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Introduction

The ATCM 2024 (atcm.mathandtech.org) returned to Yogyakarta after ATCM 2014 and ATCM 2018. The ATCM 2024 is in a hybrid format. We accepted 12 invited papers and 27 contributed papers. There are about 35% of the presentations done virtually this year and we are happy to see many old friends and we are equally thrilled to see many new faces too.

Given a Ph.D. or BS math degree program has been eliminated in recent years, the combination of ChatGPT with the Wolfram plug-in scored 96% in a UK Maths A-level paper, we certainly need to ponder why students must choose math as a major. It is exciting to see some presentations this year providing some partial solutions to this question.

All authors and readers are encouraged to contribute their favorite ideas to the next ATCM or publish interesting articles in the Electronic Journal of Mathematics and Technology (eJMT: https://ejmt.mathandtech.org) Selected eJMT papers will be published in the Research Journal for Mathematics and Technology (RJMT: http://rjmt.mathandtech.org).

It is always nice to renew old friendships, and also exciting to make more new friends at an ATCM. We hope you will invite your colleagues to experience future ATCMs for themselves. We look forward to seeing everyone in person at ATCM 2025 in the Philippines, more details will be announced later.

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Table of Contents

Invited Papers

1 Cracking the Enigma Code: Beyond the Bombe
Adam S. Downs, Neil P. Sigmon, Richard E. Klima 1

2 Understanding Geometric Pattern and its Geometry Part 12 – Octagonal investigations
Mirosław Majewski 16

3 Illustrating the Pre-Conjecture Phases of Experimental Research mediated by Dynamic Geometry: How Generalization Can Generate a Multitude of Highly Plausible Conjectures About Semi-Regular Polygons
Jean-Jacques Dahan 38

4 Creating Planar k-isogonal Tilings
Ma. Louise Antonette N. De Las Peñas, Mark D. Tomenes, Agatha Kristel M. Abila 52

5 Development of a Question Distribution and Answer Collection System for Mathematics Classes Using Only One Line of Text
Setsuo Takato and Hideyo Makishita 67

6 The Development of Fraction Virtual Manipulatives and Their Applications to Low-achieving Elementary School Students
Yuan Yuan 79

7 Learning Experiences: Connecting A-Level Mathematics with Mathematics Used in The Real World via Machine Learning
Weng Kin Ho, Zhu Hui Joel Quek 94

8 Topological Structures Between Two Closed Surfaces Inspired by An Entrance Exam Problem
Wei-Chi Yang, Weng Kin Ho, Guillermo Dávila 109

9 Polynomials associated with bicentric polygons
Alasdair McAndrew 124

10 A Timeless Puzzle in Modern Education
Vanda Santos 138

11 Professional Development in Coding and Computational Thinking for Mathematics Teachers
Keng Cheng Ang, Marc Yi Fei Yeo 153

12 Automated generation of solved problems for self-study
José A VALLEJO 168

Contributed Papers

1 Bayesian Sequential D-Optimal Sampling Designs for Fractional Partial Differential Equations
Edward L. BOONE, Ryad A. GHANAM 181

2 Engaging Future Secondary Teachers in Technology Experiences
Marshall Lassak 191

3 Learning guidance based on analysis of symbolic comprehension
Tomohiro Washino, Tadashi Takahashi 200

4 Classroom Practices with RLA to Promote Inquiry-Based Learning
Norie AOKI, Hideyo MAKISHITA 210

5 Connecting Trigonometry to Its Geometric Roots: An Introduction to Trigonometric Values
Niroj Dahal, Binod Prasad Pant, Bal Chandra Luitel, Basanta Raj Lamichhane, Tara Paudel 218

6 The Role of Educational Robotics in Integrated STEM Learning Towards the Formation of 21st- Century Skills
Cucuk Wawan Budiyanto, Muhammad Husein Arafat, Rosihan Ari Yuana 229

7 Some Poncelet invariants for bicentric hexagons

Grant Keady and Alasdair McAndrew 240
8 Exploring Conic Sections through Desmos Artworks: Where Math, Art, and Technology Combine
Maria Digi Anna Mance Avila, Maria Alva Q. Aberin 250
9 Impact on Student Learning Outcomes in Mathematics Using GeoGebra
Werachai Pattanapiboon, Hitoshi Nishizawa 258
10 Technology Integration Frameworks in Geometry Education: A Synthesis of Recent Research
FNU Pujiyanto 268
11 Automated reasoning tools for dealing with elementary but intriguing geometric loci
Tomas RECIO and Carlos UENO 282
12 On a Study of Improvements of the Quadratic Curves Addition Method: Consideration of Theorem 8
Hideyo MAKISHITA 292
13 Mathematical Modeling in a 3D Printing Project
Leyre Gilardi, Lucía Rotger-Garcia, Alvaro Nolla, Juan M. Ribera-Puchades Angélica Benito 305
14 E-Learning material creation system that utilizes existing teaching materials
Takuya Kitamoto, Masataka Kaneko, Takeo Noda, Hiroyoshi Kihara 314
15 Towards a conceptual framework for integrating mathematical digital competencies in digital project-based learning of mathematics
Marc Helton C. Sua, Lester C. Hao 324
16 Research on the Use of Dynamic Geometry via Blended Learning
Sárka Voráčová 334
17 An Initial Analysis on the Reliability of Using Chatbots in Solving Word Problems in Probability
Reggie V. Nalupa, Lester C. Hao 344
18 Online learning incorporating a mathematics online test system and a dynamic geometry system
Chieko Komoda, Yasuyuki Kubo, Satoshi Yamashita, Masumi Kameda 354
19 Development of a Specification Management System for Multidimensional PISA-like Mathematics Items
Mark Lester Garcia, Aldrich Ellis Asuncion, Catherine Vistro-Yu, Nia Kurniasih 361
20 Constructing Multi-layered Acrylic Tiling Models Using Laser Engraving and Cutting
Mark Loyola, Ma. Louise Antonette De Las Peñas, Mark Tomenes 373
21 Exploring the Influence of Educational Robotics in Geometry to Students' Computational Thinking Skills
Cucuk Wawan Budiyanto, Kristof Fenyvesi, Nur Isnaini Hanifah, Aris Budianto 383
22 The Influence of Educational Robotics on Students' Computational Skills: The Case of Physics Learning Material on the Relationships Between Work and Energy
Cucuk Wawan Budiyanto, Kristof Fenyvesi, Alfiyah Aini Hanifah, Aris Budianto 393
23 AI, Tensors, and Math Homework
Russel O. Carlson 404
24 Teachers' Narratives on Integrating GeoGebra for Enhancing Conceptual Understanding in Mathematics
Binod Prasad Pant, Niroj Dahal, Bal Chandra Luitel 411
25 Preparing Math Teachers for the Future: Analysing the Use of Technology in a Master's Program
José Manuel Dos Santos, Jaime Carvalho e Silva, Zsolt Lavicza 420
26 Applications of GPT and Copilot in Tourism Courses: A Case Study in Chung Hua University
Ming-Gong Lee, Che-Yuan Yang 435
27 Integrating 3D Modelling into STEAM Education
Jozef Hvorecký, Vera Ferdiánová, Angelika Schmod, Lilla Korenova 447

Invited Papers

Cracking the Enigma Code: Beyond the Bombe

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Abstract: *The work of Allied codebreakers in the cryptanalysis of the Enigma cipher machine during World War II has been well-documented, and rightfully recognized as one of the supreme achievements of the human intellect. One of the trickiest parts of this analysis related to the plugboard, which contributed by far the largest factor to the total number of configurations of the entire machine. To determine the daily plugboard connections, codebreakers used electromechanical devices called the bombe and checking machine. After they found the plugboard connections though, they still needed to discover some additional machine settings. The difficulty in discovering these additional settings varied depending on which branch of the German military had created the messages under attack, since each branch used slightly different procedures. The process was in fact significantly more difficult for messages created by the German Navy, as opposed to those created by either the German Army or Air Force. In this paper, we will describe and demonstrate some of the procedures involved in recovering this additional information that was needed to fully cryptanalyze the Enigma machine. To assist in demonstrating these procedures, technology involving Maplets will be used.*

1 Introduction

In 1918, German electrical engineer Arthur Scherbius applied for a patent for a mechanical cipher machine. This machine, later marketed commercially under the name *Enigma*, was designed with electric current passing through revolving wired wheels. Scherbius offered the machine to the German

military. Only after learning that their World War I ciphers had routinely been broken did they adopt it, which they used as their primary resource for encrypted communications throughout World War II.

As we described in [3], Polish codebreakers led by Marian Rejewski successfully cracked the Germans' first implementation of the Enigma. After modifications to the machine by German cryptographers rendered the Polish techniques no longer viable, British mathematician Alan Turing identified weaknesses in the encryption process using patterns generated by *cribs*, which are small parts of plaintext corresponding to known parts of the corresponding ciphertext, and which were made easier to find through the frequent mistaken use of salutations, titles, and addresses by operators. When an electromechanical device designed by Turing called the *bombe* became operational, codebreakers began again reading German messages. For messages created by the German Army and Air Force, once one encrypted message between two specific operators had been broken, every message between any two operators on the same day could be broken due to a deficiency in how Enigma operators transmitted their rotor starting positions. However, German Navy operators used a more secure procedure for transmitting their positions, which made the codebreaking process much more difficult.

In this paper, we will examine some of the challenges and illustrate some of the techniques used by Allied codebreakers to discover daily Enigma machine settings and break Enigma messages sent by operators from all branches of the German military. We will begin with a brief review of the components of an Enigma and the basics of how the bombe was used in the initial cryptanalysis.

2 Components of Enigma and Initial Cryptanalysis

We begin by reviewing the components of an Enigma machine and the challenges it presented to Allied codebreakers. We give more detailed descriptions of these components and challenges in [2]. Figure 1 shows an image of an Enigma with several important components labeled.

Figure 1: Enigma cipher machine.

To encrypt or decrypt a letter using an Enigma, the key labeled with the letter on the keyboard was pressed, which launched electric current representing the letter into the machine. The current traveled first through the plugboard, where it could change to another letter depending on whether sockets labeled with the letter were connected by a cable to sockets labeled with the other letter. The current then passed through three rotors, in each of which it could change to another letter, and then through a reflector, where it definitely changed to another letter. After this the current went back through the rotors in the opposite direction and then back through the plugboard, in each of which it could again change to another letter. Finally, the current went to the lampboard, where it lit a lamp labeled with the encrypted or decrypted letter. We describe this process in much more detail in [4].

For the plugboard, the Germans settled on using 10 cables to connect 20 letters in pairs. Using a formula that we describe and justify in [2], we find that there are 150,738,274,937,250 different ways to choose 20 letters and connect these letters in pairs using 10 cables at the plugboard.

Around each of the three rotors in an Enigma, a ring was etched with the 26 letters (or numbers representing the letters) alphabetically clockwise when viewed from the right. Above each rotor slot, a small window was cut to make the letter (or number) at a particular location on the ring visible, called the *window letter*. While each rotor could be rotated into any of 26 different positions inside the machine, just the ring could also be rotated while the central part of the rotor was held fixed. A rotation of the entire rotor versus just the ring was distinguished using a number called the *ring setting*. The ring setting essentially indicates a rotation of just the ring, while the window letter indicates a combined rotation of the central part of the rotor with the ring attached. The window letter and the ring setting each contribute a factor of $26^3 = 17{,}576$ to the total number of machine configurations.

The window letter and the ring setting must be considered separately to account for all of the variability in the rotors because the rotors rotated in the machine during encryption and decryption, rotation which was only influenced by the ring, not the central part of the rotor. Encryption and decryption was done one letter at a time, and each time an input letter was pressed on the keyboard, the rightmost rotor (before the current reached it) would rotate one position counterclockwise when viewed from the right. Notches cut into the ring around each rotor then influenced the rotation of the rotor to its left. The German Army and Air Force used rotors labeled **I–V**, into which one notch was cut into the ring. The German Navy also used additional rotors labeled **VI–VIII**, into which two notches were cut into the ring. Since each notch was on the ring, its position in the rotor slot at any time could be identified by the window letter. For each notch, there was one particular position in the rotor slot, identified by a window letter called the *notch letter*, for which if the rotor rotated one position counterclockwise, the notch would cause a rotor to its left to also rotate one position counterclockwise. Table 1 shows the notch letters for each of the rotors **I–VIII**.

Rotor	**Notch letters**	**Rotor to left rotates when window letter changes as indicated**
I	Q	Q $\to$ R
II	E	E $\to$ F
III	V	V $\to$ W
IV	J	J $\to$ K
V	Z	Z $\to$ A
VI	M and Z	M $\to$ N or Z $\to$ A
VII	M and Z	M $\to$ N or Z $\to$ A
VIII	M and Z	M $\to$ N or Z $\to$ A

Table 1: Notch letters for Enigma rotors **I–VIII**.

Since the German Army and Air Force chose from among five rotors for the three rotor slots, there were $5 \cdot 4 \cdot 3 = 60$ ways for them to arrange rotors in the machine. Since the German Navy chose from among eight rotors though, there were $8 \cdot 7 \cdot 6 = 336$ ways for them to arrange rotors in the machine. Although rotors could only be situated with each side facing in a particular direction, current passed through them in both directions, from right to left before the reflector, and then from left to right after the reflector. Like the rotors, reflectors had electrical contacts representing the 26 letters alphabetically clockwise when viewed from the right, though only on one side, since current entered and exited reflectors on the same side. Unlike the rotors (and the plugboard), letters in reflectors were always fully connected in 13 pairs. Having the reflector in the middle of the symmetric sequence

$$\text{plugboard} \rightarrow \text{rotors (right to left)} \rightarrow \text{reflector} \rightarrow \text{rotors (left to right)} \rightarrow \text{plugboard}$$

also ensured that the machine configuration for encrypting and decrypting the same message was identical, since the decryption of a ciphertext letter was obtained by simply reversing the path through the machine when the corresponding plaintext letter had been encrypted.

Reflectors labeled B and C with two different electrical wirings were produced for the German Army, Air Force, and Navy. The reflector in use was always placed in the machine in only one way and did not rotate. As such, the number of ways in which a reflector could be chosen and placed into the machine was 2.

Combining the factors resulting from the plugboard, window letters, ring settings, rotor arrangements, and reflector, the total number of different machine configurations for an Enigma used by the German Army and Air Force was 5.59×10^{24}. In an Enigma used by the German Navy, for which the number of rotor arrangements was 336 rather than 60, the total number of different machine configurations was 3.13×10^{25}.

Allied codebreakers discovered that identifying cribs allowed them to eliminate the largest factor in the total number of different machine configurations, the one resulting from the plugboard. Through a geometric pairing of the crib and its corresponding part of the ciphertext known as a *menu*, a search was done for the reflector, rotor order, window letters, and ring settings that gave plugboard pairs that produced logical consistencies, in which each letter in the menu always had a unique plugboard partner assignment.

During the search for the machine settings that gave the plugboard pairs though, the window letters and ring settings that were found were usually not the ones that had been used in the actual encryption of the message. Instead, the window letters and ring settings that were found resulted in the same rotor core starting positions, or *rotor offsets*, that had been used in the actual encryption of the message. To understand this idea better, consider an encryption with initial left-to-right window letters NCS (which correspond to position numbers 14, 3, 19 in the alphabet) and corresponding ring settings 4, 26, 13. The rotor offsets for this encryption would then be 10, 3, 6, since $14-4=10$, $3-26=-23=3 \bmod 26$, and $19-13=6$. However, if we did the same encryption with these rotors and initial window letters ZZZ (corresponding to 26, 26, 26) and corresponding ring settings 16, 23, 20, the rotor offsets would again be 10, 3, 6, since $26-16=10$, $26-23=3$, and $26-20=6$. Then, assuming the same reflector and rotor order were used, and that there were no rotations of the middle and left rotors in either scenario, the encryption using these rotors with the initial window letters NCS and corresponding ring settings 4, 26, 13 would be the same as with the initial window letters ZZZ and corresponding ring settings 16, 23, 20. To test for the correct rotor core starting positions, a setting for the initial window letters, in many cases ZZZ, was chosen, and the ring settings were found by a brute force attack from among $26^3 = 17{,}576$ possibilities, until the correct rotor core starting positions were

found. To find a plugboard partner for the central letter that was logically consistent, a combination of a reflector (either B or C), a rotor order (of which there were 60 possibilities for Army and Air Force messages or 336 possibilities for Navy messages), and a ring setting (of which there were 17,576 possibilities) was tested, which eliminated having to test the $26^3 = 17{,}576$ possibilities for the window letters.

This process allowed Allied codebreakers to break a single encrypted message between two individual Enigma operators on any particular day. To break all of the messages sent by any Enigma operators on that same day though, more work was required. We describe this next.

3 Army and Air Force Encryption

German Army and Air Force Enigma operators were equipped with a three-rotor Enigma machine and a set of five different rotors labeled **I–V**. Each operator was also issued the same monthly setting sheet. For each day of the month, the setting sheet would identify to the operator the machine setting, or key, for the day. Included in the daily key were the specific rotors to be installed in the machine, the prescribed order for these rotors, the choice of reflector between B and C, the ring settings for the rotors, and the ten plugboard letter pairs.

When an Enigma operator needed to encrypt a message, they set up the machine using the specified daily common key. However, when placing a rotor in the machine, each individual Enigma operator would choose their own window letter for it. The three window letters they chose for all three rotors, in order from left to right, were known as the *message setting*.

Of course, the Enigma operator to whom they were sending the message needed to know the message setting as well. We will describe a procedure for providing this that was used by German Army and Air Force Enigma operators after May 1940. For the purpose of communicating their message setting, the sending operator would choose another three letter group, known as the *indicator setting*, and start by setting up their machine using the indicator setting rather than the message setting. The sending operator would then use this setup to encrypt the message setting. Then the sending operator would change the window letters to match the message setting, and encrypt the message. The result was then transmitted to the receiving operator by sending the encrypted plaintext message following by the six letter sequence consisting of the sending operator's indicator setting and encrypted message setting.

We will demonstrate this process using a Maplet[1] entitled **Encrypt/Decrypt Enigma Message** which was written by the authors and designed to encrypt and decrypt messages using an Enigma.[2] The source code for this and all of the other Maplets demonstrated in this paper, as well as directly usable versions of them, can be downloaded at [5]. The code is unique to Maple, but could easily be altered for use with any programming language.

Suppose, for example, that the daily Enigma key is the following.

Rotor Order:	**I**, **V**, **III**
Reflector:	B
Ring Settings:	19, 13, 25
Plugboard Pairs:	O/S, N/T, R/H, B/U, L/I, D/F, G/A, Q/P, C/K, J/Y

[1] A Maplet is like an applet, but uses (and requires) the engine of the computer algebra system Maple, and is written using Maple functions and syntax.

[2] The Enigma machine used a typewriter style in which one letter was encrypted or decrypted at a time. This Maplet is designed to expedite this process by encrypting or decrypting all letters with one click.

Suppose also that the sending operator chooses message setting TEX, and indicator setting RED. The sending operator would first set up the machine using the daily key with the indicator setting as the window letters, and use this setup to encrypt the message setting. Figure 2 shows the result of using the **Encrypt/Decrypt Enigma Message** Maplet to do this, resulting in the output `YXS`.

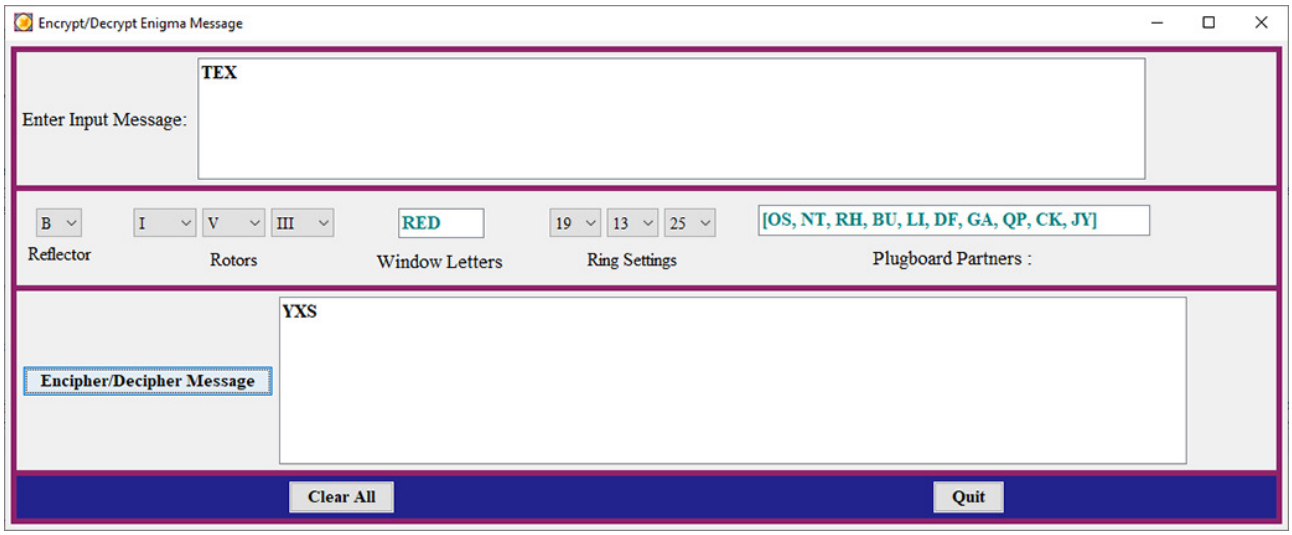


Figure 2: Encrypting a message setting.

Suppose now that the sending operator wished to encrypt the message `TODAY IS THE DAY`. The sending operator would then change the window letters in the machine to TEX, and use this setup to encrypt the message. Figure 3 shows the result of using the Maplet to do this, resulting in the ciphertext `HKMQWALFVHABJ`.

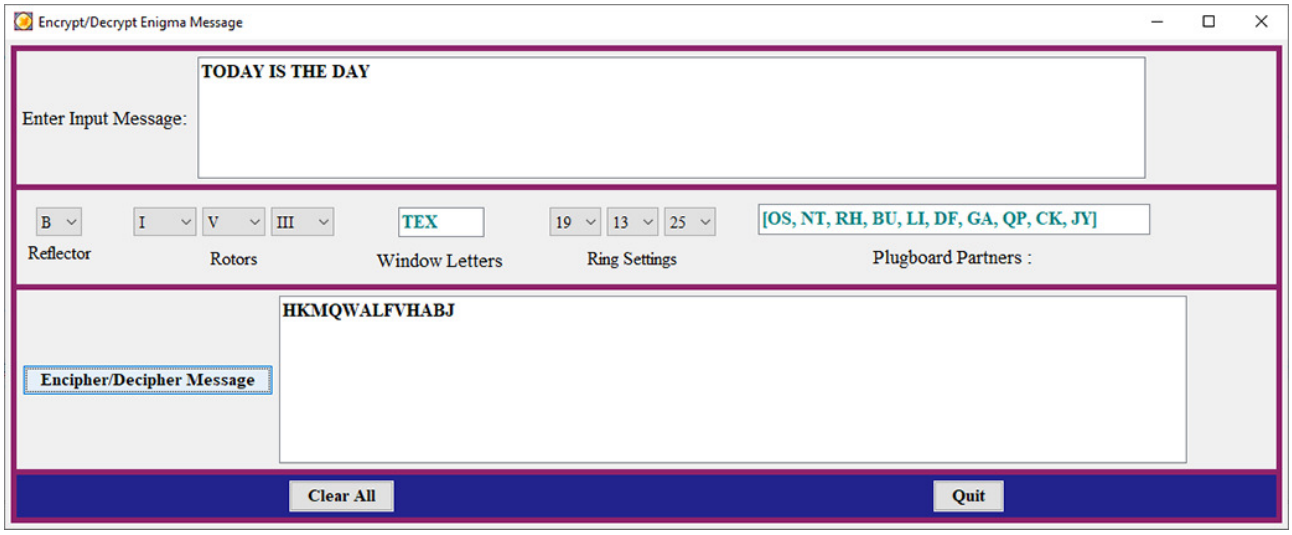


Figure 3: Encrypting a message.

Finally, the sending operator would transmit the ciphertext followed by the indicator setting and the encrypted message setting: `HKMQWALFVHABJ REDYXS`. To decrypt the message, the receiving operator would first set up their machine using the daily key with the indicator setting `RED` as the window letters, and decrypt the last three letters received. This would result in `TEX`. The receiving

operator would then change the window letters in the machine to TEX, and use this setup to decrypt the message.

4 Army and Air Force Cryptanalysis

Turing's bombe enabled Allied codebreakers to decrypt messages that had been encrypted by German Army and Air Force Enigma operators. However, there was not enough time to use the bombe to decrypt all messages sent between any two Army or Air Force operators. We will now consider how it was possible for Allied codebreakers to quickly decrypt all messages sent between any two Army or Air Force operators on the same day as one message they had decrypted using the bombe.

When an individual Enigma ciphertext was broken using the bombe, the process resulted in ring settings that gave rotor core starting positions based on the assumption that the last window letters prior to the crib were ZZZ. When the bombe was successful on one Enigma ciphertext, although it would give the reflector, rotor order, and plugboard connections from the daily key on the day the ciphertext was formed, it would only give the ring settings from the daily key if the last window letters prior to the crib had actually been ZZZ. Finding the actual ring settings from the daily key was necessary in order to be able to decrypt most or all of the other encrypted messages sent between any two Army or Air Force operators on the same day without having to use the bombe.

The first step in finding the ring settings from the daily key was to determine the actual initial window letter and ring setting of the rightmost rotor. These could be found from knowing where a turnover of the middle rotor first occurred, the notch letter of the rightmost rotor, and the rightmost initial window letter resulting from the assumption that the last window letters prior to crib were ZZZ.

To demonstrate this, we will continue an example that we began in [2], in which we assumed we knew the crib FOLLOWORDERSTO corresponded to the underlined part of the ciphertext message NU<u>ENTZERLOHHBTDS</u>HLHIYWEABHTQKC. Using this fact, we were able to determine in [2] the following parameters from using the bombe on this one ciphertext.

Rotor Order:	**I**, **V**, **III**
Reflector:	C
Ring Settings:	16, 23, 18
Plugboard Pairs:	O/S, N/T, R/H, B/U, L/I, D/F, G/A, Q/P, C/K, J/Y

We will now demonstrate how the actual initial window letter and ring setting of the rightmost rotor can be determined for this example. We will do this using a Maplet entitled **Clonking**, which was written by the authors and designed for this purpose. The result is given in Figure 4. In the Maplet, the reflector, ring settings, rotor order, and plugboard pairs discovered using the bombe are entered, as well as the crib, its corresponding part of the ciphertext, and the ciphertext in full. Clicking the **Display Plaintext/Ciphertext** button then causes the window letters, plaintext, and ciphertext to be displayed under the assumption that the last window letters prior to the crib were ZZZ. From the output shown in the Maplet, we can see that the decrypted message is legible only through the window letters ZZS. The reason the message is no longer legible after this point is very likely due to a turnover of the middle rotor at this point. Since there are two letters in the ciphertext before the start of the crib, the Maplet thus indicates that the actual encryption starts with the window letters ZZX. So to find the actual window letter of the rightmost rotor when the ciphertext was formed, note that since a turnover of the middle rotor first occurred at ZZS and the machine was set to ZZX, then a total of 21 turnovers occurred for the rightmost rotor during encryption. From Table 1, we can see that

the rightmost rotor **III** only has the notch letter V. After 21 turnovers of the rightmost rotor during encryption, the notch letter V would have been the rightmost window letter. Thus, if we go backwards on the ring of letters around the rotor (i.e., backwards in the alphabet, wrapping from the start of the alphabet to the end if necessary) 21 positions from V, the result would be the actual rightmost initial window letter. Going backwards on the ring of letters around the rotor 21 positions from V gives A, which is the actual rightmost initial window letter when the ciphertext was formed. Next, to find the ring setting of the rightmost rotor when the ciphertext was formed, we can use the fact that for the rightmost initial window letter X and ring setting 18 that resulted from the assumption that the last window letters prior to crib were ZZZ, or the actual rightmost initial window letter A and actual ring setting, the rotor core starting positions would have to be the same. For the rightmost initial window letter X and ring setting 18, since X is the 24th letter in the alphabet, the rotor core starting position would be $24 - 18 = 6$. Thus, because subtracting the rotor core starting position from the number of the initial window letter would give the ring setting, and the actual rightmost initial window letter A is the first letter in the alphabet, the actual ring setting would be 21, since $1 - 6 = -5 = 21 \bmod 26$. Clicking the **Find Actual Right Most...** button in the Maplet shows these results.

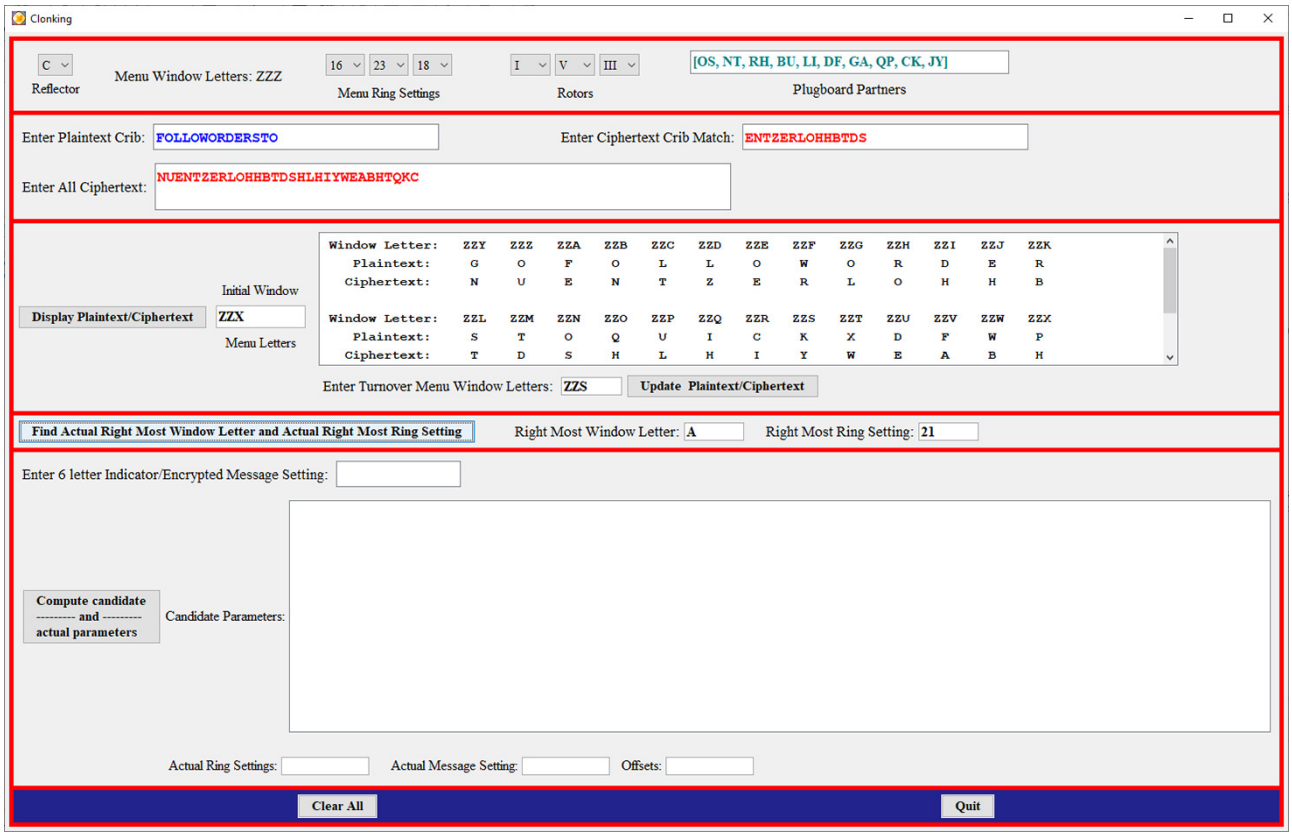


Figure 4: Finding an actual rightmost window letter and ring setting.

After finding the actual initial window letter and ring setting of the rightmost rotor, Allied code-breakers could then find the actual ring settings of the middle and leftmost rotors by exploiting how they knew Army and Air Force Enigma operators transmitted their message settings as described in Section 3. To demonstrate this, consider an Enigma configured with the correct reflector, rotor order, plugboard connections, and actual initial window letter and ring setting of the rightmost rotor, and with the rotors turned so that the indicator setting is showing through the windows. There would only

be $26 \cdot 26 = 676$ different possible combinations of ring settings of the middle and leftmost rotors. If the letters in the operator's encrypted message setting were decrypted assuming each of these 676 different possible combinations, only a relatively small number should cause the third letter to decrypt to the actual rightmost initial window letter. It would not be unreasonable to try all 676 different possible combinations to see which ones caused the third letter in the operator's encrypted message setting to decrypt to the actual rightmost initial window letter, and this is exactly what the codebreakers did. They actually did not even have to decrypt all of the letters in the operator's encrypted message setting for every trial. By rotating the rightmost rotor two positions forward to replicate the rotations that would occur when the first two letters were decrypted, they could initially test a combination by decrypting just the third letter, and then decrypt the first two letters only if the third letter decrypted to the actual rightmost initial window letter.

For example, suppose that the sequence `BOXFUW` consisting of the indicator setting and encrypted message setting was intercepted. For the sequence `BOXFUW`, the codebreakers could, with the rotors turned so that BOZ was showing in the windows, try all 676 different possible combinations of ring settings of the middle and leftmost rotors to see which ones caused `W` to decrypt as `A`. Then only for those that did, they could turn the rotors back so BOX was showing in the windows and decrypt `FU`.

The Allied codebreakers referred to the process that we just described as *clonking*. After clonking, from the combinations of ring settings of the middle and leftmost rotors that caused the third letter in the operator's encrypted message setting to decrypt to the actual rightmost initial window letter, they could identify the correct actual ring settings of the middle and leftmost rotors by checking to see which ones together with the resulting decrypted letters gave the same rotor core starting positions as the initial window letters and ring settings that had worked in the cryptanalysis process on the individual ciphertext.

As an demonstration of clonking, consider the example we began in Figure 4, in which we found that the actual initial window letter and ring setting of the rightmost rotor when the ciphertext was formed were A and 21, respectively. Clonking for the sequence `BOXFUW` would involve trying all 676 different possible combinations of ring settings of the middle and leftmost rotors to see which ones caused `FUW` to decrypt as `..A`, and then for each combination that did, finding the resulting rotor core starting positions. By entering BOXFUW and clicking the **Compute candidate...** button, the **Clonking** Maplet displays all ring settings of the middle and leftmost rotors that decrypt `FUW` as `..A`. The results are shown in Figure 5. By examining the output, we can see for example that the ring settings 3, 23, 21 cause `FUW` to decrypt as `ZFA`. Since ZFA are the alphabet letters in positions 26, 6, 1, the ring settings 3, 23, 21 with these initial window letters result in the rotor core starting positions 23, 9, 6, since $26-3=23$, $6-23=-17=9 \bmod 26$, and $1-21=-20=6 \bmod 26$. The initial window letters and ring settings that had worked in the cryptanalysis process on the ciphertext in this example were ZZX and 16, 23, 18, respectively. Since ZZX are the alphabet letters in positions 26, 26, 24, the ring settings 16, 23, 18 with these initial window letters result in the rotor core starting positions 10, 3, 6, since $26-16=10$, $26-23=3$, and $24-18=6$. Note that in the Maplet output, the ring settings 10, 15, 21 and decrypted letters `TRA` give these same rotor core starting positions. Thus, the actual initial window letters and ring settings when the ciphertext in this example was formed were TRA and 10, 15, 21. As a result, 10, 15, 21 must be the ring settings in the daily key for all Enigma ciphertexts formed on the same day as the ciphertext in this example. This result is verified in the Maplet. With the correct ring settings found for the daily key for the particular day, the Allied codebreakers could use the indicator setting transmitted by each Enigma operator, decrypt their message setting, and then decrypt the ciphertext message sent by the operator.

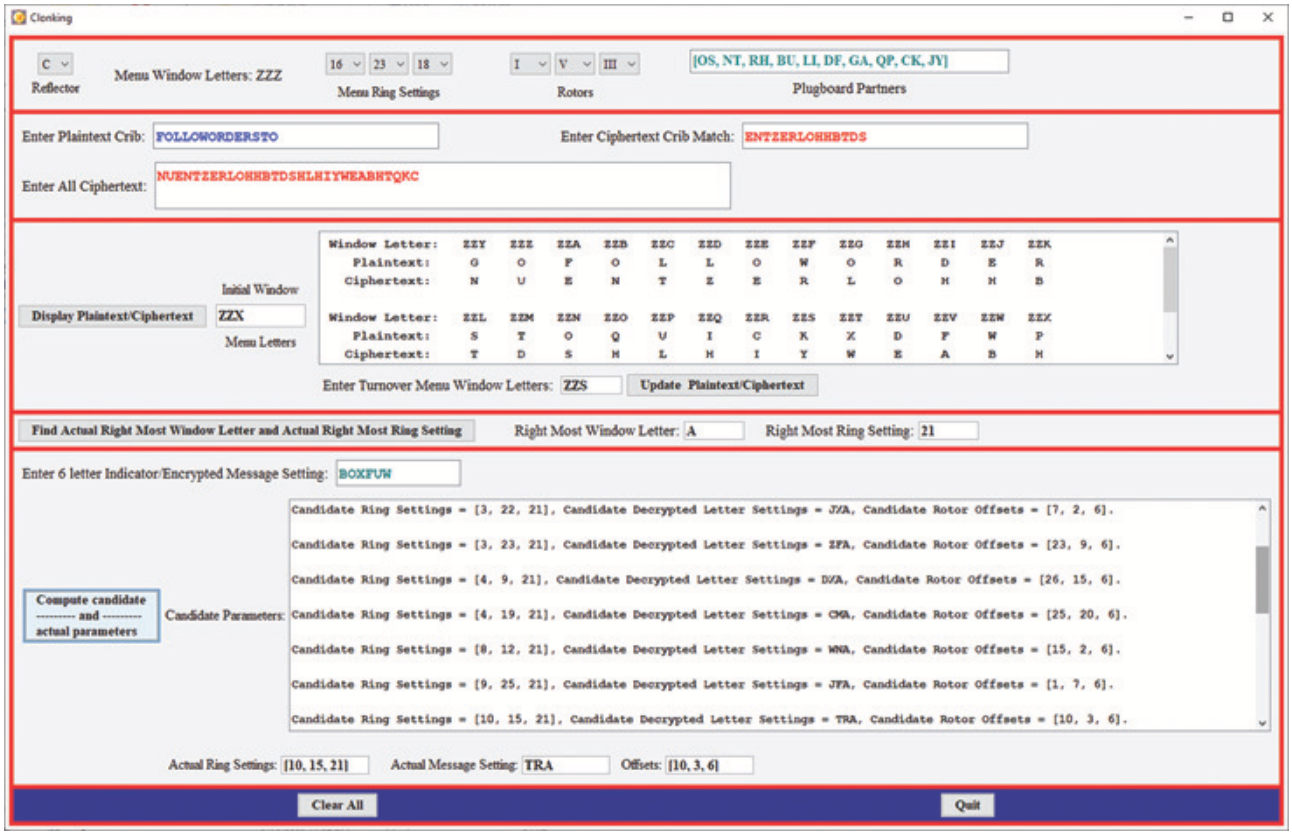


Figure 5: Clonking to find actual message setting and ring settings.

5 Navy Encryption

At the beginning of World War II, the German Navy used the same three-rotor Enigma machine as the German Army and Air Force. However, in addition to the five rotors used by the Army and Air Force, Navy Enigma operators were equipped with three additional rotors, for a total of eight. Like Army and Air Force Enigma operators, Navy operators were issued a monthly setting sheet with instructions on how to configure their machine on any given day. Unlike their Army and Air Force counterparts though, Naval Enigma daily keys were issued in two parts. First, the *inner settings* specified the rotor order and ring settings that were to be used for pairs of days for the month covered by the setting sheet. Then, the *outer settings*, which were changed daily, specified the plugboard pairs and a collection of special positions specified by the rotor window letters known as the *Grundstellung* (in English, *initial position*).

In contrast to their counterparts in the German Army and Air Force, German Navy Enigma operators did not choose their own message settings. Instead, each operator selected a group of three letters called the *procedure indicator group* from a designated section of a printed book called the *K-book*. Then, after configuring their Enigma machine with the specified rotor order, ring settings, plugboard pairs, and window letters specified by the Grundstellung, the sending Enigma operator encrypted the three letters specified by the procedure indicator group. The three letters resulting from this encryption represented the message setting. The process was referred to by the Allied codebreakers as *Dolphin*.

Suppose, for example, that the daily key is the following.

Inner Settings:	Ring Settings:	16, 10, 22
	Rotor Order:	**VIII**, **IV**, **I**
Outer Settings:	Reflector:	B
	Plugboard Pairs:	O/S, N/T, R/H, B/U, L/I, D/F, G/A, Q/P, C/K, J/Y
	Grundstellung:	YES

To create the message setting, suppose the operator uses the K-book to look up procedure indicator ATS. The sending operator would first set up the machine using the inner and outer settings with the Grundstellung as the window letters, and use this setup to encrypt the procedure indicator. Figure 6 shows the result of using the **Encrypt/Decrypt Enigma Message** Maplet to do this, with output `SJK`.

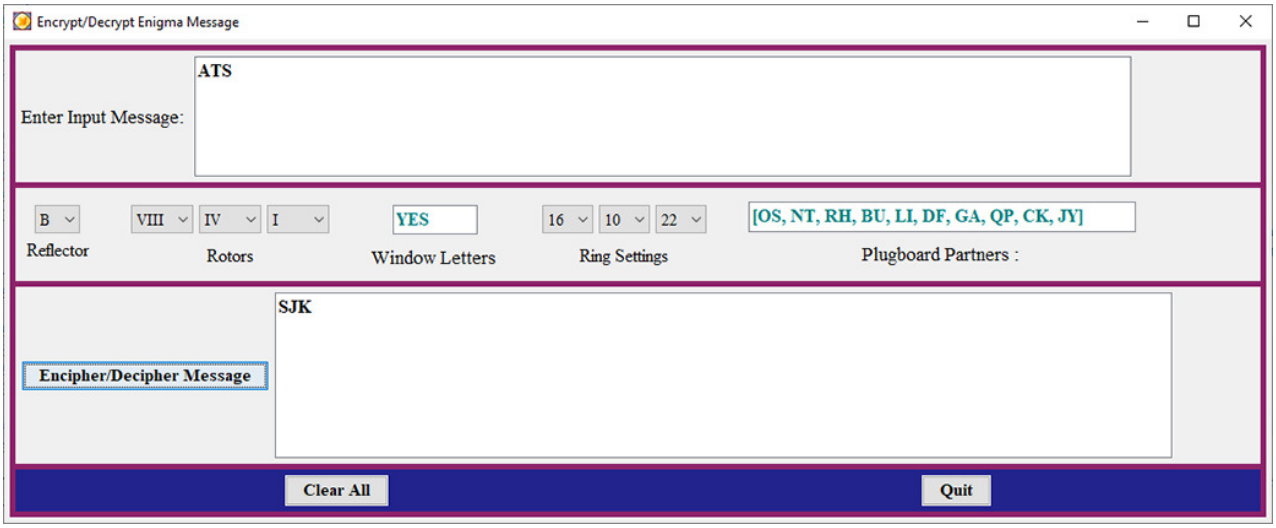


Figure 6: Encrypting a procedure indicator group.

Suppose now the sending operator wished to encrypt `TOMORROW IS BETTER`. They would then change the window letters in the machine to SJK, and use this setup to encrypt the message. Figure 7 shows the result of using the Maplet to do this, resulting in the ciphertext `CQCLWTIXNNLKFGGS`.

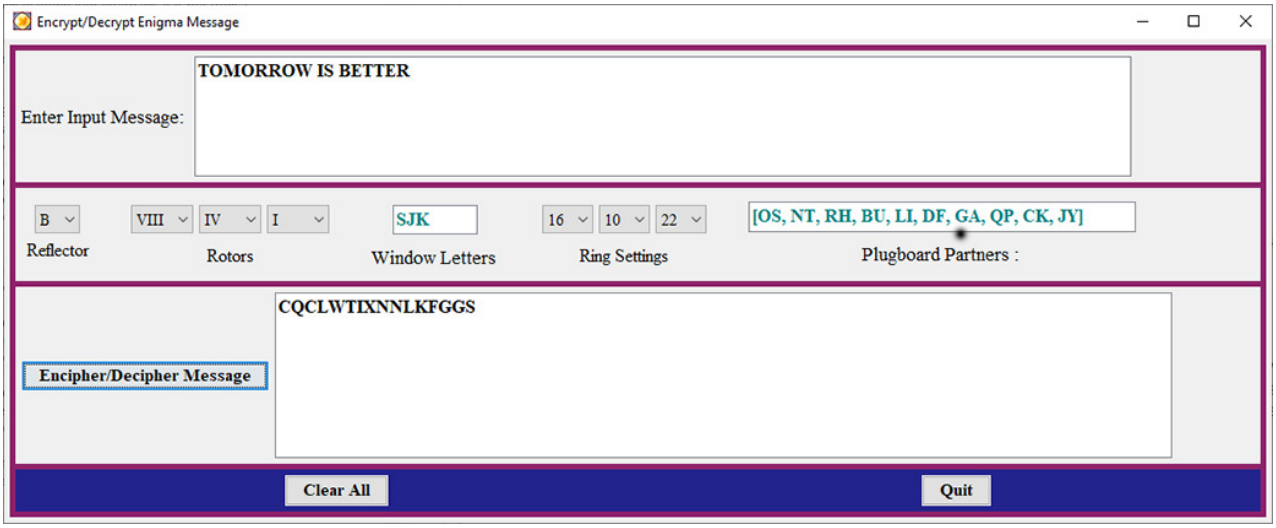
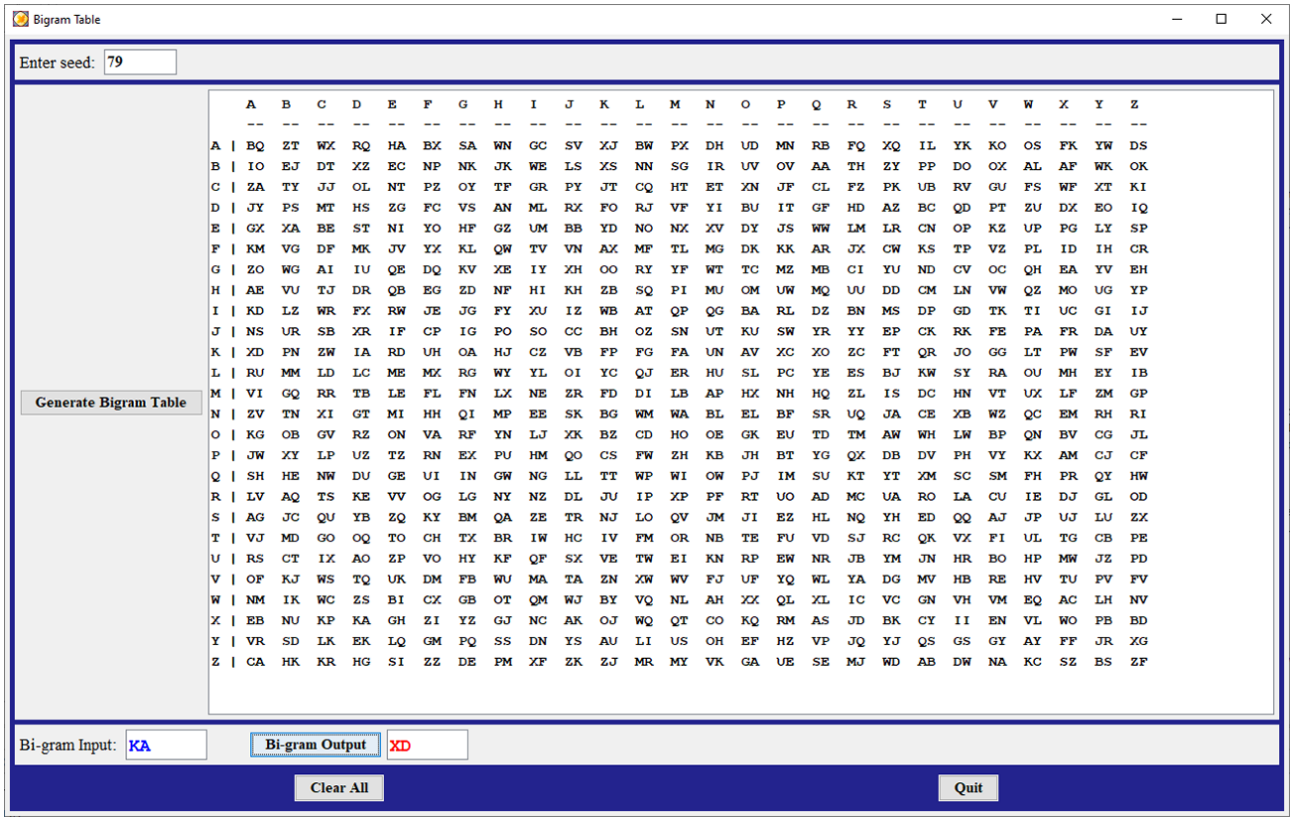


Figure 7: Encrypting a message.

