Do geometrical constructions in a Dynamic Geometry environment affect student's algebraic expressions?

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Abstract: *The present paper attempts to bridge the world of digital technology and the world Euclid bequeathed us in his "Elements". The role of the design process of activities in an dynamic geometry interactive environment such as that of Geometer's Sketchpad v4 is examined, along with ways in which students can be assisted to understand algebraic concepts through geometrical reconstructions. The ways in which it can facilitate the understanding of geometrical concepts are examined, along with the bridging, between the fields of algebra and geometry, and strategies for overcoming obstacles. Two constructional examples are presented which can lead to the visual proving process and the formulation of the algebraic expression through the theoretical calculation of the areas of shapes.*

1. Introduction

The geometrical representations of identities and algebraic expressions in general were initially developed in Euclid's "Elements" (325-265). Stamatis (see [9]) declares: "The second book of Euclid's "Elements"… includes the application of Geometry to Algebra and is ascribed for the most to Pythagoreans. The first 10 theorems relate to algebraic identities, which we are able to represent in the following way: if with the letters a, b, c mentally consider by the equal segments "(my translation from the Greek text). Namely, representations were the first empirical mode leading to the proving process in Ancient Greece, too (see, for example, Socrates and Meno) although the process observed in Euclid's "Elements" does not display a transition from visual representation to rigorous reasoning. The visual representations can proof only specialized cases, while the Euclidean proof can empower every case by reinforcing the initial visual proof. In the Platonic dialogue (Socrates: slave), the slave's incorrect answers are restructured with a shape: a concrete representation, with the particular shape functioning as the visual proof of the accuracy of Socrates' proposition to the slave. The phrase "ei mi voulei arithmein alla deixon" which means "if you don't want to measure, just prove" (my translation from the ancient Greek text) can be considered an interpretation of what the ancient Greeks meant by 'proof', making use of the concrete expression of the ancient verb "deiknymi". Furinghetti and Paola (see [11]) draw on the application of Lakatos' ideas to mathematical problem solving. As they write "The didactical suggestion implicit in Lakatos' words is that it is advisable to recover the spirit of Greek geometers." They continue "According to Szabo, this is the original concept of proof held by Greeks, called deiknimi. The "deiknimi" be developed in two ways, which correspond to analysis and synthesis (see [11])." How meaningful can the act of "deiknimi" be nowadays, and how do students understand it? How can this be achieved through modelling such as composing and decomposing of shapes (see [24])? How important is the mediation of the tools or the materials /artefacts for students to construct knowledge?

These are the issues /questions that will be addressed in the paper, as follows: a) section 2 will deal with the rationale which underlay the design of the activities, bearing in mind the learning theories mentioned in section 3 b) the methodology when designing the activities in the software in section 4.

2. Decomposing Euclid's proposition using structural algebraic units

The first ten propositions of Book II can be easily interpreted in modern algebraic notation, and for this reason the subject matter of Book II is usually called "geometric algebra". The proposition 4 of Euclid's Elements (Book II) supports that: "If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments." One may of course interpret proposition 4 in algebraic terms, but strictly speaking one cannot say that Euclid developed "geometrical representations of identities and algebraic expressions"; it would be much better to say that "after Euclid mathematicians developed algebraic representations of his geometrical propositions", and by so doing they extended the domain of application to any kind of numbers (and consequently their meaning): for instance the identity $(a + b)^2 = a^2 + 2ab + b^2$ is also valid when a and/or b are negative numbers (or even complex numbers).

For instance in Euclid's proof of proposition 4 (book II) numbers do not appear as such (and even less algebraic expressions), since this proposition deals only with lengths. It is not just a question of translating into "modern notation" but a question of what Euclid had in mind. Nowadays, the historical process is reversed since in classrooms geometrical representations are used to introduce or illustrate algebraic formulas. The statement of the proposition can be interpreted in modern algebraic notation as saying that if $x = a + b$, then $x^2 = a^2 + b^2 + 2ab$ or as an identity, it says $(a + b)^2 = a^2 + b^2 + 2ab$. In fact, this modern interpretation of Euclid's statements should be put in the correct perspective. I would not try to show that this interpretation can be useful and effective because it is the 'original', but on the opposite because this is the way that it can be interpreted nowadays.

