of problem solving strategies available to students. One possible solution to this difficulty is to encourage students to use the metastrategy of choosing an exact algebraic strategy or an approximate numerical or graphical strategy on the basis of whether or not an algebraic strategy is valid and justified or required. In so doing, teachers can take full advantage of the time saved by the use of an algebraic calculator to provide students with enhanced opportunities to develop and refine their modelling and problem solving skills.

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