Hyperbolic Geometry as a View Screen in Minkowski Space

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Abstract: Let $\angle APB$ be an angle in the three dimensional Euclidean space. When we look at this angle from various view points, the angle $\angle APB$ changes its appearance, which we call "visual angle". Mori and Maeda [5] studied the relations among three visual angles of three dimensional orthogonal axes. They found out several geometric properties of these visual angles. In this paper, we will make a similar discussion in Minkowski space. In general, visual angle is realized on an equidistance surface (view screen) from the view point. For example, in the Euclidean space the view screen is a sphere centered at the observer. On the other hand, the view screen in Minkowski space is a (Euclidean) hyperboloid of two sheets centered at the observer. We will make clear the difference of visual angles of three dimensional orthogonal axes between Euclidean and Minkowski spaces.

1. Introduction

We see various angles in our daily life. An angle has only one size in degrees (or radians) measured by protractor, however, how many degrees is it measured from your view point? We call this angle ''visual angle''. Mori and Maeda found out relations among three visual angles of three dimensional orthogonal axes in Euclidean space. Then, what relations are there among three visual angles of three dimensional orthogonal axes in other spaces? In this paper, we investigate three visual angles of three dimensional orthogonal axes in Minkowski space. Figure 1.1 shows the typical cases of visual angles of orthogonal axes in Euclidean and Minkowski spaces. Everyone in Euclidean space $\mathbf{E}^3$ can intuitively find out that orthogonal axes do not look like the right picture of Figure 1.1 but the left one. Conversely, in Minkowski space $\mathbf{M}^3$, orthogonal axes do not look like the left one but the right one. The aim of this paper is to make clear the difference of visual angle between $\mathbf{E}^3$ and $\mathbf{M}^3$. In Section 2, we will give the definition of visual angle and how to measure it in $\mathbf{E}^3$ and $\mathbf{M}^3$. In Section 3, let us review the relations among three visual angles of three dimensional orthogonal axes in $\mathbf{E}^3$(see, [5] and [6]). In Section 4, we will find out a very important theorem which shows the relations among three visual angles of three dimensional orthogonal axes in $\mathbf{M}^3$. Figures in this paper are drawn by dynamic geometry software *Cabri II* and *Cabri 3D*. They are very useful tools for providing ideas and giving hints for proofs.

2. Definition of Visual Angle

In this section, let us define visual angle, and introduce how to measure the size of the visual angle.

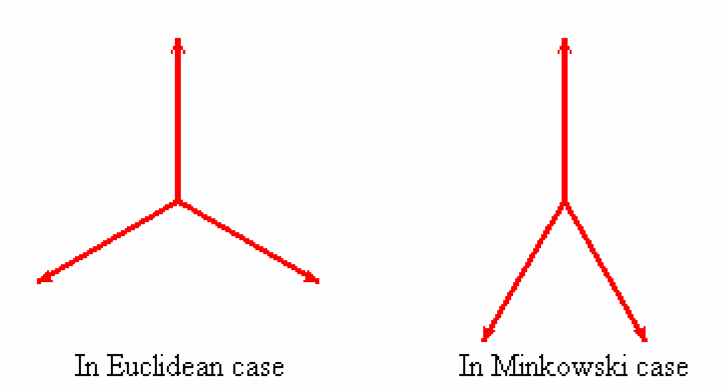

Figure 1.1 Visual angles of three dimensional orthogonal axes in $\mathbf{E}^3$ and $\mathbf{M}^3$.

Definition 2.1(Visual Angle) Let $\angle APB$ be an angle in the three dimensional $\mathbf{E}^3$ or $\mathbf{M}^3$. For a viewpoint O, let us denote by

$$\angle oAPB$$

the dihedral angle of the two faces OPA and OPB of the (possibly degenerate) tetrahedron $OPAB$ (see, Figure 2.1) (see, [5]). This angle $\angle oAPB$ is called the visual angle of $\angle APB$ from the viewpoint O. Its size (measure) is called the visual size of $\angle APB$ from O, and denoted by the same notation $\angle oAPB$ as the visual angle without any confusion.