Working backwards from Euclid's method, we see that the terms a^2, ab can be translated into areas by composing the initial shape out of its component parts or, in other words, by modelling the algebraic expression geometrically. Rahim in [24] writes "mathematically and in a Euclidean sense, this combination allows us to extend the concept of congruence from rigid shapes congruence (same intact shapes) to congruence by pieces (different shapes) thus opening a wider range of interesting possibilities". At the same time, these areas can also serve as algebraic units (as can, for instance, the terms x^2, x, 1), allowing the form of an identity —or, more generally, a polynomial— to be constructed (or structuring an identity or polynomial). They are, in other words, '*structural algebraic units*'. These algebraic units are only meaningful when they are given substance, shape and form during the learning and teaching of algebra; it is generally accepted that the use of materials and tools during the learning process can help the learner visualize the meanings involved.

The artefacts used to model or represent mathematical processes of concepts are made out of materials constructed by man, either they be palpable (a card construction, for example) or digital (a dynamic geometry software construction). In that sence the abstract meanings are rendered concrete by means of the figurative /schematic material. This is not a specificity of the DGS tool, but can be achieved as well through cutting up paper shapes. For example the students could be helped by the constructions of the basic shapes, and this could be the same if they had constructed them with instruments on a sheet of cardboard before cutting them up. But the representation of structural algebraic units using sheet of cardboard is a time-consuming process, and is not recommended for the construction of algebraic constructions in class, at least with High School students. On the other hand the students in the computer environment, are drawn into a process when they can see it develop effectively and rapidly in which they are working. As Mariotti in [20] writes "The effect is so strong to lead someone to speak of a new experiential mathematical realism. Originally abstract and formal conceptions get more and more accessible and usable through their computational

counterpart, a sort of reification which gives the impression of a direct manipulation of mathematical objects and relations." Moreover, the sloppiness and inaccuracy of man-made constructions can also be avoided through the use of technology (Figure 2.1). Researchers around the world concur in the view that greater emphasis should be put on activities in software that actively involve students (see for instance [6]).
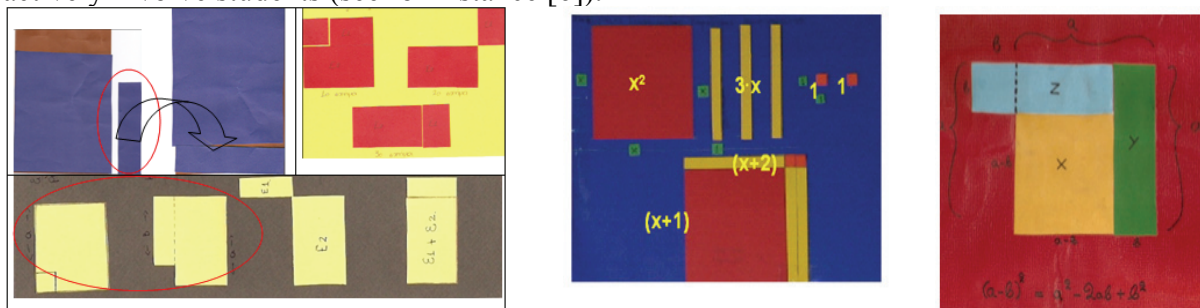

Figure 2.1 Students constructions with instruments on a sheet of cardboard

Activities should therefore be designed to motivate students and encourage them to actively construct knowledge. Designing learning activities, and making them effective in their static and dynamic dimensions, is a complex pedagogical task that lies at the heart of teacher practices. In this sense, Euclid's propotitions become meaningful for the active teaching and learning of algebra when they are presented as constructional activities and open constructional problems whose structural units have to be assembled and reassembled in a dynamic geometry enviroment. Taking this approach, students acquire an interest in the history of mathematics, understand that algebraic expressions of identities are meaningful in reality because they are representative of the areas of shapes, and understand the area of a complex geometric shape to comprise of the areas of the component shapes of which it is composed. This aspect will be examined below. For the design of activities, I took into account the theories referred to in the next section relating to knowledge, learning and teaching during the design/construction and implementation of the activities.