German Navy Enigma sending operators used a different and more secure method than their counterparts in the Army and Air Force to transmit their message settings and ciphertexts to the receiving operators. The sending operator would encrypt the procedure indicator group by means of a *bi-gram* table. Bi-gram tables, which were arranged in 26×26 grids, were issued in a book to all Navy Enigma operators, with instructions for which bi-gram table was to be used on any given day. The **Bigram Table** Maplet, which was written by the authors, is designed to generate a random bi-gram table resulting from a given seed. For example, Figure 8 shows the bi-gram table generated with a random seed of 79.



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
A	BQ	ZT	WX	RQ	HA	BX	SA	WN	GC	SV	XJ	BW	PX	DH	UD	MN	RB	FQ	XQ	IL	YK	KO	OS	FK	YW	DS
B	IO	EJ	DT	XZ	EC	NP	NK	JK	WE	LS	XS	NN	SG	IR	UV	OV	AA	TH	ZY	PP	DO	OX	AL	AF	WK	OK
C	ZA	TY	JJ	OL	NT	PZ	OY	TF	GR	PY	JT	CQ	HT	ET	XN	JF	CL	FZ	PK	UB	RV	GU	FS	WF	XT	KI
D	JY	PS	MT	HS	ZG	FC	VS	AN	ML	RX	FO	RJ	VF	YI	BU	IT	GF	HD	AZ	BC	QD	PT	ZU	DX	EO	IQ
E	GX	XA	BE	ST	NI	YO	HF	GZ	UM	BB	YD	NO	NX	XV	DY	JS	WW	LM	LR	CN	OP	KZ	UP	PG	LY	SP
F	KM	VG	DF	MK	JV	YX	KL	QW	TV	VN	AX	MF	TL	MG	DK	KK	AR	JX	CW	KS	TP	VZ	PL	ID	IH	CR
G	ZO	WG	AI	IU	QE	DQ	KV	XE	IY	XH	OO	RY	YF	WT	TC	MZ	MB	CI	YU	ND	CV	OC	QH	EA	YV	EH
H	AE	VU	TJ	DR	QB	EG	ZD	NF	HI	KH	ZB	SQ	PI	MU	OM	UW	MQ	UU	DD	CM	LN	VW	QZ	MO	UG	YP
I	KD	LZ	WR	FX	RW	JE	JG	FY	XU	IZ	WB	AT	QP	QG	BA	RL	DZ	BN	MS	DP	GD	TK	TI	UC	GI	IJ
J	NS	UR	SB	XR	IF	CP	IG	PO	SO	CC	BH	OZ	SN	UT	KU	SW	YR	YY	EP	CK	RK	FE	PA	FR	DA	UY
K	XD	PN	ZW	IA	RD	UH	OA	HJ	CZ	VB	FP	FG	FA	UN	AV	XC	XO	ZC	FT	QR	JO	GG	LT	PW	SF	EV
L	RU	MM	LD	LC	ME	MX	RG	WY	YL	OI	YC	QJ	ER	HU	SL	PC	YE	ES	BJ	KW	SY	RA	OU	MH	EY	IB
M	VI	GQ	RR	TB	LE	FL	FN	LX	NE	ZR	FD	DI	LB	AP	HX	NH	HQ	ZL	IS	DC	HN	VT	UX	LF	ZM	GP
N	ZV	TN	XI	GT	MI	HH	QI	MP	EE	SK	BG	WM	WA	BL	EL	BF	SR	UQ	JA	CE	XB	WZ	QC	EM	RH	RI
O	KG	OB	GV	RZ	ON	VA	RF	YN	LJ	XK	BZ	CD	HO	OE	GK	EU	TD	TM	AW	WH	LW	BP	QN	BV	CG	JL
P	JW	XY	LP	UZ	TZ	RN	EX	PU	HM	QO	CS	FW	ZH	KB	JH	BT	YG	QX	DB	DV	PH	VY	KX	AM	CJ	CF
Q	SH	HE	NW	DU	GE	UI	IN	GW	NG	LL	TT	WP	WI	OW	PJ	IM	SU	KT	YT	XM	SC	SM	FH	PR	QY	HW
R	LV	AQ	TS	KE	VV	OG	LG	NY	NZ	DL	JU	IP	XP	PF	RT	UO	AD	MC	UA	RO	LA	CU	IE	DJ	GL	OD
S	AG	JC	QU	YB	ZQ	KY	BM	QA	ZE	TR	NJ	LO	QV	JM	JI	EZ	HL	NQ	YH	ED	QQ	AJ	JP	UJ	LU	ZX
T	VJ	MD	GO	OQ	TO	CH	TX	BR	IW	HC	IV	FM	OR	NB	TE	FU	VD	SJ	RC	QK	VX	FI	UL	TG	CB	PE
U	RS	CT	IX	AO	ZP	VO	HY	KF	QF	SX	VE	TW	EI	KN	RP	EW	NR	JB	YM	JN	HR	BO	HP	MW	JZ	PD
V	OF	KJ	WS	TQ	UK	DM	FB	WU	MA	TA	ZN	XW	WV	FJ	UF	YQ	WL	YA	DG	MV	HB	RE	HV	TU	PV	FV
W	NM	IK	WC	ZS	BI	CX	GB	OT	QM	WJ	BY	VQ	NL	AH	XX	QL	XL	IC	VC	GN	VH	VM	EQ	AC	LH	NV
X	EB	NU	KP	KA	GH	ZI	YZ	GJ	NC	AK	OJ	WQ	QT	CO	KQ	RM	AS	JD	BK	CY	II	EN	VL	WO	PB	BD
Y	VR	SD	LK	EK	LQ	GM	PQ	SS	DN	YS	AU	LI	US	OH	EF	HZ	VP	JQ	YJ	QS	GS	GY	AY	FF	JR	XG
Z	CA	HK	KR	HG	SI	ZZ	DE	PM	XF	ZK	ZJ	MR	MY	VK	GA	UE	SE	MJ	WD	AB	DW	NA	KC	SZ	BS	ZF

Figure 8: Finding a bi-gram table.

Bi-gram tables were used to replace groups of two letters. For example, with a given bi-gram table, the replacement for the two-letter bi-gram KA would be the two letter sequence in the position in the table where the row labeled with the letter K intersected the column labeled with the letter A. For the table shown in Figure 8, this replacement for KA is XD. Note that bi-gram tables were symmetric in the sense that for the two-letter sequence XD, the position in the table where the row labeled X intersects the column labeled D is KA.

German Navy Enigma sending operators used bi-gram tables to encrypt their procedure indicator groups. To demonstrate how this worked, as a continuation of the example in this section, recall that the procedure indicator ATS was chosen from the K-book. The operator would first attach a random letter, say J, to the end of this procedure indicator to obtain ATSJ. From the K-book, the Enigma sending operator would choose another 3-letter sequence, say PRO, and attach a random letter, say K, to the beginning of it to obtain KPRO. These two four letter sequences would then be stacked to obtain the following.

K	P	R	O
A	T	S	J

With, for example, the bi-gram table shown in Figure 8, the operator would make the replacements KA $\rightarrow$ XD, PT $\rightarrow$ DV, RS $\rightarrow$ UA, and OJ $\rightarrow$ XK to obtain the following.

X	D	U	X
D	V	A	K

The Enigma sending operator would then transpose this array into the following.

X	D	D	V
U	A	X	K

Finally, the sending operator would string these rows together as `XDDVUAXK`, and transmit this to the receiving operator twice with the ciphertext `CQCLWTIXNNLKFGGS` generated in Figure 7 in between: `XDDVUAXK CQCLWTIXNNLKFGGS XDDVUAXK`.

To decrypt the message, the receiving operator would begin by reversing the process of the bi-gram table to recover the procedure indicator group ATS. Then, using the Grundstellung setting of YES and the rest of the machine settings that were given in the daily key, the receiving operator would encrypt ATS to recover the message setting of SJK. Finally, using this message setting, the receiving operator would be able to decrypt the ciphertext `CQCLWTIXNNLKFGGS` to recover the plaintext `TOMORROW IS BETTER`.

6 Navy Cryptanalysis

The fact that the German Navy had 336 possible rotor orders as compared to 60 for the German Army and Air Force alone vastly increased the amount of Bombe time needed to discover the initial plugboard pairs, reflector, rotor order, and ring settings needed to break Enigma messages. Some of this additional challenge was overcome through a technique developed by Alan Turing called *banburismus* which eliminated certain rotor orders. The construction of more bombes to test rotor orders also helped. Even when an Enigma daily key was found with the successful breaking of a single message though, the Navy's use of bi-gram tables to encrypt procedure indicators better prevented the direct recovery of the message settings used by the other Enigma operators on any particular day.

With these added difficulties, it became imperative that the Allies capture German monthly setting sheets. The first such capture occurred in 1941, and as the war progressed, other captures occurred. This enabled progress to be made, since it allowed the daily key to be known for a limited number of days. However, without knowledge of the bi-gram tables used by each operator, it was still difficult to recover the message setting used by each Enigma operator. This problem was initially resolved by constructing each day what was known as an *EINS catalog*.

The German word *eins*, which translates in English to *one*, was suspected to occur most often in German Navy plaintext messages. This suspicion was later verified since it was confirmed that approximately 90% of German Navy messages contained at least one occurrence of *eins*. Using the daily key provided by the captured setting sheets, a daily catalog of four letter groups obtained by encrypting `EINS` at all of the $26^3 = 17{,}576$ possible window letter position groups was constructed. The daily construction of the catalog was an arduous task, and was first carried out by hand. Later, a machine known as *the baby* was used, which recorded the entries on punch cards.

The **EINS Encryption Finder** Maplet, which was written by the authors, is designed to generate EINS catalogs. The Maplet takes daily key parameters as input, and returns after the **Generate EINS Table** button is clicked the EINS catalog resulting from these parameters. As an example, Figure 9 shows the catalog table that results from a daily key. The table displays all possible ways in which `EINS` could be encrypted with the given daily key followed by the window letters that generate each encryption. For example, the result `AABC NIB` in the upper left corner of the table indicates that `EINS` encrypts as `AABC` when the window letters are NIB.

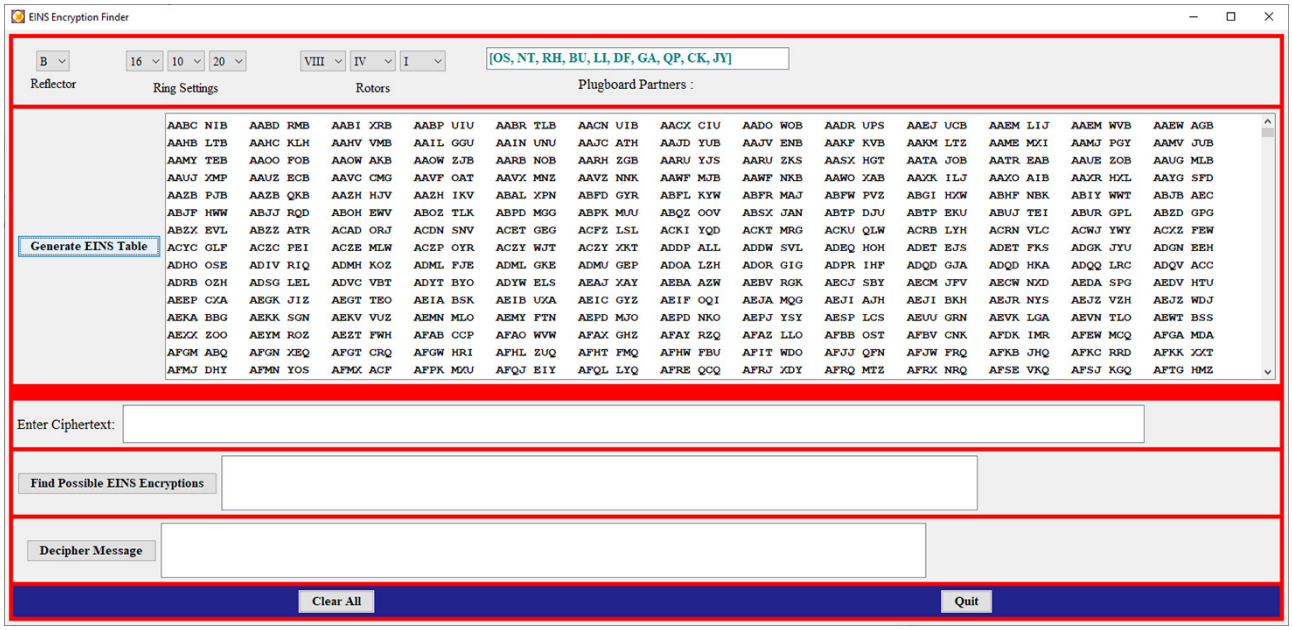


Figure 9: Generating an EINS table.

Ciphertexts intercepted from German Navy Enigma operators were compared with the entries in the catalog to see if any of the four letter groups constructed by encrypting `EINS` also occurred in the ciphertexts. For a particular rotor position where a match occurred, the four characters in the ciphertext would decrypt to `EINS`. Once a match was located, further letters of the ciphertext would be decrypted to see if a legible plaintext was obtained.

As a demonstration of this process, and as a continuation of Figure 10, suppose the ciphertext `BPDXHXWOPKPRCDKLWFGXVUTYL` was intercepted. After entering this ciphertext in the **EINS Encryption Finder** Maplet, clicking the **Find Possible EINS Encryptions** button initiates a search for possible encryptions of `EINS` in the generated table. Figure 10 on the next page shows that there are three matching occurrences in the table, the locations in the ciphertext where each of these encryptions of **EINS** begins, and the window letters that give each one. Clicking the **Decipher Message** button then causes the full ciphertext to be decrypted with each possible set of window letters, and shows the full plaintext as the only legible option: `ATCM OCCURS EINS TIME PER YEAR`. Note that the Maplet also gives the actual window letters from the start of the encryption, FBI, which it finds by reverse-rotating the rotors the number of positions in the ciphertext prior to the encryption of `EINS` from the window letters that produced the encryption of `EINS`.

If approximately 200 message settings were recovered on any particular day, it became possible for Allied codebreakers to recover the bi-gram tables that were used. This involved a very laborious procedure though, and will not be discussed here.

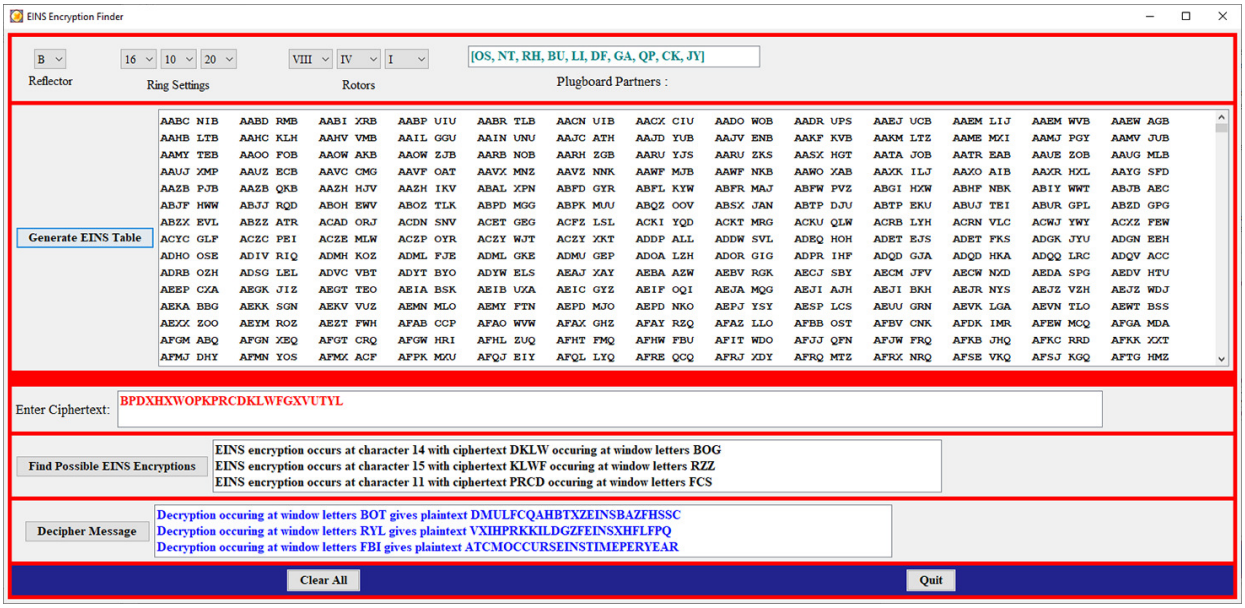


Figure 10: Finding a message setting and decrypting a message.

7 Conclusion

In this paper, we extending our discussion that can be found in [2] to show how the German Army, Air Force, and Navy versions of the three-rotor Enigma machine could be cryptanalyzed and messages sent by all Enigma operators on any particular day read. Due to the more involved and secure method used by Navy operators to transmit their message settings, breaking Navy messages provided a much greater challenge than Army and Air Force messages.

Allied codebreaking efforts were complicated in February 1942 by the introduction of a four-rotor version of the machine by the German Navy. Called *Shark* by the Allies, keys for the four-rotor Enigma required further efforts to find, and required faster four-rotor bombes which were developed by the United States in 1943. A brief description of this process can be found in [1].

References

[1] Frank Carter, *Breaking Naval Enigma*, Bletchley Park Trust, Milton Keynes, UK, 2008.

[2] Richard Klima and Neil Sigmon, 2018. The Turing Bombe and Its Role in Breaking Enigma. Proceedings of the 22nd Asian Technology Conference in Mathematics (ATCM 2017). Available at: https://atcm.mathandtech.org/EP2017/invited/4202017_21528.pdf.

[3] Richard Klima and Neil Sigmon, 2021. Recognizing the Polish Efforts in Breaking Enigma. Proceedings of the 25th Asian Technology Conference in Mathematics (ATCM 2020). Available at: https://atcm.mathandtech.org/EP2020/invited/21799.pdf.

[4] Richard Klima and Neil Sigmon, *Cryptology, Classical and Modern, Second Edition*, Taylor & Francis, Boca Raton, FL, 2019.

[5] Neil Sigmon, 2024. Maplet Download Page for Cracking the Enigma Code: Beyond the Bombe. Available at: https://sites.radford.edu/~npsigmon/beyondthebombe/paper.html.

Understanding Geometric Pattern and its Geometry Part 12 – Octagonal investigations

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Abstract: *This paper aims to discuss some basic examples of octagonal tessellations and patterns made with them. We will limit our investigations to a group of patterns with 90-degree intersections of lines. We will discuss the 'square side and diagonal' ratio and its use in architectural adornments.*

Introduction

In the last few years, my research has been mostly related to Persian and Ottoman geometric patterns, mainly based on decagonal geometry. However, a few other groups of patterns are worth exploring and researching their properties. One of them are octagonal adornments seen both in Central Asia and Iran and in Maghreb. There is a huge diversity of octagonal patterns worth sorting out and investigating. In this paper, we will concentrate on a few octagonal designs from Uzbekistan and Iran.

Geometer's Sketchpad

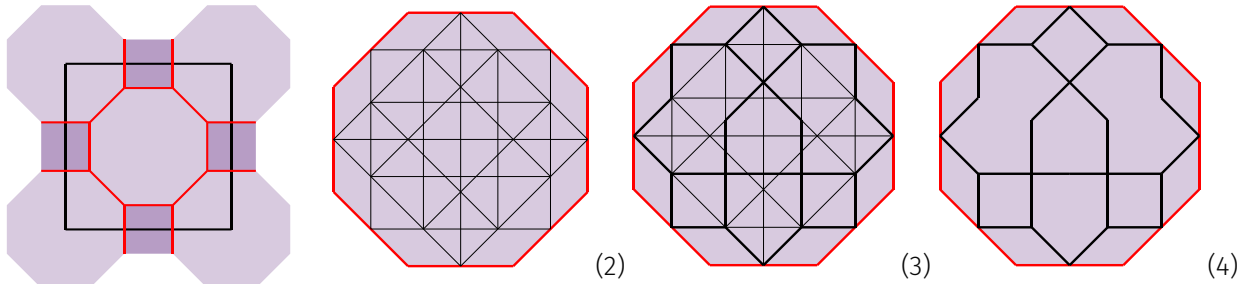
In all my papers, all examples were constructed with Geometer's Sketchpad, a software from KCP Technologies Inc. Recently, GSP is a free tool that one can download and use. GSP contains the same tools as those in traditional pattern design, i.e., compasses and ruler.

In GSP, we have the so-called custom tools. Thus, if we create a useful geometric construction, we can save it as a custom tool. Then we can use it in any place we need it. This reminds us of the copy-and-paste technique frequently used when drawing patterns with traditional non-electronic tools. Custom tools in GSP are more flexible than the traditional copy-and-paste technique. Another advantage of GSP is that it can handle huge amounts of data – points, segments, arcs, and polygons.

The vocabulary of geometric pattern design

Like in any other science, we need terminology that will help us discuss pattern properties and design functionality. Thus, in the following table, we will show samples of geometric elements with their names. It is important to note that we deal only with samples here. There is an infinity of tessellations and an infinity of geometric motifs.

Patterns can be created by constructing them directly without a tessellation or using tessellations and motifs filling them. You have to choose which method works better for you.



Vocabulary of design: (1) a sample tessellation; (2) an octagon with a grid inside; (3) a sample motif for an octagon created on a grid; (4) a sample motif for an octagon.

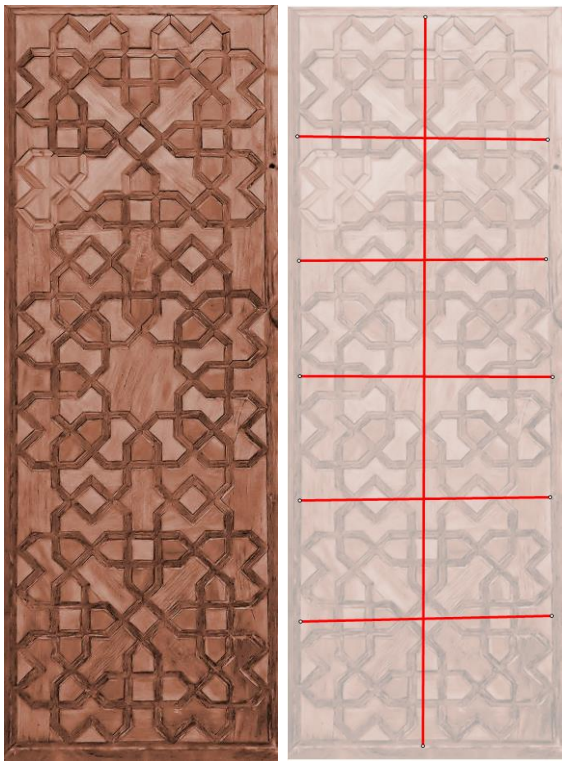
(1) It doesn't matter if you know each shape in a tessellation or a pattern; you have to construct it using a proper geometric construction. Not by assembling polygons. Tessellation shown here uses octagons and squares. The black segments show the future edges of a pattern created on this tessellation.

(2) For some patterns, grids are very useful devices. We often use them in hexagonal designs. Sometimes in octagonal designs.

(3) A motif can be constructed on a grid-like basis, as shown here. Motifs can be very simple or complex. After creating a motif, we wipe out the grid used to create the motif. The motif is constructed, not drawn.

(4) A motif for a polygon is one of the components of the pattern. It can be repeated.

Pattern from Shah Mosque in Isfahan (example 1)

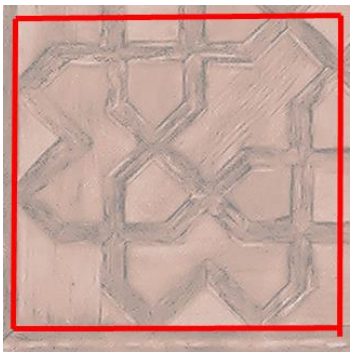


Isfahan, a fragment of doors from Shah Mosque

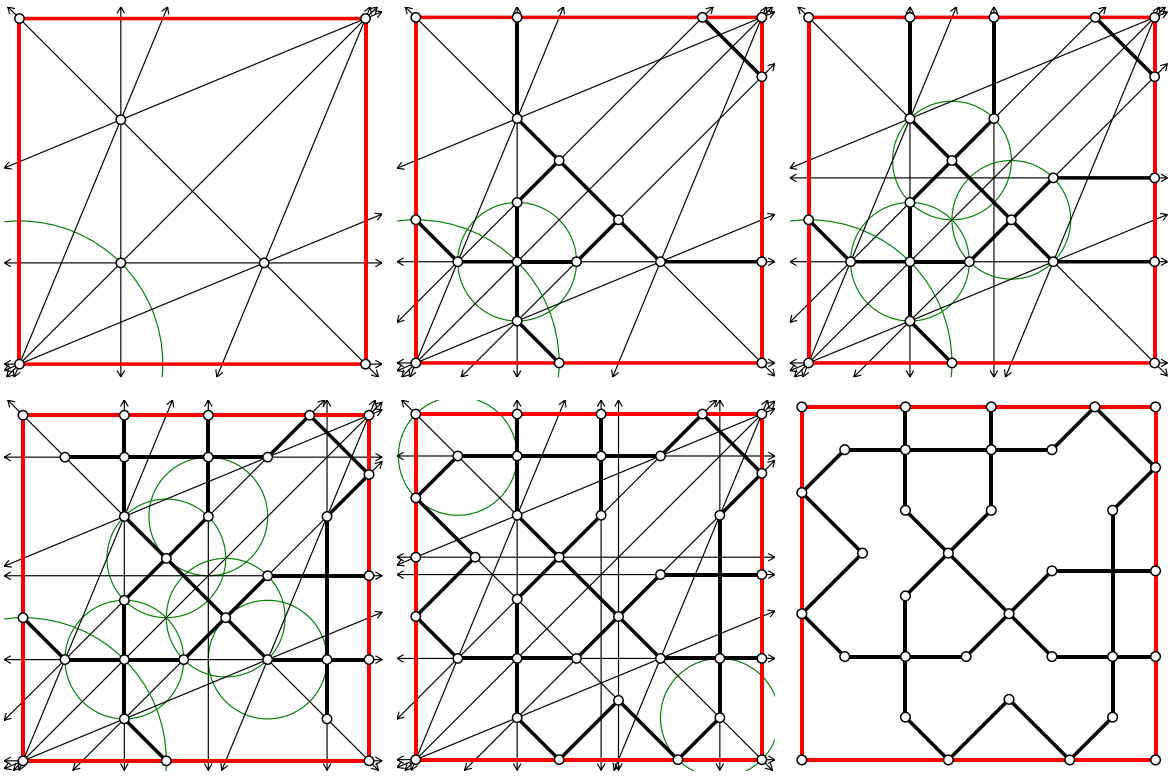
This is one of the most frequently seen octagonal patterns in Iran.

The middle illustration shows how the pattern can be divided into 12 identical squares. Each can be used to create the whole design by taking its copies or reflections about one of its edges.

Below, we show one of these squares. The series of following drawings will explain how one can construct it directly.

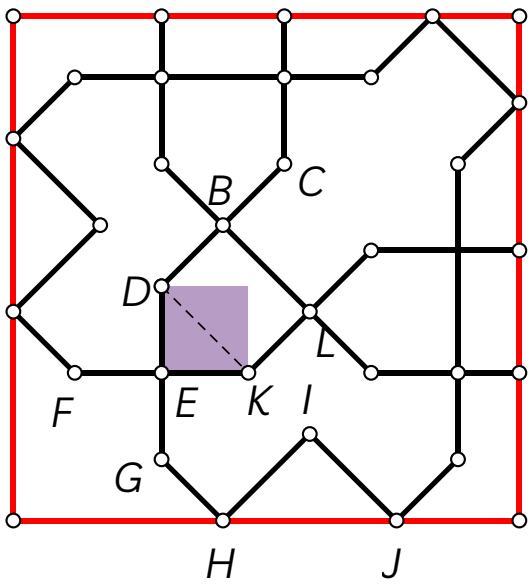


Direct construction of the pattern from Shah Mosque in Isfahan



The drawings show all the steps in creating this pattern. This is a construction often presented by some gurus of pattern design. It is complex and not easy to follow.

We used Geometer's Sketchpad to show all construction points and lines. Lines are usually perpendicular or parallel to another line. Intersection points or circles restrict the lengths of segments. The pattern is very regular, and we have only two segment lengths here. Note – this technique is often called a point-to-point design.



$m\,\overline{BC} = 1.33916$ cm

$m\,\overline{DB} = 1.33916$ cm

$m\,\overline{DE} = 1.33916$ cm

$m\,\overline{FE} = 1.33916$ cm

$m\,\overline{EG} = 1.33916$ cm

$m\,\overline{DE} = 1.33916$ cm

$m\,\overline{EK} = 1.33916$ cm

$DK = 1.89386$ cm

$BL = 1.89386$ cm

Lengths of segments of the pattern

The lengths of segments were calculated using GSP. But we can easily calculate these lengths by hand.

We have here two lengths of segments and

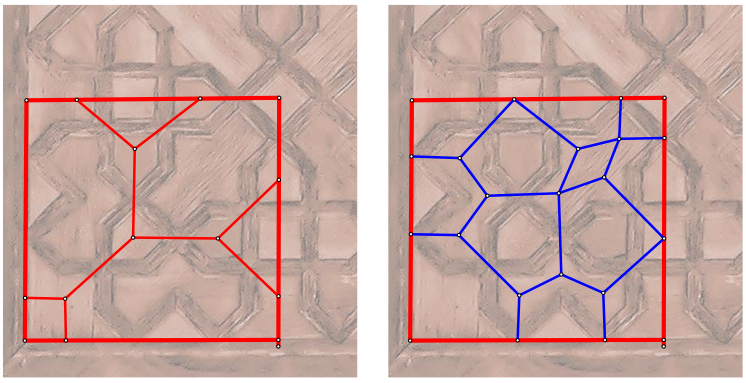
$$BL = DK = \sqrt{ED^2 + EK^2}$$

Here comes a very important conclusion: long segments are equal to the diagonal of a square with short segments as its edges.

Polygons in octagonal geometry

There are also other techniques for designing geometric patterns. In the following examples, we use the polygonal method. It is a structural method. It allows us to simplify the number of objects for our design. We deal with structures, which are usually very simple. Let us explore this method.

We can learn more interesting mathematical facts by drawing from the photograph. The two illustrations shown below are what we may get.



Polygons

In any geometric pattern, we may get a few different families of polygons and tessellations.

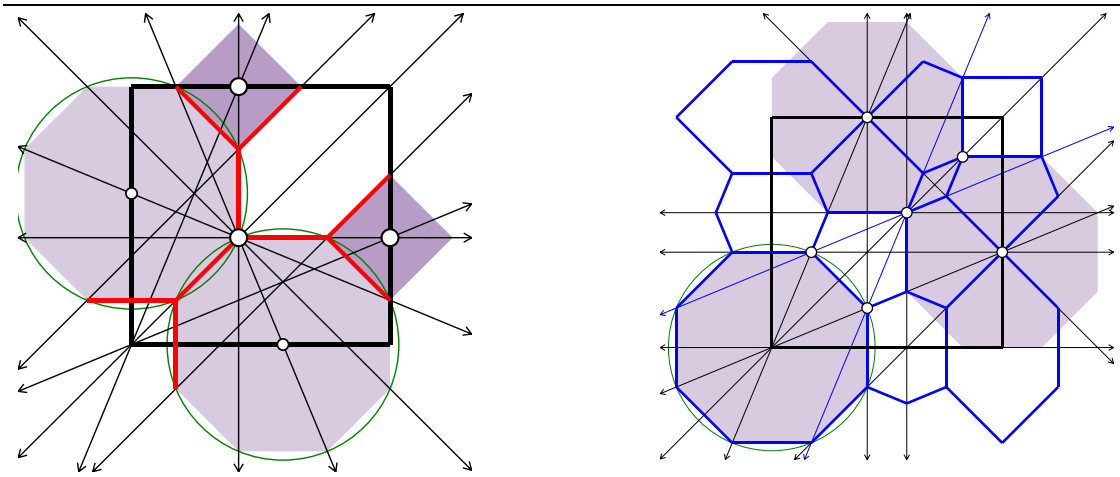
Here we have two tessellations – the red one and the blue one. Each contains a slightly different set of polygons; polygons from one may occur in the other one.

NOTE – We deal with tessellations and patterns here. Each pattern itself is also a tessellation. But its polygons may not be convex. In tessellation for a pattern, all polygons must be convex and have at least one symmetry line, while polygons in a pattern may not be convex.

Thus, we must construct one of the two tessellations to create a pattern. We choose the one that is easier to construct. Usually, tessellations with a smaller number of polygons are easier to construct. In our example, red tessellation can be produced in very few steps. Each of these two tessellations contains fewer polygons than the original pattern. In our example, we have five red and 12 blue polygons, while the pattern contains 16. Note – in octagonal tessellations, we often get gaps between tiles.

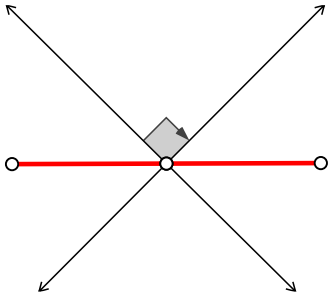
On the next page, constructions of both tessellations are shown.

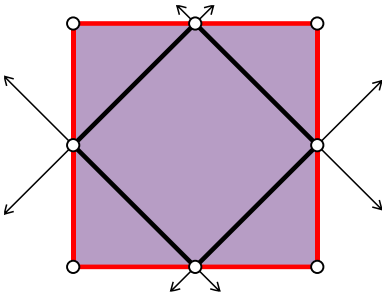
Structure of both tessellations



Elements of the red tessellation

Creating motifs for the tessellation polygons shown above requires knowledge of each tile's shape and the principal angle. Here, it is 90 degrees.

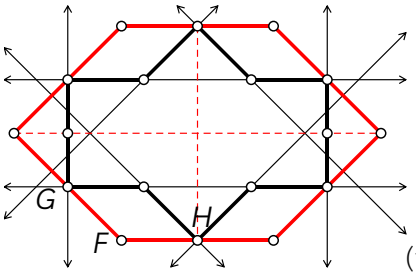




Creation of a motif for a square

For each edge, we draw two lines passing through the midpoint of the edge, with 90 degrees between them.

We will use 90 degrees in other patterns' motifs.



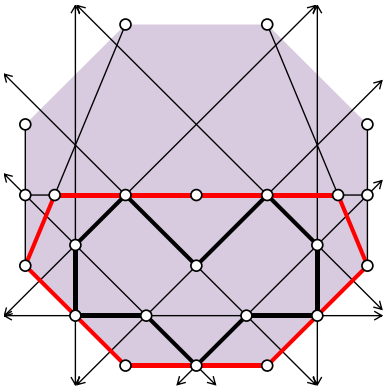
(1)

Long hexagon with pattern

The large angles of the rhombus are 135 degrees, and the two others are 90 degrees. All edges of it are equal.

The motif inside the rhombus has angles equal to 90 degrees and short edges of the same length. The relation between long and short edges is the same as the diagonal and side of a square.

The dashed red lines are mirror lines of the figure.



(2)

A motif for the half of the octagon

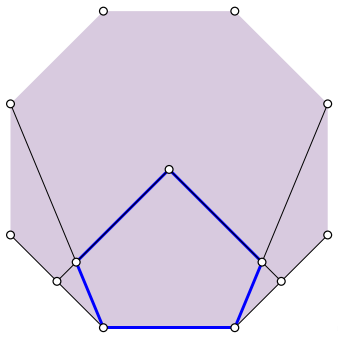
Here, we created a motif for half of the octagon.

The drawing presents only one out of a few pattern designs for an octagon. We will examine some of them later.

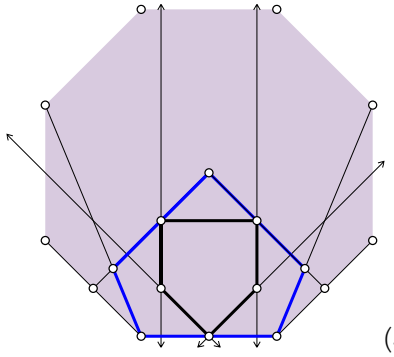
Some polygons occur in both types of tessellation.

The blue tessellation

The blue tessellation contains a regular octagon, a flat hexagon, a square, and a very specific shape with a motif similar to a house with a slanted roof. We will call it "yurt," the same name we will use for the motif inside it. Let us examine these shapes and their motifs.



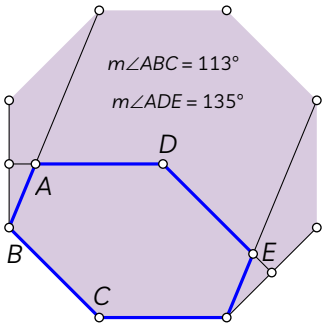
(3)



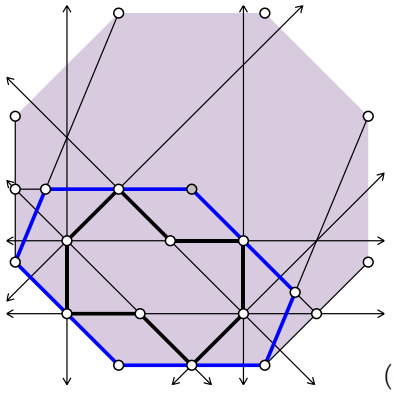
(4)

The yurt tile and its motif

This construction is very easy to redraw.



(5)



(6)

A flat hexagon

In the long hexagon, all edges were the same length. This one has two short edges; all angles are the same, as in a long hexagon.

Each of the tiles shown here can occur in red and blue tessellation. Thus, red and blue colors are useful only for distinguishing between tessellations, but it does not matter which color we use. There is also an easy way to transform one tessellation into another one.

In various regions and workshops, particular shapes had their names. Names were usually in Farsi and its local dialects. Some of them can be found in Russian literature ([3] [7]). Here are some examples.

Kapchuk – a bag, or shirt,

Barg-e-chanar – a leaf of the platanus tree (there are a few shapes with this name),

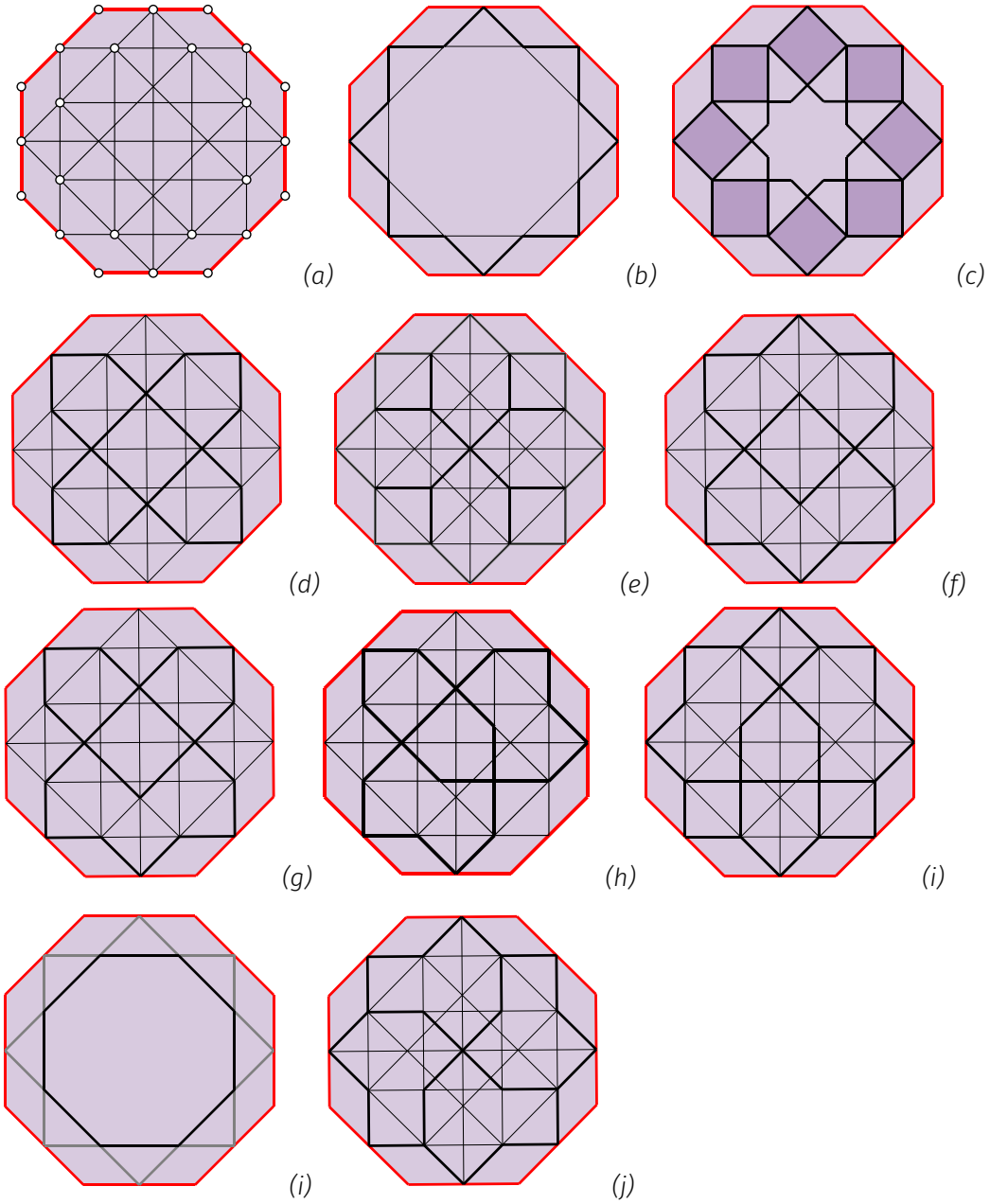
Taul – a drum.

In this paper, we discussed only a few shapes that the reader can spot in simple octagonal designs. However, the quantity of these shapes is larger than one could mention in a brief paper. Each region has a variety of designs that may occur in a given part of the world. For example, in Uzbekistan, we may find a unique collection of patterns that do not exist in other parts of the world. Some of them are discussed in [1] [7] . It seems that many octagonal designs originated in old Uzbekistan (XII century). Another great collection of octagonal designs can be found in the monograph by Mustafa Bulut [2] . His collection of octagonal designs is admirable. In his book, we can find patterns even from remote Anatolian villages and towns. There are no catalogs of octagonal patterns from Iran or Egypt.

Finally, a huge collection of octagonal designs can be found in Jean-Marc Castera's book *Arabesques, the decorative arts in Morocco*. But patterns from the Maghreb represent a completely different style. We may return to them in the future.

Filling an octagon with a motif

There are a few ways to fill a regular octagon with a motif. Here are some of them. For their construction, we use the grid (*a*). We could also fill octagons using polygons and their motifs presented in the previous pages.



Motifs for a regular octagon

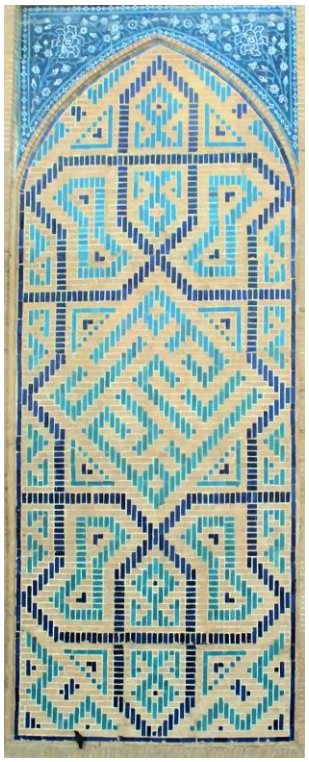
The motifs presented here were made on a grid, filling a regular octagon (*a*). The construction of each of them is obvious. Some explanation is needed for *(i)*. Here, we have segments in black and gray. Some of the gray segments can be removed. This particular motif can be found in the Magoki Attari Mosque in Bukhara (XII century).

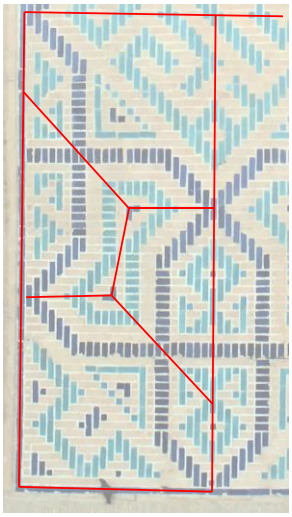
NOTE: motifs *(b)* and *(c)* have eight mirrors; *(d)* and *(e)* have four mirrors; *(f)* has two mirrors; *(g)*, *(h)*, and *(i)* have one mirror. Motif *(j)* has no mirrors.