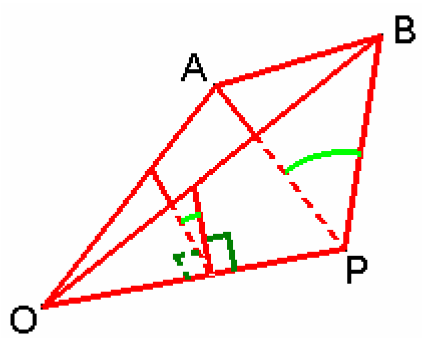

Figure 2.1 Definition of visual angle.

Proposition 2.2(Visual Size in $\mathbf{E}^3$) For an angle $\angle APB$ in $\mathbf{E}^3$, let P', A' and B' be the central projected points of P, A, and B on the unit sphere $\mathbf{S}^2$ centered at the view point O. Then the visual size $\angle oAPB$ is equal to the angle $\angle P'$ of the spherical triangle $\Delta P'A'B'$(see, Figure 2.2).

Proof. Let $\vec{a}$ (resp. $\vec{b}$) be a vector tangent to the arc $P'A'$ (resp. $P'B'$) at P'. The angle between the vectors $\vec{a}$ and $\vec{b}$ is equal to the spherical angle $\angle P'$ by the definition of spherical triangle. Note that both vectors $\vec{a}$ and $\vec{b}$ are perpendicular to the line OP which is the intersection of the planes OPA and OPB. Therefore, the visual size of $\angle APB$ is equal to the angle $\angle P'$ of the spherical triangle $\Delta P'A'B'$. ■

Here, let us briefly review three dimensional Minkowski space $\mathbf{M}^3$ (see, [2] p.177). This space is three dimensional space with the metric

$$ds^2 = dx^2 + dy^2 - dz^2,$$

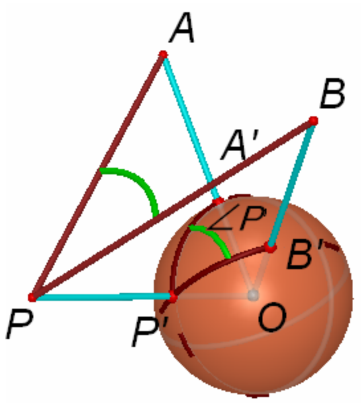

Figure 2.2 Central projection of $\angle APB$ on $\mathbf{S}^2$.

in other words, the inner product $<\vec{a},\vec{b}>$ of two vectors $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)$ is defined by

$$<\vec{a},\vec{b}>= a_1b_1 + a_2b_2 - a_3b_3,$$

which is called Minkowski product. Transformations, which preserve the Minkowski product is called Lorentz transformations. The quadratic hypersurface

$$\mathbf{Q} := \{(x,y,z) \in \mathbf{M}^3 \mid x^2 + y^2 - z^2 = -1, z > 0\},$$

is well known as a model of hyperbolic geometry. The distance $d(A,B)$ between two points A and B on $\mathbf{Q}$ is satisfies (see, [4])

$$\cosh(d(A,B)) = -<\overrightarrow{OA},\overrightarrow{OB}>.$$

Note that $\mathbf{Q}$ is an equidistant surface from $O=(0,0,0)$ which plays an important role as a view screen in $\mathbf{M}^3$.

Proposition 2.3(Visual Size in $\mathbf{M}^3$) For an angle $\angle APB$ in $\mathbf{M}^3$, let P', A' and B' be the central projected points of P, A, and B on $\mathbf{Q}$ of the hyperboloid of two sheets($x^2+y^2-z^2=-1, z>0$) centered at the viewpoint O. Then the visual size $\angle oAPB$ is equal to the angle $\angle P'$ of the hyperbolic triangle $\Delta P'A'B'$(see, Figure 2.3).

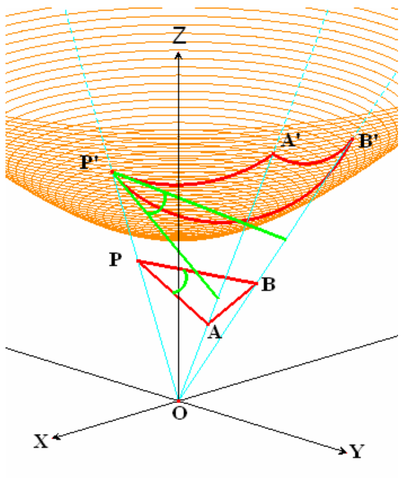

Figure 2.3 Projection of $\angle APB$ on $\mathbf{Q}$.