3. Theoretical framework

Mathematics visualization and connections, links and relations between representations have appeared in recent literature as fundamental aspects to understanding students' construction of mathematical concepts, as well as important characteristics of learning and problem solving (see for example [13] and [15]). The constructivist view of representation as conceptual knowledge is consistent with the notion that learners actively construct new knowledge in problem solving situations "when their current knowledge results in obstacles, contradictions, or surprises" (see [2] p. 92). The subject of this study is clearly linked with the notion of semiotic register developed by Duval (see [3]). The semiotic registers used in the mathematical activities are the algebraic, the graphical, the figurative and the natural language. A semiotic register, according to Duval (see [3]), constitutes a system of representation if it allows three cognitive fundamental characteristics: its production, a treatment, and a conversion between different semiotic registers. According to Duval (see [4]) "the kind of operative connections we expect to be made when learning is not between deductive and empirical mathematics, proofs and constructions, nor between mathematical structures and symbol structures, but between the different registers of semiotic representation"

Another possible point of view could have been to refer to conceptual metaphors (see for instance [8]). In the present case, a geometrical diagram can be considered as a metaphor for the corresponding algebraic formula. Sfard (see [28]) draws on the work of Lakoff and Johnson, reporting that a metaphor is "a mental construction which plays a constitutive role, in structuring our experience and in shaping our imagination and reasoning" (p. 46).

Kaput, Noss, and Hoyles in [17] analysed how *new representational infrastructures*, namely "the ways we use to present and re-present our thoughts to ourselves and to others, (in order) to create and communicate records across space and time, and to support reasoning and computation" and how "the associated artifacts and technologies have, over long periods of time, gradually externalized aspects of knowledge and transformational skill that previously existed only in the minds and practices" quoted in [17]. The interaction with representations in a computing environment has two aspects: "the action upon a representation by the user through the intermediary of a human-computer interface, and the representation communicating back through some form of reaction or response" quoted [26].

The theoretical framework includes the notions of instrumental genesis (see [30]) and the distinction between phases of instrumentation and instrumentalisation (see [7]), which are fundamental in teaching in computer environment. During the instrumental genesis both the phases (instrumentation and instrumentalization) coexist and interact. Then the user structures that Rabardel (see [23]) calls utilization schemes of the tool/artefact. Utilization schemes are the mental schemes that organize the activity through the tool/artefact. This process involves many studies, among them, for example the one of Artigue (see [1]), based on the research of Verillon & Rabardel (see [30]) about the ways by which an artefact becomes an instrument for a student. According to Artigue (see [1]), "An instrument is thus seen as a mixed entity, constituted on the one hand of an artefact and, on the other hand, of the schemes that make it an instrument for a specific person. These schemes result from personal constructions but also from the appro-priation of socially pre-existing schemes." One must also bear in mind the teacher's role, as introduced by Mariotti (see [20]), in this approach: it can in no way be considered certain that an artifact will function in the construction of this meaning, even if it does incorporate a mathematical meaning and schemes of use. That said, an artifact can encourage students to fully utilize their communication strategies and guide them towards the desired meaning if it is placed between the teacher and the students.

In the following section, we shall examine the reasons why the proposed constructional processes which make use of software tools are superior to the ways in which they can be constructed in a static environment. The advantages of the former, in terms of the way in which students construct knowledge, are: a) whether they themselves are operating the software when constructing shapes/constructions like those described in subsection 4.1 using the tools provided in the menu or b) working on diagrams (semi) predesigned by the teacher and activated by the students themselves using interactive techniques like those described in subsection 4.2 which can facilitate students guiding them towards the correct way to link the sequential constructional steps and the visual proof of the problem. These processes are presented in the next section using two problems which are reworkings of propotitions 4 & 5 in Book II of Euclid's Elements.