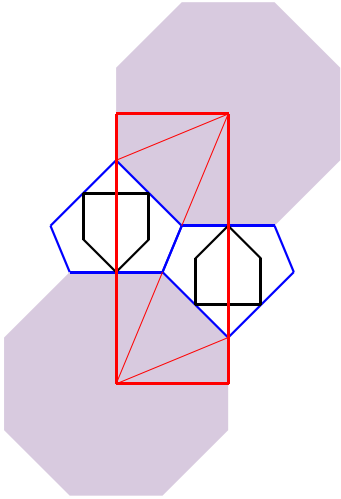
Some of the motifs for octagons are often used, and some are rare. The motif (*j*) occurs only in a few places, e.g., Isfahan, Charbagh Madrasah, and India. All other motifs for the octagon occur more frequently.

All presented here polygons and motifs were created with Geometer's Sketchpad. We can go further and create custom tools for each of them. Thus, we can fill each polygon with an appropriate motif after creating a tessellation. Finally, as in this paper, we could use a single line for each motif. We could use double lines or even triple lines like it was done in real examples.

Example 2: A pattern from Bibi-Khanum Mosque in Samarkand







Wall pattern from Bibi-Khanum Mosque in Samarkand

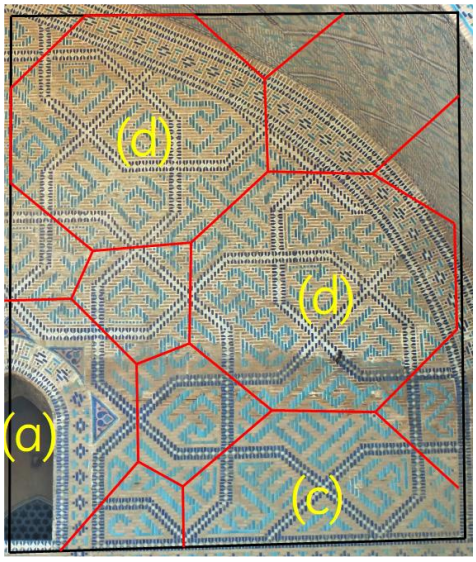
This is a large panel made from rectangular ceramic tiles. Uzbek builders frequently used this technique. Above, we show a quarter of the pattern and its geometry. Some parts of the pattern are fixed. The two octagons can be filled using any of the motifs presented in the previous illustration. The construction of this pattern is obvious. More about this place at https://en.wikipedia.org/wiki/Bibi-Khanym_Mosque

Example 3: A pattern from Ulugh Beg Madrasah in Regestan Square in Bukhara

saic from Ulugh Beg Madrasah in Regestan

estan is a large square in Samarkand with three major dings - Ulugh Beg Madrasah, Tilya-Kori Madrasah, and r-Dor Madrasah. Each of them has some decorations made ctagonal geometry.

one shown here comes from Ulugh Beg Madrasah. Like previous example, it is made using ceramic tiles. Parts of pattern (top corners) are missing; thus, we will have to onstruct them.



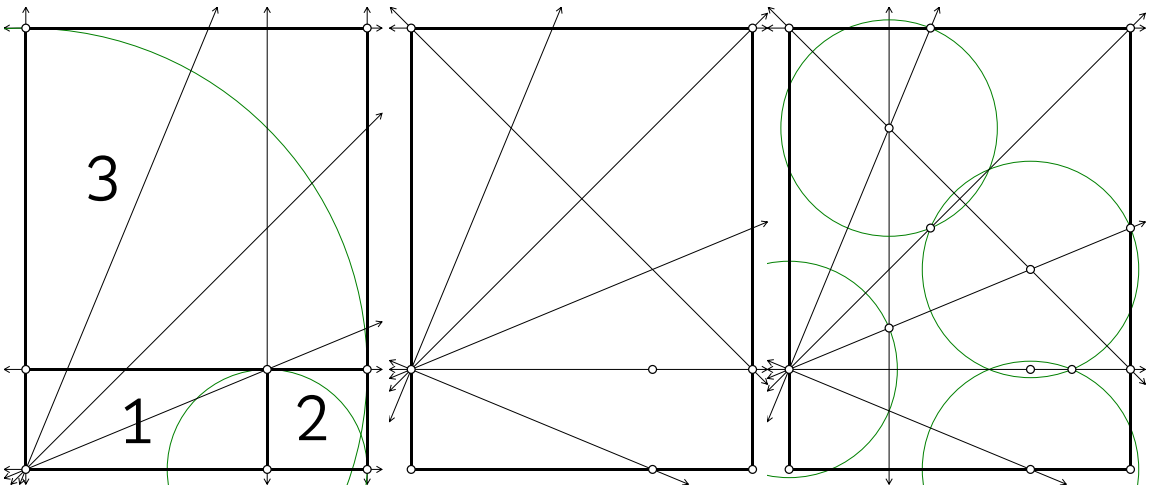
Geometry of the mosaic

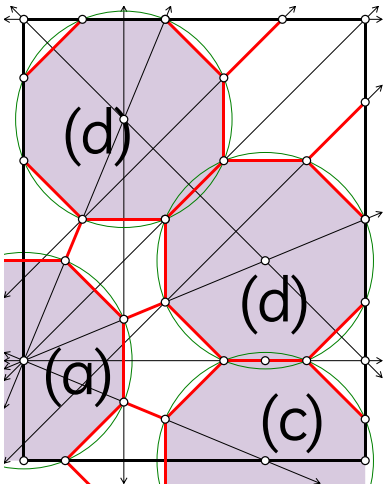
At the beginning of this paper, we created motifs for octagons. This way, we can make our designs more complex and sophisticated.

The illustration shows our pattern with four tangent octagons. Two of them are filled with fill (d), one with (c) and one with (a). We could use different motifs to get different patterns. Some of them could probably be seen in other places.

Now, we should construct the contour and then octagons with appropriate motifs. We will ignore the calligraphy in the background.

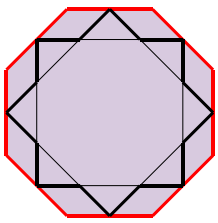
Creation of contour, tessellation, and pattern



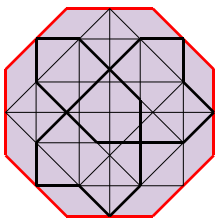


The drawing shows the construction of four identical tangent octagons, one of which will be the center of the pattern.

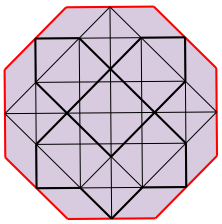
Here are motifs that were used in the original pattern.



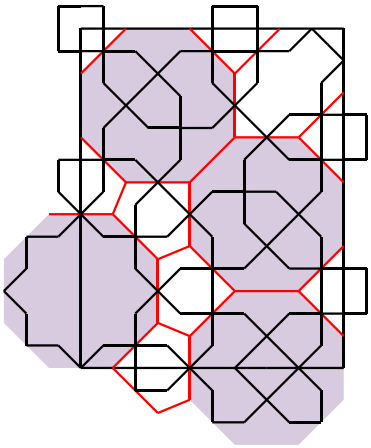
(*b*)



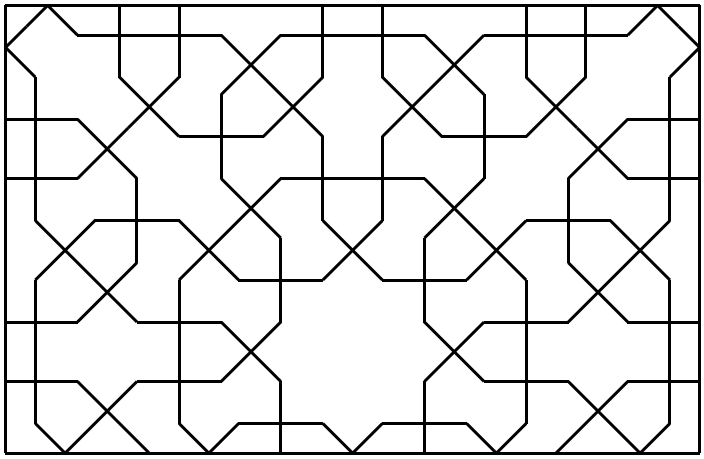
(*h*)



(*g*)



Pattern and tessellation



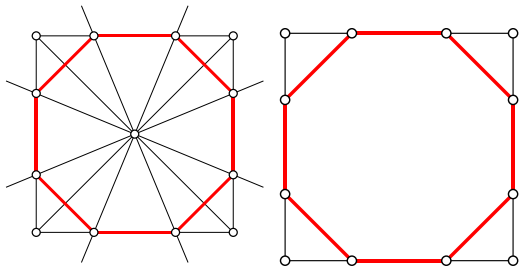
A complete pattern

Designing octagonal patterns piece-by-piece

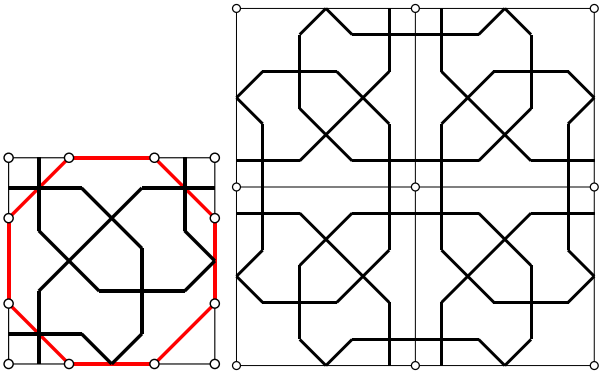
The idea is simple – design a few simple patterns in rectangles with common width and patterns on horizontal edges matching somehow. Then, combine these rectangles in a large block. This way, we can produce large designs with almost any size.

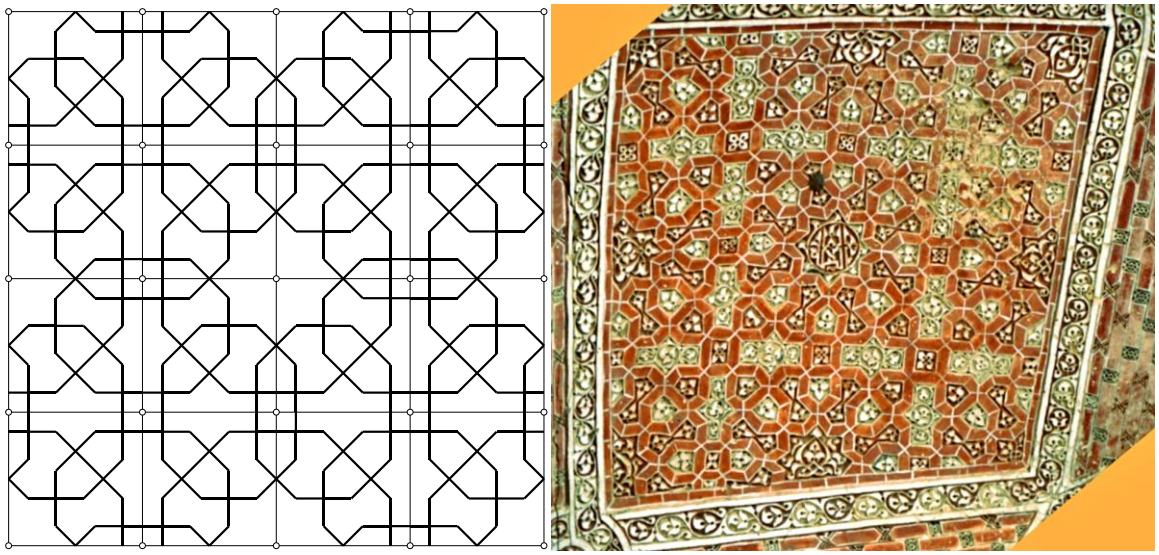
Let us check how it can be done in practice. We will create three octagonal tessellations with the same width and possibly different lengths. The challenge will be constructing each contour and then fitting octagons onto this contour. We also assume each rectangle will have squares or octagons centered on its corners.

Example 3: Ceiling from Sultaniyeh



Here, we have one octagon inscribed in a square. In the corners, we have small squares. The octagon can be filled with one of the motifs we created before. Thus, with motifs (*a*) – (*h*) applied, we may construct any number of patterns. Persian designers used this specific technique. They created magnificent ceilings based on this idea. Below is an example: a ceiling from the Mausoleum of Sultan Oljeitu, Sultaniyeh, Iran.

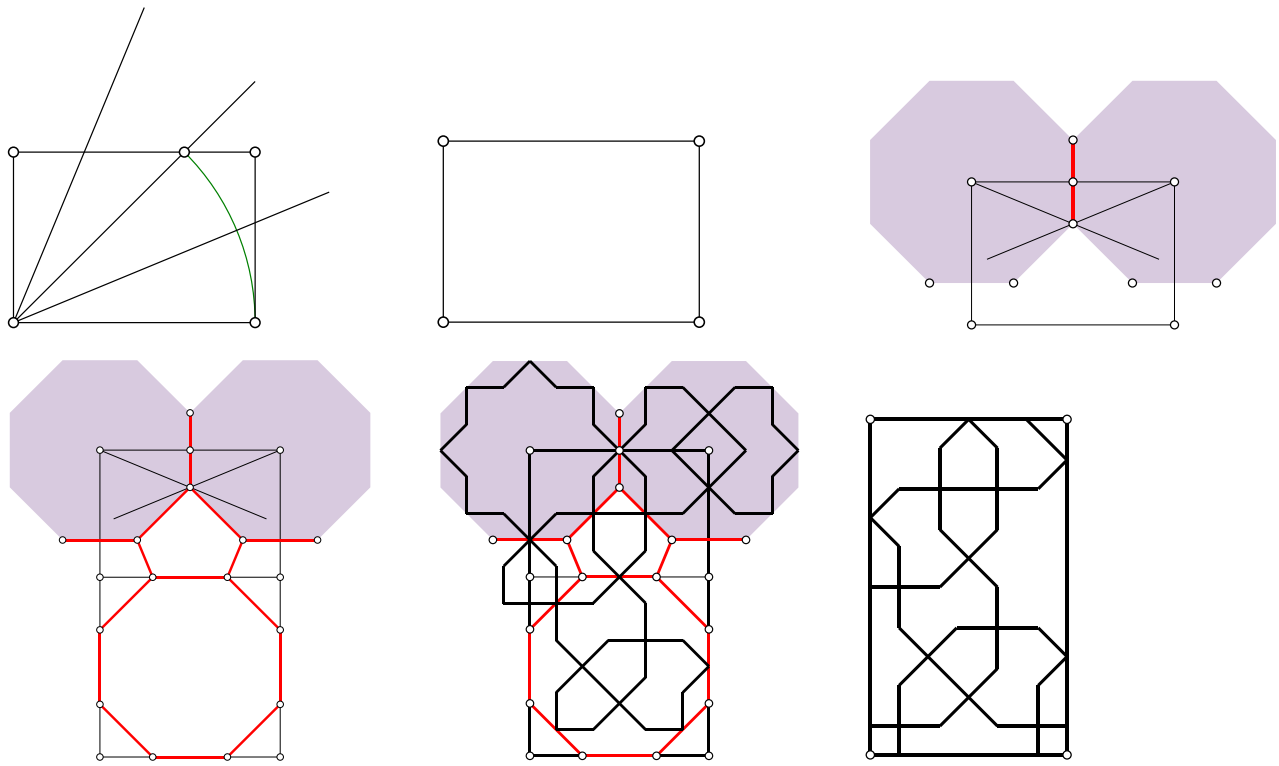




The example from Sultaniyeh is very impressive. But its spectacular view is the result of the very thick frames. The construction is very simple. We used the motif (h) for the octagon. We could use any other motif, and this way, we could get another interesting design.

Example 4: A flat rectangle

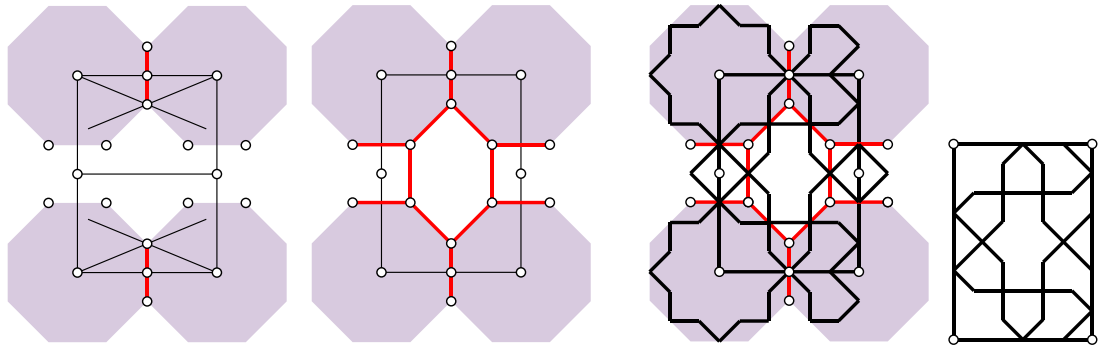
Let us imagine that we still follow the same idea. But this time, we will introduce another rectangle and combine it later with a square.



These drawings show step-by-step how two contours can be used to make another contour and how patterns from both contours (square and flat square) can be joined into one pattern.

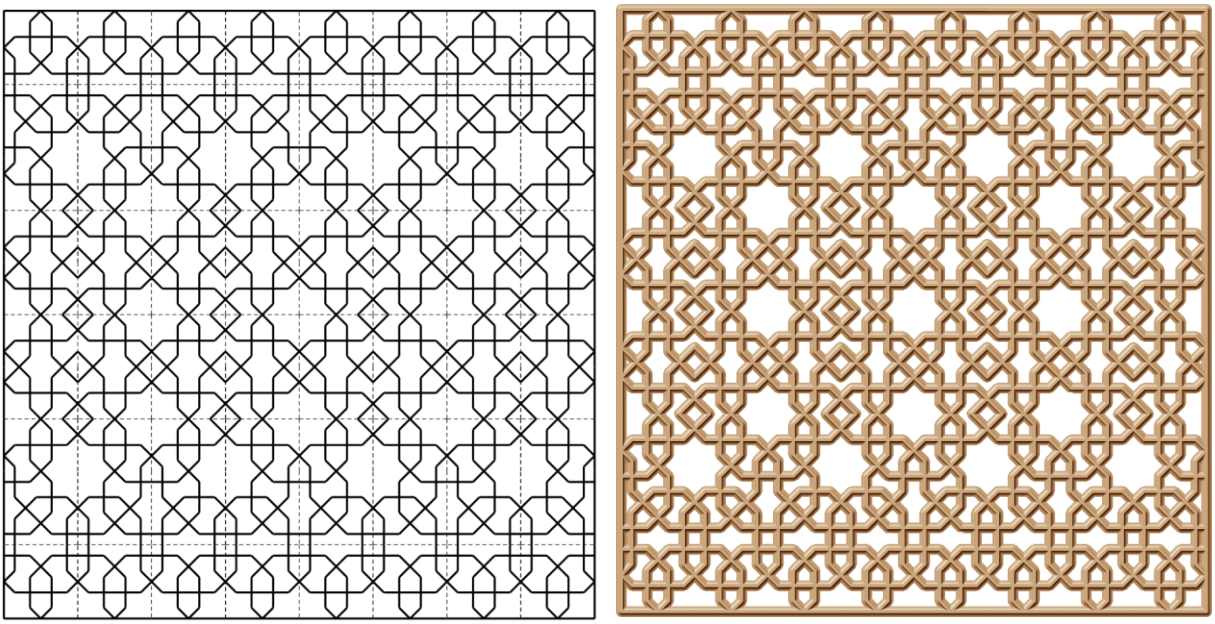
IMPORTANT – In octagonal patterns, we can leave some tessellation polygons without a motif. Here, one of the tent polygons was left empty. This is a specific feature of octagonal designs.

Example 6: Another way of using the flat rectangle



Example 7: Large ceiling pattern

Finally, we can use all three designs to create a bigger ceiling pattern.



Two versions of the ceiling pattern

Left: the complete drawing. Dashed lines show places where squares and rectangles were joined together.

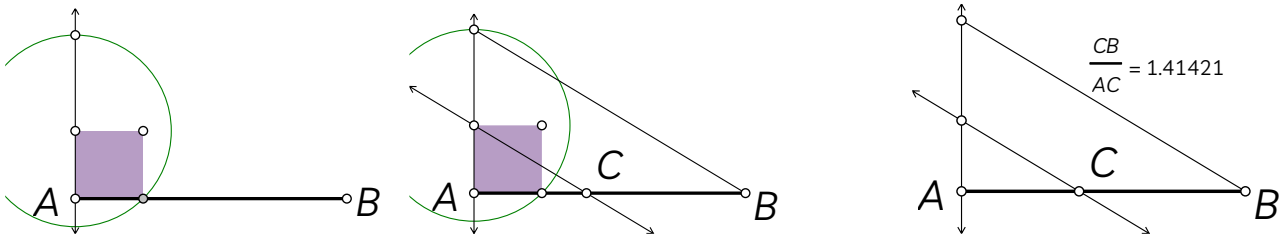
Right: the same pattern rendered as a woodworking object.

Assuming that the square's edge is equal to 1, the lengths of the two remaining rectangles are equal to 1.70711 and 1.41421. Thus, the whole design is size 8 (width)*8.24264(length). This way, we are able to calculate precisely how many rectangles we need for our design.

Similarly, many other patterns were created using this piece-by-piece method.

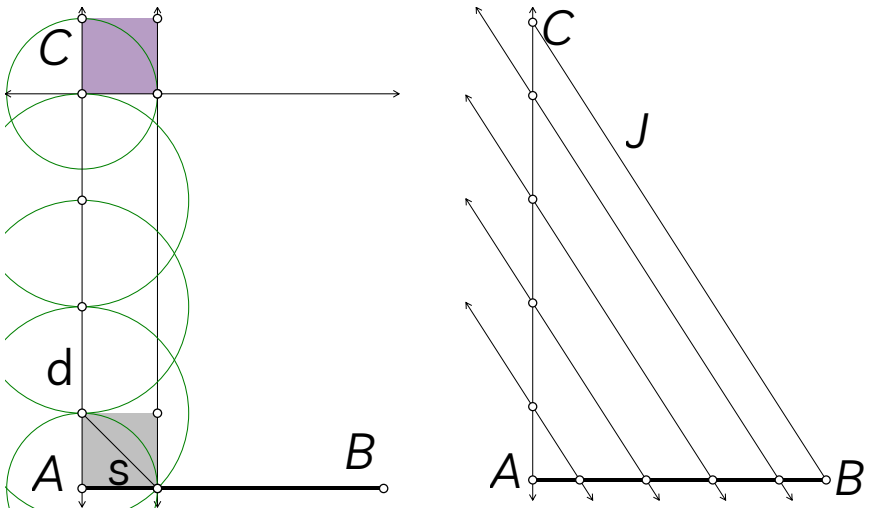
Square side-and-diagonal ratio

A few patterns can be created using the square's side-and-diagonal ratio. The ratio is fixed and equal to d/s=1.41421... In a few cases, splitting a segment into smaller segments with ratios [d,s,s] or [s,d,s] or [s,d,d,d,s] is convenient. All these constructions can be made in the same way. It is just a generalization of a well-known construction to split a segment into n equal parts.



Division of a segment into [s,d] ratio

Start with a segment AB and line perpendicular to it. Choose a fixed point on AB and draw a square with this point as the corner. Draw a circle with the square diagonal as its radius. This way, we get two points on the vertical line. Connect the top point with point B and draw a parallel line to it through the middle point.



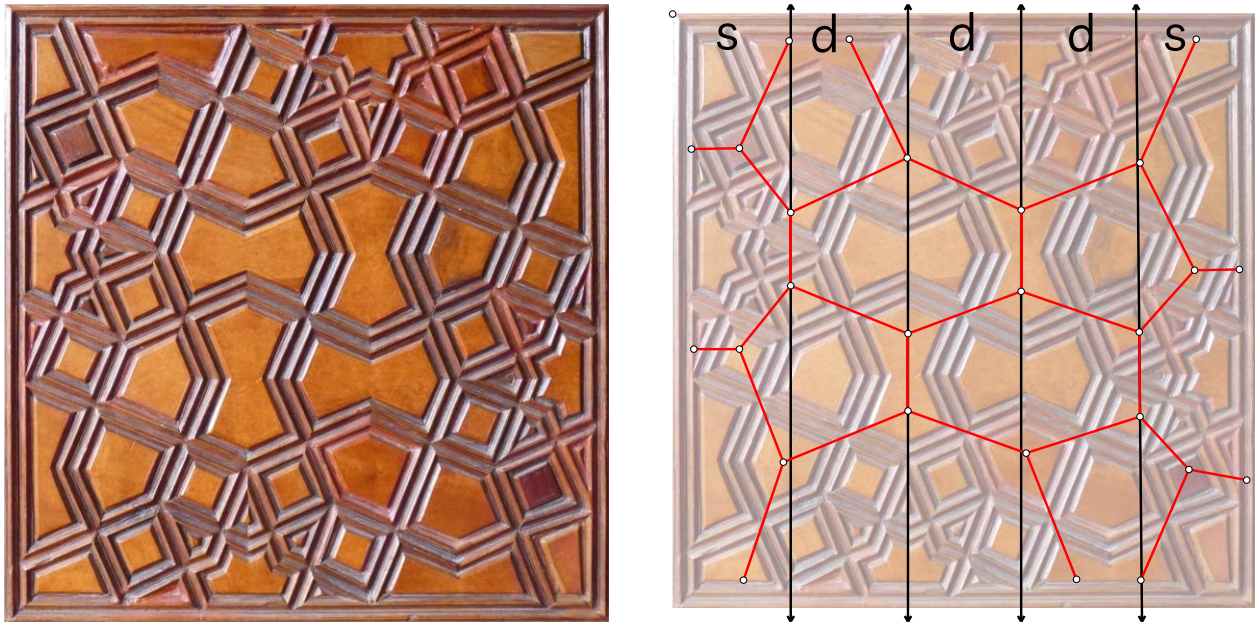
Construction of the ratios [a,d,d,d,a]

Start with segment AB and draw a perpendicular line to it. Choose a fixed point on AB and draw a small square. Then, create five new points on the vertical line using circles. Connect point C with B, and draw the other four lines parallel to it. Intersections of AB with these lines give us the required sequence of ratios.

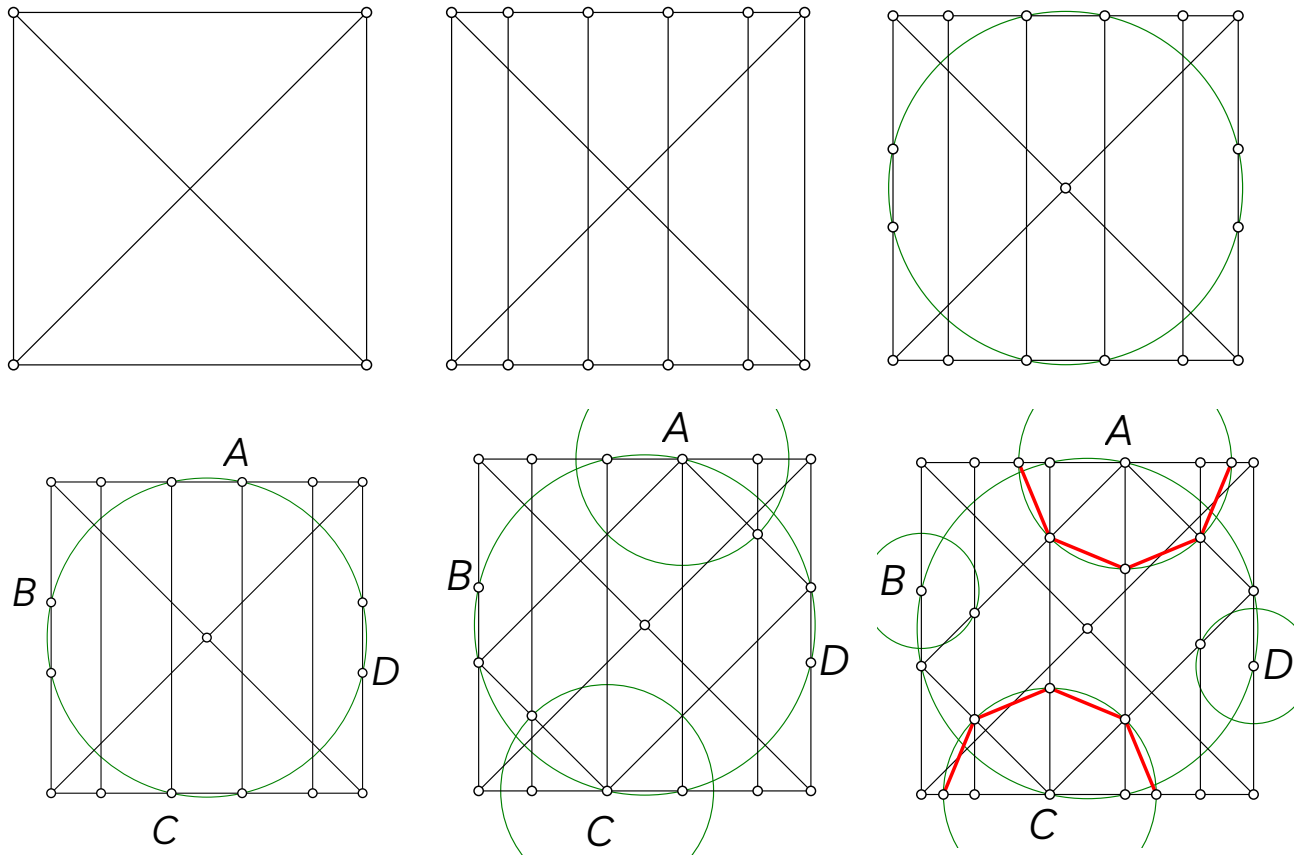
We will use this construction for a kundekari pattern from Konya, Turkey.

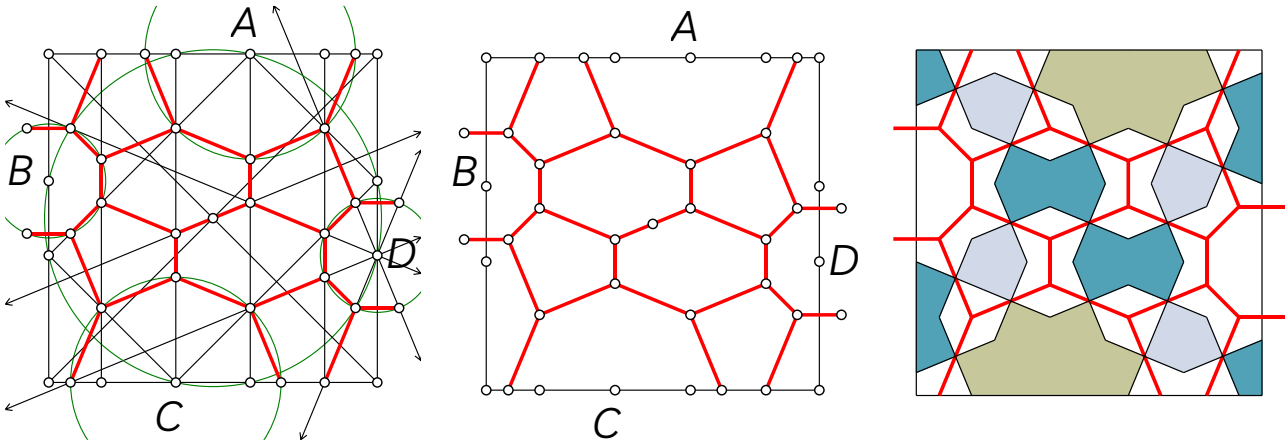
Example 8: Pattern from Beyazid Mosque in Konya, Turkey

There exist a few versions of this pattern. One of them contains a swastika in the central part. As we see in the next photo – it could be convenient to split the top and bottom edges into smaller segments with ratios [s,d,d,d,s]. The four large stars on the edges of the pattern contain small squares with an extra star in the middle. Usually, finding the center of each star is the main challenge.



The following series of drawings shows us the whole construction step-by-step.

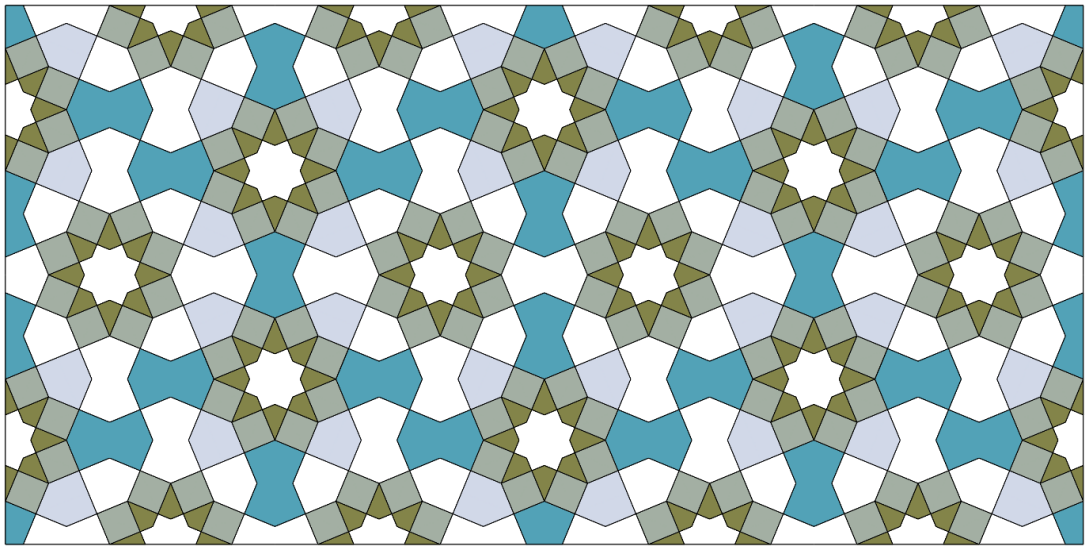




This design has three particularly interesting features. There are two types of octagonal tiles – the large one (A, C) filled with motif (*a*). The other octagons (B, D) are empty and smaller than the large ones. The stars around them are created by the shapes located near them.

Inside each large star, we have rings of squares and smaller stars. The space inside large stars was too large, and we needed something inside them to properly balance shapes.

In the pattern's center, we have a specific combination of shapes and the 90° rotation center.

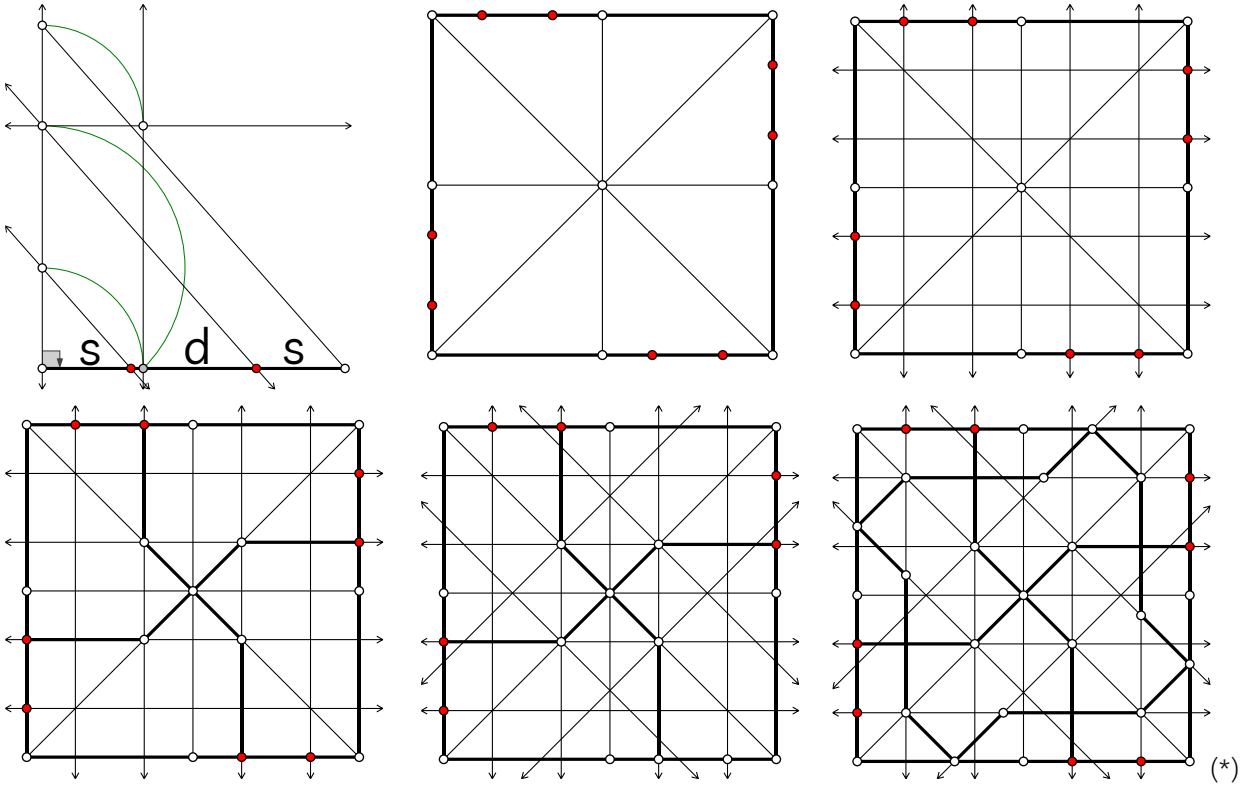


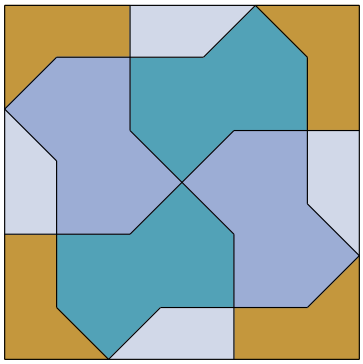
Geometric pattern expanding the pattern from Beyazid Mosque in Konya

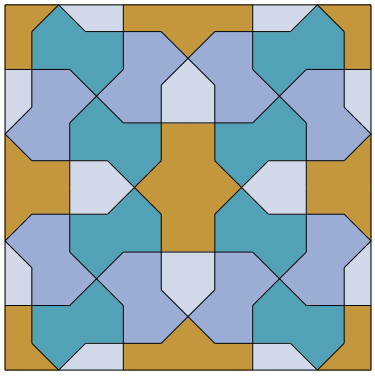
Seljuks frequently used the combination of blue and white and identical shapes between four stars in Anatolia. We can find a few designs of this type in Amasya (Gokmedrese Camii), Kirsehir Melik Gazi tomb, etc.

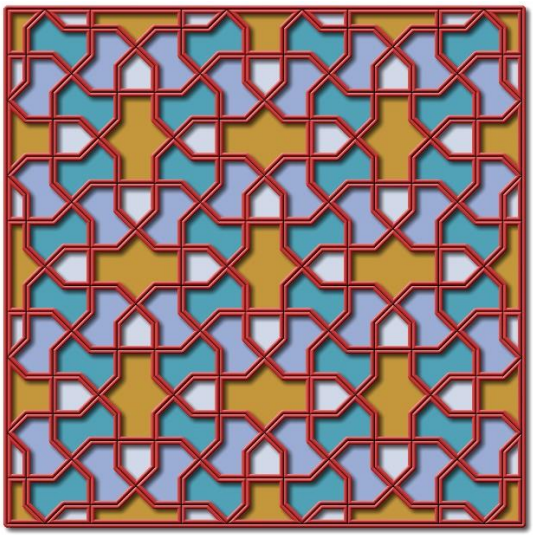
Example 9: Creating a pattern directly with s,d,s ratio

This pattern is located in Turkey: Haci Kilic Mosque and Hunat Hatun Tomb. We can probably find it in other places in Anatolia as well. This example aims to show how one can create an octagonal pattern without tessellation and using the [*s*,*d*,*s*] ratio.









The foundation of this design is ratio [*s*,*d*,*s*]. The red points on the first image define this ratio.

We use the [*s*,*d*,*s*] ratio to divide each half of the square edge into three parts. Through the red points, we draw lines perpendicular to the edges of the square.

Vertices of the central cross are used to add some slant lines parallel to the square's diagonals.

We use all these lines to draw the whole pattern. No tessellation was involved.

The drawing (*) shows one interesting thing. The whole design is covered by rectangles with horizontal and vertical edges. Each rectangle has edges equal to:

s,*s*; or *s*,*d*; or *d*,*s*; or *d*,*d*.

This drawing explains why the whole construction makes sense. We can try other ratios to develop more examples.

Left – larger design obtained from our construction.

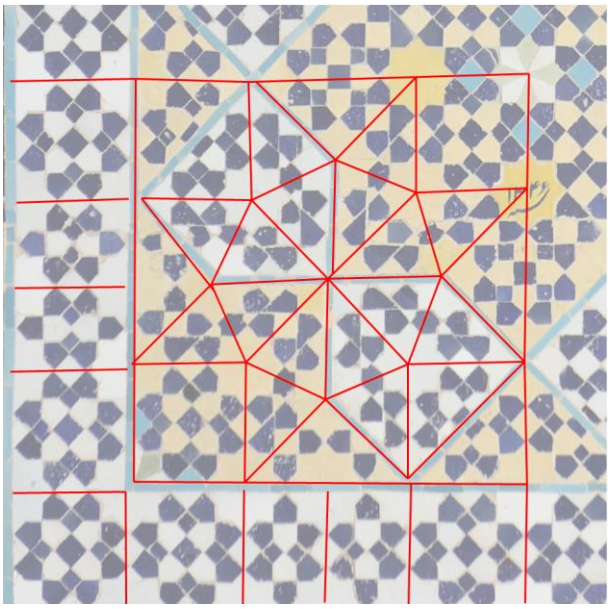
Geometric inflation in octagonal patterns

Geometric inflation is not known in Western mathematics. But it was frequently used by Central Asia designers. We find beautiful examples of geometric inflation in Uzbekistan (Nakshband), Isfahan, and a few other places.

From a mathematical point of view, geometric inflation can be considered a translation of a given pattern into a new, more complex design with the previous one as a framework. Thus, an inflated pattern can be considered an overlapping combination of the old and new designs. The new design will be more dense, following the style of the old design or another one with the same geometry. For a mathematician, an interesting task would be to find the inflation coefficient, check for different inflations for a given pattern, or calculate the ratio of the number of polygons in a new design (N) versus those in an old design (O). We may denote by Ω the ratio $\Omega = {}^{N}/_{O}$. Patterns with larger Ω may be more difficult to create.

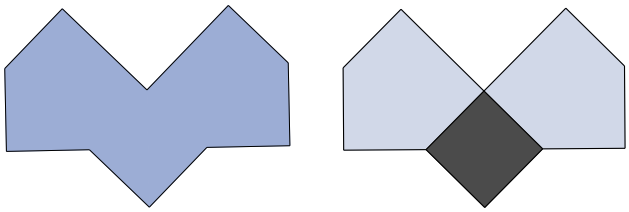
Example 10: Pattern from Chahar Bagh Theological School in Isfahan

A fragment of a pattern from Chahar Bagh Theological School. The original pattern is large, and it is a repetition of the fragment shown in this photo.



The same fragment with drawn on it polygons of the geometric structure of the design after inflation. One can easily notice that we must draw only some inflated fragments and copy them elsewhere.

The pattern from Chahar Bagh Theological School is a typical inflation design. But its designer made a few interesting inventions. One of them was replacing the chanar motif as follows:



Modification of the chanar motif.

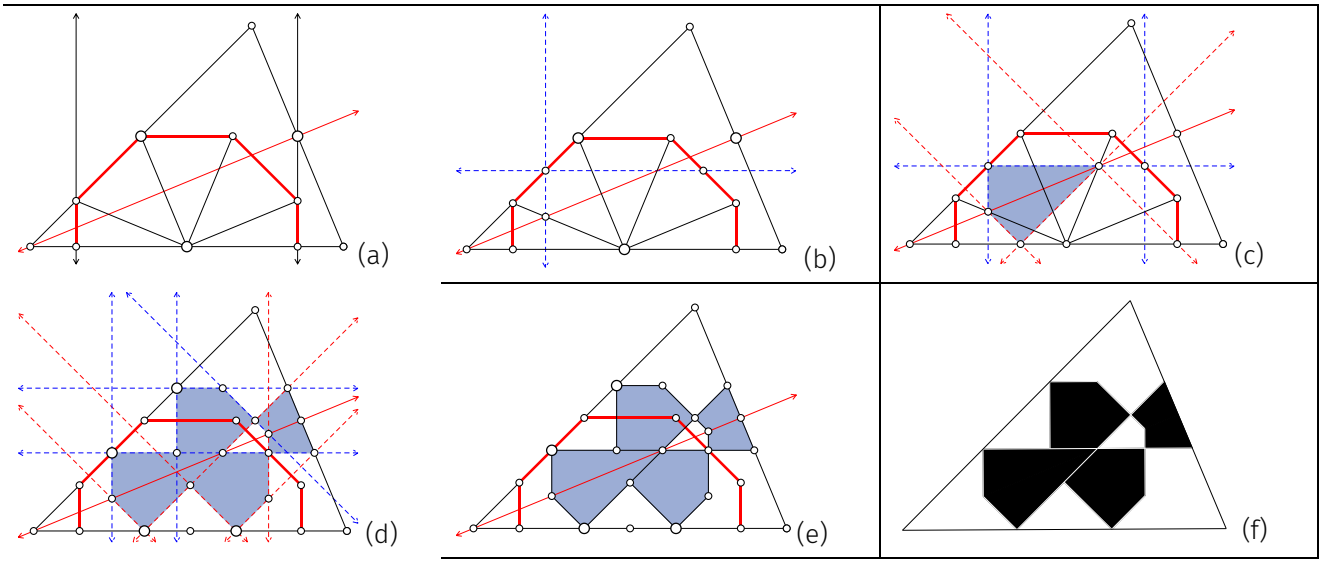
It is useful from an artisan's point of view. He has to cut three shapes instead of one, but each of them is convex. Cutting nonconvex shapes is more difficult.

Another invention was the use of long triangles and square blocks. So, this way, we get a modular design with only two modules. The border for this pattern was also created using square and rectangular blocks.

On the drawing, the original or primary pattern is represented by the thick green-blue lines. The pattern resulting from the inflation is represented by white lines (mortar between tiles), and most pattern elements are symmetric tiles: black, yellow, blue, and white.

We will start by constructing the triangular module. Then, we will design the square and rectangular modules for the border.

Construction of the triangular module



The large points on (a) are the midpoints of the triangle's edges. This is an equilateral triangle with the sharp angle of 45 degrees. The red line is the symmetry line for the pattern.

The thick red segments form half of an octagon, with its center in the middle of the horizontal edge of the triangle and two vertices on the slant edge. The two dashed lines in (b) form 90 degrees angle.

The red dashed lines in (c) are reflections of blue dashed lines about the symmetry line.