Proof. This proof is similar to Proposition 2.2. Let $\vec{a}$ (resp. $\vec{b}$) be a vector tangent to the geodesic $P'A'$ (resp. $P'B'$) at P' on **Q**. The angle between the vectors $\vec{a}$ and $\vec{b}$ is equal to the hyperbolic angle $\angle P'$ by the definition of hyperbolic triangle. Note that both vectors $\vec{a}$ and $\vec{b}$ are perpendicular to the line OP which is the intersection of the planes OPA and OPB. Therefore, the visual size $\angle APB$ is equal to the angle $\angle P'$ of the hyperbolic triangle $\Delta P'A'B'$.
■

Proposition 2.3 indicates that visual angle in $\mathbf{M}^3$ is measured as an angle in hyperbolic geometry.

3. Visual Angle of Three Dimensional Orthogonal Axes in Euclidean Space

In this section, let us review the relations among the three visual angles of three dimensional orthogonal axes due to [5]. Set up a system of orthogonal coordinates $O-XYZ$ on $\mathbf{E}^3$ with the origin at the viewpoint $O=(0,0,0)$. Let us consider to see another three dimensional orthogonal axes $P-ABC$ with the origin $P(\neq O)$ from the viewpoint O. For simplicity, let us assume that both orthogonal axes $O-XYZ$ and $P-ABC$ are positive oriented which is determined by using the ''right hand rule''. Then, with appropriate rotations centered at O, we can arrange the orthogonal axes $P-ABC$ parallel to $O-XYZ$ without changing the visual sizes $\angle oAPB$, $\angle oBPC$ and $\angle oCPA$. Let V_{AB}, V_{BC} and V_{CA} be the three visual sizes of $\angle oAPB$, $\angle oBPC$ and $\angle oCPA$, respectively. As in Figure 3.1, these visual sizes are realized around P' which is the central projection of P on $\mathbf{S}^2$. Then the projection of $A-axis$ passes through $A_\infty=(1,0,0)$ which corresponds to the projection of the vanishing point of $A-axis$. In the same way, the projection of $B-axis$ passes through $B_\infty=(0,1,0)$, and that of $C-axis$ passes through $C_\infty=(0,0,1)$. In the following argument, we consider the visual angles as oriented, which are measured in the counterclockwise direction looking from the outside of $\mathbf{S}^2$.

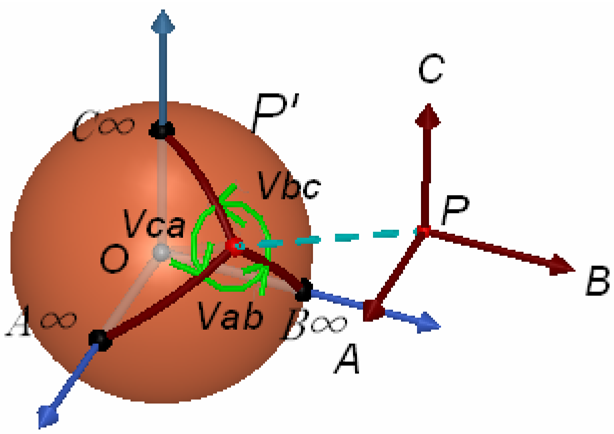

Figure 3.1 Central projection of three dimensional orthogonal axes $P-ABC$ on $\mathbf{S}^2$.

Theorem 3.1(Law of Visual Angles of Three Dimensional Orthogonal Axes in Euclidean Space) If the point $P'=(x_0,y_0,z_0)$ is not on the axes of $O-XYZ$ $(x_0,y_0,z_0\neq\pm1)$, then three visual sizes V_{AB}, V_{BC}, and V_{CA} are given by

$$\tan V_{AB}=-\frac{z_0}{x_0y_0},\ \tan V_{BC}=-\frac{x_0}{y_0z_0},\ \tan V_{CA}=-\frac{y_0}{z_0x_0}. \tag{3.1}$$

Conversely, if V_{AB}, V_{BC}, nor V_{CA} are not equal to 0 and π, then the point P' (the direction of the origin P of $P-ABC$) is determined as the following the equation

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} sign(\sin V_{BC})\sqrt{\cot V_{CA} \cdot \cot V_{AB}} \\ sign(\sin V_{CA})\sqrt{\cot V_{AB} \cdot \cot V_{BC}} \\ sign(\sin V_{AB})\sqrt{\cot V_{BC} \cdot \cot V_{CA}} \end{pmatrix}, \tag{3.2}$$

where $sign(t) = \pm 1$ is the sign of t.