4. The Constructivist approach

4.1 . Composing the shapes
Problem1: Construct 4 shapes: *a square of side a, a square of side b and two rectangles in different colours with sides a, b. Then construct a larger shape matching the shapes which you have already constructed in order to construct an quadrilateral.*
In order to comprehend the advantages (and disadvantages) of the construction mode in the software, it is necessary to examine the differences between it and the mode of construction using static means. This will allow us to compare the two modes.
a) Working using static means, we can use a compass and a straightedge (ruler with measurements). Dynamic geometry systems such as the Geometer's Sketchpad (see [12]) or Cabri II (see [18]) (or any other DGS software) are microworlds. They were designed to facilitate the teaching and learning of Euclidean geometry. The basic tools of such a learning enviroment are a) Circle

(equivalent to Compass) b) Segment/Ray/Line (equivalent to Unmarked Straight Edge) c) Point (which simply enables us to place one of the fundamental 'objects' of Euclidean geometry) d) Pointer (which crucially enables us to drag objects) (see [19]). When these tools are combined with the software's options menu, they allow the user to produce constructions which must conform with the principles of Euclidean geometry if they are to function and pass the dragging test. This means that the student has to know the theory of geometry if s/he is to make a correct geometric construction. And while we have explained that, in the software, the constructions can contain the same mathematical logic as the constructions on paper, there are substantial differences in the manner in which the tools are used.

For instance, in the use a straightedge with measurements. The mode of constructing a figure in the software (e.g a square of side a) could be different from the mode students use to construct it on paper. For example: When a pupil works using static means, s/he is able to measure the length of side 'a' with a ruler. Afterwards s/he is able to use this measurement to construct either the next side of the square given that s/he knows the geometrical properties of the shape or of another shape (for example a rectangle) whose one side is equal to a. This "measurement" method is not the method the software demontrates for constructing a shape which does not mean that the software faces a constructional inability/disadvantage comparing it with static means. According to Duval (see [4]) "We can have even a conflict between the figure and the measures leading to a paradox".

Duval (ibid.) argues that "Visualization consists only of operative apprehension. Measures are a matter of discursive apprehension, and they put an obstacle in the way not only for reasoning but also for visualization." By forcing students to think of ways of constructing an equal segment, this methodological weakness can thus provoke a cognitive conflict in students, and in so doing raise the level of difficulty. One such way would be to define side 'a' as an arbitrary segment on the screen and then use it as a radius of a circle in the construction. This construction method inducts a different mental perception in the students with regard to construction in the software.

In this way, the sides of the square cannot be modified from the vertices of the shape using the dragging modality. Instead, they depend on the modification of the initially defined segment a. The arbitrary segment 'a' could thus be confined as a non-collapsible compass to either the square or any shape whose a side is equal to 'a'. This construction procedure depends on the students' level of conceptual knowledge and cognitive abilities. In Fig. 4.1.1, the hidden elements have appeared in the construction and we are able to demonstrate the construction process of the shapes. As a consequence, the construction of the shapes depends either on segments a & b--both of which are arbitrary defined --and the relationship among them, or on the students' geometrical knowledge of the relationships between and properties of shapes.

Another important point about this construction is that students can use it to verify and to construct arguments, and in so doing overcome an epistemological obstacle which one frequently finds obscuring comprehension in pupils taking tests using static means, and in which they mistakenly write that $(a+b)^2 = a^2+b^2$. Obstacles like this can be seen as an opportunity for students to reflect on their own learning rather than a barrier to achieve understanding of mathematical ideas. Through the software constructions the abstract meanings will be rendered concrete by means of the schematic digital material on the computer screen. The students can act on the visual diagrams, and—through the instrumentation process activated by the tool—they can overcome obstacles and shape their mental schemes, based on schmes of use of the tools/artefacts.