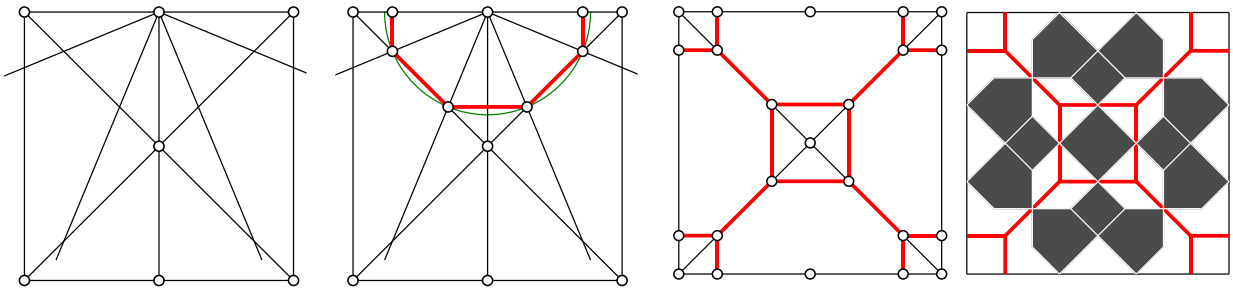
All dashed lines in (d) form the outline of the motif. We fill it with color polygons. The top house-like shape is a reflection of the bottom one.

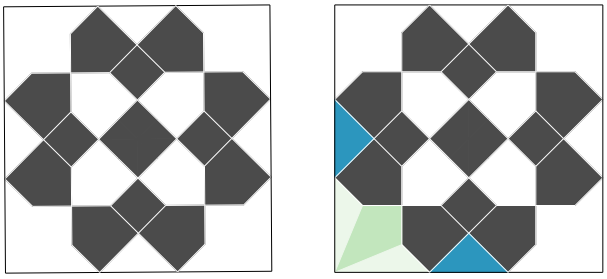
In (e), we show the whole motif with half of the octagon and all necessary construction points.

Drawing (f) shows only the motif; all other design elements were removed.

All colors can be changed, and the interiors of polygons can be removed. In GSP, polygonal fills are decorations only. Points and segments are the construction elements.

Construction of the square module



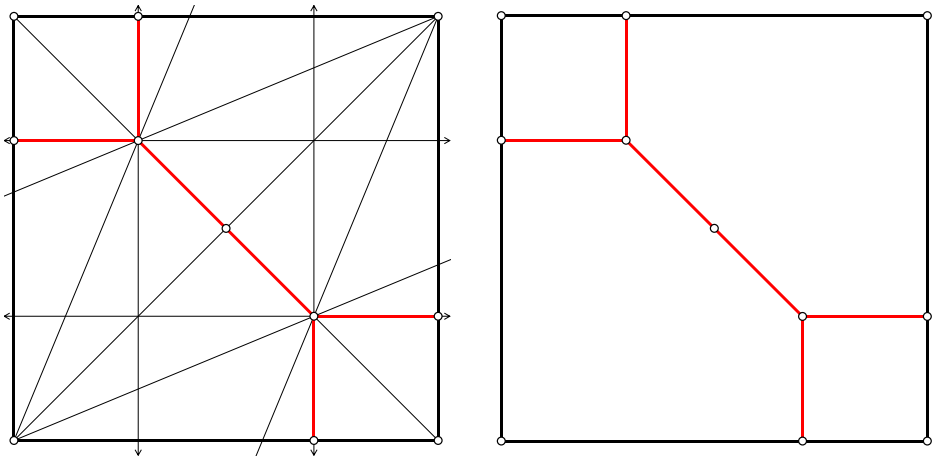


The half of the octagon created in the second picture was copied or rotated to each side of the square.

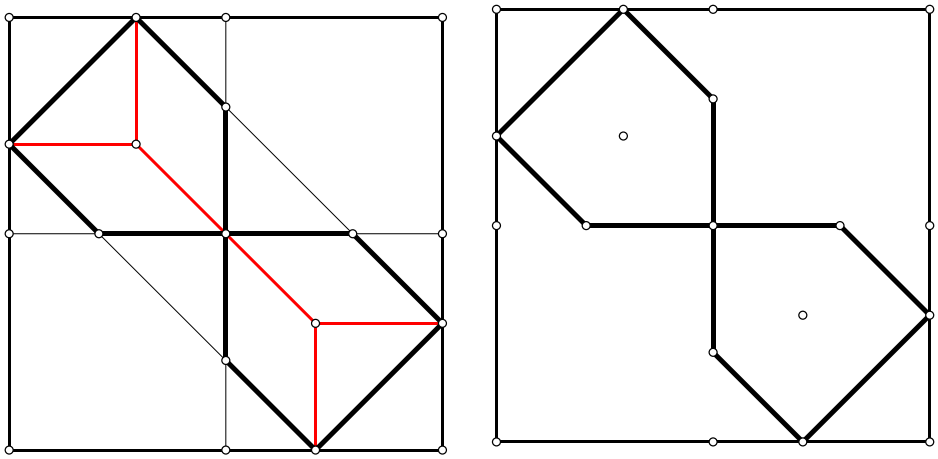
We created two versions of the square module. This way, we introduce colors around each corner and center of the final pattern.

Note also – the small squares in the corners of the large square were left empty.

We have all the components for the large pattern ready. Now, we can proceed with the main pattern and its inflated modules.



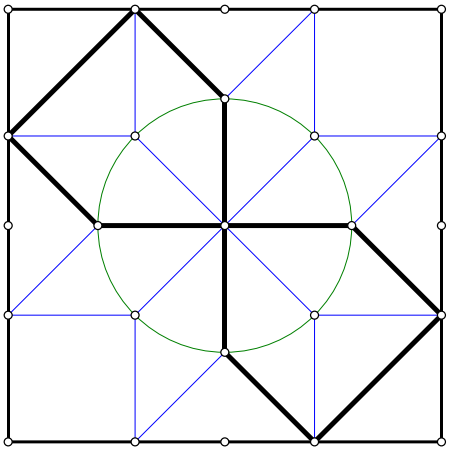
Tessellation of the primary pattern

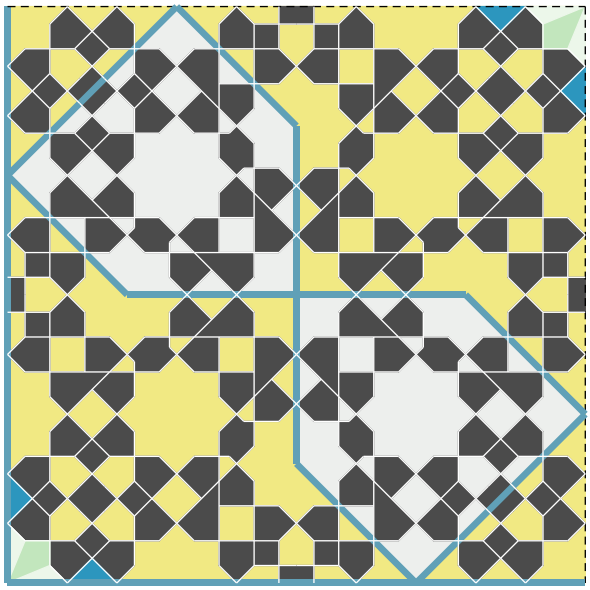


The primary pattern

The construction shows how we can draw our main pattern. It is a very simple construction and is often used with many patterns. The last image is a quarter of a pattern that can be inflated in many ways.

The pattern from Istafan uses a very simple coloring palette. There are three main colors – yellow, black, and white. In corners, we have some blues and light green colors used sparingly. Readers can change them to anything they want.



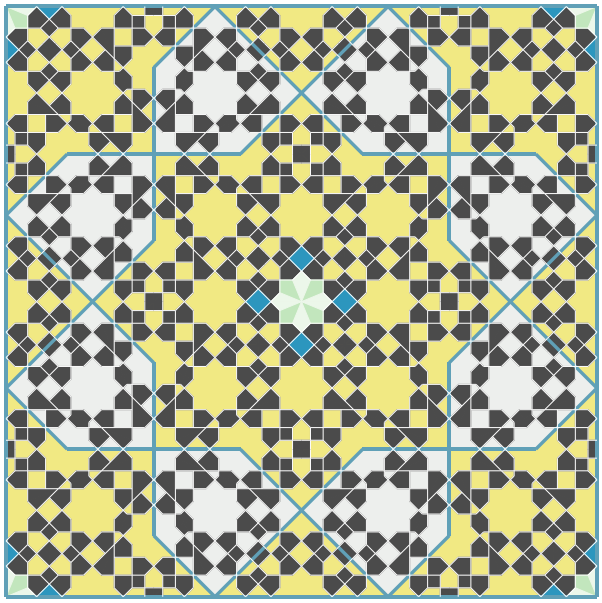


The top-left photo shows a triangulation of the pattern area. We have here only rhombi and squares. Each rhombus is a combination of two long triangular modules.

The dashed segments are the mirror lines.

The central part of the pattern uses the extra square module with blue squares and green two colored stars.

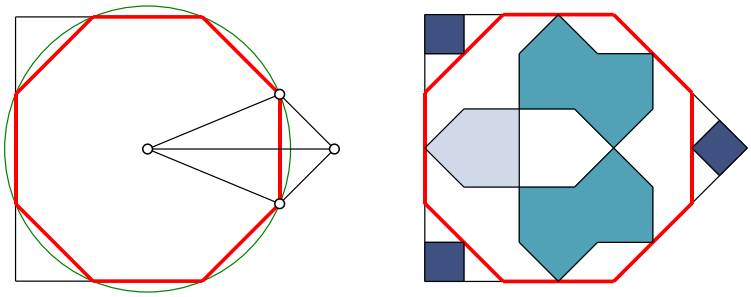
The inflation type presented here is not very popular. The long triangles/rhombi can be seen only in very few patterns. However, there exist a few other ways of inflating this design. We will present one of them.



Example 11: A very minimalistic inflation

Any geometric inflation may have more or fewer elements. The one presented here is quite elaborate. Thus, we may ask if we can inflate the primary pattern and obtain the secondary pattern with fewer elements. This way, the artisans could have less work. A mathematician would search for an absolute minimal inflation. We could also check for inflation with a smaller or larger inflation coefficient. Here, it is L/s=6.82843. This means that the edges of polygons in the primary pattern are 6.82843 longer than those in the secondary pattern.

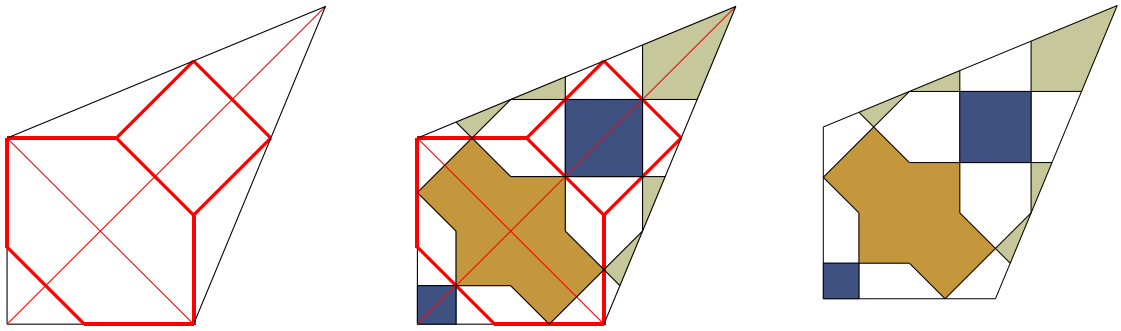
Let us consider the inflation shown in the following images.

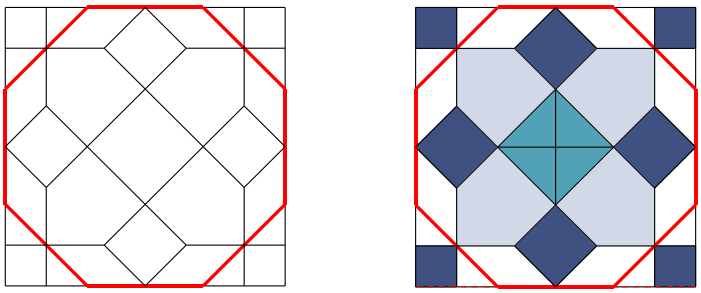


Inflation of the tent polygon

This is the largest octagon that can be inscribed in octagonal pentagon. We fill it with one of the motifs for the octagon and the left space with quarters of a square.

Below is the inflation of the kite with angles 90, 45, and 112.5 degrees.

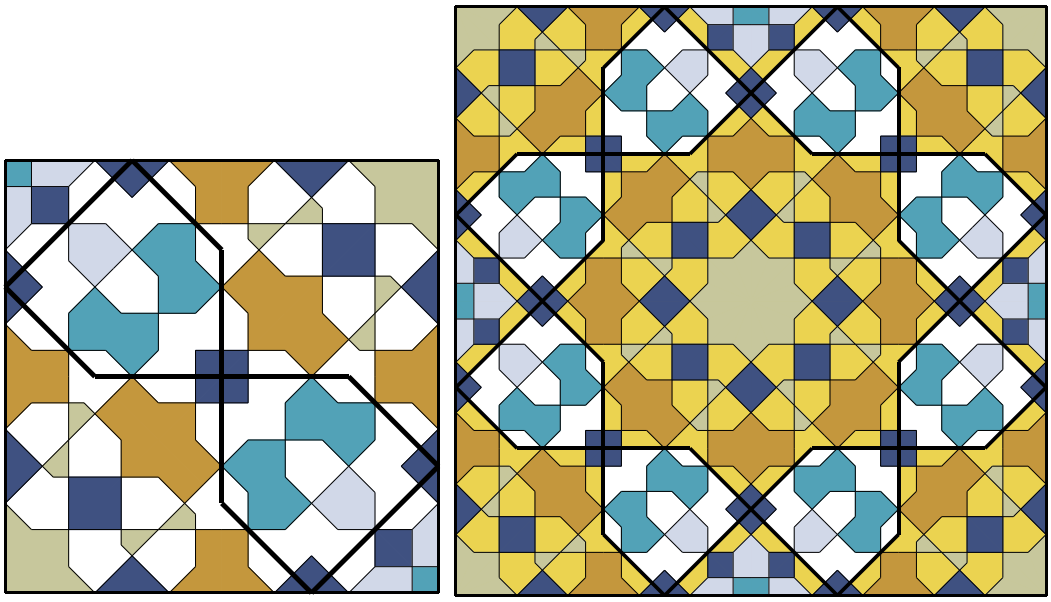




Inflation of a square

There are a few other ways to inflate the square and other polygons. Below is the final construction.

The inflation coefficient is equal to 3.41421.



Summary

In this paper, we touched only a few features of octagonal geometry and examples of octagonal patterns. From a mathematician's point of view, two features were important – the side-and-diagonal ratios in octagonal pattern design and the octagonal inflation techniques. But there is a lot more waiting for further explorations.

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Illustrating the Pre-Conjecture Phases of Experimental Research mediated by Dynamic Geometry: How Generalization Can Generate a Multitude of Highly Plausible Conjectures About Semi-Regular Polygons

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Abstract: *The origin of this work lies in the result concerning the area of the Steiner ellipse of a triangle: it asserts that the Steiner ellipse is the largest-area ellipse inscribed in a triangle. The first comprehensive proof of this result was provided by Minda and Phelps in 2008 ([5]). Finding their proof, which used complex numbers, not very geometric, I began an investigation using dynamic geometry. This led me to present an original proof, which, unfortunately, contained a part justified only experimentally ([6], [1']). Unable to find a formal proof for this part, I passed the task to my colleague Michel Carral, who found a purely geometric solution and asked me to present it in Prague ([8], [3']). Today, I decided to revisit the search for the missing proof from my 2019 attempt. This is the story of that work (see [3]). I show how my inability to prove the decrease of a complicated function on the interval [0, π/6] led me to reexamine the problem in various ways, generating, to my great surprise, unexpected results. This erratic search subsequently led me to explore a problem that generalizes the one that had been obstructing me. These new investigations proceeded methodically and systematically, producing both geometric and functional conjectures, will be detailed below. Although the proofs remain elusive, I discovered results that will be the focus of future research. This research journey perfectly illustrates the pre-conjecture phases of experimental research, as described in my 2005 thesis ([4]). It is worth noting that I have already demonstrated the crucial role of the generalization process in my article on tritangent conics ([9], [4']).*

1. The starting point of this investigative work

1.1. The initial figure (Figure 1)
(E) is the ellipse whose foci are *f1* and *f2* (*f1f2* = 2*c* with *c* < 1) and where *(C)* is the director circle associated to *f1* (radius *f1U=2*). The excentricity of *(E)* is *c*. Figure 1 shows the constructions of all the equilateral triangles (in red in figure 1) circumscribed around *(E)*, each triangle varies depending on the position of point *m1* on *(C)*. Note that ∠*m1f2m2* = ∠*m2f2m3* = ∠*m3f2m1*=120°. Note also that the sides of these triangles are defined by the perpendicular bisectors of [*f2m1*], [*f2m2*] and [*f2m3*].

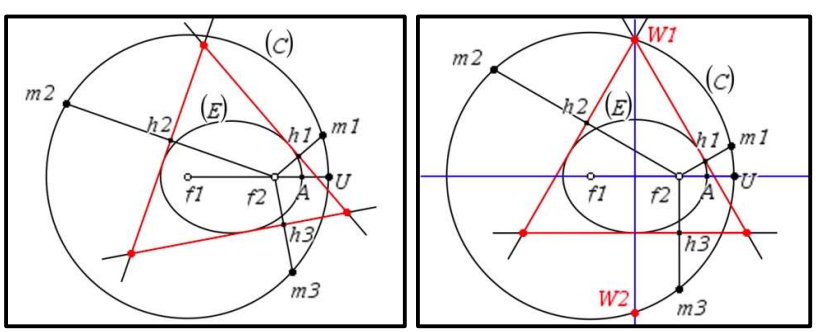

Figure 1: Equilateral triangles circumscribed around a given ellipse

1.2. The problem to solve (Figure 1)**:**
We aim to determine the positions of *m1* on *(C)* that result in the corresponding equilateral triangle having its minimum area. The position on *(C)* is specified by *x* which is angle $\angle Uf2m1$.
Knowing that the *area* of an equilateral triangle varies with its height *h* (*area* = $\frac{h^2}{\sqrt{3}}$). Knowing that *h* = $(f2h1 + f2h2 + f2h3) = \frac{1}{2}(f2m1 = f2m2 + f2m3)$, we need to understand the variations of the function $B(x) = f2m1 + f2m2 + f2m3$ which was evaluated in my paper of ATCM 2019 ([6]).
As $f2h1 = -c.cos(x) + \sqrt{1 - c^2.(\sin(x))^2}$, $f2h2 = -c.cos(x+\frac{2\pi}{3}) + \sqrt{1 - c^2.(\sin\left(x + \frac{2\pi}{3}\right))^2}$
and $f2h3 = -c.cos(x+\frac{4\pi}{3}) + \sqrt{1 - c^2.(\sin\left(x + \frac{4\pi}{3}\right))^2}$, then:

$$B(x) = \sqrt{1 - c^2.(\sin(x))^2} + \sqrt{1 - c^2.(\sin\left(x + \frac{2\pi}{3}\right))^2} + \sqrt{1 - c^2.(\sin\left(x + \frac{4\pi}{3}\right))^2}$$

Because $cos(x) + cos(x+\frac{2\pi}{3}) + cos(x+\frac{4\pi}{3}) = 0$.
In this paper, we have proven that this function is periodic (period = $\frac{\pi}{3}$) and is an even function. To understand its variations, it is sufficient to examine its behavior on the interval $[0\ \frac{\pi}{6}]$. Experimentally, using a CAS, we have conjectured that this function is decreasing on this interval and thus the minimum is achieved at $x = \frac{\pi}{6}$. However, I have been unable to formally prove this result.

1.3. The present work
After a hiatus of several years, I decided to resume the search for a proof of what seemed to be a simple problem. However, after several months of unsuccessful efforts, I decided to stop focusing solely on this single problem. Instead, I chose to generalize the question, which led to many new results that will be presented here. Although I do not provide formal proofs for most of these results, we will see how the process of generalization has contributed positively to uncovering unknown properties. Additionally, we will find that the initial phase of this work was not entirely unproductive, as some unexpected conjectures emerged.

1.4. The theoretical framework of this research
The approach taken in this work vividly illustrates the model of pre-conjecture experimental discovery facilitated by technology, as outlined in my 2005 thesis ([4]. Specifically, we will illustrate the three highlighted phases:
- **The erratic research phase**: This phase involves producing and observing data that initially do not provide clear interpretations related to the problem under investigation. However, we will explore how this phase led to the emergence of unexpected conjectures.
- **The methodical research phase**: During this phase, the process of generalization prompts us to conduct experiments in an organized manner. The goal is to generate interpretable data that can lead to conjectures about issues related to the original problem.
- **The acceleration phase**: This phase involves swiftly testing conjectures through experimental validation.

2. Erratic research and colateral conjectures

2.1. Special points viewing a given ellipse under an angle of 60°

When I resumed my research, I had to replicate the same figure multiple times to generate data that might guide me towards a solution. As I have mentioned, these attempts were unsuccessful. However, at one point, I decided to observe the figure without focusing on the problem I needed to solve, and then I discovered the following: If my conjecture was correct, the positions of the equilateral triangle with the minimum area circumscribed around the given ellipse would be those where the principal axis of the ellipse is parallel to one side of the triangle. It was easy to observe that the equilateral triangle had a vertex on the secondary axis of the ellipse, and moreover, this vertex was located on the director circle associated with *f1* (first colateral result).
This result suggests a simple construction of the two tangent lines to an ellipse from the two points where the secondary axis intersects any of its director circles (*W1* and *W2* in Figure 1 right).

2.2. An original construction of a directrix of an ellipse (Figure 2 left)
Here is a second exploration of the erratic phase. Note the distinction I make between exploration and investigation: an exploration involves generating data without expecting any interpretation that could lead to a conjecture or to the experimental validation or invalidation of a conjecture. In contrast, an investigation also involves generating data using recognized techniques, with the hope that the interpretation will lead to a conjecture or allow for the experimental validation or invalidation of a conjecture.
During the various manipulations of the same figure where the starting ellipse (*E*) is constructed from one of its director circles (*m1* can move on circle *(C)* and generate the point *n1* which defines the given ellipse), I curiously constructed the point *p*, the intersection of the rays [*On1*) and [*f2m1*), and particularly observed its trace when *m1* moved along *(C)*, meaning when *n1* moved along *(E)*. I immediately conjectured that the set of points *p* formed a line perpendicular to the major axis of *(E)*. The technique I used to validate this conjecture is as follows: I used my software to construct the locus *(L)* of *p* when *m1* moves along (*C*); I placed a point *q* on this locus and drew the line (*D*) passing through *q* and perpendicular to *(f1f2)*. Then, I placed a point *r* on *(L)* and displayed the distance from *r* to *(D)*; I obtained results on the order of 10^{-14} or 10^{-15} when moving *r* along *(L)*, which I interpreted to mean that the locus *(L)* is indeed a line perpendicular to the major axis of *(E)*.
I immediately wondered if this line could be specifically associated with *(E)*, and given its position, I immediately conjectured that it could be one of its directrices.
The experimental verification I conducted is as follows: I constructed a variable point *m* on *(E)* and drew a line parallel to *(f1f2)* that intersects *(D)* at point *h*. I measured the distances *mf2* and *mh* and found that their ratio displayed close to $\frac{a}{b}$ in figure 2. It can be observed that this ratio remains constant up to the ninth decimal place as *m* is moved along *(E)*.

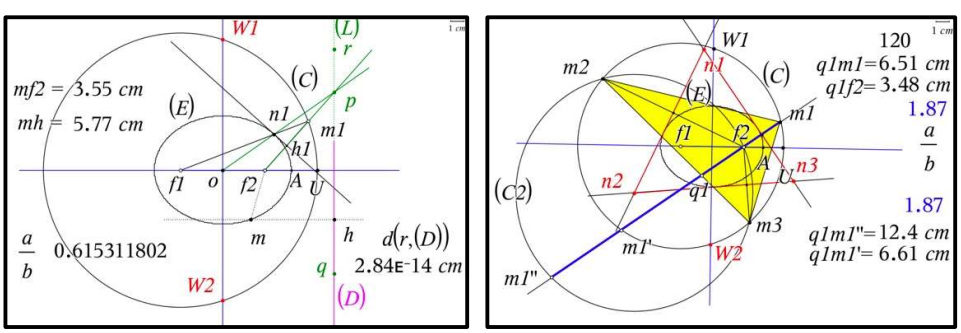

Figure 2: Two colateral conjectures related to the ellipse

Finally, the discovered property can be stated as follows:

For a given ellipse *(E)* with foci *f1* and *f2* and center *o*, if *n1* is a point on this ellipse and *(T)* is the tangent line at this point, the line *(on1)* and the perpendicular to *(T)* intersect at a point on a directrix of *(E)*. For greater accuracy, a formal approach is required.

2.3. Fermat point and Napoleon triangle (Figure 2 right)
By carefully observing the figure I was working on, I quickly noticed that the triangles *m1m2m3* all had a common circumcircle *(C)* with center *f1* and shared the same Fermat point, *f2*, which lies inside circle *(C)*. Indeed:
We can also see that the circle *(C2)* centered at *n2* and passing through *f2* is the circumcircle of the the triangle *m2f2m3* making the angle *m2n2m3* equal to 120°. Additionaly, *n2* lies on the perpendicular bisector of [*m2m3*] making *n2* the center of the equilateral triangle based on [*m2m3*]. A similar demonstration yields analogous result for *n1* and *n3*. Therefore, the triangle *n1n2n3* is the Napoleon triangle of the triangle *m1m2m3*.
I then attempted to use the known property involving the sum *f2m1*+ *f2m2*+ *f2m3* which is evaluated by *m1m1''*, but without success. However, my search for invariants in this figure led to the discovery of the following property: $\frac{q1m1}{q1f2} = \frac{q1m1''}{q1m1'}$. Experimental validation is straightforward: simply move *m1* along the circle *(C)* and observe that this equality persists.

3. Methodical research. Semi-regular polygons circumscribed about a given ellipse

3.1. A new direction for this research

From now on, I have decided to focus on variations in the areas of semi-regular polygons circumscribed to a given ellipse, rather than on those of equilateral triangles. This new research aims to generalize the previous work done with equilateral triangles and will therefore focus on polygons (with *n* sides) constructed using the same technique. We will see that this technique leads us to construct polygons where the angles between two consecutive sides are constant and equal to $\frac{(n-2)\pi}{n}$ (equal to $\pi - \frac{2\pi}{n}$).
This phase of my research is referred to as methodical research, as constructions, displays, and observations are planned in advance. We will conduct this phase by focusing on the following semi-regular polygons, quadrilaterals, pentagons, hexagons, heptagons, octagons, nonagons, decagons, undecagons and dodecagons, with the goal of generating a conjecture for *n*-sided semi-regular polygons.
We will construct these polygons, display the values of the angle on which they depend, and calculate their areas. By varying the defining angle, we will attempt to identify patterns in the variations of their areas, hoping to observe a conjecturable periodicity. We will also aim to highlight how the area changes over each period. Additionaly, we will investigate wether these variations are similar to those we encountered for $n = 3$.

3.2. The first case: that of the semi-regular quadrilateral (the rectangle)

Figure 3 left shows the construction of such a quadrilateral with the method used for the equilateral triangle.

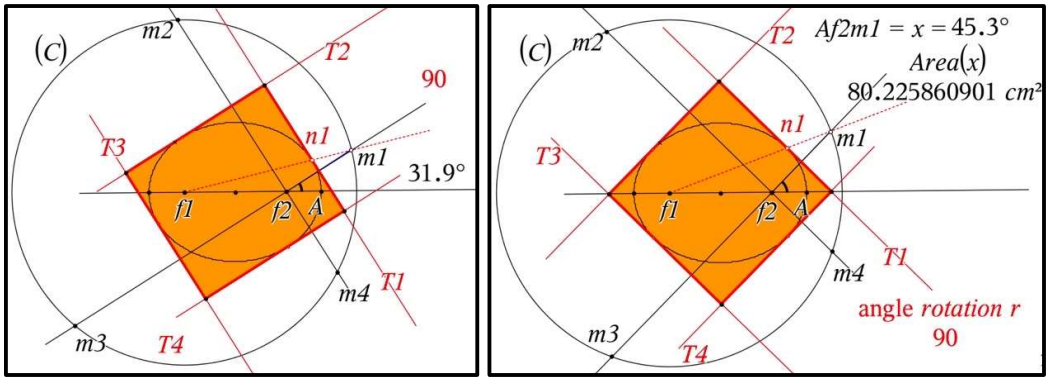

Figure 3: 4-sided semi-regular polygons

Construction algorithm: After constructing the ellipse with foci *f1* and *f2* and displaying the number 90° (which is $\frac{(n-2)\pi}{n}$ in radians with $n = 4$), we choose a point *n1* on the ellipse and draw its tangent *T1* at *n1* which is the perpendicular bisector of [*f2 m1*]. Here *m1* is the intersection point of the ray [*f1 n1*) and the circle *(C)*, the director circle of the ellipse associated with *f1*. If we call *r* the 90° rotation centered at *f2*, the tangent lines *T2*, *T3* and *T4* defining with *T1* our semi-regular quadrilateral (a rectangle in fact) are the perpendicular bisectors of [*f2 m2*], [*f2 m3*] and [*f2 m4*]; the points *m2*, *m3* and *m4* are the intersection points of *(C)* with the images of the ray [*f2 m1*) iterated by *r*.

First investigations with this figure: we move *m1* and observe the variations in the displayed area when angle $\angle Af2m1$ varies from 0 to 2π. This corresponds to *n1* traveling once around the ellipse or *m1* traveling once around the circle *(C)*. It is easy to see that the area increases over the interval [0°, 45°] (or [0, $\frac{\pi}{4}$] in radians) and then decreases over the interval [45°, 90°] (or [$\frac{\pi}{4}$, $\frac{\pi}{2}$] in radians). Additionaly, the area at 0 appears to be the same as at $\frac{\pi}{2}$. When *x* continues to increase, the variations in the area seem to repeat periodically every 90° (or $\frac{\pi}{2}$ in radians). Thus, the minimum area seems to be reached for $x = k\frac{\pi}{2}$ (k belonging to Z) and the maximum area seems to be reached for $x = \frac{\pi}{4} + k\frac{\pi}{2}$. Due to the symmetry of the constructions with respect to the line (*f1f2*), the area function is necessarily even. We can also see that the rectangle appears to be a square when the area is at its maximum, with its diagonals aligned with the axes of the ellipse (case showed in Figure 3 right).

3.3. The second case: that of the semi-regular pentagon (*n* = 5) illustrated by figures 4 and 5

First, let's see how to modify the previous figure to easily obtain the semi regular pentagon circumscribed around the same ellipse. These modifications can be iterated to transform the figure illustrating the case of the semi-regular *n*-sided polygon into the figure of the semi-regular (*n+1*)-sided polygon.

Iteration construction: first, change the number 90 (which is $\frac{2\pi}{4}$ in radians) to 72 (which is $\frac{2\pi}{5}$ in radians) Then delete the 4-sided semi-regular polygon. Use *r* to rotate the ray [f2m4) and call its intersection point with *(C)*, *m5*. The fifth tangent line *T5* to the ellipse is the perpendicular bisector of [*f2m5*] (Figure 4 left). Next, create the pentagon defined by the 5 tangent lines *T1*, *T2*, *T3, T4* and *T5* and display its area.

Investigations with this iterated figure: By moving *m1* along the ellipse and conducting investigations similar to those with the 4-sided semi-regular polygon (the rectangle), we arrive at the following conjectures: the area decreases over the interval [0°, 18°] (or [0, $\frac{\pi}{10}$] in radians) and then increases over the interval [18°, 36°] (or [$\frac{\pi}{10}$, $\frac{\pi}{5}$] in radians). Additionaly, the area at 0 appears to be

the same as at $\frac{\pi}{5}$. When x continues to increase the variations in the area seem to repeat periodically every 36° (or $\frac{\pi}{5}$ in radians). Thus, the maximum area seems to be reached for $x = k\frac{\pi}{5}$ (k belonging to Z), and the minimum area seems to be reached for $x = \frac{\pi}{10} + k\frac{\pi}{5}$. Due to the symmetry of the constructions with respect to the line ($f1f2$), the area function is necessarily even.

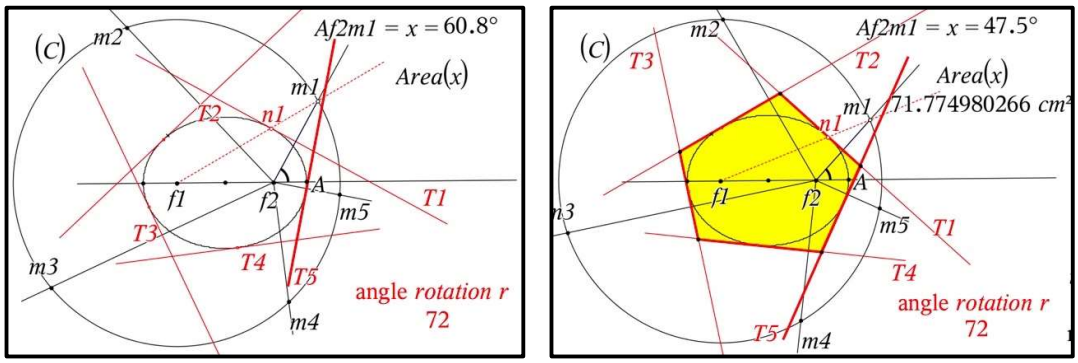

Figure 4: From the semi-regular 4-sided polygon to the semi-regular pentagon

Other conjectures: the maximum area appears to be reached when one vertex of the pentagon lies on the major axis of the ellipse and one side is perpendicular to this axis (Figure 5 left). The minimum area appears to be reached when one vertex of the pentagon lies on the minor axis of the ellipse and one side is perpendicular to this axis (Figure 5 right).

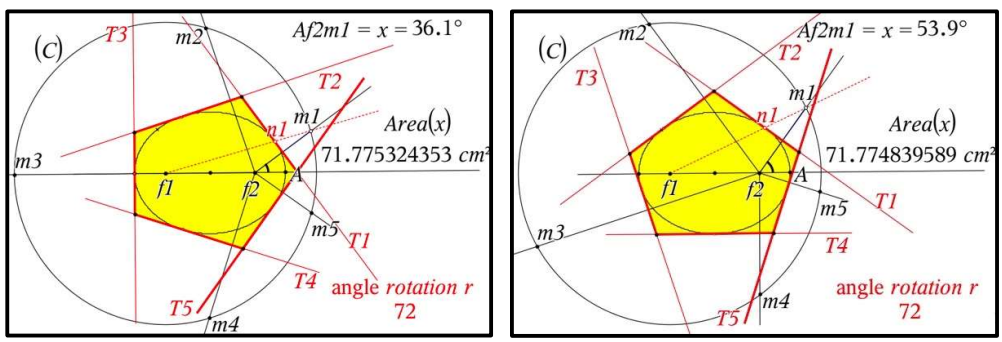

Figure 5: Semi-regular pentagons of maximum and minimum areas

3.4. Other investigated cases: from the semi-regular hexagon to the semi-regular dodecagon

Here are the conjectures we have derived from iterating the process of constructing semi-regular polygons with n sides (for n ranging from 6 to 12).

Case of the semi-regular hexagon (n = 6) illustrated by figure 6

Figure 6 (left) depicts the semi-regular pentagon obtained using the iteration process described above. Investigations with this dynamic figure lead to the following conjectures:

The area function is periodic (period equal to 60° or $\frac{\pi}{3}$ in radians) and even; it decreases on [0, 30°] and increases on [30°,60°].

Figure 6 (center) shows the positions of hexagons with maximum area, characterized by two sides perpendicular to the principal axis of the ellipse and two vertices on the minor axis. Figure 6 (right) shows the positions of hexagons of minimum area, characterized by two sides parallel to the principal axis of the ellipse and two vertices on the major axis.

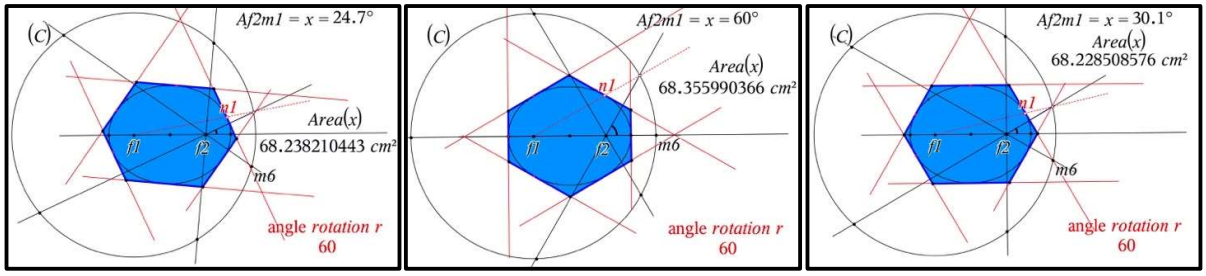

Figure 6: semi-regular hexagons

Case of the semi-regular heptagon (*n* = 7) illustrated by figure 7 (left)

The area function is periodic (with a period equal to 25.7° or $\frac{\pi}{7}$ in radians) and even; it decreases on [0, 12.85°] ([0, $\frac{\pi}{14}$] in radians) and increases on [12.85°,25.71°] ([$\frac{\pi}{14}$, $\frac{\pi}{7}$] in radians).

Figure 7 (center) shows the positions of pentagons of maximum area, characterized by a side perpendicular to the principal axis of the ellipse and a vertex on this axis. Figure 7 (right) shows the positions of pentagons of minimum area, characterized by a side perpendicular to the minor axis of the ellipse and a vertex on this axis.

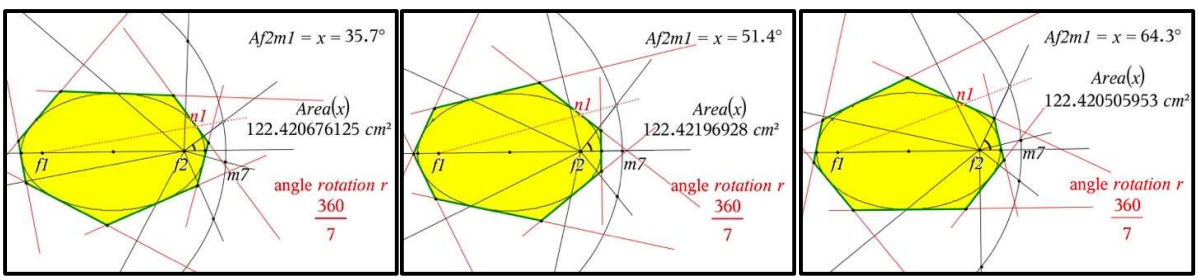

Figure 7: semi-regular pentagons

Case of the semi-regular octagon (*n* = 8) illustrated by figure 8 (left)

The area function is periodic (period equal to 45° or $\frac{\pi}{4}$ in radians) and even; it increases on [0, 22.5°] ([0, $\frac{\pi}{8}$] in radians) and decreases on [22.5°,45°] ([$\frac{\pi}{8}$, $\frac{\pi}{4}$] in radians).

Figure 8 (center) shows the positions of octagons of maximum area, characterized by two vertices belonging to the major axis of the ellipse and two vertices belonging to the minor axis. Figure 8 (right) shows the positions of octagons of minimum area, characterized by two sides parallel to the principal axis of the ellipse and two other sides parallel to the other axis.

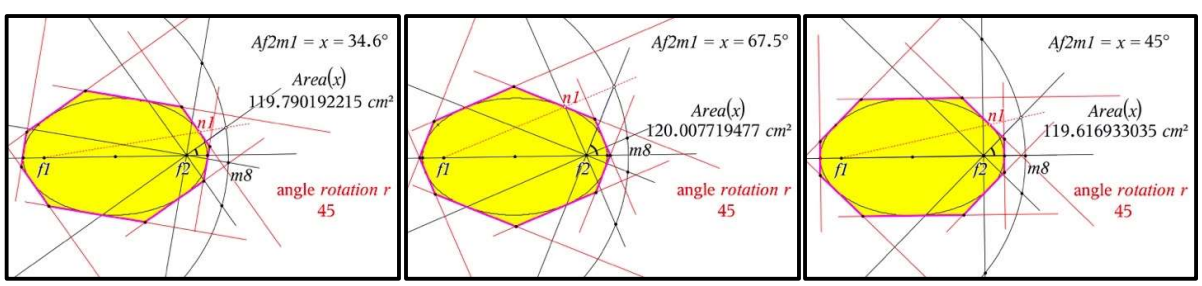

Figure 8: Semi-regular octagons

Case of the semi-regular nonagon (*n* = 9) illustrated by figure 9 (left)

The area function is periodic (period equal to 20° or $\frac{\pi}{9}$ in radians) and even; it decreases on [0, 10°] ([0, $\frac{\pi}{18}$] in radians) and increaseses on [10°,20°] ([$\frac{\pi}{18}$, $\frac{\pi}{9}$] in radians).
Figure 9 (center) shows the positions of nonagons of maximum area, characterized by one vertex on the principal axis of the ellipse and a side perpendicular to this axis. Figure 9 (right) shows the positions of nonagons of minimum area, characterized by one vertex on the second axis of the ellipse and a side perpendicular to this axis.

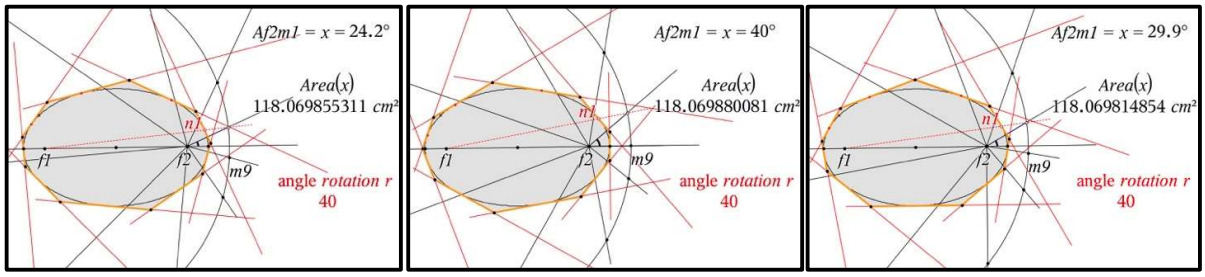

Figure 9: Semi-regular nonagons

Case of the semi-regular decagon (*n* = 10) illustrated by figure 10 (left)

The area function is periodic (period equal to 36° or $\frac{\pi}{5}$ in radians) and even; it decreases on [0, 18°] ([0, $\frac{\pi}{10}$] in radians) and increases on [18°,36°] ([$\frac{\pi}{10}$, $\frac{\pi}{5}$] in radians).
Figure 10 (center) shows the position of decagons of maximum area, characterized by two vertices on the second axis of the ellipse and two sides parallel to this axis. Figure 10 (right) shows the position of decagons of minimum area, characterized by two vertices on the principal axis of the ellipse and two sides parallel to this axis.

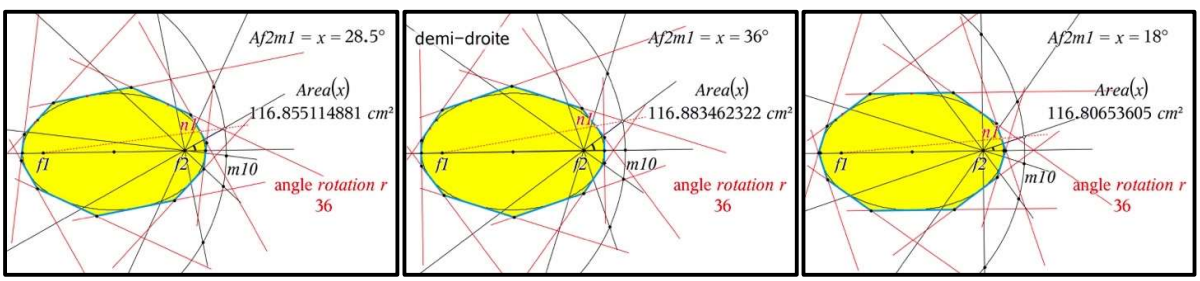

Figure 10: Semi-regular decagons

Case of the semi-regular undecagon (*n* = 11) illustrated by figure 11 (left)
The area function is periodic (with a period equal to 16.36° or $\frac{\pi}{11}$ in radians) and even; it decreases on [0, 8.18°] ([0, $\frac{\pi}{22}$] in radians) and increases on [8.18°,6.36°] ([$\frac{\pi}{22}$, $\frac{\pi}{11}$] in radians).
Figure 11 (center) shows the positions of undecagons of maximum area, characterized by one vertex on the principal axis of the ellipse and one side perpendicular to this axis. Figure 11 (right) shows the positions of undecagons of minimum area, characterized by one vertex on the second axis of the ellipse and one side perpendicular to this axis.

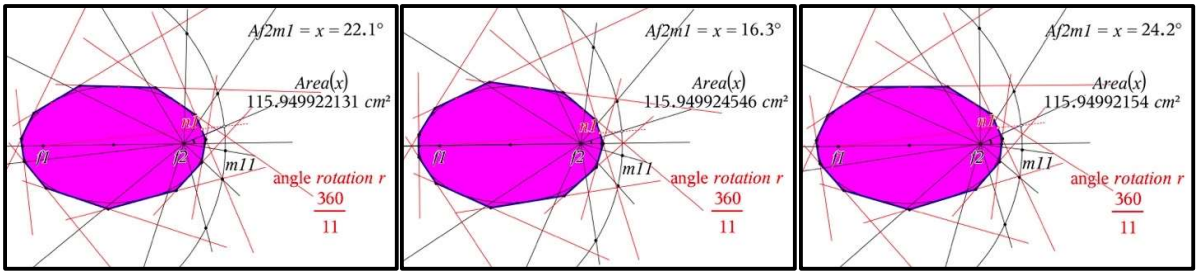

Figure 11: Semi-regular undecagons

Case of the semi-regular dodecagon (n = 12) illustrated by figure 12 (left)

The area function is periodic (period equal to 30° or $\frac{\pi}{6}$ in radians) and even; it increases on [0, 15°] ([0, $\frac{\pi}{12}$] in radians) and increases on [15°,30°] ([$\frac{\pi}{12}$, $\frac{\pi}{6}$] in radians). Figure 12 (center) shows the positions of dodecagons of maximum area, characterized by two vertices on the principal axis of the ellipse and two vertices on the other axis. Figure 12 (right) shows the positions of dodecagons of minimum area, characterized by having two sides perpendicular to the principal axis of the ellipse and two other sides perpendicular to the other one.