Proof. To prove this theorem, it is very useful to represent the coordinates of P' with direction cosine, such as $P' = (\cos\alpha, \cos\beta, \cos\gamma)$ $(\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1)$ where α (resp. β, γ) is the spherical distance of $P'A_\infty$ (resp. $P'B_\infty, P'C_\infty$), respectively.

First, we can easily check that $x_0 \cdot \sin V_{BC} \geq 0$ as in Figure 3.2.

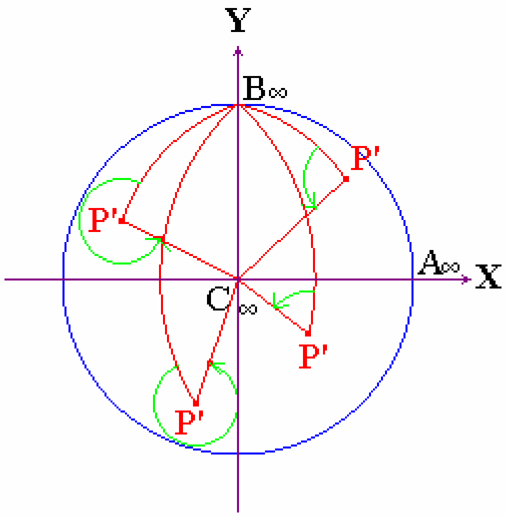

Figure 3.2 The relation between the sign of x_0 and that of $\sin V_{BC}$.

Since $y_0, z_0 \neq \pm 1$, we apply *the law of cosines for sides* (see, [3] p.54) to the spherical triangle $\Delta P'B_\infty C_\infty$,

$$\cos\frac{\pi}{2} = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos V_{BC}.$$

So,

$$\cos V_{BC} = -\frac{\cos\beta\cdot\cos\gamma}{\sin\beta\cdot\sin\gamma}, \text{ and } \sin V_{BC} = \frac{\cos\alpha}{\sin\beta\cdot\sin\gamma},$$

where we use the fact that $\sin V_{BC}$ has the same sign as $\cos\alpha (= x_0)$. Hence,

$$\tan V_{BC} = -\frac{\cos\alpha}{\cos\beta\cdot\cos\gamma} = -\frac{x_0}{y_0 z_0}.$$

The other the equations of (3.1) follow in a similar way.

Next, we will prove Equation (3.2). $V_{AB}, V_{BC}, V_{CA} \neq 0, \pi$ implies $\tan V_{AB} \cdot \tan V_{BC} \cdot \tan V_{CA} \neq 0$. Then multiplying the first and the third equations of Equations (3.1), we get

$$\cot V_{CA} \cdot \cot V_{AB} = x_0{}^2 .$$

Since $\sin V_{BC}$ and x_0 have the same sign,

$$x_0 = sign(\sin V_{BC})\sqrt{\cot V_{CA} \cdot \cot V_{AB}} .$$

In the same way, we get the rest of Equation (3.2). ■

Equations (3.1) indicate that the tangent values of three visual sizes have the same sign for any cases (see, Figure 1.1(left)). We will discuss this property again in the last section.

4. Visual Angles of Three Dimensional Orthogonal Axes in Minkowski Space

In this section, let us investigate the relations among the three visual angles of three dimensional orthogonal axes in $\mathbf{M}^3$. Set up a system of orthogonal coordinates $O-XYZ$ on $\mathbf{M}^3$ with the origin at the viewpoint $O=(0,0,0)$. Let us consider to see another three dimensional orthogonal axes $P-ABC$ with the origin $P(\neq O)$ from the view point O. For simplicity, let us assume that both orthogonal axes $O-XYZ$ and $P-ABC$ are positive oriented, and the vector $\overrightarrow{OP}$ is a time-like, future pointing vector such that the central projected point P' of P is on $\mathbf{Q}$. Then, with appropriate Lorentz transformations centered at O, we can arrange the orthogonal axes $P-ABC$ parallel to $O-XYZ$ without changing the visual sizes $\angle oAPB$, $\angle oBPC$ and $\angle oCPA$. Let V_{AB}, V_{BC} and V_{CA} be the three visual sizes of $\angle oAPB$, $\angle oBPC$ and $\angle oCPA$, respectively. We prepare two lines in $\mathbf{Q}$,