b) On the computer, on the other hand, the student can construct the shapes using the potentialities offered by the software. This means s/he can use transformation tools like rotation or reflection in addition to the ruler and compass. In this point we are limited to refer the effects of the construction through rotation. We follow these next steps to create a rotation in Sketchpad v4: to begin with, we select the point which will act as the center for rotation and define it on the transform menu as

'mark center'. Then we select the object we would like to rotate based on an angle, choosing the specified/fixed angle (for example 90°). When the command runs, a new object is created which is a rotated image of the original object. The rotation of the rectangle for 90 degrees in the software shapes a utilization scheme which leads the students to conceptually grasp the meaning of a) perpendicularity/a right angle; b) congruent shapes. This transformation has a significant impact: during the instrumental approach, the student structures a utilization scheme of the tool, and consequently a mental image of the functional/operational process of rotation, since any modification/ transformation of the initial figure (input) results in the modification/transformation of the final figure (output). (see [22])

c) The processes can be added to through the construction of custom tools. According to Straesser (see [29]) macros /custom tools "can help to structure a geometrical construction by condensing a complicated sequence of construction steps into one single command". As a result of the construction and application of the custom tool the direct perception of the user is attained in regard to the steps in the development of the construction pertaining to (see [21]): 1) the repetitions in the measurements or calculations of the areas of initial/original shapes 2) the developmental way of the construction of the shape and 3) its orientation towards the sequential steps of the construction on the screen's diagram or in successive pages of the same file. If we save the constructions which refered in the previous subsection as scripts (customs tools in the case of the Geometer's Sketchpad), the problem acquires a different dimension (figure 4.1.2). For example in this case, the student can construct squares and rectangles using the saved tools and two arbitrary segments on the screen. Each shape can now be dragged unmodified on the screen, while one of the vertexes can be used to change its orientation. This helps in the reconfiguration (see [4]) and construction of a square or, alternatively, the known algebraic expression (propotition 4).(figure 4.1.3)
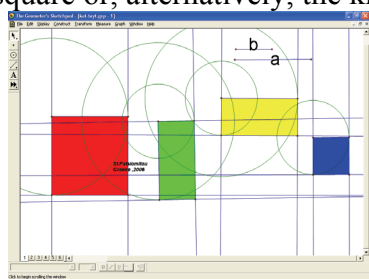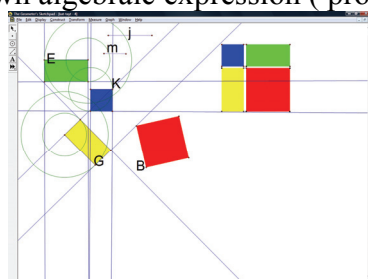

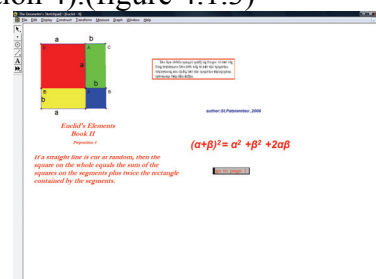**Figure 4.1.1**   **Figure 4.1.2**   **Figure 4.1.3**

4. 2 Interaction techniques and transformational processes

Problem 2: *Can a square of side α be transformed into a rectangle if we subtract a square of side b (in one of the corners of the square of side a) from it? If so, can we estimate the area of the shape that remains using only the sides of the two squares, a & b? or Are we able to manipulate the remaining area so that it can be transformated into a rectangle without altering its area?*

In order to subtract the squares we need a visual illustration to the transformation. I semi pre-designed the activities, concentrating on two aspects of the design process: 1) linking the steps in the constructional, transformational or explorative actions or processes in the software using interaction techniques (see [26]) 2) linking the steps in the proof via a sequence of pages or the same page in the DGS environment using interaction techniques. The design in the computing environment was inspired by what Freudenthal called "hand-eye performances" in [10].