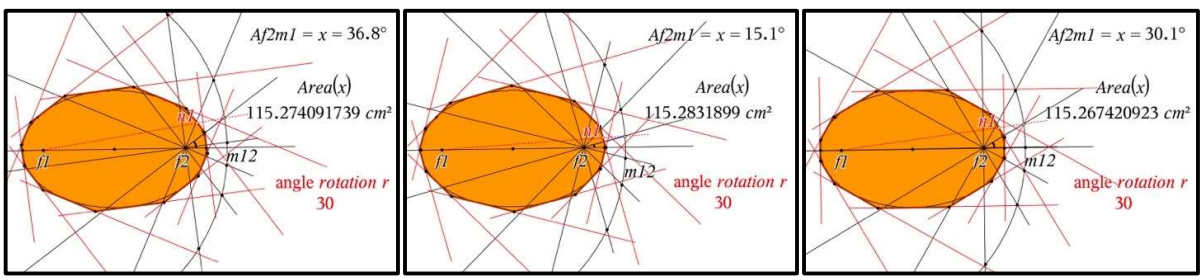

Figure 12: Semi-regular dodedecagons

4. Acceleration of the research: treatment of the previous collected data

4.1. Acceleration of the research

This phase of the experimental discovery research approach, highlighted in my thesis work ([4]), is the third pre-conjecture phase, which leads to the formulation of conjectures. During this phase, conjectures emerge, and rapid validations are attempted. In fact, the acceleration phase began after observing the data generated up to the semi-regular octagon: we were able to predict the data that the semi-polygons from n = 9 to 12 would generate and see that our conjectures were well corroborated. At this stage of my work, I assumed that a general rule might emerge from all the previous data. To clarify this rule, if it exists, I decided to present all the data in a table, which is reproduced below.

4.2. Summary table

n	semi-regular	rotation angle	period	increasing on	decreasing on
3	triangle	120° or $\frac{2\pi}{3} = \beta$	60° or $\frac{\beta}{2}$	$[\frac{\beta}{4}, \frac{\beta}{2}]$	$[0, \frac{\beta}{4}]$
4 = 4p	quadrilateral	90° or $\frac{2\pi}{4} = \alpha$	90° or α	$[0, \frac{\alpha}{2}]$	$[\frac{\alpha}{2}, \alpha]$
5	pentagon	72° or $\frac{2\pi}{5} = \beta$	36° or $\frac{\beta}{2}$	$[\frac{\beta}{4}, \frac{\beta}{2}]$	$[0, \frac{\beta}{4}]$
6 ≠ 4p	hexagon	60° or $\frac{2\pi}{6} = \alpha$	60° or α	$[\frac{\alpha}{2}, \alpha]$	$[0, \frac{\alpha}{2}]$

7	heptagon	$\frac{360}{7}$ or $\frac{2\pi}{7} = \beta$	25.7° or $\frac{\beta}{2}$	$[\frac{\beta}{4}, \frac{\beta}{2}]$	$[0, \frac{\beta}{4}]$
$8 = 4p$	octagon	45° or $\frac{2\pi}{8} = \alpha$	45° or α	$[0, \frac{\alpha}{2}]$	$[\frac{\alpha}{2}, \alpha]$
9	nonagon	40° or $\frac{2\pi}{9} = \beta$	20° or $\frac{\beta}{2}$	$[\frac{\beta}{4}, \frac{\beta}{2}]$	$[0, \frac{\beta}{4}]$
$10 \neq 4p$	decagon	36° or $\frac{2\pi}{10} = \alpha$	36° or α	$[\frac{\alpha}{2}, \alpha]$	$[0, \frac{\alpha}{2}]$
11	undecagon	$\frac{360}{11}$ or $\frac{2\pi}{11} = \beta$	16.36° or $\frac{\beta}{2}$	$[\frac{\beta}{4}, \frac{\beta}{2}]$	$[0, \frac{\beta}{4}]$
$12 = 4p$	dodecagon	30° or $\frac{2\pi}{12} = \alpha$	30° or α	$[0, \frac{\alpha}{2}]$	$[\frac{\alpha}{2}, \alpha]$

From this table we can induce the following conjectures:

If ***n* is even**, the period of the area function of the semi-regular polygon is equal to $\frac{2\pi}{n}$

If $n = 4p$, this function increases over the first half-period $[0, \frac{\pi}{n}]$ and decreases over the second half-period $[\frac{\pi}{n}, \frac{2\pi}{n}]$. The angle values that maximize the area are of the form $\frac{\pi}{n} + 2k\frac{\pi}{n}$ $(k \in \mathbb{Z})$, while the values that minimize the area are $2k\frac{\pi}{n}$ (also for $k \in \mathbb{Z}$)

If $n \neq 4p$, this function decreases over the first half-period $[0, \frac{\pi}{n}]$ and increases over the second half-period $[\frac{\pi}{n}, \frac{2\pi}{n}]$. The angle values that maximize the area are of the form $2k\frac{\pi}{n}$ $(k \in \mathbb{Z})$, while the values that minimize the area are $\frac{\pi}{n} + 2k\frac{\pi}{n}$ (also for $k \in \mathbb{Z}$)

If ***n* is odd**, the period of the area function of the semi-regular polygon is equal to $\frac{\pi}{n}$. This function decreases over the first half-period $[0, \frac{\pi}{2n}]$ then increases over the second half-period $[\frac{\pi}{2n}, \frac{\pi}{n}]$. The angle values that maximize the area are of the form $k\frac{\pi}{n}$ $(k \in \mathbb{Z})$, while the values that minimize the area are $\frac{\pi}{2n} + k\frac{\pi}{n}$ (always for $k \in \mathbb{Z}$).

Remark: for a n-sided semi regular polygon the constant angle between two consecutive sides is equal to $\pi - \frac{2\pi}{n}$ or $\frac{(n-2)\pi}{n}$.

4.3. Conjectures on the shape of semi-regular polygons with maximum and minimum area

By meticulously observing the figures of semi-regular polygons with maximum and minimum areas in section 3.4, we arrive at the following conjectures regarding the characteristics of such polygons circumscribed around a given ellipse:

If ***n* is even and *n* = 4*p***:

Polygons with maximum area are characterized by having two vertices on the principal axis of the ellipse and two vertices on the other axis.

Polygons with minimum area are characterized by having two sides perpendicular to the principal axis of the ellipse and two other sides perpendicular to the other one.

If ***n* is even and *n* ≠ 4*p***:

Polygons with maximum area are characterized by having two vertices on the principal axis of the ellipse and two sides perpendicular to this axis.

Polygons with minimum area are characterized by having two vertices on the second axis of the ellipse and two sides perpendicular to this axis.

If ***n* is odd**:

Polygons with maximum area are characterized by one vertex on the principal axis of the ellipse and one side perpendicular to this axis.

Polygons with minimum area are characterized by one vertex on the second axis of the ellipse and one side perpendicular to this axis.

4.4. Other formal generalizations corroborated experimentally

4.4.1. Link with a sum of *n* distances:

In the initial problem, we observed that the area of the equilateral triangle (3-sided semi-regular polygon) circumscribed around the given ellipse varied with its height. This height is equivalent to the sum of the distances from *f2* to the tangent lines to the ellipse defining the equilateral triangle. Extending this idea, we can conjecture that the area of *n*-sided semi-regular polygons varies with the sum of the distances from *f2* to the tangent lines to the ellipse defining this polygon, expressed as $\sum_{i=1}^{n} f2hi$.

To corroborate such a result experimentaly, we can use the previous dynamic figures for *n* from 4 to 12. We display the *n* distances *f2hi* and then evaluate their sum $s_n(x)$. By moving point *n1* (thereby changing the values of *x*) along the ellipse, we can observe that this sum varies exactly like the area of the *n*-sided semi-regular polygon. To our surprise, our three attempts at corroboration for *n* equal 7, 8 and 9 were successful, albeit experimentally. From this, I was confident that checks for other values of *n* would confirm our generalization.

Here is a visualization (Figure 13) for *n* = 8, showing the simultaneous and identical variations of the area of the semi-regular octagon and the sum of the distances from *f2* to all its sides. Note that the variations of the aera of the semi-regular octogon $\mathcal{A}(x)$ and the sum of the distances from *f2* to the eight sides of the octagon $\mathcal{S}(x)$ are visualized by displaying the curves of ($\mathcal{A}(x)$-13.3949).10000 (in blue) and that of ($\mathcal{S}(x)$-16.10537).100000 in purple). These curves have been obtained as loci depending on *m1* on the director circle associated with the focus *f1* of the ellipse. Visually we can state that the variations are identical. The maxima and minima are reached for the same values of *x* ($k\frac{\pi}{8}$ in radians or k.22.5° in degrees).

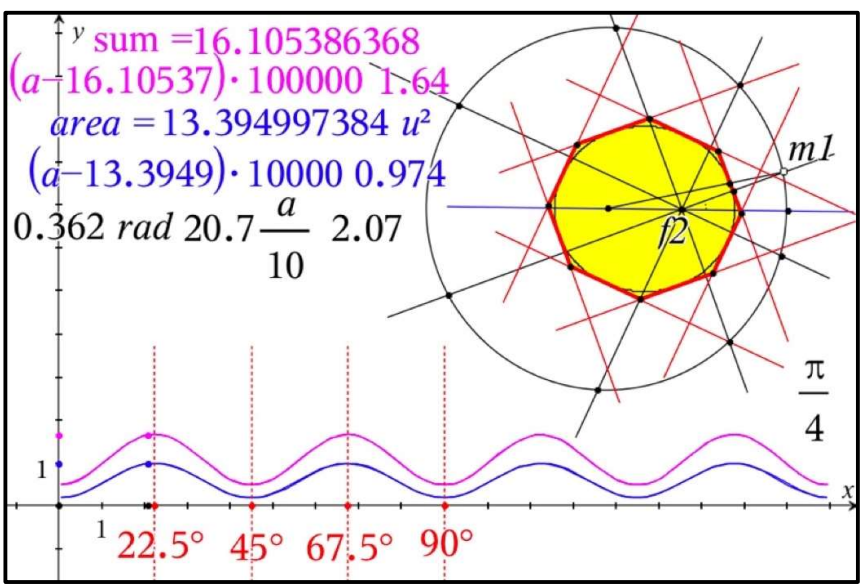

Figure 13: Simultaneous variations of the area and a sum of distances

4.4.2. Link with the variations of functions depending on parameter *n*:

Using the same technique employed to evaluate the sum of the distances from *f2* to the three sides of an equilateral triangle circumscribed around a given ellipse (a 3-sided semi-regular polygon), we now

evaluate the sum $\boldsymbol{v}$ **(*x,n*)** of the distances from *f2* to the *n* sides of a *n*-sided semi-regular polygon. Using the same previous notations with *c* between 0 and 1:

If $u(x) = \sqrt{1 - c^2(\sin(x))^2}$ and $\boldsymbol{v(x,n)} = \sum_{p=0}^{n-1} \boldsymbol{u}(\boldsymbol{x} + \frac{2p\pi}{n})$, the variations and some propreties of functions $\boldsymbol{v}$ are given in the summary table of section 4.2.

This result generalizes the one who aimed to prove for $n = 3$. It is a more comprehensive conjecture that the one we failed to prove. Here are the properties conjectured for the functions *v*:

→ For any value of *n*, function *v* is even and periodic with period *T*.

If *n* is even, $T = \frac{2\pi}{n}$

If *n* is odd, $T = \frac{\pi}{n}$

→ Variations of *v* are summarized in Figure 14 below.

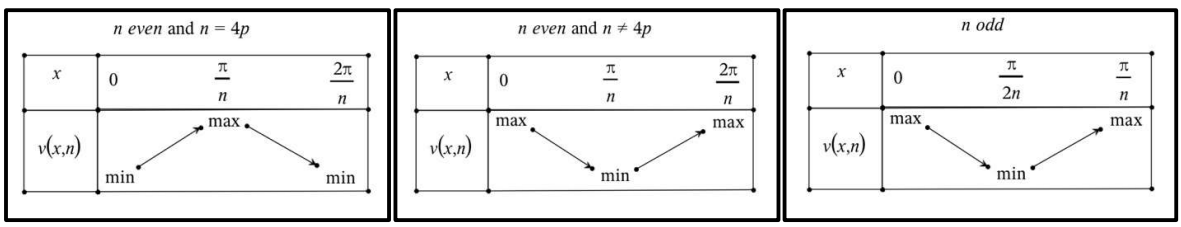

Figure 14: Variations of functions *v*(*x*,*n*)

4.4.3. Some formal corroborations

Although it is straightforward to demonstrate it, we can verify with CAS that for any value of *n* between 3 and 12, we have *v*(-*x*,*n*) - *v*(*x*,*n*) = 0 reflecting the parity of functions *v*. This verification can even be extended up to *n* = 300.

Regarding periodicity, we can verify that:

For any odd number *n* between 3 and 11, we obtain $v(x + \frac{\pi}{n}, n) = v(x, n)$.

For any even number *n* between 4 and 12, we obtain $v(x + \frac{2\pi}{n}, n) = v(x, n)$.

We also can conduct checks for larger values of *n*.

Remark: we have yet to prove the result concerning the variations of the function representing the sum of the distances from *f2* to the three sides of the equilateral triangles circumscribed around a given ellipse. However, by generalizing this problem, we were able to conjecture with a high degree of plausibility the variations of the function representing the sum of the distances from *f2* to the *n* sides of *n*-sided semi-regular polygons circumscribed around a given ellipse. Additionally, note that these variations are exactly the same as those of the function representing the area of such polygons: this highly plausible conjecture, if studied meticulously, would allow us to understand the generalization of calculating the area of semi-regular polygons from their *n* heights.

4.5. Final remark

Note that the family of semi-regular polygons circumscribed around a given ellipse, which we have focused on in our research, is a subset of the set of all polygons circumscribed around a given ellipse. Carral proved that the polygons circumscribed around a given ellipse with minimum area are those whose side midpoints are the points of contact with the given ellipse ([8], [3']). In our work, we characterized the semi-regular polygons with minimum area, which are not the ones identified by Carral. Additionally, we characterized those with maximum area, a case that does not occur for general polygons as their areas are not bounded.

5. Conclusion

Popper believed it was impossible to reconstruct the path to discovery. If he was referring to the great discoveries in the history of science, I largely share his opinion. However, when it comes to localized research like the one behind this work, one can partly dispute his view. This work, as I have presented it, may constitute an attempt to reconstruct the path toward the conjectured discoveries. It showcases many of the constituents of a researcher's journey, starting with the most important: failure in solving a problem, the motivation to overcome it, and, above all, attempting to encompass the problem within a more general context. Even if success is not achieved at the end of this process, other problems emerge thanks to a dynamic approach whose solutions are easily surmised.

Leibniz said that discovery is more difficult than its proof: "Give me a result, and I will deal with its proof." During this research, I highlighted the properties of *n*-sided semi-regular polygons circumscribed around a given ellipse, which allowed me to "discover" the variations of functions generalizing those of the original function that inspired this work. The experimental work, enabled by dynamic geometry, allowed me to generate a large number of figures (which are the data for my work), leading to the conjectures discovered.

I took the opportunity to show how the stages of my work could fit into the model presented in my 2005 thesis ([4]): erratic, methodical phases, and the acceleration phase before the expression of conjectures or the invalidation of previous ones. I have deliberately not mentioned the levels of geometry under which the researcher operated during each stage, so as not to burden the reader with didactic considerations. Interested readers can refer to my thesis for more details ([4]).

The problem initially posed in this article may not have found a solution, but it generated everything that was presented, particularly the results concerning the semi-regular polygons on which I focused this experimental research work.

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YouTube links

[1'] ATCM 2019 JJ DAHAN
https://www.youtube.com/playlist?list=PLOIs4xavv0zFCmU_ANuf2IPCdIQfoV3Ob
[2'] ATCM 2020 JJ DAHAN
https://www.youtube.com/playlist?list=PLOIs4xavv0zFXIEs_boQkk0scPh_ZWZeV
[3'] ATCM 2022 JJDAHAN (second presentation)
https://www.youtube.com/playlist?list=PLOIs4xavv0zEC8-LRy47siRdL5WVtoAmM
[4'] ATCM 2023 JJ DAHAN
https://www.youtube.com/playlist?list=PLOIs4xavv0zEWRqmEyGiIADau4kfOpKf-

Software

TI-Nspire™ CX CAS Premium Teacher Software version 5.2.0.771 by Texas Instruments

Creating Planar *k*-isogonal Tilings

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Abstract: *This paper addresses the research problem of the characterization of a class of tilings or tessellations called k-isogonal tilings in the Euclidean and hyperbolic plane. Tilings that satisfy the property of having k transitivity classes or orbits of vertices under the action of their respective symmetry groups are called k-isogonal tilings. We contribute to answering the research problem by giving various constructions of k-isogonal tilings carried out with the aid of the dynamic geometry software GeoGebra.*

1. Introduction

A *tiling* or *tessellation* of the plane $\mathbb{X}$ (Euclidean plane, hyperbolic plane, or 2-dimensional sphere) is a collection of closed topological disks called *tiles* whose interiors do not intersect and whose union is the entire $\mathbb{X}$. Families of tilings and their characterization in terms of symmetry groups and transitivity properties can be found in the classic textbook *Tilings and Patterns* [1]. One can also refer to online databases such as the *Tilings Encyclopedia* [2], or sites such as Jos Leys' *Mathematical Imagery* [3] to retrieve beautiful images of tilings and examine their properties.

Tilings with vertex, edge or tile transitivity conditions remain a topic of interest in Discrete Geometry. These tilings have been studied not only for their interesting mathematical properties, but also to model periodic and non-periodic crystals [4,5]; and nanomaterials [6,7,8]. These tilings are also used to create aesthetically pleasing artwork [9,10]; as well as serve as instructional materials for students to understand transformation geometry. Grünbaum and Shepard in their works [11,12,13] discussed a tiling called *isogonal*, *isotoxal,* or *isohedral*, where the group of symmetries of the tiling acts transitively on either its vertices, edges, or tiles, respectively. The authors also discussed methods in constructing these tilings and provided complete lists of these tilings. If in a given tiling, k transitivity classes of vertices, edges or tiles are formed under the action of its symmetry group, then the tiling is called *k-isogonal*, *k-isotoxal,* or *k-isohedral,* respectively [1,14].

In the literature, not much has been discussed on k-isogonal tilings. There have been previous studies on Euclidean k-uniform tilings; these are k-isogonal tilings consisting solely of regular polygons. The enumeration of the complete list of 2-uniform tilings and 3-uniform tilings, respectively, has been completed by Krötenheerdt [15] and Chavey [16]. The 4-, 5- and 6-uniform tilings have been found using a computer program and published by Galebach in [17]. His method is not described, and no explicit proof is available that his lists are complete.

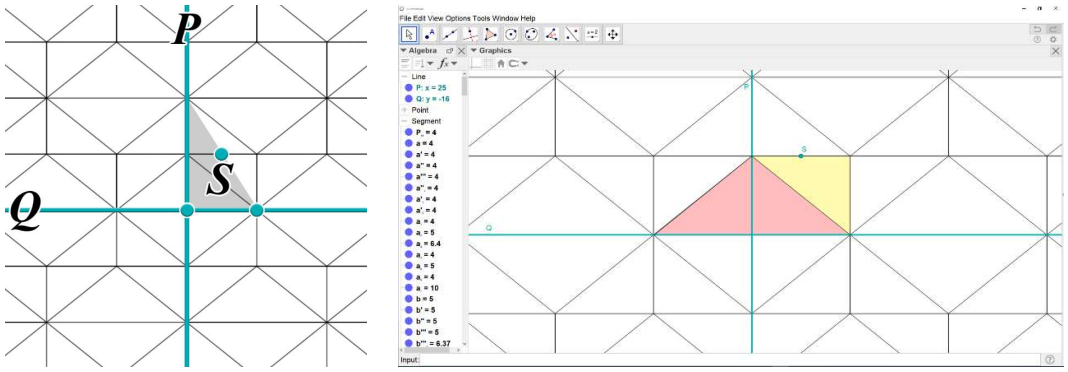
This short note contributes to the open problem of a complete characterization and enumeration of k-isogonal tilings in the Euclidean and hyperbolic plane by featuring constructions of k-isogonal tilings for particular values of k carried out using the dynamic geometry software *GeoGebra* [18]. The geometry software allows for the construction of the tilings with a degree of precision, and facilitates an accurate display of their transitivity features. In the experimentation stage, dynamic geometry tools facilitate understanding the different symmetries of a tiling and determining the effect of these symmetries on its elements such as its tiles, vertices, or edges. Moreover, given a tiling, through the flexibility of dynamic geometry tools, one can perform explorations by introducing new elements to the tiling or varying existing elements of the tiling to lead to the construction of a new tiling with different transitivity properties.

The paper is organized as follows. In Section 2, we discuss the construction of k-isogonal tilings from k-isohedral tilings. An example of a hyperbolic k-isogonal tiling derived from a $[p^n]$ tiling is given in Section 3. Finally, two methods to obtain a k-isogonal tiling from a j-uniform tiling ($j \leq k$) is presented in Section 4. Section 5 gives the conclusion and future outlook of the study.

2. k-isogonal tilings from k-isohedral tilings

One way to construct k-isogonal tilings is to start with known k-isohedral tilings. We illustrate a method of construction by presenting the process of arriving at a 2-isogonal tiling from a given 2-isohedral tiling.

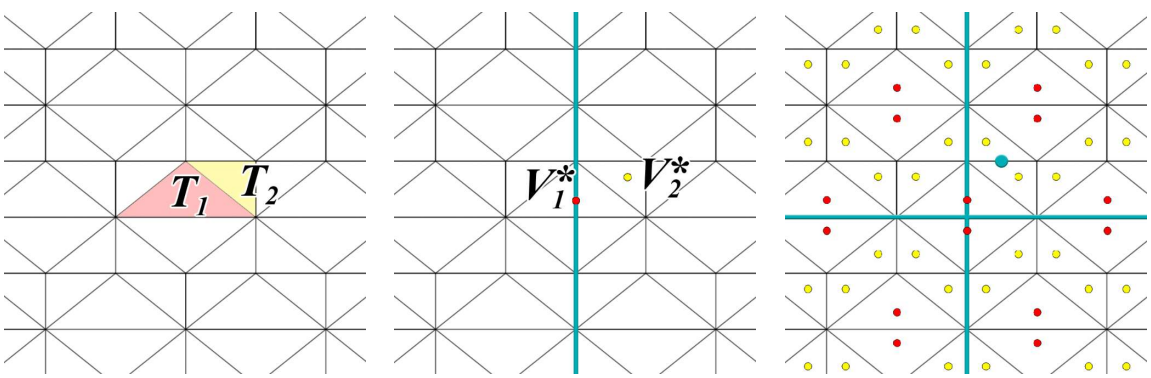
An interested reader can refer to the list of 2-isohedral tilings in [19]. One can choose any of the given tilings provided, such as the tiling shown in Figure 2.1(a) with symmetry group generated by two reflections P, Q and a 2-fold rotation (180° rotation) S. The axes of P, Q and center of S are shown on a fundamental tile of G, which is a smallest region of the tiling wherein upon application of symmetries of G to the region obtains the whole tiling. The symmetry group is described as $2 * 22$ (in Conway notation [20]) to denote the presence of a 2-fold rotational symmetry not lying on an axis of reflection and two other distinct 2-fold rotational symmetries lying on an axis of reflection on the fundamental tile. Aside from the center of S, the other centers of 2-fold rotations are also shown on the fundamental tile in Figure 2.1(a). This 2-isohedral tiling has two transitivity classes of tiles, one class consisting of isosceles triangles (or 3-gons) and the second class consisting of right triangles. The tiling can be reconstructed by a dynamic geometry software such as *GeoGebra* by first constructing tiles that are representative from the two transitivity classes, for example, an isosceles triangle (colored red) and a right triangle (colored yellow) shown in Figure 2.1(b). To generate new tiles, the reflections P and Q, as well as the 2-fold rotation S, are applied to these starting tiles. This can be accomplished using *GeoGebra's* "Reflect about Line" and "Rotate around Point" tools. By systematically applying these symmetries to the starting tiles, additional tiles can be generated. The iterative application of the transformations P, Q and S to constructed tiles will complete the construction of the 2-isohedral tiling.



(a) (b)

Figure 2.1: (a) A 2-isohedral tiling with the axes of reflections P, Q, the center of rotation S and two other centers of 2-fold rotations lying on an axis of reflection; and (b) a *GeoGebra* screenshot showing the constructed tiling.

Given a 2-isohedral tiling $\mathcal{T}$ with symmetry group G, the construction of a 2-isogonal tiling $\mathcal{T}^*$ with symmetry group G, can proceed through a "dualization procedure" where the vertices of $\mathcal{T}^*$are constructed at the interiors of the tiles of $\mathcal{T}$. Two points V_1^* and V_2^*, respectively are first constructed in the interiors of representative tiles T_1 and T_2 from the two transitivity classes of $\mathcal{T}$. Then an application of the symmetries of G to V_1^* and V_2^* will generate all the vertices of $\mathcal{T}^*$. It is important to note that the construction of the vertices has to be carried out in such a way that only one point will result in the interior of every tile of $\mathcal{T}$ when the symmetries of G are applied to V_1^* and V_2^*.



(a) (b) (c)

Figure 2.2: Construction of the vertices of a 2-isogonal tiling of $\mathbb{E}^2$: (a) 2-isohedral tiling with two types of tiles T_1, T_2; (b) V_1^* in the interior of T_1 lying on the axis of P and V_2^* in the interior of T_2; and (c) the resulting vertices after application of the symmetries of G to V_1^* and V_2^*.

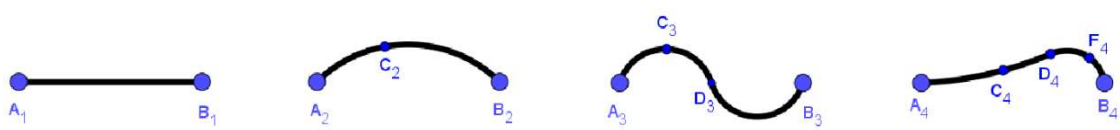
In the 2-isohedral tiling given earlier with representative tiles T_1 and T_2 (Figure 2.2 (a)), the points V_1^* and V_2^* are constructed respectively, in the interior of the tiles such that there will be exactly one point inside every tile of $\mathcal{T}$ when the symmetries in G are applied to V_1^* and V_2^*. Using *GeoGebra*, exploration on the possible locations of V_1^* can be carried out and an appropriate choice will be to situate V_1^* at the axis of P (red point in Figure 2.2(b)). Choosing a point not lying on this axis will result to two points in the interior of T_1 upon application of P. In deciding the appropriate location of V_2^* in the interior of T_2 it can be observed that there is no axis of reflection symmetry passing

through T_2 and there is also no rotational symmetry with center in the interior of T_2. Thus, V_2^* can be situated anywhere in the interior of T_2 (yellow point in Figure 2.2(b)). Once V_1^* and V_2^* are constructed, the transformations P, Q, S, and other symmetries of G are used to construct all other points inside each tile of $\mathcal{T}$ (Figure 2.2(c)). These points will serve as the vertices of $\mathcal{T}^*$. As in the construction of the 2-isohedral tiling, the construction of the complete set of vertices can be achieved using the repeated application of the "Reflect about Line" and "Rotate around Point" tools in *GeoGebra*.

The next step in arriving at $\mathcal{T}^*$ involves constructing its edges. An important consideration at this point is the isotoxality property of the starting tiling $\mathcal{T}$. For each representative edge E selected from each of the orbits of edges of $\mathcal{T}$ under G, a corresponding edge E^* of $\mathcal{T}^*$ is constructed by connecting the points in the interior of tiles adjacent to E. After the construction of the representative edges, the symmetries of G will be applied to form the skeleton of $\mathcal{T}^*$. Applying the same criteria used when constructing the vertices, the construction of the edges has to be carried out in such a way that there will be no overlapping of edges when the symmetries of G are applied to the representative edges. The choices for the types of edges to be used are given in Figure 2.3. We show the constructed edges using *GeoGebra* in Figure 2.4.

(a) (b) (c) (d)

Figure 2.3: Different types of edges to be used in the construction of k-isogonal tilings: (a) straight edge; (b) edge with reflection symmetry; (c) edge with 2-fold rotational symmetry; and (d) edge with trivial symmetry.



(a) (b) (c) (d)

Figure 2.4: The edges constructed using *GeoGebr*a tools: (a) "Segment" using endpoints A_1, B_1; (b) "Circumcircular Arc" using endpoints A_2, B_2, and a third point C_2 which can be varied to adjust the curvature of the edge; (c) "Midpoint or Center" to obtain midpoint D_3 of endpoints A_3, B_3 then "Circumcircular Arc" for A_3, C_3, D_3 followed by 2-fold rotation about D_3 using the "Rotate around Point" to complete the edge; and (d) "Circumcircular Arc" for A_4, C_4 and D_4; D_4, F_4 and B_4.

One can observe that there are four distinct edges E_1, E_2, E_3 and E_4 in the two representative tiles T_1 and T_2 as shown in Figure 2.5(a). By two edges being *distinct*, we mean, there are no elements of G that can send one edge to the other. Note that aside from $\mathcal{T}$ being 2-isohedral, it is also 4-isotoxal.

Consider E_1 and its adjacent tiles. We construct the first edge E_1^* by connecting the points in the interiors of the tiles (Figure 2.5(b)). One can check after some experimentation using *GeoGebra* that the most suitable edge to use is the one given in Figure 2.3(c), positioning E_1^* in such a way that the edge is fixed by the 2-fold rotational symmetry of $\mathcal{T}$ centered at the midpoint of E_1. Now, take E_2. Observe that there is no axis of reflection symmetry of $\mathcal{T}$ intersecting E_2 nor a center of rotational symmetry of $\mathcal{T}$ located in E_2. In this case, the points in the interior of the adjacent tiles of E_2 are connected by an edge E_2^* (Figure 2.5(c)) using the edge type with trivial symmetry shown in Figure 2.3(d). In the case of edge E_3, the corresponding edge E_3^* (Figure 2.5(d)) is constructed of type shown

in Figure 2.3(b) positioned in such a way that the reflection with axis through E_3 becomes its reflection symmetry. Finally, corresponding to edge E_4, the edge E_4^* (Figure 5(e)) is constructed as a straight edge as shown in Figure 2.3(a). This is due to the presence of two perpendicular reflections passing through E_4. We show all the constructed edges in Figure 2.5(f). We then apply the symmetries in G to form the skeleton of $\mathcal{T}^*$(Figure 2.5(g)). Finally, different colors are assigned to distinguish between two transitivity classes of vertices, four transitivity classes of edges and the two types of tiles – a 5-gon with a single reflection symmetry and an 8-gon with two perpendicular reflection symmetries in the 2-isogonal tiling $\mathcal{T}^*$ formed in Figure 2.5(h).

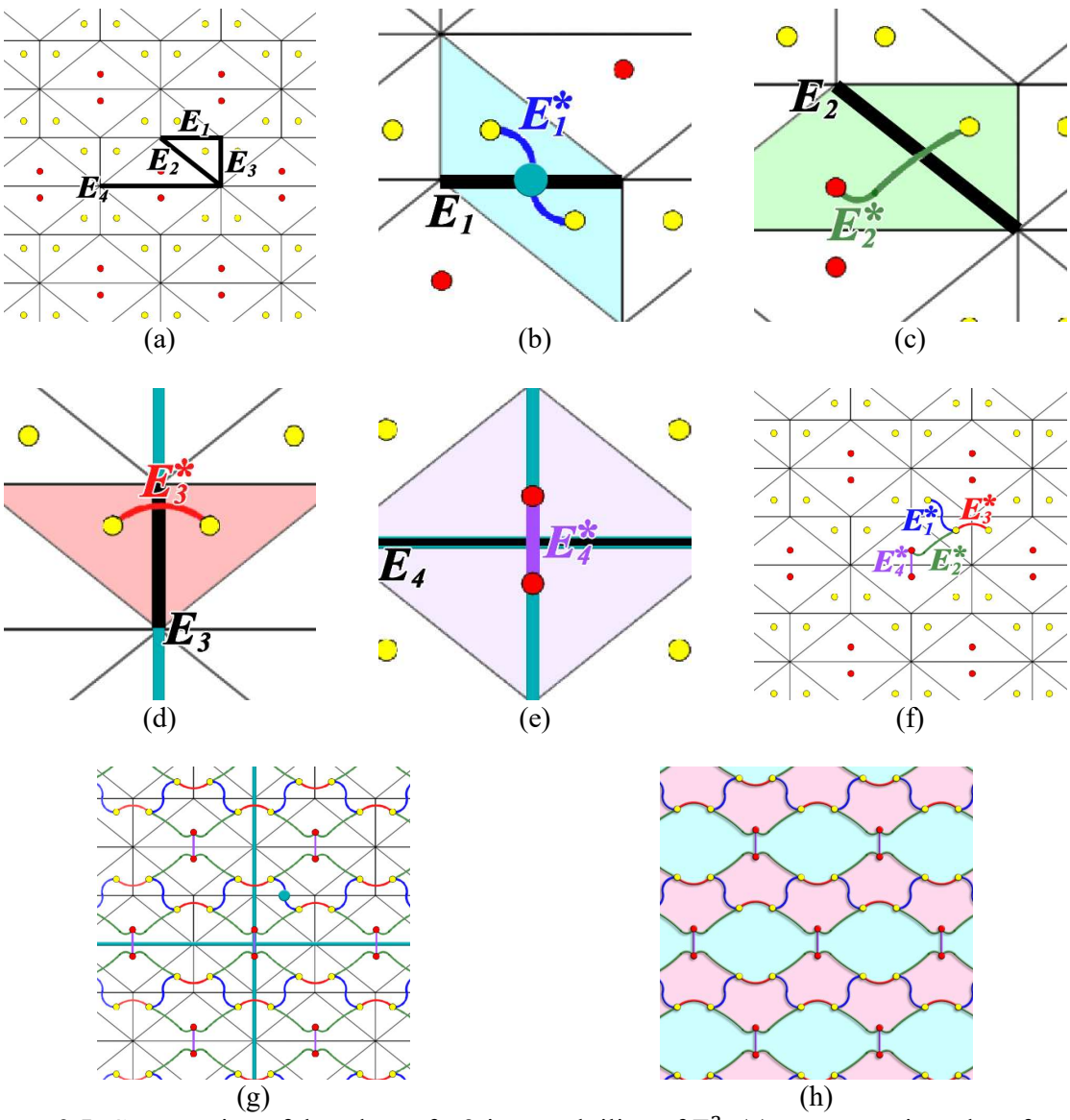
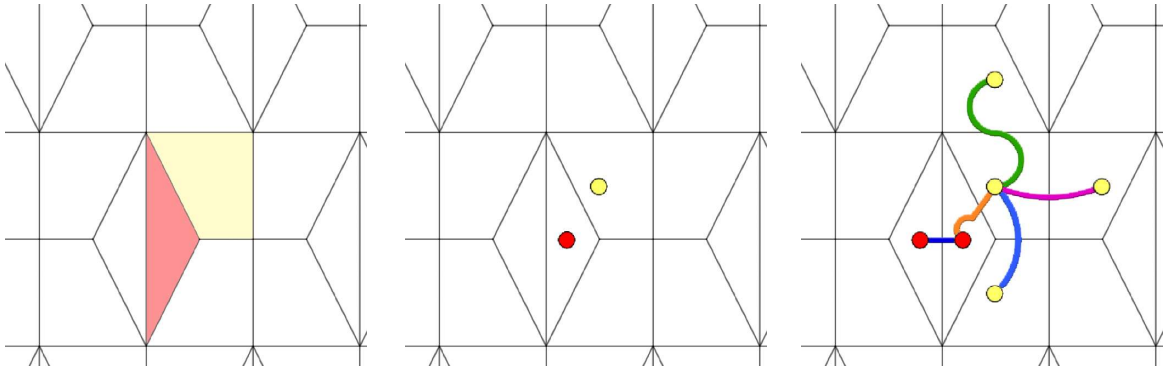


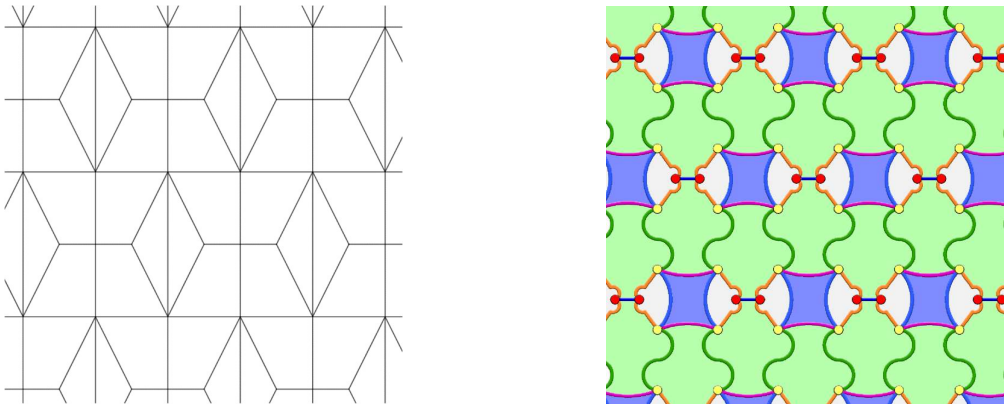
Figure 2.5: Construction of the edges of a 2-isogonal tiling of $\mathbb{E}^2$: (a) representative edges from each edge orbit of $\mathcal{T}$; (b) E_1^* with 2-fold symmetry; (c) E_2^* with trivial symmetry; (d) E_3^* with reflection symmetry; (e) straight edge E_4^* ; (f) edges E_1^*, E_2^*, E_3^*, E_4^*; and (g) skeleton of $\mathcal{T}^*$after application of symmetries of G to E_1^*, E_2^*, E_3^*, E_4^*; and (h) the 2-isogonal tiling $\mathcal{T}^*$.

The construction of a 2-isogonal tiling from a 2-isohedral tiling is an activity that can be given to senior high school students or undergraduate mathematics students as part of exposure to Discrete Geometry research. We asked a group of senior high school students to carry out the activity as part of their internship program in our mathematics laboratory. In creating the tiling, the students were able to understand applying the appropriate transformations (rotation, reflection, translation or glide reflection) to construct the vertices and edges after a series of exploratory tasks using *GeoGebra*. The students found "the construction of edges most challenging as initial constructions caused overlapping edges". Most of the students also experience for the first time creating polygons with curved edges.



(a) (b) (c)

Figure 2.6: Details of the *GeoGebra* construction of the 2-isogonal tiling shown in Figure 2.7(b).



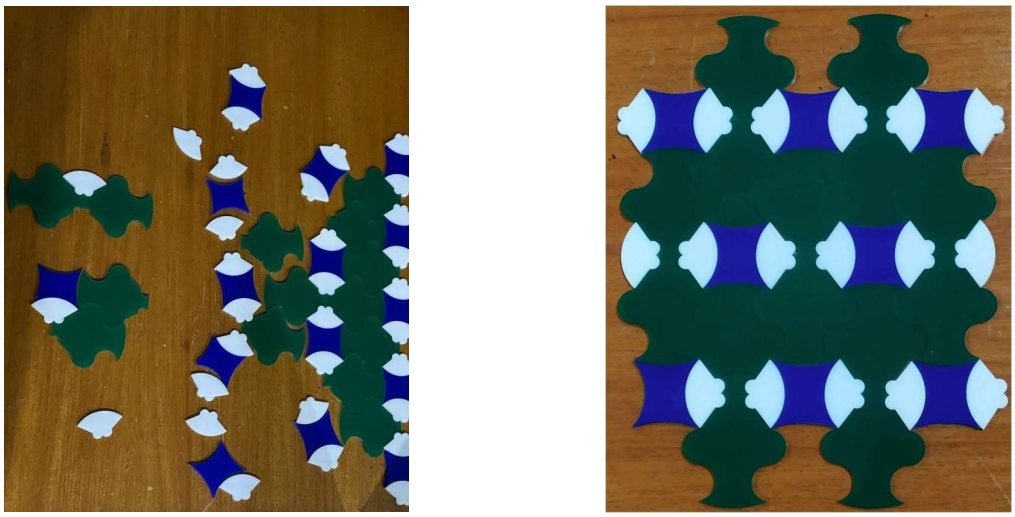
(a) (b)

Figure 2.7: (a) A starting 2-isohedral tiling and (b) the constructed 2-isogonal tiling.

An additional activity that we asked the students to carry out is the realization of the construction of a 2-isogonal tiling by printing and cutting the tiles at the Ateneo de Manila University Eugenio Lopez Jr Makerspace, a digital innovation laboratory (Figure 2.8). The students create versions of their tilings with fine edges enough to be traced by a laser. A choice of different colored acrylic sheets may be used for the various types of tiles present in the tiling. The manipulatives become physical realizations of the tilings constructed using *GeoGebra*. The tiles can be treated as parts of a jigsaw puzzle that the students can play and experiment with. This hands-on activity helped the

students understand how the tiles fit together to form the tiling and allowed them to observe the two distinct ways the tiles fit around a vertex, characteristic of a 2-isogonal tiling.

We present in Figure 2.6 (a)-(c) the construction of another 2-isogonal tiling highlighting the choice of vertices in the tile representatives and the edges. The starting 2-isohedral tiling is shown in Figure 2.7 (a) and the resulting 2-isogonal tiling is shown in Figure 2.7 (b). In Figure 2.8, we show the realization of the 2-isogonal tiling as a manipulative. In Figure 2.8(a), the two vertex coronae are displayed. The entire tiling is shown in Figure 2.8(b).



(a) (b)

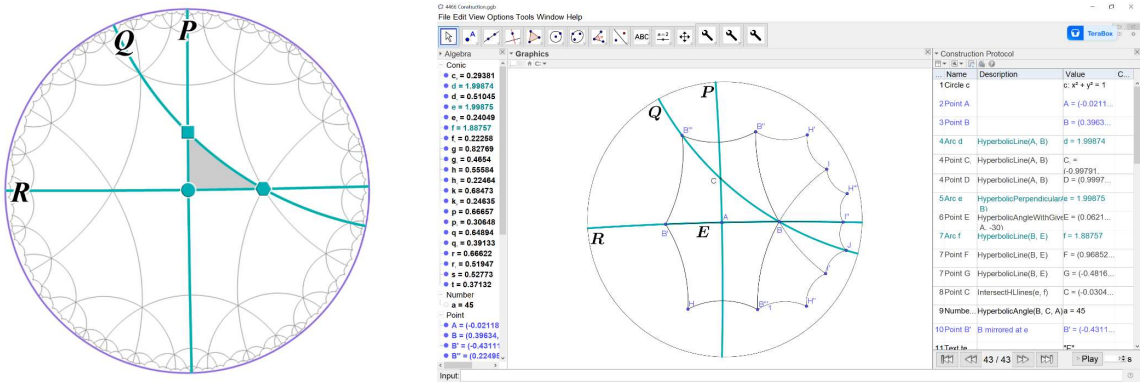
Figure 2.8: Acrylic tiles printed using a laser cutter showing: (a) the two types of vertex coronae; and (b) put together as the realized 2-isogonal tiling shown in Figure 2.7(b).

3. k-isogonal hyperbolic tilings from p^n tilings

Tilings of the hyperbolic plane ($\mathbb{H}^2$) having k transitivity classes of vertices remain scarce in the literature. In this section, we present the construction of a k-isogonal tiling from a well-known hyperbolic tiling, called the $[p^n]$ tiling.

A $[p^n]$ tiling of $\mathbb{H}^2$ is an isohedral tiling with p regular n-gons meeting at a vertex where p, n satisfy the inequality $1/p + 1/n < 1/2$. It has symmetry group G generated by three reflections P, Q and R with pairs of axes forming π/n, π/p and $\pi/2$ angles. To arrive at a k-isogonal tiling, the approach would be to consider a proper subgroup H of G that forms k transitivity of tiles in the $[p^n]$ tiling. Then a dualization procedure is carried out to form a k-isogonal tiling using elements of H.

Consider for example the $[6^4]$ tiling with symmetry group $G = \langle P, Q, R | P^2 = Q^2 = R^2 = (PQ)^4 = (QR)^6 = (PR)^2 = id \rangle$ generated by three reflections P, Q and R where P and Q intersects at a $\pi/4$ angle, Q and R intersects at a $\pi/6$ and angle, and P and R intersects at $\pi/2$ angle. The group G is denoted by $*642$ to indicate that the centers of 6-fold, 4-fold and 2-fold rotational symmetries lie on reflection axes as shown in Figure 3.1(a). The construction of this tiling is shown in the *GeoGebra* interface together with GeoGebra's Construction Protocol Panel in Figure 3.1(b).



(a) (b)

Figure 3.1: The $[6^4]$ tiling (a) shown in the Poincaré disk model of $\mathbb{H}^2$ together with the axes of reflections and centers of rotations of the generators of its symmetry group; and (b) its construction using *GeoGebra.*

We consider the subgroup $H = \langle Q, R, PQRQRQP,\ PQPRQRQRPQRPQP \rangle$ of G of index 10 generated by the reflections Q, R, and $PQRQRQP$ and the 90° rotation $PQPRQRQRPQRPQP$. This subgroup is denoted by $4 * 62$ in Conway notation indicating the presence of a 6-fold rotation and a 2-fold rotation lying on an axis of reflection as well as a 4-fold rotation not lying on an axis of reflection on the fundamental tile of H in Figure 3.2(a). There are three orbits of tiles and three orbits of edges in the $[6^4]$ tiling under the action of H. The representative tiles from each orbit are shown in Figure 3.2(b) and the representative edges from each orbit are given in Figure 3.2(e). Using H we obtain a 3-isogonal tiling. The details of the construction is presented in Figures 3.2(c)-(g).

In contrast with the method discussed in the previous section where symmetries in G are applied to the vertices, the construction of vertices in the representative tiles in this case is carried out such that there will be exactly one point inside every tile of $\mathcal{T}$ when the symmetries in H are applied to these vertices. In the construction of the new edges, there are also four possibilities to be used for an edge. The first one is an arc which is a geodesic connecting the two endpoints of the edge, that is, the shortest possible arc connecting the endpoints having two perpendicular reflection symmetries. This can be constructed using a "Hyperbolic Segment" tool. The other three possible edges are similarly constructed as in Figure 2.4.

In Figure 3.2(c), an appropriate choice for the location of V_1^* and V_2^* is to situate these points in the axes of Q and $PQRQRQP$, respectively while an appropriate choice for the location of V_3^* is in the incenter of tile T_3. Choosing any other point will result to some tile having two points in the interior upon application of symmetries in H. Once V_1^*, V_2^* and V_3^* are constructed, all other vertices are obtained by applying symmetries in H to these vertices. Since E_1 is fixed by the reflection symmetry R in H, E_1^* is constructed such that it is also fixed by this reflection. An edge of type shown in Figure 2.3(b) is used in this case. Since E_2 and E_3 are not fixed by any symmetry in H, E_2^* and E_3^* are constructed to have trivial symmetry such the one shown in Figure 2.3(d). Applying symmetries in H to these edges results to the skeleton of the 3-isogonal tiling $\mathcal{T}^*$ shown in Figure 3.2(g). Different colors are assigned to distinguish between two transitivity classes of vertices, three transitivity classes of edges and the three types of tiles – a 6-gon with six distinct reflection symmetries, a 6-gon with two reflection symmetries with perpendicular reflection axes and another 6-gon with a single reflection symmetry (Figure 3.2(h)).