$$L_X := \{(x,y,z) \in \mathbf{Q} \mid x=0\},$$
$$L_Y := \{(x,y,z) \in \mathbf{Q} \mid y=0\},$$

which are the base lines in the following argument. As in Figure 4.1 drawn in the Poincare model, the projection of $A-axis$ intersects L_X orthogonally, and let A_0 be the intersection. In the same way, the projection of $B-axis$ intersects L_Y orthogonally, and let B_0 be the intersection. The projection of $C-axis$ passes through C_∞=(0,0,1) which corresponds to the projection of the vanishing point of $C-axis$.

In the previous section, we have already seen that direction cosine is useful to calculate visual size. Hence, let us consider a similar thing in $\mathbf{Q}$ as in the following proposition.

Proposition 4.1(Parametric Representation) For an arbitrary point $P'=(x_0,y_0,z_0)\in \mathbf{Q}$, let us define three parameters α, β, and γ such as

$$\alpha := sign(x_0)\,d(A_0,P'),\quad \beta := sign(y_0)\,d(B_0,P'),\quad \gamma := d(C_\infty,P').$$

Then the coordinates of P' is written as

$$P'=(x_0,y_0,z_0)=(\sinh\alpha,\sinh\beta,\cosh\gamma).$$

Proof. First, let us prove $x_0=\sinh\alpha$. The reflection of P' in L_X is the point $P''=(-x_0,y_0,z_0)$, and $d(A_0,P'')=d(A_0,P')$. Then

$$\cosh(d(P',P''))=-(-x_0{}^2+y_0{}^2-z_0{}^2)$$
$$\cosh(2d(A_0,P'))=-(-x_0{}^2-x_0{}^2-1)$$
$$2\sinh^2(d(A_0,P'))+1=2x_0{}^2+1.$$

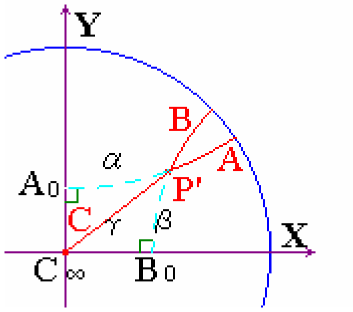

Figure 4.1 Visual angles in the Poincare model with three parameters α, β and γ.

Therefore, $\sinh|\alpha|=|x_0|$, that is, $x_0=\sinh\alpha$. The equation $y_0=\sinh\beta$ is shown as a similar way. As for z_0,

$$\cosh(d(C_\infty,P'))=-<\overrightarrow{OC_\infty},\overrightarrow{OP'}>=z_0 \qquad \blacksquare$$

In this paper, we consider the visual angles as oriented which are measured in the counterclockwise direction looking from the upper side of **Q**.

Theorem 4.2(Law of Visual Angles of Three Dimensional Orthogonal Axes in Minkowski Space) If the point $P'=(x_0,y_0,z_0)$ is not C_∞ ($z_0\neq 1$), then three visual sizes V_{AB}, V_{BC}, and V_{CA} are given by

$$\tan V_{AB}=\frac{z_0}{x_0y_0},\ \tan V_{BC}=-\frac{x_0}{y_0z_0},\ \tan V_{CA}=-\frac{y_0}{z_0x_0}. \qquad (4.1)$$

Conversely, if V_{AB}, V_{BC}, nor V_{CA} are not equal to 0 and π, then the point P' (the direction of the origin P of $P-ABC$) is determined as the following the equation

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}=\begin{pmatrix} sign(\sin V_{BC})\sqrt{-\cot V_{CA}\cdot\cot V_{AB}} \\ sign(\sin V_{CA})\sqrt{-\cot V_{AB}\cdot\cot V_{BC}} \\ \sqrt{\cot V_{BC}\cdot\cot V_{CA}} \end{pmatrix}. \qquad (4.2)$$