Specifically, the process of proving a problem or theorem is made up of a series of steps which can function as a response, anticipating the questions posed explicitly or implicitly by the teacher (or student). The completion of a step could correspond to a different page in the software connected to the previous page via link action buttons, while on every page different constructional

or exploratory actions are linked with different software techniques. The further into the pages we click, the more complex or sophisticated and closer to the problem solving the representation depicted. Thus, a problem would be solved by breaking it down into a series of questions whose answers gradually distil the proof the students seek. The questioning process thus helps students determine and extend their underlying knowledge. Bringing this analysis of the logic of the design process, it should be noted that I bore the following in mind when designing, constructing and implementing the activities: a) the process should be active to keep the students interested and promote discovery; b) students should be guided to reach conclusions. The teacher can guide the students through elucidation or questions to conclusions which compose a step by step visual proof.
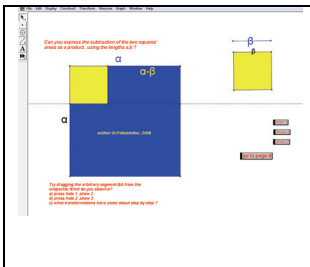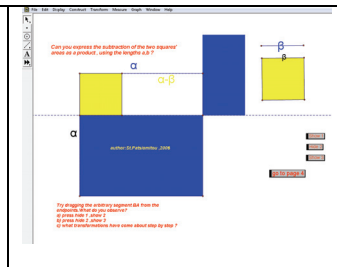

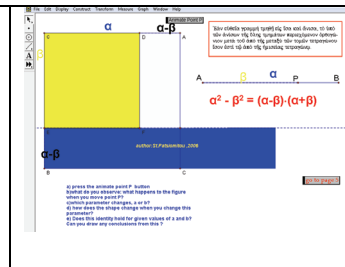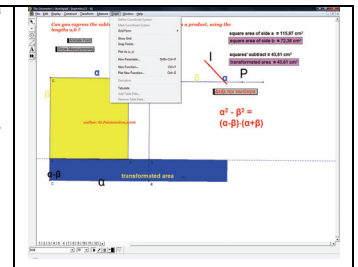**Figure 4.2.1 :**step 1   **Figure 4.2.2:** step 2, 3   **Figure 4.2.3:** step 4   **Figure 4.2.4**

Linking visual representations of the construction steps

   Also significant is the role of the software's successive pages, which could be considered as a vivid section of a book revealing the parts of the proof. In this way, the sequence of increasingly sophisticated construction steps in the activity could correspond to the numbering of the action buttons, allowing the student to interact with the tool on his/her own volition or on being encouraged to do so by the teacher during the time allotted for the activity. The link buttons can join different pages and later, more evolved steps in the construction of the representation of the problem. This leads to a cognitive linking of the representations which (see [14]) "creates a whole that is more than the sum of its parts...It enables us to see complex ideas in a new way and apply them more effectively". In that way the pupils can improve their knowledge by having a mental schema elicited from them. As Mariotti in [20] declares "the temporal sequence of the constructions' steps represents the counterpart of the logic hierarchy between the geometric properties of a figure". The cognitive linking of the representations, could come about through a series of actions which model the problem in the software: Step 1) Placing the square of side b on top of the square of side a.(figure 4.2.1) Step 2) Rotating the rectangle P_1: this action is connected/completed with a hide/show action button (figure 4.2.2) Step 3) Displaying rectangle P_2 vertically by transforming rectangle P_1 by rotating it through 90° (figure 4.2.2) Step 4) Hiding rectangle P_2 and showing rectangle P_3 which resulted from a reflection by line e (figure 4.2.3). The squares are constructed on the screen using the parametrical mode described above with linear segments 'a', 'b' which can be modified by manually dragging or using animation. For example the animation process of segment P allows users to observe the corresponding modification: 1) of the subtraction of the two squares' areas: a^2-b^2 2) the area of rectangle (figure 4.2.3, 4.2.4), which has dimensions a-b, a+b and that they have the same result. Despite the modification of the segments a, b the empirical visualization of the equality of the areas--the modified shape, and the initial squares areas' subtraction--can be investigated in the next page of the same file. Segments 'a', 'b' have been constructed intedependently: segment b belongs on segment a. For this reason, one of the endpoints of segment b (named point P on the diagram)- is located on segment a. Manually dragging point P (or using the animation tool to do so) modifies the length of segment b, and hence both the dependant area of the square and the subtraction between the squares' areas or in algebraic terms: a^2-b^2. (figures 4.2.3, 4.2.4) The student can observe the alteration of the initial shape as this