Using various subgroups of the symmetry group G of the $[6^4]$ tiling and applying the symmetries in each subgroup (as observed in the construction of the hyperbolic 3-isogonal tiling) can give rise to other tilings with vertex transitivity properties.

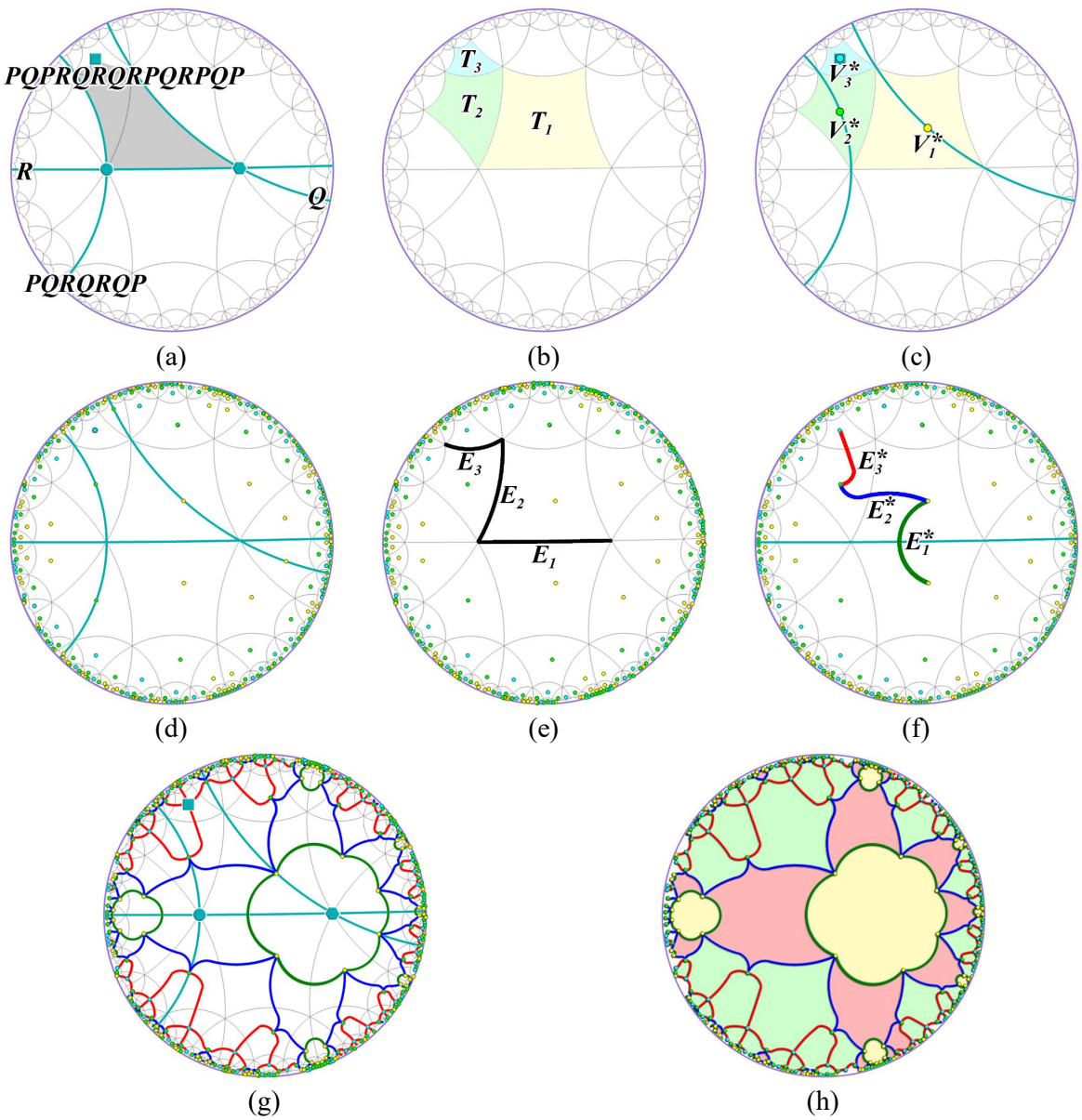


Figure 3.2: Construction of 3-isogonal tiling of $\mathbb{H}^2$ from the $[6^4]$ tiling: (a) $[6^4]$ tiling with axes of reflections and centers of rotations of elements of H on a fundamental tile of H; (b) representative tiles T_1, T_2, T_3 from each tile orbit of the $[6^4]$ tiling; (c) V_1^*, V_2^*, V_3^* constructed in the interiors of T_1, T_2, T_3, respectively; (d) applying symmetries of H to V_1^*, V_2^*,V_3^* give exactly one point in the interior of every tile of $\mathcal{T}$; (e) representative edges E_1, E_2, E_3 from each edge orbit of the $[6^4]$ tiling; (f) edges E_1^*, E_2^*, E_3^*; (g) skeleton of the 3-isogonal tiling after application of H to E_1^*, E_2^*, E_3^*; and (h) the 3-isogonal tiling.

The computer software Groups, Algorithms and Programs (GAP) [21] is used to arrive at the subgroup structure of hyperbolic symmetry groups. A sample GAP output of a list of subgroups of a given index of the symmetry group G of the $[6^4]$ tiling is provided in Figure 3.3.

```
Subgroups of Index 10

No. of Conjugacy Classes of Subgroups: 11

Conjugacy Class 1
  (1) Generators: [ P, Q, R*Q*P*R*Q*R^-1*Q^-1*P^-1*Q^-1*R^-1 ]
  (2) Generators: [ P, R*Q*R^-1, Q*P*R*Q*R^-1*Q^-1*P^-1*Q^-1 ]
  (3) Generators: [ Q*P*Q^-1, P*R*Q*R^-1*Q^-1*P^-1, Q*R*Q*R^-
  (4) Generators: [ Q*P*Q^-1, R*Q*R^-1*Q^-1, P*Q*R*Q*R^-1*Q^-1
*P^-1 ]
  (5) Generators: [ P, R*Q*R^-1*Q^-1, Q*P*Q*R*Q*R^-1*Q^-1*P^-1
*Q^-1 ]

Conjugacy Class 2
  (1) Generators: [ P, R, Q*P*Q^-1, Q*R*Q*P*Q*R*Q^-1*P^-1*Q^-1
*R^-1*Q^-1, Q*R*Q*P*R*Q*R^-1*P^-1*Q^-1*R^-1*Q^-1 ]
  (2) Generators: [ P, Q*P*Q^-1, Q*R*Q^-1, R*Q*P*Q*R*Q^-1*P^-1
*Q^-1*R^-1, R*Q*P*R*Q*R^-1*P^-1*Q^-1*R^-1 ]
  (3) Generators: [ P, R*Q*P*Q^-1*R^-1, R*Q*R*Q^-1*R^-1,
Q*P*Q*R*Q^-1*P^-1*Q^-1, Q*P*R*Q*R^-1*P^-1*Q^-1 ]
  (4) Generators: [ Q*P*Q^-1, P*Q*R*Q^-1*P^-1, P*R*Q*R^-1*P^-1,
R*Q*R*Q^-1*R^-1, Q*R*Q*P*Q^-1*R^-1*Q^-1 ]
  (5) Generators: [ Q*P*Q^-1, Q*R*Q^-1, R*Q*R^-1, P*R*Q*R*Q^-1
*R^-1*P^-1, P*Q*R*Q*P*Q^-1*R^-1*Q^-1*P^-1 ]
  (6) Generators: [ P*Q*P^-1, Q*R*Q^-1, R*Q*P*Q^-1*R^-1,
P*R*Q*R*Q^-1*R^-1*P^-1, (R*Q)^2*P*(Q^-1*R^-1)^2 ]
  (7) Generators: [ P, R, Q*R*Q*R^-1*Q^-1, Q*P*R*Q*R*Q^-1*R^-1
*P^-1*Q^-1, Q*P*Q*R*Q*P*Q^-1*R^-1*Q^-1*P^-1*Q^-1 ]
  (8) Generators: [ Q, P*Q*R*Q^-1*P^-1, R*Q*P*Q^-1*R^-1,
R*Q*R*Q^-1*R^-1, P*(R*Q)^2*P*(Q^-1*R^-1)^2*P^-1 ]
  (9) Generators: [ R, P*Q*P^-1, Q*R*Q*P*Q^-1*R^-1*Q^-1,
Q*P*R*Q*R*Q^-1*R^-1*P^-1*Q^-1,
  (Q*R)^2*Q*P*(Q^-1*R^-1)^2*Q^-1 ]
  (10) Generators: [ Q, R, P*Q*R*Q*P*Q^-1*R^-1*Q^-1*P^-1,
P*Q*P*R*Q*R*Q^-1*R^-1*P^-1*Q^-1*P^-1,
  P*(Q*R)^2*Q*P*(Q^-1*R^-1)^2*Q^-1*P^-1 ]
```

Figure 3.3: Portion of the GAP output of the list of subgroups of $G = \langle P, Q, R | P^2 = Q^2 = R^2 = (PQ)^4 = (QR)^6 = (PR)^2 = id \rangle \cong *642$ showing the subgroups of index 10.

4. k-isogonal tilings from j-uniform tilings ($j \leq k$)

The objective of this section is to illustrate the construction of a k-isogonal tiling from a j-uniform tiling, $j \leq k$.

Given a j-uniform tiling $\mathcal{T}$ with symmetry group G, a k-isogonal tiling $\mathcal{T}^*$ ($j \leq k$) may be constructed using a subgroup H of G satisfying the property that the vertices of $\mathcal{T}$ form k transitivity classes under the action of H. In the construction, the edges of $\mathcal{T}$ are replaced by new edges such that H becomes the symmetry group of the constructed tiling $\mathcal{T}^*$.

As an example, consider the 5-uniform tiling $\mathcal{T}$ with symmetry group G generated by reflections R, S and a 2-fold rotation T with axes of reflections and center of rotation shown in Figure 4.1(a). The symmetry group G is described as $2 * 22$ to denote the presence of a 2-fold rotational symmetry not lying on an axis of reflection and two other distinct 2-fold rotational symmetries lying on an axis of reflection in the fundamental tile of G. We show the construction of an 8-isogonal tiling $\mathcal{T}^*$ from $\mathcal{T}$.

The list of subgroups of G is then obtained by using the software GAP [21]. We consider the subgroup H of G of index 2 generated by the reflection S and 2-fold rotations T and RTR shown in Figure 4.1(b). The subgroup H is described as $22 *$ to denote the presence of two distinct 2-fold rotational symmetries not lying on an axis of reflection in the fundamental tile of H. Under the action of H, the vertices of $\mathcal{T}$ form eight transitivity classes with representatives $V_1, \ldots, V_8$ (Figure 4.1(c)). The edges of $\mathcal{T}$ form 21 transitivity classes with representatives $E_1, \ldots, E_{21}$ (Figure 4.1(d)).

The construction of an 8-isogonal tiling $\mathcal{T}^*$ is facilitated as follows: For each representative edge $E_i, i = 1, \dots, 21$ from each transitivity class of edges of $\mathcal{T}$ under H, a corresponding edge E_i^* is constructed. Figure 2.3 provides the possible edge types for E_i^*. For the construction, we choose an appropriate edge E_i^* and we position the edge such that the rotations or reflection symmetries, if any, of E_i that are found in H also become symmetries of E_i^*. For example, edge E_{14} has reflection symmetry $S \in H$ (Figure 4.1(d)) so we choose the edge type in Figure 2.3(b) as E_{14}^* and construct it appropriately such that it is also fixed by S (see Figure 4.1(e)). Similarly, E_2 is fixed by the 2-fold rotation $RTR \in H$ and thus, we use the edge E_2^* with the 2-fold rotational symmetry in Figure 2.3(c) and position it in such a way that it is also fixed by RTR.

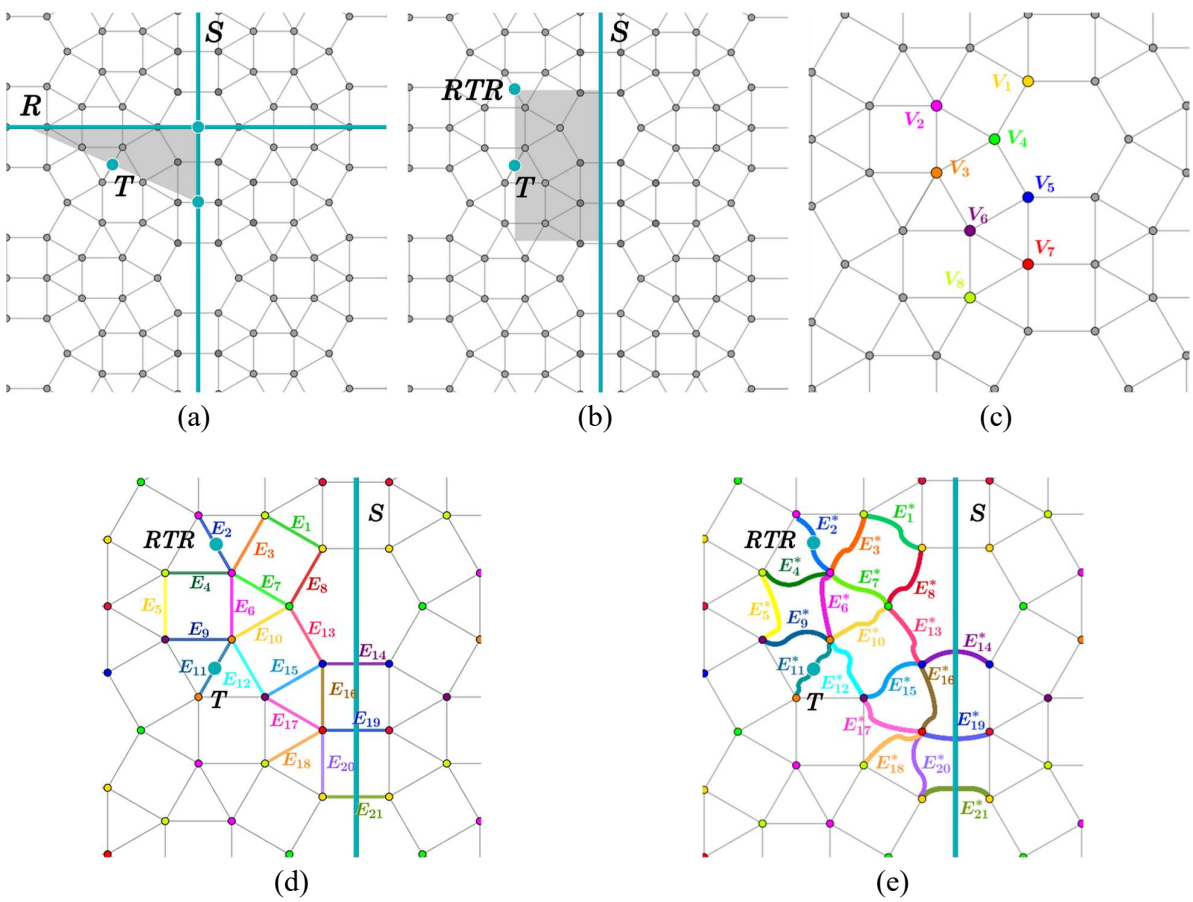


Figure 4.1: Construction of an 8-isogonal tiling of $\mathbb{E}^2$ from a 5-uniform tiling of $\mathbb{E}^2$: (a) 5-uniform tiling with axes of reflections R and S, center of 2-fold rotation T, and two other centers of rotations lying on an axis of reflection on a fundamental tile of G; (b) axis of reflection S and centers of 2-fold rotations T and RTR on a fundamental tile of H; (c) representative vertices $V_1, \dots, V_8$ from each transitivity class of vertices of $\mathcal{T}$ under H; (d) representative edges $E_1, \dots, E_{21}$ from each transitivity class of edges of $\mathcal{T}$ under H; and (e) newly constructed edges $E_1^*, \dots, E_{21}^*$ corresponding to $E_1, \dots, E_{21}$, respectively.

Each of the edges $E_1, \dots, E_{21}$ will then be replaced by the edges $E_1^*, \dots, E_{21}^*$ respectively (Figure 4.1(e)). One can observe that the 21 new edges $E_1^*, \dots, E_{21}^*$ are distinct under H. Applying the symmetries in H to these new representative edges, we obtain the tiling $\mathcal{T}^*$ in Figure 4.2 whose vertices are the vertices of $\mathcal{T}$, the edges are the union of the orbits of the E_i^*'s under H, and the tiles are the regions bounded by these edges. The symmetry group of $\mathcal{T}^*$ is H and since there are 8 transitivity classes of vertices under H, the tiling $\mathcal{T}^*$ is 8-isogonal.

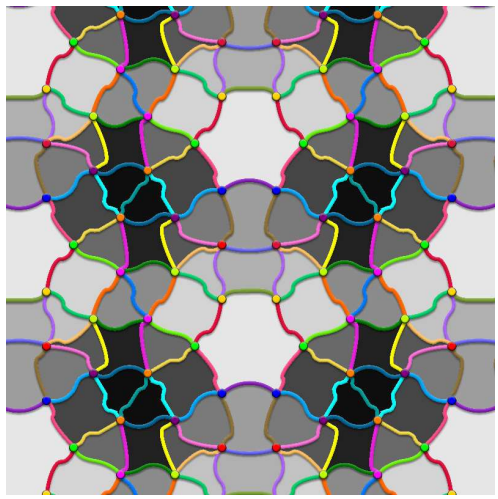


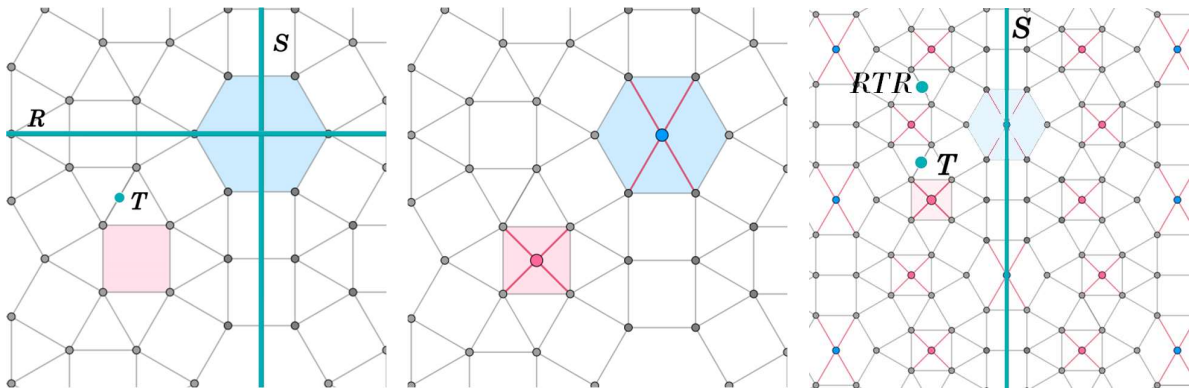
Figure 4.2: The 8-isogonal tiling $\mathcal{T}^*$ with symmetry group H.

In the previous example, we discussed how to construct a k-isogonal tiling using a subgroup H that forms k transitivity classes of vertices in the given j-uniform tiling $\mathcal{T}$. At this point, we discuss how we can extend this scenario to construct a $(k+r)$-isogonal tiling from the same j-uniform tiling $\mathcal{T}$ using the same subgroup H. We include in the method a process of dissection of the tiles of $\mathcal{T}$. If the tiles of $\mathcal{T}$ form s transitivity classes under the subgroup H, then we can take r tiles, representative from r distinct transitivity classes of tiles ($r \leq s$), and perform the dissection to these selected tiles such that r new vertices will be introduced. Applying the elements of H to these new vertices, together with their incident edges, we obtain another tiling $\mathcal{T}'$ having $j+r$ transitivity classes of vertices under H. Now, from the tiling $\mathcal{T}'$, we construct the $(k+r)$-isogonal tiling $\mathcal{T}'^*$ by applying the same method in constructing a k-isogonal tiling from a j-uniform tiling.

To illustrate, consider the 5-uniform tiling $\mathcal{T}$ described in the previous example. We consider the same subgroup H of the symmetry group G of $\mathcal{T}$ where the vertices of $\mathcal{T}$ form eight transitivity classes under H. Upon verification, we find that there are 12 transitivity classes of tiles under H in $\mathcal{T}$. It is possible to obtain more than eight transitivity classes of vertices by dissecting representative tiles coming from the 12 transitivity classes of tiles to produce more vertices. To obtain 10 transitivity classes of vertices for example, we take two tile representatives from two distinct transitivity classes, for example, a 6-gon (colored blue) and a 4-gon (colored pink) shown in Figure 4.3(a). These selected tiles are then dissected in such a way that new vertices and edges are formed (see Figure 4.3(b)). Applying the elements of H to these newly formed vertices and edges, we obtain the tiling $\mathcal{T}'$ with symmetry group H (see Figure 4.3(c)).

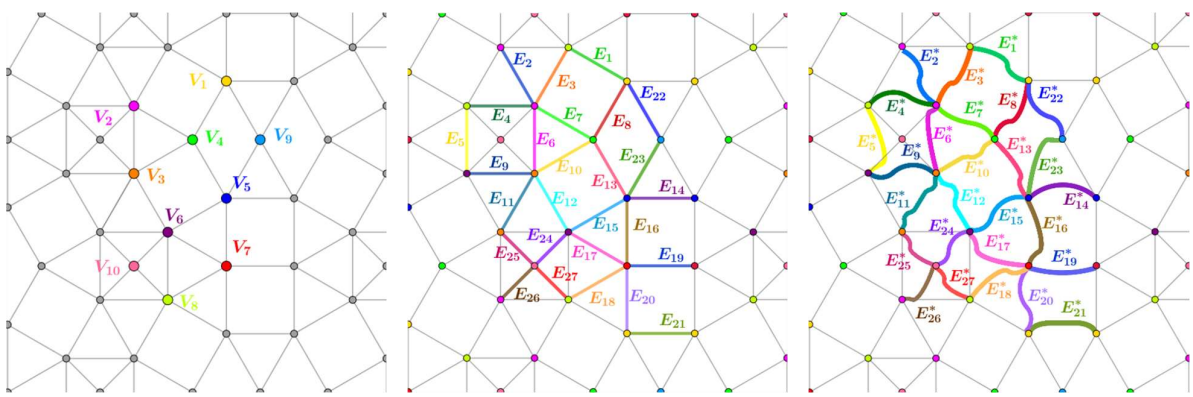
In the constructed tiling $\mathcal{T}'$, observe that under the subgroup $H = \langle S, T, RTR \rangle$ of G, the vertices form 10 transitivity classes with representatives $V_1, \dots, V_{10}$ (Figure 4.4(a)) and the edges form 27 transitivity classes with representatives $E_1, \dots, E_{27}$ (Figure 4.4(b)). From $\mathcal{T}'$, a 10-isogonal tiling $\mathcal{T}'^*$

is then constructed by introducing a new set of edges $E_1^*, \dots, E_{27}^*$ following the same criteria as discussed in the previous example. The edges $E_1, \dots, E_{27}$ of $\mathcal{T}'$ is then replaced by the edges $E_1^*, \dots, E_{27}^*$, respectively to form a new set of representative edges as shown in Figure 4.4(c). Applying the symmetries of H to these new representative edges, we obtain the 10-isogonal tiling $\mathcal{T}'^*$ in Figure 4.5 with symmetry group H.



(a) (b) (c)

Figure 4.3: (a) Two tiles of $\mathcal{T}$, a 6-gon (colored blue) and a 4-gon (colored pink) from two distinct transitivity classes of tiles under H; (b) the vertices and edges formed by dissection of these two tiles; and (c) the tiling $\mathcal{T}'$ obtained by applying H to the vertices and edges shown in (b).



(a) (b) (c)

Figure 4.4: Construction of a 10-isogonal tiling of $\mathbb{E}^2$ from the tiling $\mathcal{T}'$: (a) representative vertices $V_1, \dots, V_{10}$ from each transitivity class of vertices of $\mathcal{T}'$ under H; (b) representative edges $E_1, \dots, E_{27}$ from each transitivity class of edges of $\mathcal{T}'$ under H; and (c) the newly constructed edges $E_1^*, \dots, E_{27}^*$ corresponding to $E_1, \dots, E_{27}$.

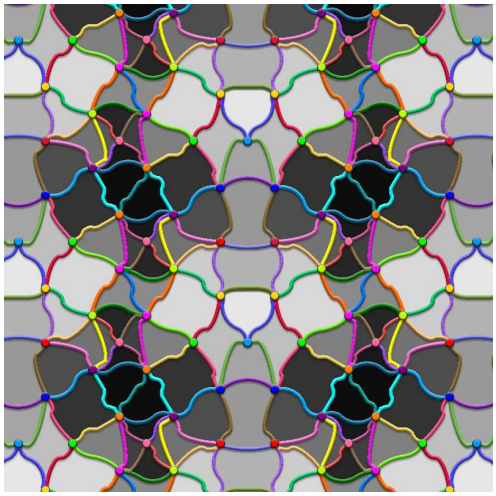


Figure 4.5: The 10-isogonal tiling $\mathcal{T}'^{*}$ with symmetry group H.

5. Conclusion and Outlook

In this paper we present methods to arrive at k-isogonal tilings applied to existing tilings with tile or vertex transitivity properties. In Sections 2 and 3, we present an approach to create a k-isogonal tiling from a starting tiling where the tiles form k orbits under a group of symmetries, and then proceed through a dualization procedure. In Section 4, the method involves starting with a tiling where the vertices form j orbits under a group of symmetries (such as a j- uniform tiling) and form the k- isogonal tiling ($j \leq k$). Herein, edges of the starting tiling are replaced with new edges. It is also possible to create more transitivity classes of vertices by a process of dissection of the tiles of the original tiling.

The approach presented in Section 2 involves applying the elements of the symmetry group of the starting tiling, and the other methods in Sections 3 and 4 involve applying elements of a chosen subgroup of the symmetry group of the starting tiling. Understanding the group and subgroup of symmetries of the starting tiling is key to arriving at tilings with new vertex, edge and tile transitivity properties. The use of a digital geometry software is necessary to arrive at these tilings. A computer software like GAP helps in determining the subgroup structure of the symmetry groups.

We included a short discussion in Section 2 on the physical realization of the k-isogonal tilings outputs from *GeoGebra* using laser printing technology. These manipulatives have the potential to be effective instructional materials.

The outlook on the study of k-isogonal tilings for larger values of k will definitely require more technological tools. The creation of computer programs involving the algorithms provided in this paper will help towards arriving at a complete enumeration of these tilings.

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Development of a Question Distribution and Answer Collection System for Mathematics Classes Using Only One Line of Text

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Abstract

KeTCindy and KeTCindyJS are macro packages we developed for Cinderella, a dynamic geometry system. They allow the interactive creation of files of TeX figures and HTML. Since the pandemic began, we have been developing an LMS, KeTLMS, that can be used in online or blended classes. In mathematics classes, a major issue is how to communicate mathematical expressions. For teachers, it is easy to write them in TeX, but not so easy for students. They often take screenshots of their notebooks and submit them, which makes it difficult for teachers to grade them. In KeTLMS, questions and answers including mathematical formulas are exchanged as follows:

1. *We have defined KeTMath rules, which make TeX math expressions easier. For example,*
   `fr(sin(x)+cos(x),x)` means `$\dfrac{\sin x+\cos x}{x}$`.
2. *These formulas are instantly displayed on the HTML screen as two-dimensional formulas.*
3. *KeTMath keyboard is placed on the HTML screen.*
4. *Teachers create a text file of questions using KeTMath rules, and generate HTML of the questions using the Cinderella file named 'toolketmathE.cdy'. They distribute it via a platform such as Google Classroom.*
5. *Students answer the questions using KeTMath rules and submit them as a one-line text generated with the button 'Rec'.*
6. *Teachers collect students' answers, grade them, and communicate the results to students.*

The text files are small in size and easy to process. For example, teachers can also easily create CSV files for table formats. KeTLMS can handle questions of various types, including not only standard questions that ask students to answer mathematical expressions, but also questions to prove an equation and post-class surveys.

1 Introduction

In 2020, when the coronavirus pandemic began, online classes became mainstream in many schools in Japan. Mathematics classes were no exception, but teachers faced the big issue of how to exchange mathematical formulas. During the lecture part, they can explain their lectures using slides or videos of mathematical formulas on the blackboard and distribute exercise files created using TEX. However, it is difficult for teachers and students to exchange assignments, especially submitting answers for students. They often send photos of their handwritten notebooks, which makes it very tedious for teachers to collect and grade their answers. Therefore, we decided to create a system using KETCindyJS, which we have been developing, that would allow easy exchange of mathematical formulas between teachers and students. Here, we briefly explain KETCindyJS and its development history. We first released KETpic[1] in 2006. This is a package for mathematical software such as Maple, R, and C; currently, R is used. For example, write scripts in R on the left below. Then the Tpic codes on the right will be generated.

Script 1 Scripts of KETpic and codes of Tpic

```
setwd('...')
source(ketpic.r')
  # Loading of KeTPic package
Ketinit()
Setwindow(c(-2,2), c(-2,2))
pA=c(-2,2);pB=c(1,2);pC=c(2,-2)
sg1=Listplot(c(pA,pB,pC,pA))
Openfile('fig/example1')
  Drwline(sg1)
  Letter(pA,"w","A")
  Letter(pB,"n","B")
  Letter(pC,"e","C")
Closefile("0")
```

```
{\unitlength=1cm%
\begin{picture}(5,5)(-2.5,-2.5)%
\special{pn 8}%
\special{pa -787 787}\special{pa 394 -787}\special{pa 787 787}%
\special{pa -787 787}\special{fp}%
\settowidth{\Width}{A}\setlength{\Width}{-1\Width}%
\settoheight{\Height}{A}\settodepth{\Depth}{A}%
\setlength{\Height}{-0.5\Height}\setlength{\Depth}{0.5\Depth}%
\addtolength{\Height}{\Depth}\put(-2.050,-2.000){\hspace*{\Width}
\raisebox{\Height}{A}}%
\settowidth{\Width}{B}\setlength{\Width}{-0.5\Width}%
\settoheight{\Height}{B}\settodepth{\Depth}{B}%
\setlength{\Height}{\Depth}%
\put(1.000,2.050){\hspace*{\Width}\raisebox{\Height}{B}}%
\settowidth{\Width}{C}%
\setlength{\Width}{0\Width}%
\settoheight{\Height}{C}\settodepth{\Depth}{C}%
\setlength{\Height}{-0.5\Height}\setlength{\Depth}{0.5\Depth}%
\addtolength{\Height}{\Depth}%
\put(2.050,-2.000){\hspace*{\Width}\raisebox{\Height}{C}}%
\end{picture}}%
```

We can insert the figure into a TEX document by

`\input{fig/example1}`.

Note that the command is not `\includegraphics`, because the file `example1.tex` is a text file that contains drawing codes.

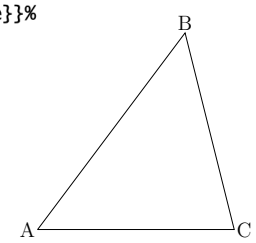

Fig.1 Generated Figure

In 2014, we had a research meeting with Professor Kortenkamp, one of the lead developers of a dynamic geometry software Cinderella2[2]([1]). One can download Cinderella for free from `https://beta.cinderella.de`. Cinderella has a general-purpose programming language, CindyScript. Therefore, we have created and released a macro package for CindyScript called KETCindy[3]([2],[3]). We have uploaded KETCindy at `https://github.com/ketpic/ketcindy`.

[1]The name comes from Kisarazu(our city) educational Tpic. Tpic is a graphics preprocessor for TEX.
[2]In this paper, we refer to Cinderella2 simply as Cinderella or Cindy.
[3]KETCindy supports pict2e and Tikz in addition to Tpic.

To run KETCindy, one also need R and TEX. Cinderella supports Java, so batch processing is possible. This feature allows us to run Maxima and GCC from KETCindy if one install them. The drawing procedure is as follows:

1. Write scripts for drawing on the CindyScript screen. Many of them are of KETCindy. Moreover, interactive Cinderella elements are available.
2. Push the button 'Figures' on the Cindy screen, and then the following batch processing will be performed.
   (1) To create an R file to generate two TEX files, one for the drawing codes and one for the main TEX file, and execute R.
   (2) To compile the main file with TEX, then a PDF file will be generated.
   (3) To display the PDF.

In 2016, the development team led by Professor Gebert of the Technical University of Munich, another main developer of Cinderella, released CindyJS([4]), a framework to create interactive web content almost compatible with Cinderella. Therefore, we developed KETCindyJS([5]) to create HTML using KETCindy and CindyJS. The difference between it and KETCindy is that the final result is an HTML or a PDF, but the scripts are almost identical. However, KeTCindyJS will use more of the Cinderella geometric elements to create interactive and dynamic HTML, such as:

1. Freely movable geometric points
2. Text boxes for running a script sequence
3. Boxes for entering editable text
4. Buttons for animation

The two figures on the right are the Cindy screen and HTML for one example, and the script is as follows:

Script 2 Scripts of KETCindy

```
Ketinit();
Plotdata("0","sin(x)","x",["dr,2"]);
Putoncurve("A","gr0"); //Put 'A' on the 'gr0'
if(addflg==1, //Button 'Add' executes addflg=1
  tmp=Textedit(1,"");
  if(length(tmp)>0,
    funL=append(funL,tmp); //list of functions
  );
  addflg=0;
);
forall(1..(length(funL)),n,
  Plotdata(text(n),funL_n,"x");
);
Windispg();
```

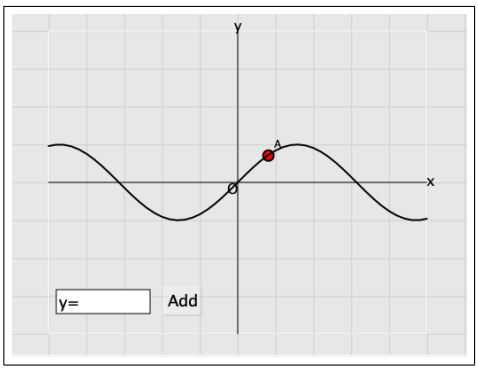

Fig.2 Cindy Screen of the example

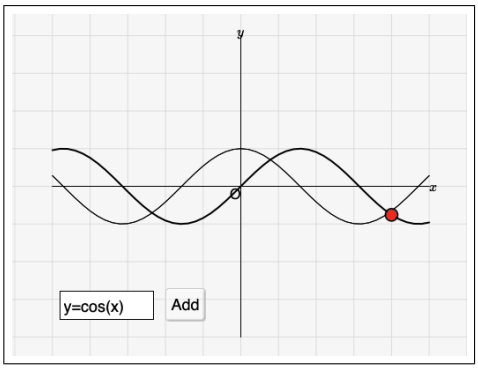

Fig.3 HTML of the example

To create the HTML,

1. select the menu 'Export as HTML' in the Cindy menu,
2. push the button 'KeTJS' on the Cindy screen.

Our systems, KeTMath and KeTLMS, which allow students who are not familiar with TEX to easily exchange mathematical formulas, have been developed using KETCindyJS. In the following, Section 2 will explain KeTMath, and Section 3 will explain KeTLMS.

2 Development of KeTMath

We first established rules for expressing mathematical expressions called KeTMath rules in order for teachers and students to exchange mathematical expressions in a single line of text. The rules are based on TEX, but are simpler as shown below.

Table 1 KeTMath rules

KeTMath	TEX	Examples	KeTMath	TEX	Examples
fr(a,b)	$\frac{a}{b}$	fr(1,2)+fr(3,4) $\Rightarrow \frac{1}{2}+\frac{3}{4}$	adot()b a(cross)b	$a{\cdot}b$ $a{\times}b$	(adot()b)^(2)+(a(cross)b)^(2)
a^(b)	a^b	fr(1-r^(n),1-r) $\Rightarrow \frac{1-r^n}{1-r}$	sq(a) sq(n,a)	$\sqrt{a}$ $\sqrt[n]{a}$	sq(sq(3,a))=sq(6,a)
sin(x)	$\sin x$	fr(sin(x),cos(x)) $\Rightarrow \frac{\sin x}{\cos x}$	sin(2,x) sin(x)^(2)	$\sin^2 x$	1+tan(2,x)=fr(1,cos(2,x))
pi	π	S=pir^(2) $\Rightarrow S=\pi r^2$	(deg)	$\circ$	tan(30(deg)+45(deg))
log(x)	$\log x$	log(fr(x,y))=log(x)-log(y)	log(a,x)	$\log_a x$	log(10,x)=fr(log(10),log(x))
10@cm@	10cm	When x=fr(3,2),2x-3=0	br(x,a,b)	$\Big[x\Big]_a^b$	br(-cos(x),0,pi)=2
int(a,b, x,dx)	$\int_a^b xdx$	int(0,pi,sin(x),dx)=2	diff(y,x) par(z,x)	$\frac{dy}{dx}$ $\frac{\partial z}{\partial x}$	dz=par(z,x)dx+par(z,y)dy
lim(x,a, f(x)	$\lim_{x\to a}(f(x))$	lim(x,0,fr(sin(x),x))	sum(k=1, n,k)	$\sum_{k=1}^{n} k$	sum(k=1,n,k^(2))
mat(a,b; c,d)	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	mat(1,2,3;4,5,6)	case(a,b; n=1,n=2)	$\begin{cases} a & n=1 \\ b & n=2 \end{cases}$	f(x)=case(0,x<0;x,x(geq)0)
det(a,b; c,d)	$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$	det(a-t,b;c,d-t)	dot() dot(3)	$\cdot$ $\cdots$	a_(1),a_(2),dot(3),a_(n)
(pm) (mp)	$\pm\ \mp$	x=fr(1(pm)sq(3),2)	(leq),(geq) (neq)	$\leq\ \geq\ \neq$	f(x)=fr(1,x)(sp)(x(neq)0)
a_(n)	a_n	fr((a_(1)+a_(n))n,2)	{\alpha} {\beta}	$\alpha\ \beta$	cos({\beta})+isin({\beta})

We added three functions `Totexform`, `Tocindyform` and `Tomaxform` to KETCindy library. These functions convert KetMath formulas to TEX, Cinderella and Maxima formulas respectively as follows.

```
kstr="sq(x)sin(fr(pix,3))";
Totexform(kstr) ⟹ \sqrt{x}\sin (\displaystyle\frac{{\pi}x}{3})
Tocindyform(kstr)⟹ sqrt(x)*sin((pi*x)/(3))
Tomaxform(kstr)⟹ sqrt(x)*sin(((%pi*x)/(3)))
```

CindyJS supports KaTeX, which displays TEX formulas instantly. Therefore, for the above example, one can display the TEX formulas in KETCindyJS by writing the following:

```
Expr(pos,"e",Totexform(kstr));
```

$$\Longrightarrow \sqrt{x}\sin(\frac{\pi x}{3})$$

On the other hand, `Tocindyform` is used when drawing graphs in KETCindy(JS).

```
Plotdata("1",Tocindyform(kstr),"x=[0,10]");
```

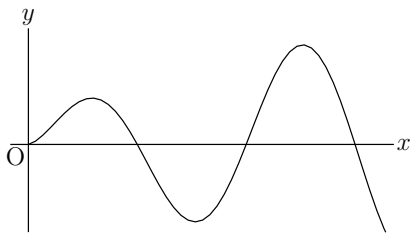

Fig. 4 Figure by KETCindy

The other, `Tomaxform`, is used for Maxima.

```
Mxfun("1","diff",[Tomaxform(kstr),"x"]);
⟹sin((%pi*x)/3)/(2*sqrt(x))+(%pi*sqrt(x)*cos((%pi*x)/3))/3
```

'ketmath.html' is an app developed with KETCindyJS that allows teachers and students to view documents containing mathematical formulas in TEX format when exchanging them.

1. Teachers or students make a question or an answer accoding to KeTMath rules.
2. They copy and paste the document into the input box in 'ketmath.html'.
3. The two-dimensional TEX formula will then be displayed, so they can check whether it is the same as what they intended.

Initially, as shown in Figure 5, there was only an input window for KetMath formulas and a display space for two-dimensional formulas.

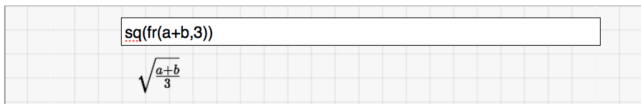

Fig.5 Original version of 'ketmath.html'

We then improved it by adding keys consisting of ASCII characters and displaying TEX formulas in the input box so that they could be copied.

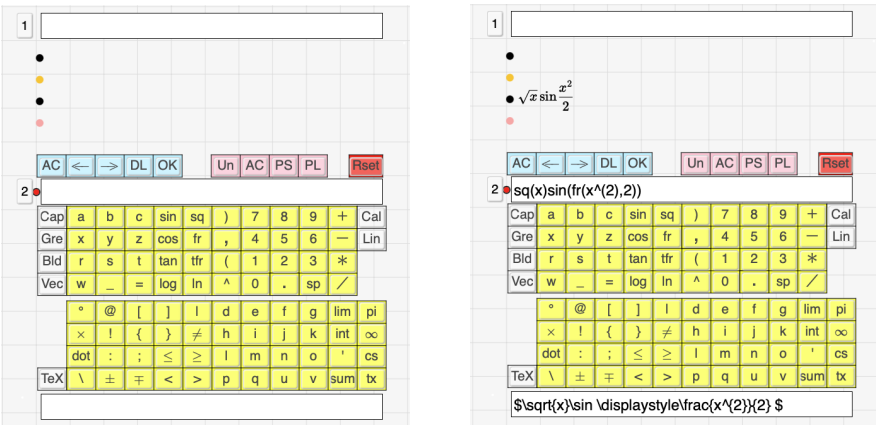

Fig.6 Current version of KeTMath.html

'ketmath.html' can be used by both teachers and students to exchange mathematical expressions. Teachers often need to check whether the KeTMath formulas in their questions are correct. In such cases, it is useful and indispensable.
However, when students submit their answers, they need to (1) check the questions sent to them using 'ketmath.html', (2) write their answers in KeTMath format, and (3) check again using 'ketmath.html' to see if the answers are correct before submitting them. These tasks were somewhat cumbersome for students, so some of them submitted their answers without using 'ketmath.html', which resulted in them sending answers in the incorrect format. Therefore, we developed a system called KeTLMS that integrates the series of tasks of teachers creating and distributing questions, students receiving them, submitting answers, and teachers collecting and grading the answers. The next section explains this system.

3 Development of KeTLMS

KeTLMS is a system based on KeTMath that exchanges questions, answers, and submissions written in KeTMath rules through a single HTML file, e.g. 'kettaskv001E.html'.

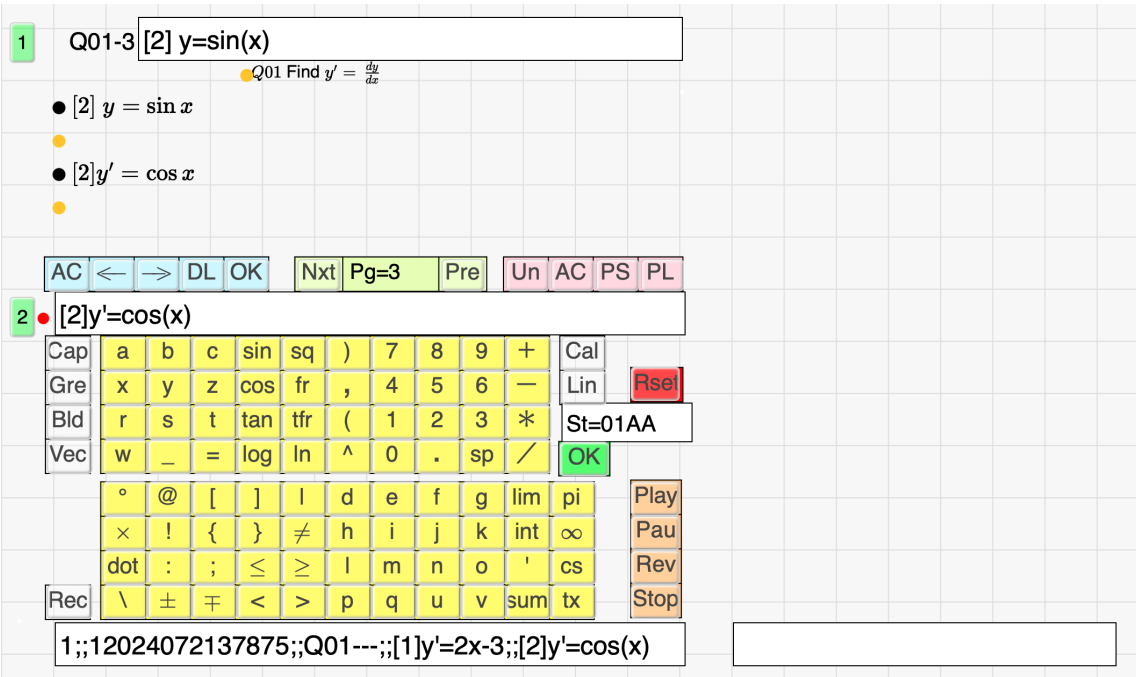

Fig.7 'kettaskv001E.html' screen when submitting answers

Here, column $\boxed{1}$ is for displaying the question, column $\boxed{2}$ is for students to write their answers, and the bottom column is where all of the student's answers with submission information will be entered when they push $\boxed{\text{Rec}}$. Note that all column data is a single line of text containing KeTMath formulas. The space on the right side is for embedding images and interactive figures using CindyScript, as described later. We have uploaded the KeTLMS files to the following page on Github:

`https://github.com/s-takato/ketmath`,

so the ZIP file is available for free download by anyone. In the unzipped folder, one can find the subfolder `subjectE01`. We will explain the main files and folders in it.

- `kettaskorgvE.html`
  This file is the template for the question HTML 'kettaskv.html' that will be distributed to students. It only loads files in `ketcindyjs` and does not require any other files, so that students can open it with just a web browser.
- `data`
  This folder contains 'student2024.txt' and 'question001-1.txt'.

  01AA // AA is the student's ID.
  02BB
  03CC
  ...

```
Q    //The number is added from the file name.
Differentiate
[1] y=sin(2x)
[2] y=fr(e^(x),x)
Sheet
[1]y'= ::5
[2]y'= ::5
Ans
[1] 2cos(2x)
[2] fr(e^x,x)-fr(e^x,x^2)
//Add blank line for the next question
```

  For more information about the question file, refer to 'indexketlmsquesE.html'.[4]
- `toolketmathE.cdy`
  It performs various operations on KeTLMS. To run this file, only download Cinderella from `https://beta.cinderella.de`. Install it and double-click the `cdy` file.