Proof. To calculate $\tan V_{BC}$, we prepare a signed distance α',

$$\alpha':=sign(x_0)\,d(C_\infty,B_0).$$

Let φ be the angle between the positive $X-axis$ and $C_\infty P'$, which is measured in the counterclockwise direction. And let θ be the acute angle $\angle P'$ of $\Delta B_0P'C_\infty$ (see, Figure 4.2). Figure 4.2 shows that $|\tan V_{BC}|=\tan\theta$ for every quadrant. Appling the trigonometric function (see [1], p.147) to the right triangle $\Delta B_0P'C_\infty$,

$$\tan\theta=\frac{|\tanh\alpha'|}{|\sinh\beta|}.$$

And also, from the right triangles $\Delta B_0P'C_\infty$ and $\Delta A_0P'C_\infty$,

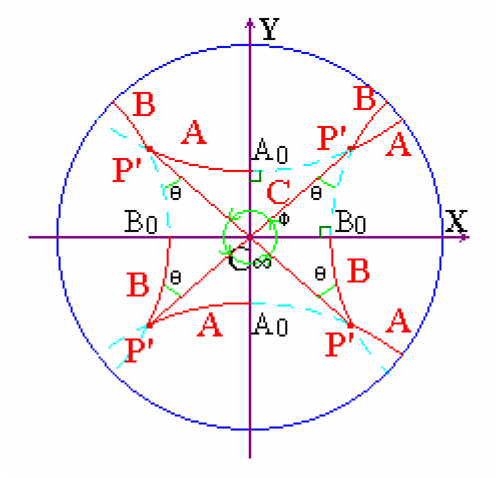

Figure 4.2 Two parameters θ and φ in the Poincare model.

$$\cos\varphi = \frac{\tanh\alpha'}{\tanh\gamma},\ \sin\left(\frac{\pi}{2}-\varphi\right) = \frac{\sinh\alpha}{\sinh\gamma}.$$

Therefore, combining the equations above,

$$\tanh\alpha' = \frac{\sinh\alpha}{\cosh\gamma}.$$

Hence,

$$|\tan V_{BC}| = \left|\frac{\sinh\alpha}{\sinh\beta\cdot\cosh\gamma}\right| = \left|\frac{x_0}{y_0 z_0}\right|.$$

Since $\tan V_{BC}\cdot x_0\cdot y_0 < 0$ as in Figure 4.2, we get

$$\tan V_{BC} = -\frac{x_0}{y_0 z_0}. \tag{4.3}$$

In a similar argument, we get

$$\tan V_{CA} = -\frac{y_0}{z_0 x_0}. \tag{4.4}$$

To calculate $\tan V_{AB}$, note that $V_{AB}+V_{BC}+V_{CA} = 2\pi, 4\pi$ for every quadrant. Using the fact that $a+b+c = n\pi$ ($n\in\mathbf{Z}$), if and only if $\tan a + \tan b + \tan c = \tan a\tan b\tan c$,

$$\tan V_{AB} = -\frac{\tan V_{BC}+\tan V_{CA}}{1-\tan V_{BC}\tan V_{CA}} = \frac{({x_0}^2+{y_0}^2)/(x_0 y_0 z_0)}{1-1/{z_0}^2} = \frac{z_0}{x_0 y_0}.$$

Next, we will prove Equation (4.2). From Figure 4.3 one can see that $\sinh\alpha\cdot\sin V_{BC}\geq 0$. Multiplying the first and the third equations of Equations (4.1),

$${x_0}^2 = -\cot V_{CA}\cdot\cot V_{AB}.$$

So, the fact that $x_0\cdot\sin V_{BC}\geq 0$ implies that

$$x_0 = sign(\sin V_{BC})\sqrt{-\cot V_{CA}\cdot\cot V_{AB}}.$$

In a similar argument, we get

$$y_0 = sign(\sin V_{CA})\sqrt{-\cot V_{AB}\cdot\cot V_{BC}}.$$

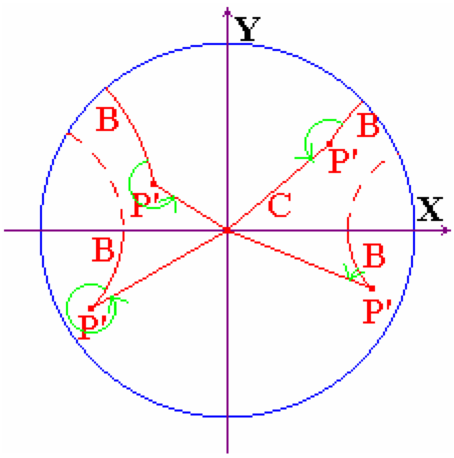

Figure 4.3 The relation between the sign of x_0 and that of $\sin V_{BC}$.