is made up of equivalent areas and the remaining area as equivalent with a transformated area of a rectangle of dimensions a-b, a+b. On the other hand, the subtraction of the squares is meaningless if we don't investigate the relationship between the squares' sides 'a', 'b'. The activity helps students to form hypotheses and investigate with regard to: 1)The equality or inequality of the segments a, b 2)The relationship between the modification of segment 'b' and the modified final area of the rectangle. Dragging segment 'a' simultaneously and proportionally modifies segment 'b'. This leads to the geometrical concept being transformed into an algebraic one, since segments 'a', 'b' operate as parameters. Consequently the students visual verify that $a^2-b^2 = (a-b)(a+b)$ and that this can be generalized to include any parameters 'a', 'b'. In Fig. 4.2.4, the dynamic link between the parameter and the area of the shape—and thence with the graphic representation leads to the cognitive linking of the parameters in algebra —initially appears the point I as plot of the selected coordinates: of parameter b and of area's $\alpha^2-\beta^2$. Animated point P which is connected with a trace, produces the diverging red line on the screen.

5. Conclusions

Dynamic geometry systems have been described as computational environments that embody some subdomain of mathematics or science, generally using linked symbolic and graphical representations. The idea of designing a didactical engineering in order to help students bridge algebra and geometry is fundamental in mathematics teaching. Although the paper does not provide objective data (e.g the results of the research) to support the idea that working in a DGS environment fulfils this purpose, in other words, that the reported constructions in the software operate as *structural algebraic units* for the students interacting with them, it is obvious that the processes in the software are more efficient, when students are structuring algebraic expressions, than in a paper-pencil-scissors environment. It is for this necessary to understand the way in which one can form and express with the different semiotic registers the specific mathematical meanings. In the present case we have acchieved a transformation on the semiotic registers and we for this transform also the semiotic representations. So we have a treatment and a conversion between different semiotic registers. The construction of the action buttons can make easier to note the relationship between the sequential linking representations and to react by reflection, noting the equality of the areas and hence, using algebra, the equality of the relationships. Sfard (see [27]) claims an interaction between a process or an object is indispensable for a deep understanding of mathematics whatever the definition of "understanding" may be. This process can be repeated, meaning that the level to which one concept is understood can function/operate as a base for developing another. Thus, the tools comprising DGS may well constitute a channel whereby children extend their imagination and conceive mathematics like a source of mathematical models and representations. Freudenthal in [5] supports that "Socrates did not teach a ready made solution but the way of reinventing the solution." The same approach two millenia later was formulated by Comenius (quoted in [5]): "The best way to teach an activity is to show it." According to Freudenthal "this is a socratic idea, though it involves more than a Socratic lesson." Freudenthal supports that modern educators are likely to subscribe to a variation of Comenius' device while "The best way to teach an activity, is (not) to show it" but rather "The best way to learn an activity, is to perform it." Overall, when the student uses a dynamic geometry tool and interacts with it, s/he can be encouraged to develop relational conclusions, which in turn helps him/her to develop his geometrical thinking. In other words, when these factors are imposed on logical organization and thinking, conceptual comprehension reaches the level of abstract thought processes as dynamic diagrams can be considered as a metaphor for the corresponding algebraic meanings. In terms of our historical perspective "this current work is a part of a large transition towards a much more broadly learnable mathematics. These representational innovations …constitute steps toward the

development of a new "alphabet" for… mathematics which might do for mathematical representation what the phonetic alphabet did for writing" (see [16]).

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