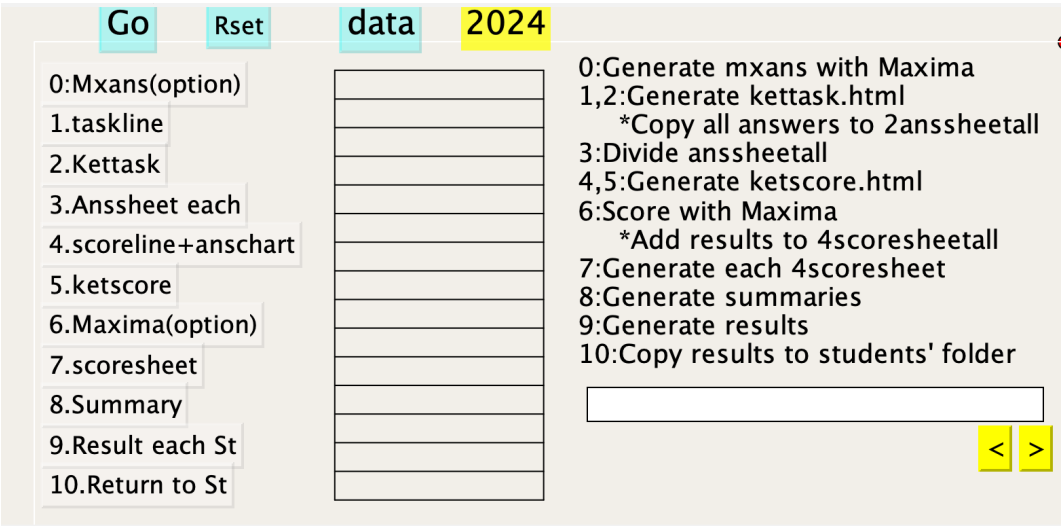

Fig.8 Screen of 'toolketmathE.cdy'

We will explain how 'toolketmathE.cdy' is used in the following subsection.

- `toolembed.cdy`
  It is used to embed images or scripts in the original kettask.html. Use Cinderella installed as above. We will explain this in the next session.

3.1 Use of toolketmathE.cdy

After saving the question file, launch 'toolketmathE.cdy'.

[4]We have posted an instructional video on YouTube on how to create kettaskfiles([6]).

The HTML requires `ketcindyjs` library folder in the same folder, so the link should be distributed instead of the file itself. We use Github Pages as the web server for this. After uploading the file and the folder, we distribute the URL to students.
The important thing is that the questions and answers are a single line of text, with no character limit, and the platform can be anything. We usually use Google Classroom(hereafter abbreviated as GC). The image below shows the GC question screen, and the one distributed to students also has an answer column. Teachers only need to enter the question title and URL. The size of the data to be uploaded is small, so if one create only the HTML before the class, teachers can send it at the appropriate time during the class. Most students use smartphones, and they can immediately receive and start answering questions.

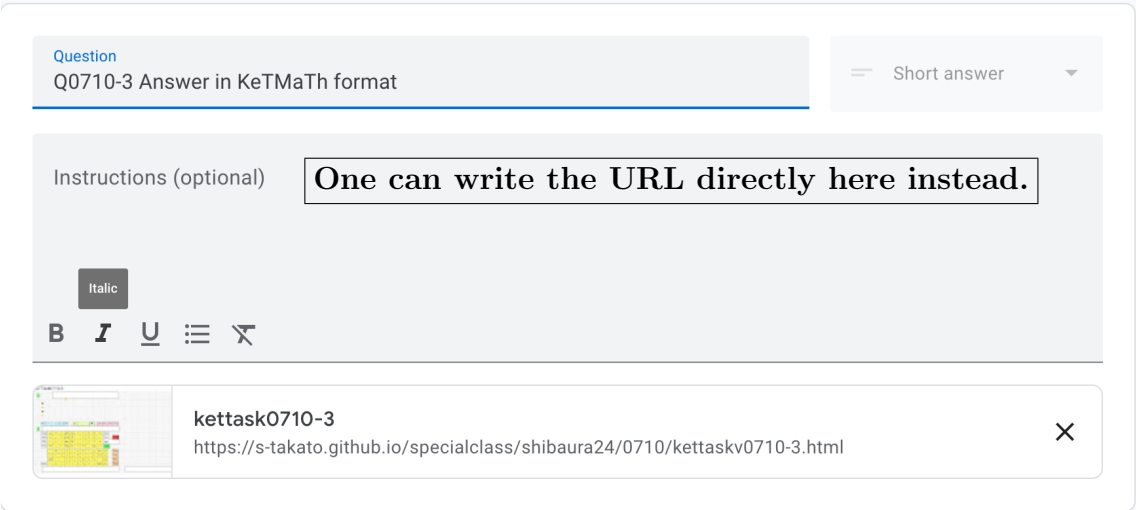

Fig.9 GC question screen

The answers submitted by students are displayed in real-time on the GC, so teachers can view them and learn about the progress and content of the answers. In addition, students' answers can be collected by simply copying them onto a pre-prepared answer sheet 'anssheetall.txt' in the folder `data`. Teachers sometimes ask multiple questions in a class, and even in this case, the results can be easily collected.
To split the lines in 'anssheetall.txt' into separate files for each question, push 3.anssheet each.

After that, push

4. scoreline+anschart

to create a CSV file of the answers.

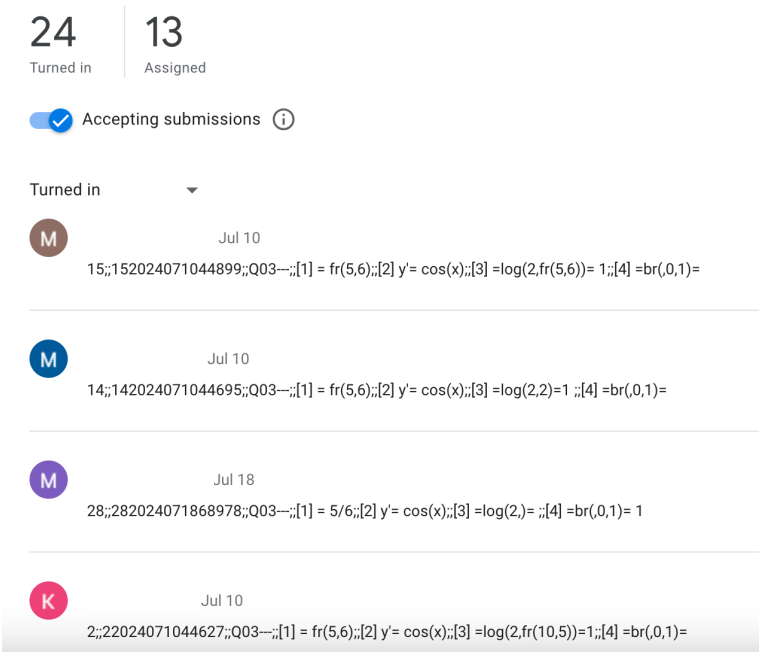

Fig.10 GC answers screen

The following Figure 11 shows a part of ‘anschart.csv’. The first line shows the correct answer. Figure 12 shows the answer displayed in two dimensions in TEX. This file is used to make the meaning of the formulas explicit when grading.

CA	2024	710		Q03---	[1]= fr(5:6)	2	[2]y'= cos(x)	2	[3]=log(2:fr(10:5))=log(2:2) =1		2
1	0001AC			Q03---	[1]na	0	[2]na	0	[3]na		0
2	0005AH	710	12:23:47	Q03---	[1]= fr(5:6)		[2]y'= cos(x)		[3]=log(2:fr(10:5))=1		
3	0013AH	710	12:29:54	Q03---	[1]= fr(5:6)		[2]y'= cos(x)		[3]=log(2:fr(10:5))=log(2:2) =1		
4	0028AH	710	12:23:8	Q03---	[1]= fr(5:6)		[2]y'= cos(x)		[3]=log(2:fr(10:5))= 1		
5	0034AH	710	12:25:47	Q03---	[1]= fr(5:6)		[2]y'= cos(x)		[3]=log(2:fr(10:5))=1		
6	0047AH	710	12:27:20	Q03---	[1]= fr(5:6)		[2]y'= cos(x)		[3]=log(2:2)=1		

Fig.11 ‘anschart.csv’

CA	2024	Q03—	$[1] = \frac{5}{6}$	2	$[2]y' = \cos x$	2	$[3] = \log_2(\frac{10}{5}) = \log_2 2 = 1$		2
1	0001AC	Q03—	$[1]na$	0	$[2]na$	0	$[3]na$		0
2	0005AH	Q03—	$[1] = \frac{5}{6}$		$[2]y' = \cos x$		$[3] = \log_2(\frac{10}{5}) = 1$		
3	0013AH	Q03—	$[1] = \frac{5}{6}$		$[2]y' = \cos x$		$[3] = \log_2(\frac{10}{5}) = \log_2 2 = 1$		
4	0028AH	Q03—	$[1] = \frac{5}{6}$		$[2]y' = \cos x$		$[3] = \log_2(\frac{10}{5}) = 1$		
5	0034AH	Q03—	$[1] = \frac{5}{6}$		$[2]y' = \cos x$		$[3] = \log_2(\frac{10}{5}) = 1$		
6	0047AH	Q03—	$[1] = \frac{5}{}$		$[2]y' = \cos x$		$[3] = \log_2 2 = 1$		

Fig.12 Anschart with TEX

Although it is possible to grade using Maxima, there are some issues with responding to questions in various formats, partial marks, error handling, etc. It is more reliable and smoother to grade while looking at ‘anschart’ such as Figure 11 and Figure 12.

The following Figure 13 shows a list of grades when a final exam was conducted in a small class using KeTLMS. This was created using 8.Summary.

			Q01		Q02		Q03		Q04		Q05		Q06		Q07			Q08		Sum	%
			[1]	[2]	[1]	[2]	[1]	[2]	[1]	[2]	[1]	[2]	[1]	[2]	[1]	[2]	[3]	[1]	[2]		
1	01TI	OK=061715:18:6	6	6	6	6	3	6	6	6	6	0	6	6	4	6	0	6	0	79	79
2	02KZ	OK=061715:23:23	6	6	6	6	3	6	6	6	6	6	6	6	4	6	0	6	6	91	91
3	03SK	Not=1	6	6	6	3	6	6	6	6	6	0			0	3	3	0	6	63	72
4	04TK	OK=061715:55:51	6	6	6	3	6	0	6	6	3	6	0	6	4	6	3	6	6	79	79
5	05TK	OK=061714:59:25	6	6	6	6	3	6	6	6	6	6	6	6	4	6	0	6	0	85	85
6	06IS	OK=061715:11:7	6	6	6	3	6	6	6	6	6	6	0	3	4	6	0	6	6	82	82
7	07AT	Not=2	6	6					6	6	6	0	6	6	4	6	6	6	3	67	67
8	08W	OK=061714:47:19	6	6	6	3	6	6	6	6	6	6	6	6	4	6	3	6	6	88	88
9	09YT	OK=061715:13:19	6	6	6	3	6	6	6	6	6	6	6	6	4	6	0	6	6	85	85
10	10YH	OK=061714:47:29	6	6	6	3	6	6	6	6	6	0	6	6	4	6	3	0	0	70	70
11	11YY	Not=1	6	6	6	6	3	6	6	6	6	6			4	6	3	6	0	70	80
																				Avg=	79.82

Fig.13 List of student scores

Menu 9. Result each St is for creating report cards to return the results to students.

```
Result of 0617ex-1
1 01TI 617 14:30:10
Q01 Transform it into the form a(x+b)^2+c.
[1] y=x^2+2x+2
  Correct answer  (x+1)^2+1
  Your answer  y= (x+1)^(2)+1
```

```
Your score  6
...
```

10.Return to St is for actually returning them. We create a folder for each student in our Dropbox and provide the students with a link to the folder in advance.

4 Embedding of scripts or images to KeTTask

KeTTask HTML files are useful for communicating mathematical formulas in a single line of text between teachers and students. Still, it would be more effective to include images and interactive scripts. For this reason, we developed 'toolembed.cdy'. The next figures are its screens.

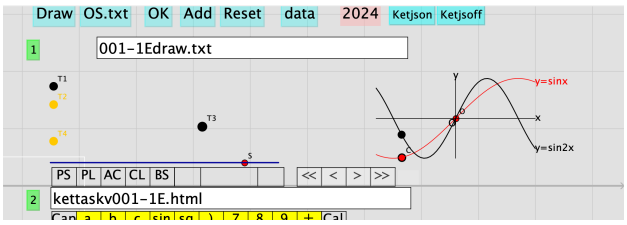

Fig.14 'toolembed' screen(Scripts)

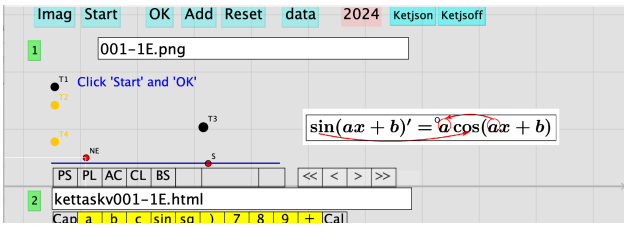

Fig.15 'toolembed' screen(PNG)

Figure 14 is a screen for embedding scripts, and the steps are as follows:

1. Create a KeTTask file (we'll call it 'kettaskv001-1E.html') as explained in Chapter 3.
2. Create '001-1Edraw.txt' in the folder `embeddata` in `subjectE0`(See indexketlmsE' in `subjectE01`).
3. Change the name of the second button from Start to OS.txt. Then, a figure created by the scripts will appear, and one can change its position and size by moving O and S.
4. Push OK ⇒ Embed, then 'kettaskv001-1Ed.html' will be generated.

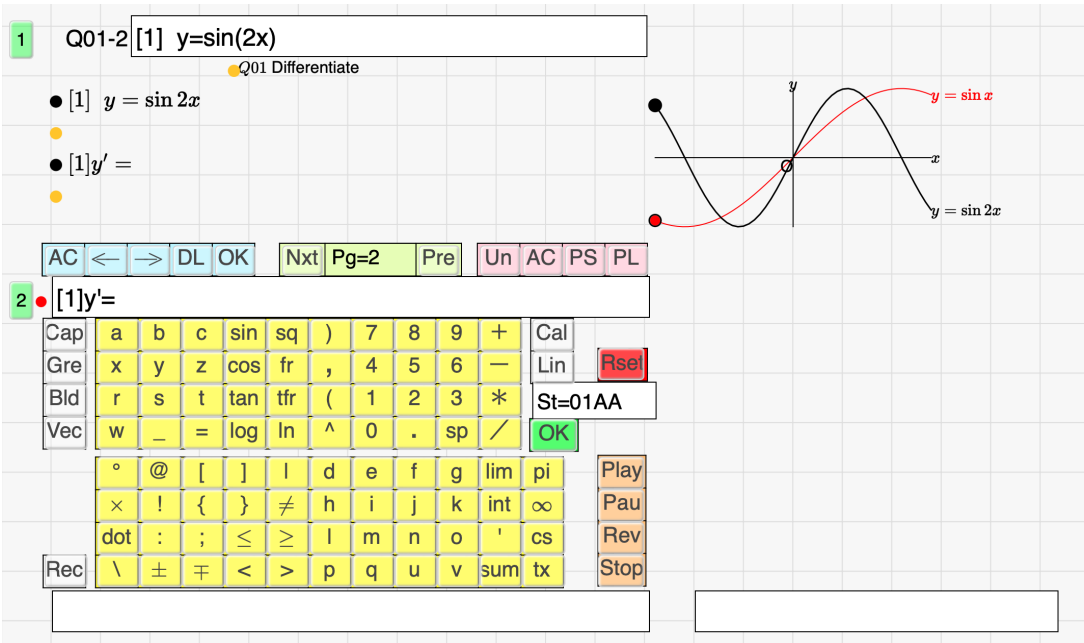

Fig.16 'kettask001-1Ed.html' with interactive figures

Figure 15 is a screen for embedding PNG and the steps are as follows:

1. Put a PNG file in folder `embeddata`.
2. Change the name of the left button from Draw to Imag.
3. From the Cinderella top menu, select 'File/MediaBrowser' and add the PNG file. Select 'Export as html'.
4. Change Start ⇒ png.txt ⇒ OK ⇒ OS.txt. Then, the PNG figure will appear, and one can change its position and size by moving T3 and S.
5. Push OK ⇒ Embed, then 'kettaskv001-1Ei.html' will be generated.

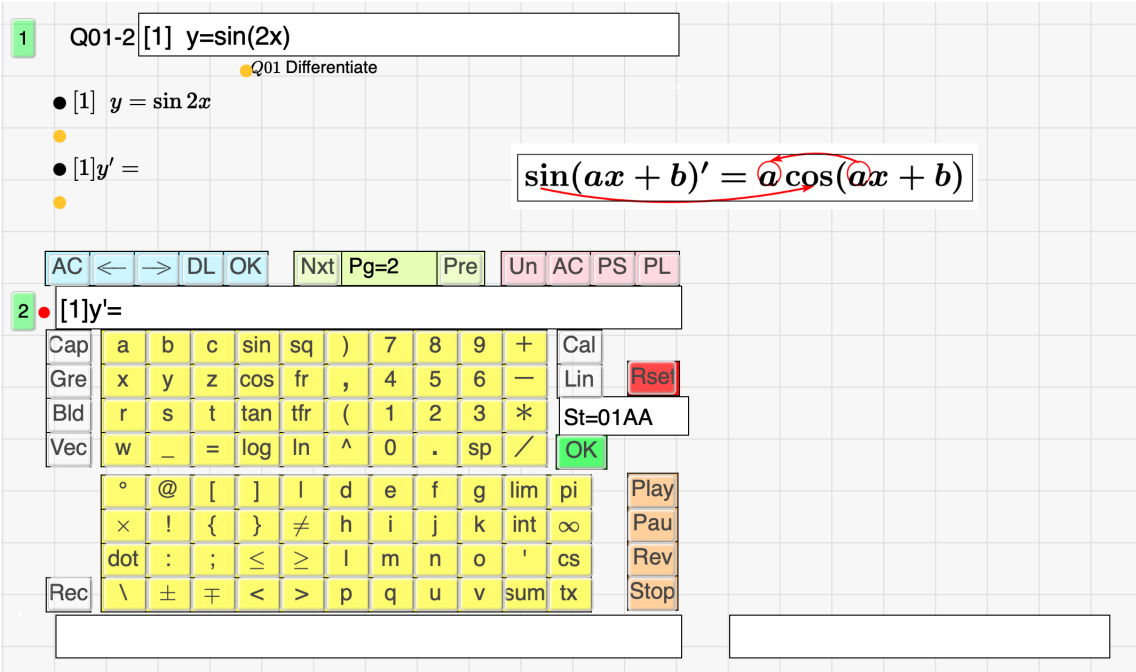

Fig.17 'kettaskv001-1Ei.html' with PNG figure

5 Acknowledgements

The authors would like to thank Professor J.Gebert and Professor U.Kortenkamp for developing and providing Cinderella2 and CindyJS. We would also like to express our deep gratitude to Professor Wei-Chi Yang for inviting us to ATCM2024.

6 Conclusion

KeTLMS helps teachers and students exchange questions and answers that include mathematical formulas in online or blended classes. Even students who use KeTMath rules for the first time can send correct formulas without making any mistakes after a few tries. Teachers can create questions in a short amount of time, and responses collected through platforms such as GC can be easily processed. The types of questions are not limited to simply asking for mathematical answers; they can be in various formats, including multiple-choice questions, simple proofs,

and questionnaires. KeTLMS is more effective for use in classes than for self-study. Teachers can monitor students' one-line answers during their classes. One of the authors(Takato) distributes five to eight questions per 200-minute class. We posted the question files for one of our classes on our homepage 'ketcindy home'. There are eight files, each tiny in size at around 150KB. The total size is about 6MB including the common library folder 'ketcindyJS' and each CSV file, making it easy to store. This could naturally lead to paperless classroom records. Recently, we've made it possible to add question marks to the answer box to indicate where to input.

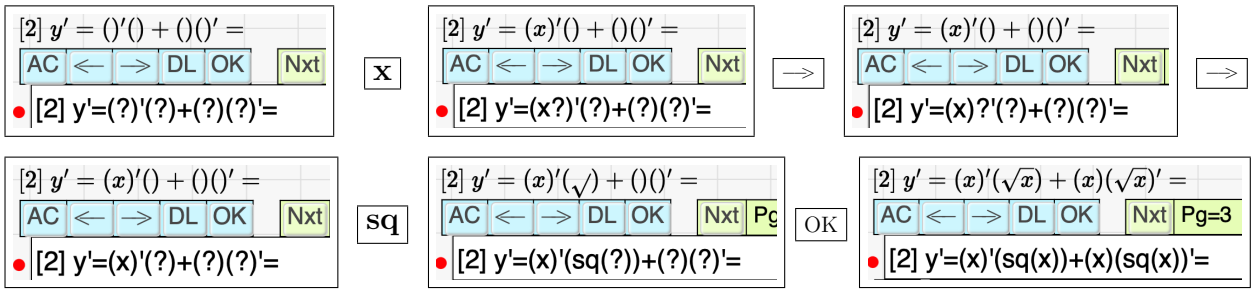

Fig.18 Use of question marks

We also incorporated functions to check correct answers and grades using Maxima. However, for many questions, the correct answers can often be found after looking at the students' answers, and the grading system does not fully address various mistakes made by students. This is a topic for future work.

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The Development of Fraction Virtual Manipulatives and Their Applications to Low-achieving Elementary School Students

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Abstract: *The concept of fractions is fundamental, but students have consistently needed help to learn it. This study aims to develop a virtual fraction teaching aid (fraction virtual manipulatives) that can be used on tablets to assist students with difficulty learning fractions. This report will discuss low-achieving math students' challenges in using graphical representations to explain the meaning of fraction calculations. The features of the developed digital tool will be discussed based on these challenges, highlighting its adaptability to different teaching methods and student needs. This application will be used in teaching low-achieving math students to explore its effectiveness. Finally, suggestions for teaching fractions will be provided.*

1. Introduction

Fractions are an essential concept and learning topic in elementary school mathematics. They are among the most complex mathematical concepts students encounter during elementary education. Numerous studies indicate that a solid foundation in fraction learning contributes to success in mathematics. Specifically, understanding the magnitude of fractions aids in the learning and conceptual development of algebra (see [3] and [4]), and algebra serves as the basis for advanced mathematics learning in the future. Knowledge of fractions is necessary when pursuing a career related to STEM or a non-STEM field such as baking. Therefore, understanding and mastering the concept of fractions helps students develop problem-solving skills in real-world situations.

However, due to fractions' complexity and abstract nature, this topic often becomes a nightmare for elementary school students (see [6]). Many students need help to grasp the basic concepts of fractions even by the time they reach middle school, hindering their subsequent mathematics learning and development. Some studies suggest that fractions predict future mathematical ability better than knowledge of whole numbers or other general cognitive abilities (see [18]), particularly algebra performance (see [1], [3] and [22]). Researchers have long emphasized the importance of learning fractions, but unfortunately, students find this topic particularly challenging and perform poorly in it. Helping students develop fraction knowledge has been a primary goal of mathematics education researchers. Despite these efforts, students' difficulties in learning fractions have yet to improve significantly (see [10]).

Twenty years ago, a technology-assisted learning tool combining physical manipulatives and computer technology, virtual manipulatives, was developed for elementary school mathematics. These virtual manipulatives are computer-based mathematical cognitive tools (see [20]). Scholars believe that supplementing mathematics teaching with physical manipulatives and these dynamic virtual manipulatives can successfully facilitate meaningful mathematical learning for students (see [18]). They also consider virtual manipulatives the most suitable mathematics learning tools for the next generation (see [11]).

Since 2011, with the advent of iPads and a wide range of handheld mobile devices (Mobile Technology, referred to as MT), interactive components that could be manipulated using a mouse on computers have gradually evolved into touch-based elements on standard handheld devices such as smartphones and iPads. In addition to being lightweight and having strong battery life, these MTs can also utilize virtual teaching tools, which can link different mathematical representations and facilitate conversions between them. These characteristics provide children with more opportunities for visual thinking when engaging in mathematical reasoning (see [8]), as well as offering students sensory experiences that make it easy to touch and move objects (see [16]), further helping them develop a deeper understanding of mathematical concepts (see [5]). This idea has led to developing and applying many apps (short for applications, which are interactive programs used on MTs) in educational settings, believing that these emerging technologies enhance students' willingness to engage in learning. Larkin (see [9]) found that many apps are similar to flashcards, mainly providing games or practice for students. Pelton and Pelton (see [17]) also noted that many apps merely present content like flashcards, calculation drills, or textbook materials. However, a good app should pay attention to the connection between three types of knowledge (conceptual knowledge, procedural knowledge, and problem-solving knowledge) and assist in constructing mathematical concepts. According to Namukasa et al. (see [13]), most existing apps do not adequately connect conceptual, procedural, and problem-solving knowledge, essential components of mathematical understanding. This finding aligns with Pelton and Pelton's (see [17]) argument that effectively integrating various visual representations is crucial for concept learning yet remains challenging for many educational apps. Highfield and Goodwin (see [7]) also believe that few interactive digital tools still allow students to construct learning content actively for mathematical concept learning. Particularly in digital tools designed for fractions, most are limited to exploring basic fraction concepts, such as fundamental fraction representations, equivalent fractions, and comparison of fraction magnitudes. More complex operations, such as fraction multiplication and division, still need practical conceptual representation tools.

In summary, fractions are challenging for many elementary school students in mathematics, yet their learning success significantly impacts future mathematical development. Recent research indicates that virtual teaching tools on touch devices can enhance student engagement, contributing to successful learning outcomes. However, most developed mathematical digital cognitive tools focus primarily on training and practice, with few interactive tools enabling students to actively construct learning content for mathematical concept development (see [7]). Therefore, this study aims to develop and implement interactive fraction virtual teaching tools and explore the effectiveness of remedial teaching tools in helping underachieving elementary school students understand the meaning of fraction multiplication and division.

2. The Issues Low-Achieving Students Face in Using Graphical Representations to Explain the Meaning of Fraction Multiplication and Division Calculations

When performing essential fraction addition and subtraction, students often apply integer system concepts to fractions (see [15]). For example, when solving fraction addition, students might add the numerators together and the denominators together as the answer. After learning fraction multiplication, students often need clarification due to the different rules for multiplication compared to addition and subtraction, resulting in errors (see [14]). Fraction division is the most abstract of the four operations because it is difficult to calculate and challenging for students to represent its meaning graphically (see [23] and [24]). When solving fraction division problems, students make various calculation errors, such as inverting the dividend instead of the divisor or

inverting both (see [2]). These errors are generally due to the algorithms' inaccurate memory. When an algorithm is perceived as meaningless, students may need to remember or alter some steps, leading to mistakes. Additionally, even if students can apply the division rules, they often need help understanding the principles behind the algorithm (see [25]).

This study first explores low-achieving students' issues using graphical representations to explain the meaning of fraction multiplication and division, serving as a reference for digital tool design. When representing the meaning of fraction multiplication and division graphically, it is necessary to depict a fraction and its operation process. Therefore, we divide the students' difficulties in graphical representation into two parts:

2.1 Issues with Representing Fractions

(1) Problems with Equal Partitioning

Students understand that the denominator represents the number of equal parts. However, when presenting it graphically, they may need help with unequal sizes of each part (Figure 1a). They might start with a strategy of halving, but upon realizing there are not enough parts, they may further subdivide the already divided parts. This phenomenon is more common in circular representations, as shown in Figure 1b.

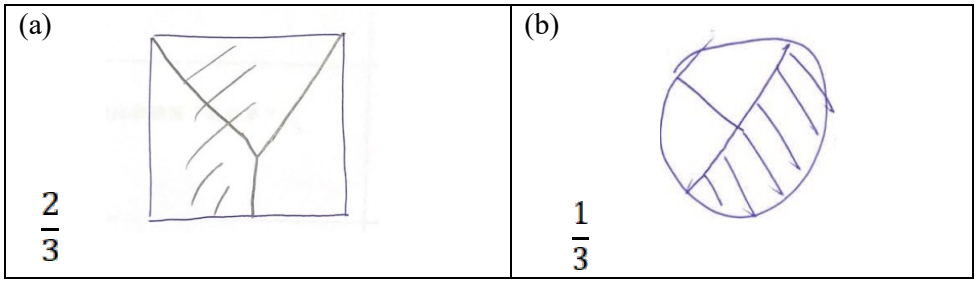


Figure 1 The problem of low-achieving students not dividing the graph equally when representing fractions

(2) Incorrect Number of Partitions

Students understand the numerator's meaning (the number of parts to take) but often must correct errors where the number of partitions does not match the denominator. For example, in Figure 4(a), the denominator is 3, but the shape is divided into four parts; in Figure 4(b), the denominator of 3/4 is 4, but the line segment from 0 to 1 is divided into five parts.

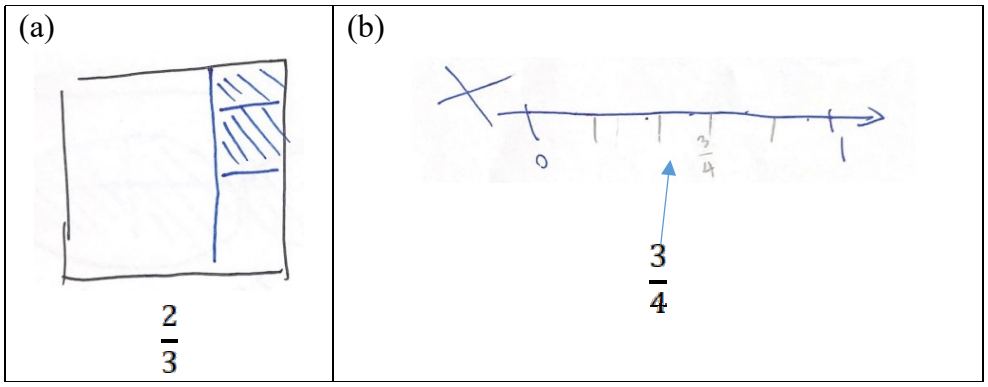


Figure 2 Problems with an incorrect number of partitions for low-achieving students when representing fractions

2.2 Issues with Representing the Calculation Process of Fractions

(1) Only Drawing the Graphical Representations of Individual Fractions in the Equation

When students use graphical representations to explain the meaning of fraction multiplication and division calculations, they often need help understanding how to illustrate the operations correctly. As a result, they only draw the individual fractions involved in the equation separately, as shown in Figure 3.

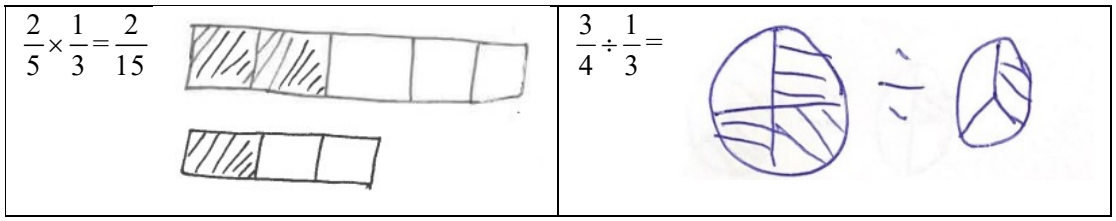


Figure 3 Low-achieving students only draw the fractions in the operation formula when explaining the meaning of fraction calculations

(2) Only Drawing the Fraction Representing the Answer

When students use graphical representations to explain the meaning of fraction multiplication and division calculations, they often need help understanding how to illustrate the entire process. As a result, they only draw the fraction representing the final answer without showing the intermediate steps or how the calculation was performed, as shown in Figure 4.

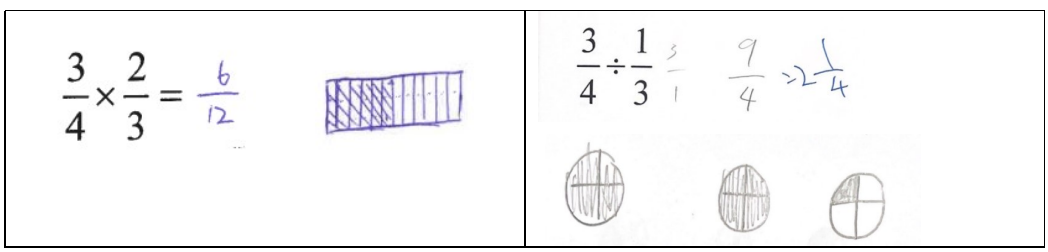


Figure 4 Low-achieving students draw the result fraction as the explanation of the meaning of fraction calculations

From observing low-achieving students use graphical representations to explain the meaning of multiplication and division of fractions, we found that students needed help to represent a fraction graphically accurately. Additionally, they needed to understand the meaning of the operations, leading to confusion about how to use graphical representations to show the obtained results or understand what the results represented. Based on this, we plan to develop fraction virtual manipulatives to help students overcome the difficulty of accurately representing fractions. We also aim to design related functions, such as re-segmentation and block segmentation, to assist low-achieving students in connecting operational rules and understanding the significance of the results.

3. The design and development of fraction virtual manipulatives

This study developed **Fraction Multiplication** (access this app from https://mathmentor.github.io/fraction-multiplication/ or app store), an app designed to assist

elementary school students in learning fractions. The app features two types of representation representations: continuous and discrete quantities. The representations include the number line, rectangle and circle, fraction wall, and discrete objects, totaling five types (Figure 5).

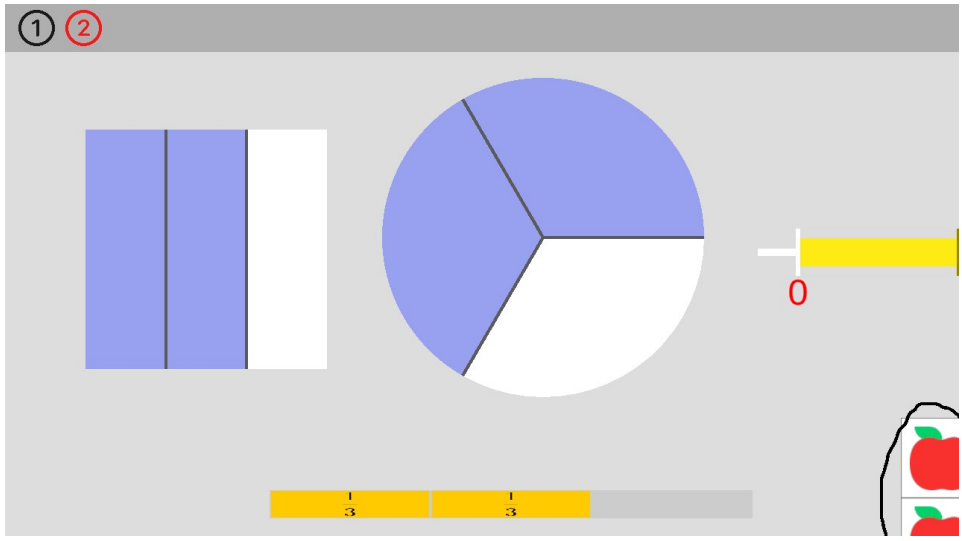


Figure 5 The five representations of fraction and interface of Fraction Multiplication

When the tool is opened, the lower bar presents five types of representations. Opting for the line segment representation, a line segment of 1 and eight unit-fraction segments is available (Figure 6). These can be dragged into the workspace, with each fraction element allowing the user to adjust the number of unit fractions by clicking '+' or '–,' which can be hidden to facilitate subsequent operations. Choosing the discrete representation (apple image) brings up a grid board and four object options on the left. The grid board allows users to determine the number of rows and columns and fill the grid with objects to swiftly create a scenario with a fixed quantity of discrete items (Figure 7). By dragging components from the toolbar into the workspace, users can apply different mathematical representations to solve problems, engaging them in problem-solving.

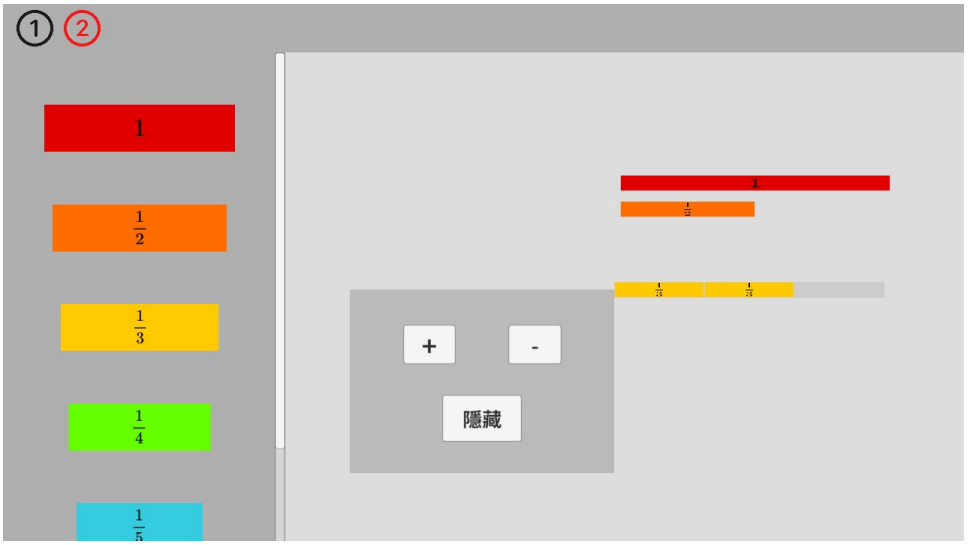


Figure 6 The line segment representation of fractions on Fraction Multiplication

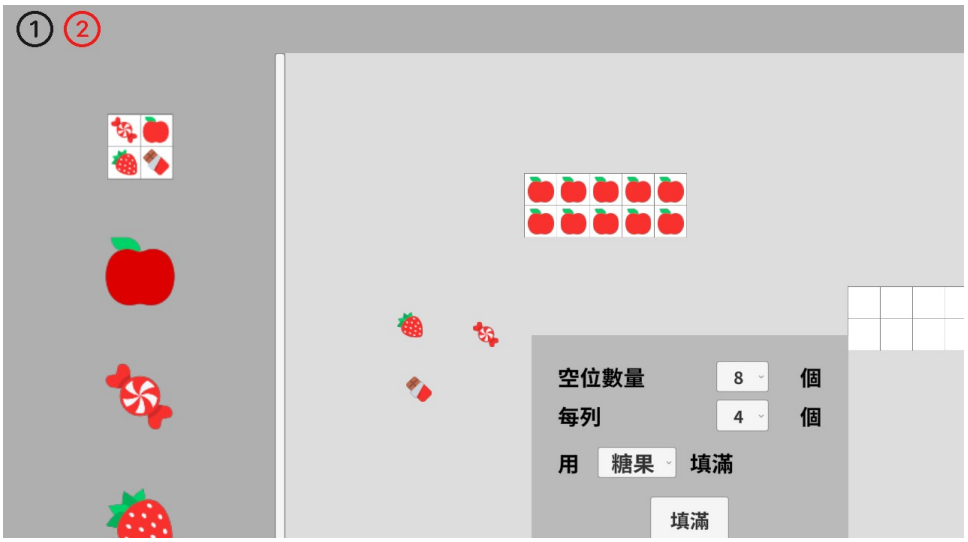
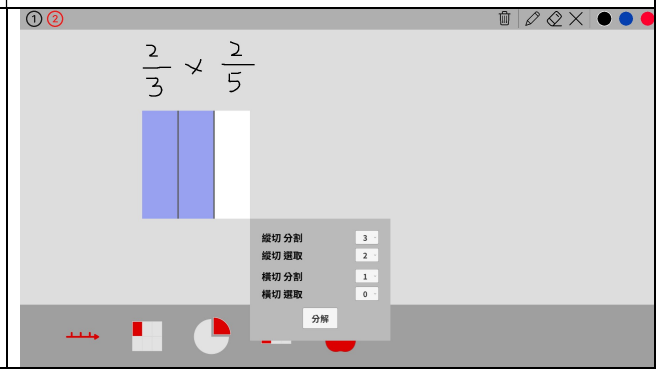


Figure 7 The discrete representation of fractions on Fraction Multiplication

Since this study focuses on the use of representations of fractional concepts, the following explains how this virtual fraction manipulative is applied to represent the characteristics of fraction multiplication and division calculations:

3.1 Connecting the Meaning of Fraction Multiplication Rules through Two Segmentation Processes

The fraction multiplication process involves the meaning of the fractional multiplier, thus requiring another segmentation action. Therefore, the exact graphical representation can undergo a secondary segmentation process to connect the meaning of fraction multiplication rules (i.e., why the numerator is multiplied by the numerator and the denominator by the denominator). When designing this digital tool, we provide students with the function to perform two equal segmentations of the exact graphical representation, as shown in Figure 8 (rectangle representation) and 9 (number line representation).

Example	$\frac{2}{3} \times \frac{2}{5}$
Take out a complete unit and make the first equal division (vertical) to complete the representation of the multiplicand.	

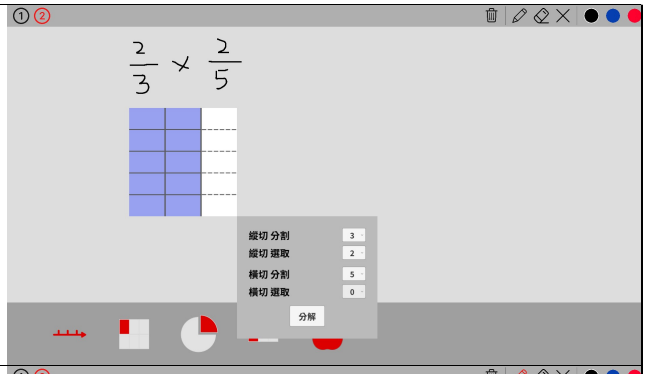
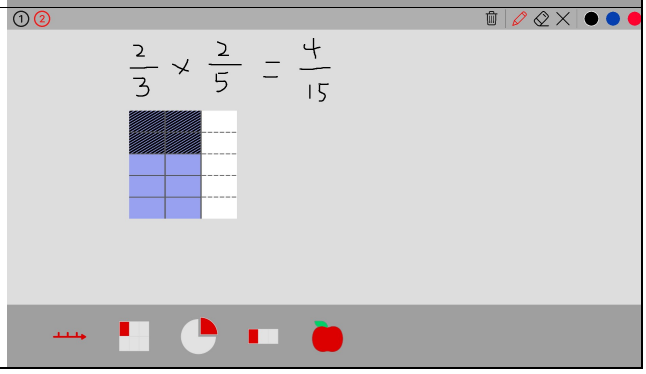
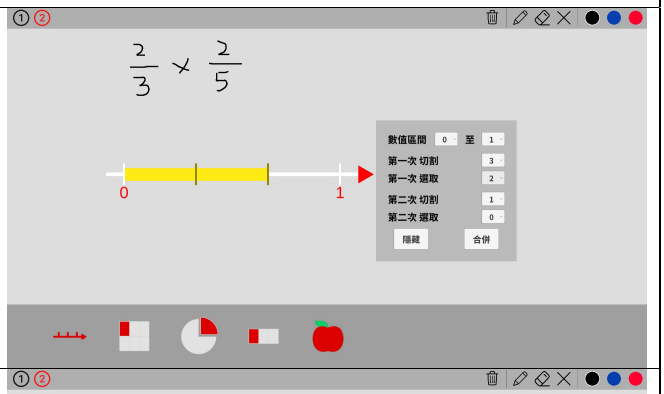
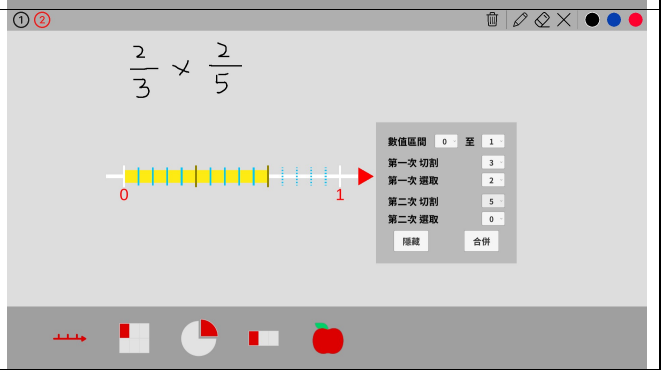
The second time, visually divide each equal part of the denominator of the multiplicand into equal parts of the denominator of the multiplier (do it horizontally). This visual aid helps see that the whole part is divided into 3 × 5 equal parts, connecting the meaning of multiplication of the denominator more engagingly.	
After the second equal division, select the number of molecules, and the digital tool will display the intersection number of the two selected numbers (the meaning of multiplication of connected molecules).	

Figure 8 The function of rectangular representation on Fraction multiplication

Example	$\frac{2}{3} \times \frac{2}{5}$
Take out a complete unit and make the first equal division to complete the representation of the multiplicand.	
The second time, divide each equal part of the denominator of the multiplicand into equal parts of the denominator of the multiplier. At this time, it can be seen that the whole number is divided into 3 × 5 equal parts (connecting the meaning of multiplication of the denominator).	