Finally, multiplying the second and the third equations of Equations (4.1),

$$z^2 = \cot V_{BC} \cdot \cot V_{CA}.$$

Since $z_0 > 0$, we get

$$z_0 = \sqrt{\cot V_{BC} \cdot \cot V_{CA}} \qquad \blacksquare$$

5. Conclusions

In the previous sections, we have already seen the relations among visual angles of orthogonal axes in $\mathbf{E}^3$ and $\mathbf{M}^3$. The discussion in $\mathbf{M}^3$ is parallel to that in $\mathbf{E}^3$, however, the crucial difference between them is the signs of tangent value. Euclidean case, the tangent values of three visual angles of three dimensional orthogonal axes have the same sign. On the other hand, in Minkowski case, the tangent values of three visual angles of three dimensional orthogonal axes do not have the same sign. Figure 5.1 shows the typical cases of visual angles in each space. The interesting point is that when three dimensional orthogonal axes are drawn on the plane at random, we can distinguish whether it is Euclidean or Minkowski cases depending on the sign of the tangent values.

In this paper, we extend the visual angle in the Euclidean space to that in the Minkowski space. It is important that hyperbolic geometry plays a role of the view screen in the Minkowski space. And we can make clear the difference of visual angles of three dimensional orthogonal axes between Euclidean and Minkowski spaces.

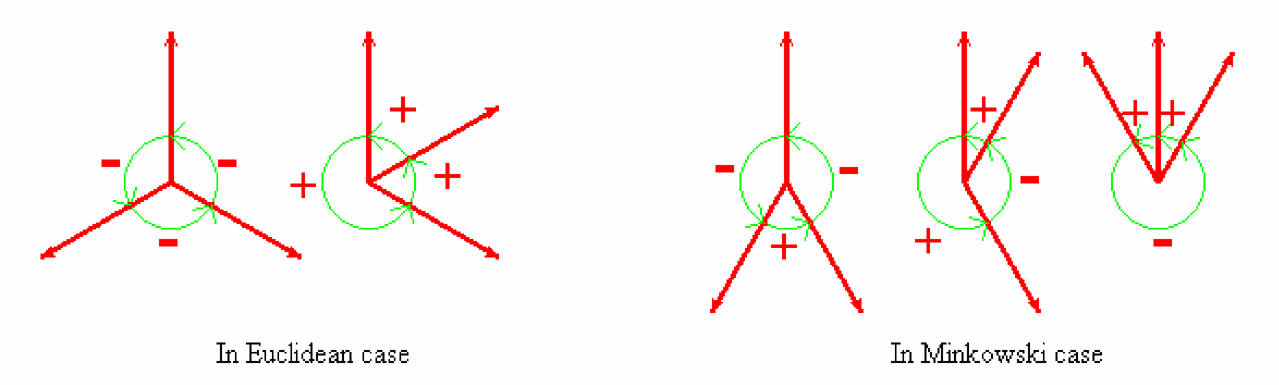

Figure 5.1 Typical cases of visual angles.

References

[1] Beardon, Alan F. (1983). *Geometry of Discrete Groups.* Springer-Verlag New York.

[2] Berger, Marcel. (2003). *A Panoramic View of Riemannian Geometry.* Springer-Verlag Berlin Heidelberg.

[3] Jennings, G. (1994). *Modern Geometry with Applications.* Springer-Verlag New York.

[4] Patrick J.Ryan. (1986). *Euclidean and Non-Euclidean Geometry.* Cambridge university press.

[5] Mori, M. and Maeda, Y. (2005). *Three Visual Angles of Three Dimensional Orthogonal Axes and Their Visualization.* Proceeding of the Tenth Asian Technology Conference in Mathematics, ATCM2005, pp. 315-321.

[6] Maeda, Y. (2003). *Viewing Cube and Its Visual Angles.* Akiyama and Kano JCDCG, LNCS 2866, pp. 192-199.