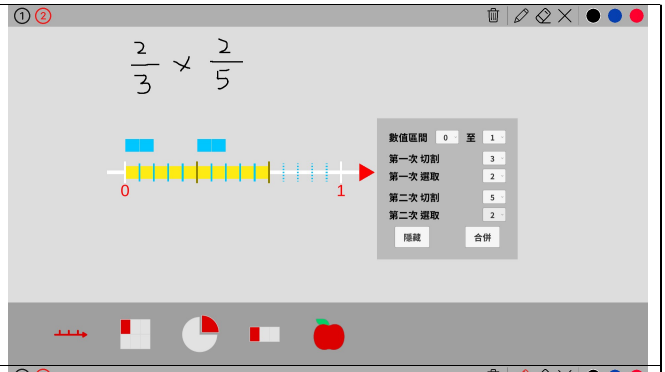
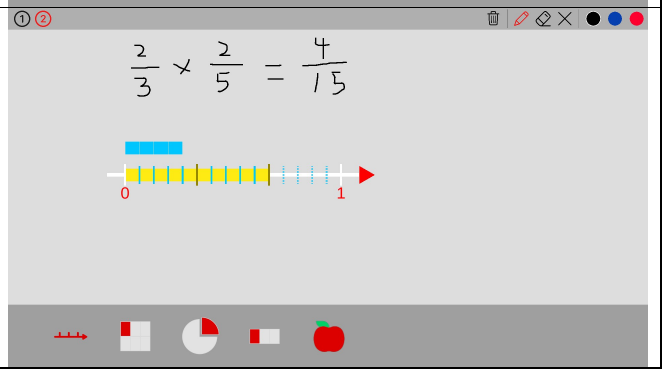
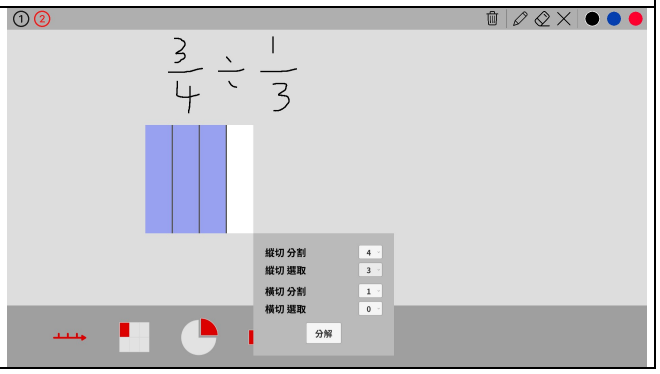
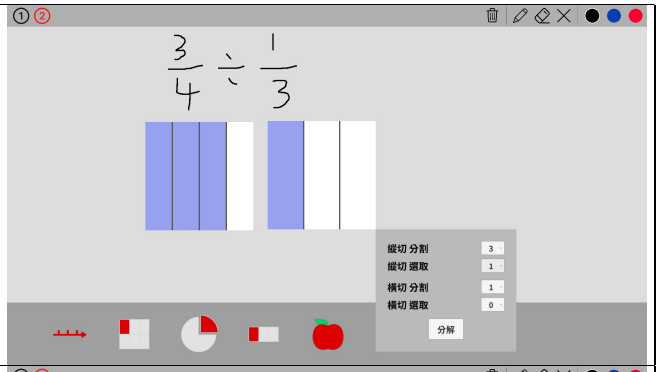
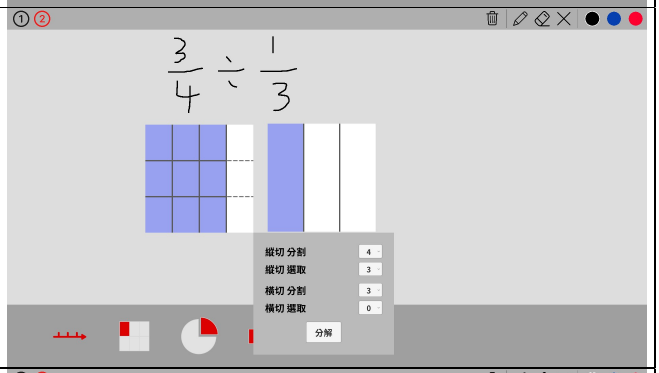
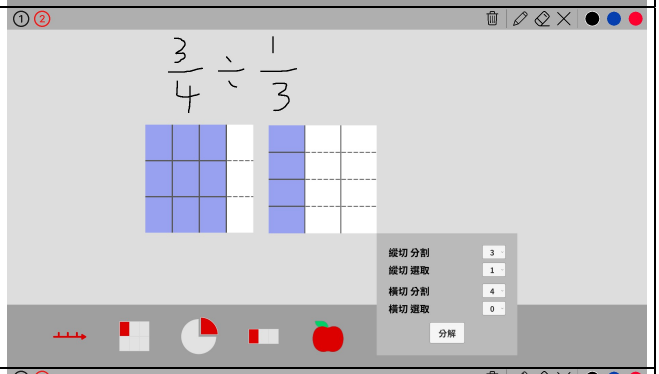
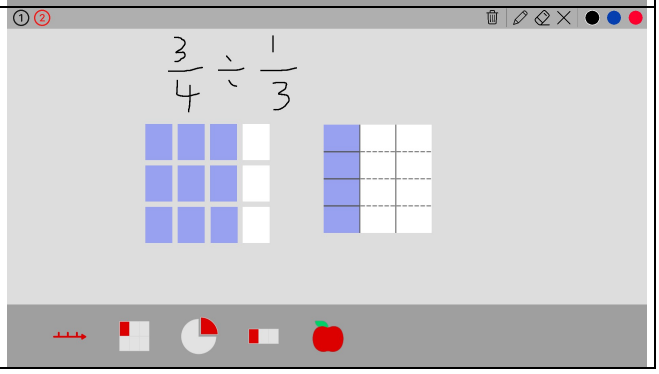
After the second equal division, select the number of molecules, and the digital tool will display the intersection number of the two selected numbers (the meaning of multiplication of connected molecules).	
Merge to display the result (the number of intersections among all copies).	

Figure 9 The function of number line representation on Fraction multiplication

3.2 Easily Explain the Meaning of Division by Fractions through Segmentation of Graphical Representations

When performing division operations where the divisor is a fraction, it is essential to guide students to understand how many times the dividend contains the divisor. This tool provides a segmentation function that helps students connect with the concept with the decomposed graphical representation. Based on this demonstration, students can quickly obtain the operational meaning and find the answer, as shown in Figures 10 (Rectangle Representation) and 11 (Number Line Representation).

Example	$\frac{3}{4} \div \frac{1}{3}$
Take out a complete unit and try to do the first equal division (vertical) to complete the representation of the dividend.	

Take out another complete unit and make the first equal division (vertical) to complete the representation of the divisor.	
Divide each equal part of the denominator of the dividend into equal parts (horizontally). At this time, it can be seen that the whole part is divided into 4×3 equal parts.	
Also, divide each equal part of the denominator of the multiplier into equal parts of the dividend denominator (horizontally). At this time, it can be seen that the whole is divided into 3×4 equal parts.	
After equal division, use digital tools to decompose the dividend by selecting the number.	

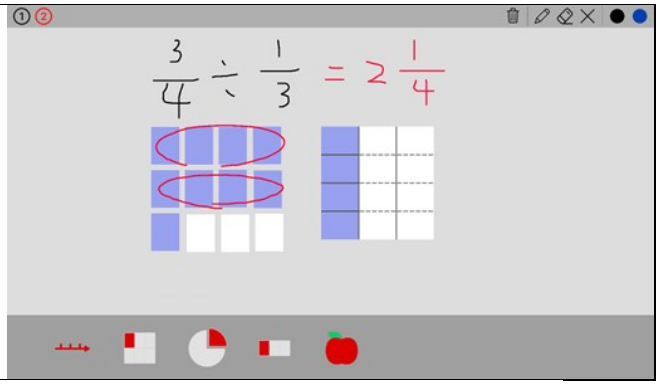
Observe how many divisors the dividend is from the numerical tool to find the result.	

Figure 10 The function of Rectangle representation on Fraction multiplication

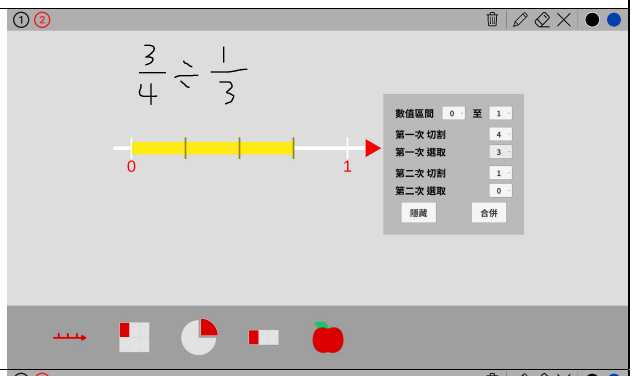
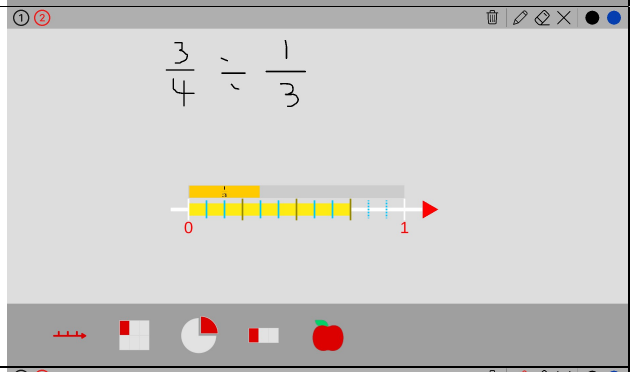
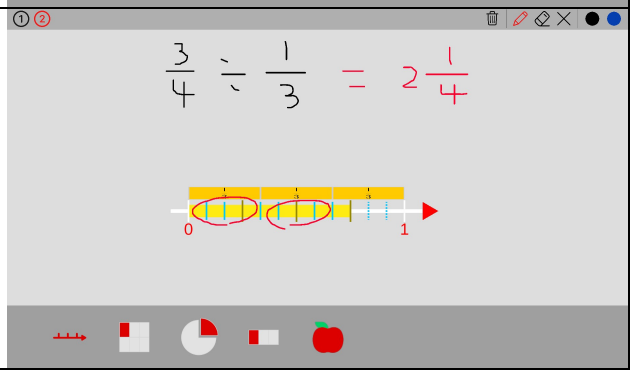
Example	$\frac{3}{4} \div \frac{1}{3}$
Take out a complete unit and make the first equal division to complete the representation of the dividend.	
Take out the line segment representation of the divisor and perform the second equal division (the denominator of the divisor).	
Observe how many divisors the dividend is from the numerical tool to find the result.	

Figure 11 The function of number line representation on Fraction multiplication

4. Investigating the Use of Virtual Tools by Low-Achieving Elementary School Students to Understand the Meaning of Fraction Multiplication and Division Calculations

This study utilized the fraction virtual manipulatives developed for sixth-grade students with low achievement in mathematics, aiming to explore its instructional effectiveness. Because the time available for teaching students is limited, and past research has shown no significant differences in student performance across different area representations, this study focuses solely on exploring rectangular and number line representations.

4.1 Experiment 1: Fraction Multiplication

4.1.1 Research Objectives and Subjects:

(1) Participants: 33 low-achieving (learning assistance) students from two elementary schools in Taichung City.

(2) Method: One-on-one instruction, allowing students to operate two graphical representations (rectangle and number line) on tablets. The meaning of fraction multiplication rules was explained and presented through two segmentations of the graphics. Student preferences for the two graphical representations were then collected.

4.1.2 Research Process:

Each student underwent a 40-minute teaching process as follows:

(1) Initial Assessment: Evaluating students' understanding of fraction multiplication. Students were asked to calculate and explain the process of proper fraction × unit fraction (e.g. $\frac{2}{5}\times\frac{1}{3}$) and proper fraction × proper fraction (e.g. $\frac{3}{4}\times\frac{2}{3}$). They also indicated their preferred graphical representation and explained their choice.

(2) Instruction: Students were taught to use the fraction virtual manipulatives to operate different graphical representations and demonstrate how these representations show the results of fraction multiplication. Practice problems were provided for students to work on.

(3) Post-Assessment: Similar to the initial assessment, students were given two questions (only the numbers differed). They could draw the graphics or use the tablet to create the representations. Students were again asked about their preferred graphical representation and the reasons for their preference.

4.1.3 Research Results:

Before the instructional intervention, 70% and 85% of students could calculate the correct answers for the two questions, but they needed help accurately explaining the process and meaning using graphical representations. After the intervention, all students could explain the meaning of fraction multiplication using graphical representations, either with the aid of the tool or by drawing themselves. Students preferred using the rectangle model to explain the meaning of fraction multiplication.

4.2 Experiment 2: Fraction Division

4.2.1Research Objectives and Subjects:

(1) Participants: 31 low-achieving sixth-grade students from four elementary schools in Taichung City.

(2) Method: One-on-one instruction, allowing students to use the virtual tool to operate two graphical representations (rectangle and number line) to learn fraction division and assess their preferences for these representations.

4.2.2 Research Process:

Each student underwent a 40-minute teaching process as follows:

(1) Initial Assessment: Evaluating students' understanding of fraction division. Students were asked to calculate and explain the process of same-denominator fraction division (e.g. $\frac{3}{4} \div \frac{1}{3}$) and different-denominator fraction division (e.g. $\frac{3}{4} \div \frac{2}{4}$). They also indicated their preferred graphical representation and explained their choice.

(2) Instruction: Fraction division was taught using fraction virtual manipulatives, demonstrating the calculation process with different representations. The same-denominator fraction division was shown with a rectangle representation, and fractions division with a different denominator was shown with a number line representation. Students were then provided a third question (same-denominator fraction division) and asked to choose a representation. If they chose the rectangle, the fourth question (different denominator fraction division) was practiced using the number line.

(3) Post-Assessment: Similar to the initial assessment, students were given two questions (only the numbers differed). They could draw the graphics or use the tablet to create the representations. Students were again asked about their preferred graphical representation and the reasons for their preference.

4.3.3 Research Results:

Before the instructional intervention, 27 out of 31 students could correctly calculate the two problems, but they needed help accurately explaining the process and meaning using graphical representations. After the intervention with the virtual tool, 25 and 26 students, respectively, could explain the meaning of fraction division using graphical representations, showing an understanding of the concept. Students preferred using the rectangle representation to explain the results of fraction division.

5. Conclusion

While most students can calculate fraction multiplication and division problems using algorithms, they often need help interpreting their answers visually. This research employed virtual teaching aids that dynamically manipulate graphical representations, especially a secondary operation on fractional representations (e.g., re-partitioning or decomposition). These functions enabled low-achieving students to visualize the rationale behind multiplying numerators and denominators in fraction multiplication and dividing the dividend into a specific number of divisors in division, which are less common in existing digital tools. The use of graphical representations facilitated a deeper understanding of the results of fraction calculations. Moreover, this study revealed that low-achieving students preferred rectangular representations when solving fraction multiplication and division problems visually, and their accuracy in solving problems using rectangular representations post-test was also higher. Thus, it is recommended that teachers incorporate area representations into their instruction and provide opportunities for students to construct their graphical representations, fostering a sense of discovery and engagement in their learning process to enhance their comprehension of fraction calculations.

Additionally, the study found that low-achieving students preferred using the rectangle representation for solving fraction multiplication and division problems. These students also showed a higher accuracy when using the rectangle representation in post-assessment problems. Therefore, it is recommended that teachers use area representations more frequently in their

instruction and provide opportunities for students to construct their graphical representations to help them understand the meaning of fraction calculations.

When using virtual tools for teaching, teachers should fully utilize the intuitive and visual nature of the graphical representations within the tools and guide students through demonstration and explanation. For example, in this study, students could often draw the dividend, divisor, or answer but could not explain their relationships. However, when teachers used the decomposition and drag-and-drop functions of the virtual tool to visually demonstrate how the dividend can be divided into several divisors, low-achieving students could concretely see the process and understand the meaning of the results of fraction division. This hands-on manipulation of graphical representations deepened their conceptual understanding.

Moreover, students often needed help with uneven segmentation and incorrect partitioning when drawing graphical representations, making it challenging to explain their calculations' meaning accurately. The precise segmentation function within the virtual tool can help students overcome these issues. Therefore, providing math virtual tools on tablets to students with learning difficulties can facilitate their learning of fraction calculations and help them grasp the underlying meanings more effectively.

Acknowledgement

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Learning Experiences: Connecting A-Level Mathematics With Mathematics Used In The Real World Via Machine Learning

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Abstract

One of the aims of the current Advanced Level H2 Mathematics Syllabus in Singapore is for students to connect ideas within mathematics and apply mathematics in the contexts of sciences, engineering and other related disciplines. A practising classroom teacher will find this very hard to achieve. On one hand, the mathematics that supports real scientific and engineering applications is often too sophisticated and lies beyond the reach of classroom mathematics – at least as perceived by the teachers. On the other hand, textbook examples that bear some semblance of a real-life application often appear contrived. The teacher's challenge is to find middle ground that caters for both accessibility and authenticity. This paper situates Gradient Descent, an elementary concept/technique commonly featured in Machine Learning, to create meaningful learning experiences with the aim of connecting topics within mathematics, and with the actual mathematics used to solve real world problems.

1 Introduction

In recent years, two types of connections in the field of mathematics education have increasingly gained research traction: one internal and the other external.

The internal connection is about linking topics *within* school mathematics through Big Mathematical Ideas. Charles defines a *Big Idea in Mathematics* (Big Ideas, for short) to be "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" ([4]). Expectant of mathematics teachers to see mathematics as a coherent and connected whole body of knowledge ([16]), Charles [4] advocates the grounding of subject content knowledge and teaching practices around Big Ideas. For students, through Big Ideas they link and see purposes in classroom activities, and construct richly linked understandings of mathematics ([9]).

The external connection *beyond* school mathematics concerns with the connection of school mathematics with the mathematics actually used in real-world contexts. Globally, mathematics is valued as an important subject in school curriculum because of its usefulness as a tool to be used by students to solve problems within personal, occupational, societal, and scientific contexts ([12]). Since the 1990s, there have been repeated calls, ranging from both educators to professional associations, for teachers to make explicit connections between school mathematics and its applications to relevant real-world situations ([11, 5]). Such demands grew out of genuine needs of both teachers and students. Studies have reported that certain middle-grade mathematics teachers valued real-world connections because these helped engage ([8]) and motivate students ([6]). Elsewhere (e.g., [2]), students have also expressed their need for a better understanding of how school mathematics connected with contexts and situations relevant to their lives.

Becoming cognizant of the importance of the two connections in mathematics education, many national mathematics curricula over the last five years responded by including official documents the need for mathematics teachers to exploit Big Ideas to present a coherent view of mathematics for their learners and to design tasks and activities, especially those that involve mathematical modeling, to link school mathematics to mathematics used in real-world contexts ([13]). Notably, textbooks in several countries (e.g., in the Netherlands, see [17]) have responded to this recent mathematics education reform embracing the so-called *Realistic Mathematics Education* (RME) —- one that emphasizes the connection of school mathematics with its applications to real-world contexts.

Despite such curricular requirements, interpreting the written curricula as intended curricula and then implementing this in enacted form are admittedly challenging processes as expressed by Stein and her colleagues in [15]:

> *Between* the written and intended phases, teachers bring their prior understandings, beliefs, and goals to bear on the written curriculum and, in the process, transform it into a form that they believe will be workable in the classroom. *Within* the enacted phase, the teacher and the students, in their interaction with each other, create something different from what exists on the pages of the book or in the teacher's mind or lesson plan.

The reality of the classroom enactment is far from being ideal: research findings reported gaps between teachers' vision of making connections and their ability to enact the vision in practice ([7]). Meaningful real-world connections were not often occurring in classrooms, especially at the middle school level. Even if these connections had occurred, they tended to be cursory and insufficient to engage students ([5, 7]). It therefore comes as no surprise that many middle-grade mathematics teachers found it challenging to first interpret the written requirement for them to establish the two kinds of connections and then to enact those interpretations into actual lessons in the classrooms. For example, teachers found themselves entrapped in several kinds of dilemma in their attempts to incorporate real-life applications of school mathematics into their lessons ([14]).

An extensive literature review of the potentialities and challenges of making connections within mathematics and with real-world applications in classrooms returned research works performed in elementary and middle school levels. There is a paucity of literature in this area at the pre-university level. Does this suggest that mathematics teachers at the pre-university level

do not face those challenges that their elementary and middle school counterparts encountered? Perhaps, the mathematics content at the pre-university level is sophisticated enough that makes it easy to connect with the advanced mathematics employed in real-world contexts. These questions do not seem to have ready answers and thus pique our interest. A small step towards understanding the situation at the pre-university level will be for the authors to start our investigation locally in their home country, Singapore.

Two research questions, which will be addressed in this paper are: (RQ1) To what the extent are Singapore pre-university mathematics teachers designing and implementing lessons that help students make connections within the topics taught in the syllabus and with the (presumably, more advanced) mathematics actually applied to solve real-world problems? (RQ2) What are the challenges that these teachers face when they attempt to do this? Additionally, we make some recommendations that pre-university mathematics teachers can use to create meaningful learning experiences that crucially help students make connections internally and externally.

We organize this paper as such. In Section 2, we initiate a case study in Singapore to understand (1) how mathematics teachers incorporate making connections *internally* and *externally* in teaching H2 Math to their students, and find out (2) what difficulties and challenges pre-university mathematics teachers encounter when they attempt to do so. A small sample of Junior College H2 Math teachers volunteered to complete an online survey designed to find out more about (1) and (2). Carrying these teachers' concerns and challenges, we revisit Tyler's curriculum model and apply it to resolve some of these issues the teachers raised in In Section 3. Then, in Section 4, we outline three learning experiences to help H2 Math students make internal connections within the mathematics they learn in the syllabus, and also external connections between H2 Math topics with the mathematics applied in real-world contexts. Specifically, we propose the Gradient Descent Method, a commonly used technique in Machine Learning, as a bridge that makes these connections. Finally, in Section 5, we make some concluding remarks and present some ideas for future work.

2 A case study in Singapore

2.1 Background information

In Singapore, Grades 11 and 12 students receive pre-university education, situated between secondary and university education. In the 2023 Joint Admissions Exercise (JAE), which involved 19,600 students, 52% of them were posted to a polytechnic, 39% to Junior Colleges or Millennia Institute, and 9% to the Institute of Technical Education. Being a pre-requisite to many science-related and engineering courses at universities, Higher 2 Mathematics (H2 Math, for short) is chosen by Junior College students who would sit for a final national A-Level examination at the end of their second year. Our paper is about H2 Math syllabus, the aims of which in [10] are:

1. acquire mathematical concepts and skills to prepare for their tertiary studies in mathematics, sciences, engineering and other related disciplines;

2. develop thinking, reasoning, communication and modelling skills through a mathematical approach to problem-solving;

3. connect ideas within mathematics and apply mathematics in the contexts of sciences, engineering and other related disciplines;

4. experience and appreciate the nature and beauty of mathematics and its value in life and other disciplines.

(3) and (4) show that the Ministry of Education, Singapore (MOE, for short) places strong emphasis on *both* the *internal* connection within the topics of mathematics in the national curriculum and the *external* one between school mathematics and the mathematics applied to real-world contexts. However, (3) and (4) – being generically phrases – do not provide explicit ways for H2 Math teachers to plan plan and enact their lessons to achieve them. It is also unknown to us what difficulties and challenges teacher face when they think of or attempt to achieve (3) or (4) in their lesson plan and enactment.

2.2 A small sample of responses to an online survey

Four H2 Math teachers (of varying lengths of teaching experience) participated voluntarily in a short survey (see Appendix A), designed to shed some insight to RQs 1 and 2. Given that this small sample gives some 'ground-sensing', the authors are aware of its limitations. In what follows, we report some of the responses to this survey, and draw some understanding towards our research questions.

2.2.1 Internal connection

All the respondents indicated that they teach towards Big Ideas as often as they appear in the Scheme of Work[1] When asked to give an example, we received Big Ideas like 'Functions', 'Vectors', 'Limits', and themes such as 'Extensions'; these being key words that appear in the H2 Math Syllabus document ([10]). None provided further descriptions. One teacher mentioned the sum to infinity for convergent geometric sequences, and offered no further explanations. Another commented that Big Ideas were useful for students to gain a deeper understanding of mathematics by saying that "The idea of function as a mapping is pervasive in math." In illustrating how students view and understand 'cross-multiplication' when performing algebraic manipulations in equations and inequalities, there is a comment: "students get to see the different scenarios and determine the conditions that are required to apply the cross multiplication, especially in inequalities." When it comes to the challenges, the most frequently occurring response is that time is a constraint. One teacher, in particular, was worried of "lower ability students not being able to draw the link as their foundation in the topic is weak". Also, one teacher mentioned about the lack of mathematical maturity among the students, and the presence of 'critics' who were fixated on students' academic performance. Another problem centres around sequencing of topics, i.e., "some chapters taught in front might be linked to chapters taught behind; students at the earlier stage might not be able to understand terminologies used in later chapters".

[1]Here, the Scheme of Work refers to a document drafted out by teachers to chart out all the topics to be taught within a given time-frame (e.g., a school term), along with pedagogical remarks and teaching tips.

2.2.2 External connection

75% of the respondents indicated that they made use of applications of mathematics in real-world contexts during their lessons as often as were indicated in the Scheme of Work. Two teachers used the examples involving the phenomenon of rainfall: "Use the example of rain appearing slanted when travelling in a bus or train to illustrate vector addition", and "Using the idea of holding an umbrella against raindrops to illustrate how the human mind uses projection to determine the direction of raindrops." Linear transformations and Magnetic Resonance Imaging (MRI) in medicine, as one teacher pointed out, are related, while bank study-loan repayment packages were used to illustrate the use of geometric progression in real life.

Interestingly, all the teachers agreed that the examples they used seemed authentic to most students, and even more, their students realized that applications of mathematics are "innate" in many aspects of life because they saw "mathematics as something that connects to real world experiences". The teacher who used the example of study loan repayment said that "study loan is going to be something that students are going to face real soon. They get to understand what the real world offers and think about questions like the assumptions we have made in our calculations (e.g., questioning when the interest rate will charged or when the repayment will be made)." When asked to state two challenges that the teachers when attempting to incorporate mathematics applied to real world contexts to externally connect H2 Math topics with the real world, all teachers mentioned about time constraints again. One teacher explained that "more time is required for a meaningful discussion. Usually we can do it for faster and better classes", while another pointed out that "while there are quite a lot of real life applications around us, we need to choose a suitable one that relates to the students and the one that we can "simplify" enough to apply the current knowledge that the students have."

2.3 Findings

Those H2 Math teachers took every available opportunity to bring in internal and external connections whenever the Scheme of Work requires of them. However, many of the Big Ideas used to establish internal connections appeared sketchy. Big Ideas that were used by these teachers, e.g., Functions, were taken directly from the H2 Math Syllabus document are different from those listed in [4]. When the teacher mentions that "function as mapping is pervasive in mathematics", such pervasiveness cannot be demonstrated in class. This same teacher attempted to link the concept of sequence with function by pointing out that a sequence is just a function of the form $f : \mathbb{N} \longrightarrow \mathbb{R}$. However, this is at best a formal definition of a real-valued sequence and does not seem at all convincing in demonstrating the pervasiveness of functions.

As for making external connections, examples mentioned by the teachers at best provide some believable links between mathematics and real world phenomenon, e.g., the rainfall looks slanted when viewed from moving vehicles. The authenticity of the real world contexts is not strongly conveyed through the mathematics. Worth mentioning is the example of the bank study-loan repayment scheme, where the teacher made it relatable to the students because these students would soon be enrolled for university education and might need to make such bank loans. From this teacher's description, one gathers that this example had been incorporated into some class activity in which students employed geometric progression to model the repayment scheme. Even in this example, the teacher was aware of the limitations of his approach because several assumptions had to be made to 'simplify' the mathematics so that it becomes accessible

to the students. This then begs the question of how authentic can such classroom examples be.

Finally, one common problem that plagued these H2 Math teachers is the lack of time to implement such enriching learning experiences. They opined that "more time is required for a meaningful discussion". The lack of mathematical maturity for the lower ability students who struggled with the basics presents yet another major deterrent to teachers who wish to enact lessons that make internal and external connections.

3 Making connections through learning experiences

In order for teachers to circumvent some of the obstacles presented in Section 2.3, it is perhaps instructive to revisit some fundamentals concerning the creation of learning experiences. For this, we look no further than the Ralph Tyler's original curriculum model, often termed as *Tyler's rationale*:

Identifying objectives. This involves defining what the learners should know or be able to do at the end of the course. The objectives should take into account the needs of the learners, societal values and aims, and the philosophy of the school.

Selecting learning experiences. This step involves identifying the educational experiences that can help learners achieve the defined objectives. The experiences should be relevant and meaningful to the learners.

Organizing learning experiences. The selected experiences are then organized in a logical and effective sequence. This organization facilitates the learning process and helps learners make connections between different parts of the curriculum.

Evaluating the curriculum. The final step involves assessing whether the objectives of the curriculum have been achieved. This evaluation helps in understanding the effectiveness of the curriculum and provides insights for future improvements.

Specializing Tyler's rationale to this case study, the above processes now read:

1. Identifying objectives:
   (a) Students demonstrate relevant mathematical competencies in connecting two (or more) seemingly different concepts and use this connection to create deeper understanding of the H2 Math topics.
   (b) Students use their knowledge of H2 Math concepts to access more advanced mathematics which are deployed in authentic real world problems, and thus appreciate and see the value of H2 Math learnt in school with life.
2. Selecting learning experiences: When teaching a certain H2 Math topic, the teacher selects an authentic real world context that involves a mathematics concept beyond the H2 Math syllabus. The teacher then designs lesson examples, tasks and activities that made advanced mathematics and authentic real world applications accessible to students, tapping on their prior knowledge of H2 Math topics.

3. Organizing learning experiences: The teacher creates a logical, effective and workable sequence of learning experiences within a given time-frame. Crucially, the teacher facilitates the learning process and helps the students make internal and external connections between different parts of the H2 Math syllabus.

4. Evaluating the curriculum: The teacher designs assessment tools to test whether the students have acquired the ability to make internal and external connections and exploit this ability to solve real world problems. This evaluative part of the assessment helps the teacher judge the effectiveness of the learning experience and make further improvements for subsequent implementations.

4 Learning experiences leveraging on ML

Both existing literature and the responses given by the teachers in the survey have informed us that teachers need to (but have difficulties) (a) select an authentic real world problem that (b) invokes mathematics beyond H2 Math syllabus that still stays within reach of these students and (c) plan and enact learning experiences within a workable, preferably short, time-frame. One likens this situation to building a bridge that connects an island to the mainland: the obvious optimal solution is to choose a pair of closest points and build in the shortest time! Translating this to the organization of learning activities, it would mean that the teacher locates an 'advanced' mathematics topic that is relatable to many topics in H2 Math syllabus, preferably those which are taught during the first year of the two-year program. Some of these topics include (1) Sequences and Series, (2) Vectors, and (3) Calculus (see Table 1 for the relevant extraction from [10]).

Backed by his background in Machine Learning (ML), the second author recognizes certain H2 topics such as Sequences and Series, Vectors, Calculus and Linear Regression as prerequisites in Machine Learning courses offered at the tertiary level. This observation gives rise to the proposition that Machine Learning be used as the suitable advanced mathematics topic (outside H2 Math Syllabus) because of its links with many H2 Math topics. Furthermore, Machine Learning is a field of study in Artificial Intelligence (AI) that finds many applications in real world contexts, such as natural language processing, computer vision, speech recognition, business, medicine. The mathematical foundations of ML are provided by mathematical optimization methods, such as Gradient Descent Method, Linear Regression, etc. ML concerns with the development and study of statistical algorithms that can learn from data and generalize to unseen data and perform tasks without explicit instructions. In view of the growing presence of Generative AI tools, teachers and students would likely find ML as an advanced mathematics topic relatable to their daily encounter with AI.

In this section, we create and organize a sequence of learning experiences that leverages on ML to equip students with the mathematical competencies to make the desired internal and external connections articulated in 1. Owing to the limitation of space here, we anchor our discussion on the *Gradient Descent Method* which is an important optimization method in ML. Each ensuing subsection contains a description of the learning experience that manifests as part of a lesson development or classroom task.

4.1 Steepest slope of a plane

4.1.1 Lesson tasks

Having learnt the sub-topic of the Cartesian equation of planes, the students are given a 3D-Desmos visualization of the plane $\Pi : x - 2y + 3z = 0$ (Figure 1).

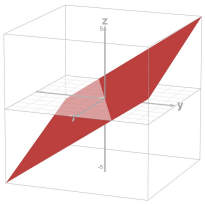

Figure 1: 3D visualization of the plane Π.

- The teacher assigns Task A to each group. As the students work on the task, they interact with the 3D visualization of Π in the Dynamic Geometry environment of 3D Desmos.

> **Task A** With respect to an origin O, the plane Π has Cartesian equation given by
>
> $$x - 2y + 3z = 0.$$
>
> (i) Verify that O lies on the plane Π and write down a vector normal to Π.
>
> (ii) Obtain a vector equation of the *line of greatest slope*, through O, lying on Π. (Recall that the line of greatest slope on a plane is one which is steepest.)

- Task B is then assigned after completion of Task A, where the students are to make connections between Tasks A and B, and based on these complete Task B.

> **Task B**
> Consider the function
>
> $$f(x, y) = -\frac{1}{3}x + \frac{2}{3}y,$$
>
> and the point $(x_0, y_0) = (0, 0)$.
> A point P moves from (x_0, y_0) in the direction of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ in such a way that f decreases fastest. Give an example of such a vector $\mathbf{u}$.

4.1.2 Tyler's rationale in action

Task A(ii) is a non-routine problem. To solve this problem, students can rely on Dynamic Geometry Software to visualize the relationships between (1) the normal vector $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, (2) the line of intersection of the plane $z = 0$ and Π, and (3) the line of greatest slope through O. Figure 2 (left) depicts the line of intersection of the plane $z = 0$ and Π, while Figure 2 (right) shows that the normal vector $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ is perpendicular to the line of intersection and the line of greatest slope through O.

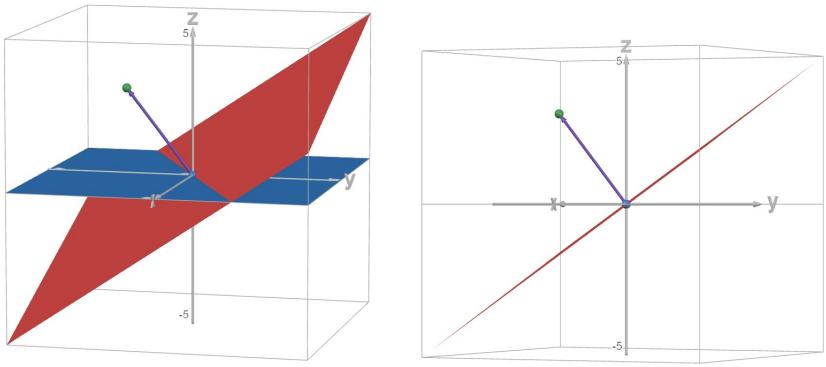

Figure 2: Two different 3D views

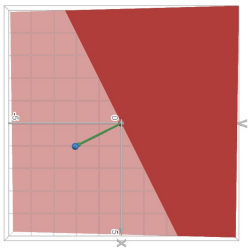

Figure 3: Top view

The learning experience explicitly requires students to relate Task A to Task B, which is an instance of making internal connections within topics in H2 Math through Big Ideas. In particular, the Big Idea of Equivalence is activated when the students realize that moving downwards along the line of greatest slope is the same as moving in the direction of greatest descent parallel to the plane $z = f(x, y)$. Based on this connection, the students can now view the projection of the normal vector $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ onto the horizontal plane, and determine that the required directional vector $\mathbf{u}$ is just this projection, i.e., $\mathbf{i} - 2\mathbf{j}$ (see Figure 3).

Remark 1 *The plane* Π *intersects* $x = 0$ *(respectively,* $y = 0$*)along the line through* O *parallel to* $-3\mathbf{i} + \mathbf{k}$ *(respectively,* $3\mathbf{j} + 2\mathbf{k}$*). Since these two vectors are the spanning vectors for* Π*, they are both perpendicular to the normal vector* $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ *of the plane* Π *(see Figure 4).*

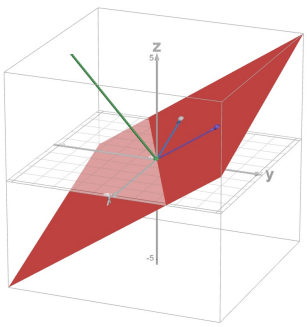

Figure 4: An alternative way of obtaining the normal vector

4.2 Greatest descent along a surface

4.2.1 Guided discovery

The teacher introduces Task C and adopts a problem solving approach. This lesson is an example of guided discovery leading the discovery of the *gradient of a function* f, i.e., ∇f.

Task C
Consider the function $f(x, y) = x^2 + y^2$ and the point $(x_0, y_0) = (1, 1)$.
A point P moves from (x_0, y_0) in the direction of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ in such a way that f decreases fastest. Determine an instance of $\mathbf{u}$.

Adopting the conception of lesson play in the classroom from [18], the teacher can anticipate a dialogue with the students given in Figure 5.

T: How would you want to approach this problem?
S1: It looks like Task B. So we can find the normal vector to the plane...
S2: But this is not a plane. 3D Desmos shows a curved surface when we plot $z = x^2 + y^2$.
T: Maybe what S1 refers to the tangential plane to the surface at $(1, 1, 2)$?
S1: You mean find the normal vector this tangential plane?
S2: Then we need two vectors parallel to this plane, which can be crossed to get the normal vector. Right?
T: From Task 2, can you recall the alternative method to find the normal vector?
S3: Intersect the plane $x = 0$ with the curved surface. But you don't get a line...
S1: This time they intersect along a curve.
T: The equation of the curve is $z = 1 + y^2$ because $x = 1$. Now the gradient at $y = 1$ is?
S2: We calculate $z'(y) = 2y$ and plug in $y = 1$.
S3: It looks like you are differentiating $f(x, y) = x^2 + y^2$ but treating x as constant because x is always 1, correct?
T: This is formally called the partial derivative of f with respect to y. So what S3 calculated is denoted by f_y.
S1: So a vector tangential to this curve of intersection is $\mathbf{j} + f_y\mathbf{k}$. This vector will also be tangential to the surface $z = f(x, y) = x^2 + y^2$.
S2: But we need another vector parallel to the surface.
T: How do you think you may get that?
S3: Can we intersect the surface $z = x^2 + y^2$ with the plane $y = 0$? Then get another curve of intersection, $z = x^2$.
T: That's a good idea.
S1: So another vector tangential to the surface must be $\mathbf{i} + f_x\mathbf{k}$.
S3: Now we have to get a normal vector to this tangential plane. But this time it is straightforward. We just cross those two tangential vectors to get the normal vector.
T: Why not you try to write down the working of what we discussed so far?

Figure 5: A sample dialogue between teacher and students.

4.2.2 Tyler's rationale in action

The above dialogue helps the teacher plan out a possible discursive road-map as he/she walks through the solution of Task C with the students. Through this discourse, the teacher has led the students to calculate a normal vector to the surface given by

$$\begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix} = \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}.$$

Again, taking the top view the students deduce that moving along $-(f_x\mathbf{i} + f_y\mathbf{j})$ gives the direction of fastest descent. The teacher may take this time to introduce the more advanced mathematical concept below:

Definition 2 (Gradient of a function) *Let $f = f(x, y)$ be a function of two real variables x and y. The* gradient *of f, denoted by ∇f, is defined to be the vector*

$$\nabla f := f_x\mathbf{i} + f_y\mathbf{j}.$$

This learning experience helps the students learn new mathematics from the H2 Math they already knew. This concept of gradient of a function is an essential ingredient of the Gradient Descent Method, which is the subject of the next learning experience.

4.3 Introducing gradient descent

4.3.1 Class discussion and activity

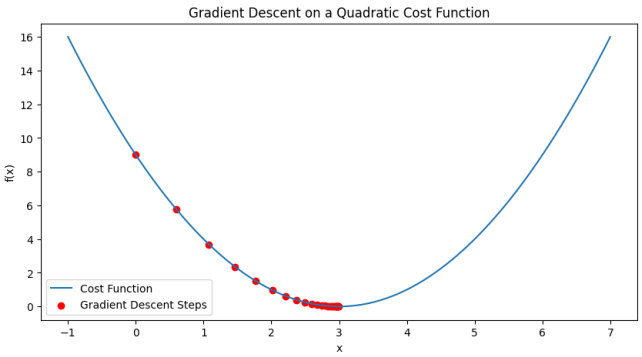

Figure 6: Gradient Descent on a Quadratic Cost Function

The teacher begins with the following scenario. Suppose you are at the point $(0, 9)$ on the curve $f(x) = (x - 3)^2$. You wish to inch your way to the lowest point on the curve by adopting the following strategy. When you are located at a point x_k you make a horizontal displacement that is proportional to the gradient $f'(x_k)$ (by a constant factor of α) which results in a drop in vertical distance (Figure 6). That is,

$$x_{k+1} = x_k - \alpha f'(x_k). \tag{1}$$

A possible discussion can emerge based on this scenario.

T: Why is this strategy a reasonable one?
S1: If the gradient is steep, the point is still quite far from the lowest point.
T: What do you mean?
S1: I mean as you approach the lowest point, the slope gets gentler ... the gradient gets near to zero, and so you inch forward with a smaller step not to go past the lowest point.
S2: Yes, the last time my friends and I were trekking to look for low-lying water bodies we do the same thing. We went down along the steepest part of the slope to get to the lowest point as quickly as we could.
T: I see ... so you can relate the math here with your own outdoor experience. Good.

The teacher then raises the students' attention on Equation 1. This equation is one of the many of its kind called *Gradient Descent Update Rule*. In this example, the gradient descent update rule associated to the function $f(x) = (x-3)^2$ is given by $x_{k+1} = x_k - \alpha \cdot \underbrace{2(x_k - 3)}_{f'(x_k)}$, and $x_1 = 0$. The constant of proportionality α is called the *learning rate*.

Given that students have some experience in using electronic spreadsheets or graphing calculators to generate recurrence sequences, the teacher then task them to engage in a hands-on computer-aided activity.

In-class activity
Use an electronic spreadsheet (e.g., Excel or Graphing Calculator) to generate the sequence defined by the recurrence equation $x_{k+1} = x_k - 2\alpha(x_k - 3)$, $x_1 = 0$ for various learning rates of $\alpha = 0.1, 0.2, 0.5, 1.0$. Which learning rate, α, is the best?

Remark 3 *Gradient descent is crucial because it is the foundation of many machine learning algorithms, enabling models to learn from data by improving their predictions over several iterations. It is especially important for training complex models like neural networks, where analytical solutions are not feasible. The efficiency and effectiveness of gradient descent make it a cornerstone in the field of optimization and machine learning.*

4.3.2 Tyler's rationale in action

It is important that the teacher helps students relate their interests and daily experience with the learning objectives. The hands-on activity involves using computer technology which makes the application of mathematics to real world situations authentic. Firstly, the mathematics – Gradient Descent – used to solve the minimization problem is closer to what people use in the real world. Secondly, the students get to appreciate that computers are essential in this activity to help them compute and visualize the convergence.

5 Conclusion

The integration of machine learning concepts, specifically Gradient Descent, into the H2 Math syllabus provides an excellent opportunity to bridge the identified gap between pre-university

mathematics in H2 Math syllabus and the mathematics applied in real-world situations. Each learning experience presented in Section 4 has been specially crafted based on Tyler's rationale, i.e., we offer students meaningful in-class interactions with authentic mathematics actually used in real-life within the context of the H2 Math topics they learn in school. These carefully designed lessons not only align with the current H2 Math syllabus but also enrich it by demonstrating practical applications, thereby enhancing students' engagement and motivation. The hands-on activities and the Python implementation offer students a chance to develop valuable 21st century competencies in mathematics and other related disciplines, preparing them for the interdisciplinary demands of higher education and modern careers.

Ultimately, the proposed approach in this paper fosters a deeper appreciation for mathematics and its relevance in today's technological landscape, helping both teachers and students make *internal* connections within H2 Math topics and *externally* with real-life applications of mathematics. By making advanced topics accessible and relatable, we hope to contributes towards "*Innovations in Mathematics and Mathematics Education in Technology*".

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A Survey questionnaire

This is a short survey that seeks to find out how you plan and enact lessons that achieve one or both of the H2 Math syllabus aims: (i) connect ideas within mathematics and apply mathematics in the contexts of sciences, engineering and other related disciplines; (ii) experience and appreciate the nature and beauty of mathematics and its value in life and other disciplines.

Internal connection: learning experiences, themes, big ideas

1. How often do you teach towards big ideas?
   Once a week/Once a month/As it appears in Scheme of Work/Incidental/Never

2. Give one example to illustrate how you help students see the connections between topics. Be specific which theme or big idea your example targets.

3. Comment about the efficacy of this particular example, e.g., do students gain an appreciation, a deeper understanding of mathematics, etc?

4. If your answer to Q1 is "Never", state briefly why.

5. Write down briefly two challenges you face in planning and enacting lessons when you think of or attempt to make connections *internally* within the topics taught in H2 Math.

External connection: mathematics applied to problems in real world contexts

1. How often do you make use of application of mathematics in real world contexts? Once a week/Once a month/As it appears in Scheme of Work/Incidental/Never
2. Give one example of how you incorporate mathematics in real world contexts in your teaching (e.g., lesson example, tasks and activities, assessment).
3. Comment about the efficacy of this particular example, e.g., are students convinced that mathematics is useful in real-life problems?
4. If your answer to Q1 is "Never", state briefly why.
5. Write down briefly two challenges you face in planning and enacting lessons when you think of or attempt to use mathematics applied to real world contexts to *externally connect* H2 Math topics with the real world.

B H2 Related Topics

Topic/sub-topic	**Content**
Sequences and Series	**Sequences and series** 1. Sequence given by a formula for nth term 2. Sequence generated by the relation $u_{n+1} = f(u_n)$, using graphing calculator or computer to generate the sequence 3. Convergence of a series and the sum to infinity 4. Formula for the nth term and sum of a finite arithmetic series 5. Formula for the nth term and sum of a finite geometric series 6. Condition for convergence of an infinite geometric series 7. Formula for sum to infinity of a convergent geometric series
Vectors	**Three-dimensional vector** 1. Vector and Cartesian equations of lines and planes
Calculus	**Differentiation** 1. Problems involving tangents and normals to curves, including cases where the curve is defined implicitly or parametrically 2. Local maxima and minima problems
Probability and Statistics	**Correlation and Linear regression** 1. Use of scatter diagram to judge if there is a plausible linear relationship between two variables 2. Concepts of linear regression and method of least squares to find the equation of the regression line

Table 1: Extracted items from H2 Math Syllabus [10].

Topological Structures Between Two Closed Surfaces Inspired by An Entrance Exam Problem

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Abstract

We shall explore with technological tools on a problem we posed in [8], which originated from a college practice entrance question (see [3]). In this paper, we are investigating whether a certain closed surface – inspired by the aforementioned college exam problem and an affine transformation – is topologically equivalent to a sphere, utilizing the three-dimensional visualization capabilities of computer technology to aid our initial analysis. This technological affordance not only makes our investigation accessible to readers with just an undergraduate mathematics background but also allows us to study this surface in detail, finally leading us to a rigorous solution to the problem. Our work highlights the essential role that technology plays in advancing and communicating mathematical research.

1 Preliminary

We first describe the college practice entrance problem (see [3]):

We are given a fixed circle and a fixed point A in the interior of the circle $(x-a)^2+(y-b)^2 = r^2$. A line passes through A intersects the circle at C and D respectively, and the point E is the midpoint of CD. Find the locus E when C moves along the circle.

This problem has been discussed using parametric equation approaches in 2D and 3D, see ([4], [5], [6], [7]). The locus E lies on CD and satisfies

$$\overrightarrow{ED} = s \cdot \overrightarrow{CD}, \tag{1}$$

where s is a given real number. For convenience, we sometimes write it as

$$E = sC + (1 - s)\, D. \tag{2}$$

We need some terminology before we start. We recall that a topological space is a non-empty set together with a topological structure that is given by the open sets of the topology. A non-empty set X can be equipped with different topological structures, and any subset $A \subset X$ can be endowed with an inherited topological structure, namely, the *relative topology*: the open sets in the relative topology on A are exactly the subsets of the form $A \cap O$ for some open sets O of X.

For the spaces $\mathbb{R}^n$, with coordinates $(x_1, \ldots, x_n)$, the usual topological structure is given by a base of open balls, which is defined by the metric space structure of $\mathbb{R}^n$. Recall that the *metric function* $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}$, for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ makes $(\mathbb{R}^n, d)$ a *metric space*. This metric gives rise to open balls $B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid d(x, y) < \varepsilon\}$, where ε is a positive real number, which are the basic open sets for the usual topology on $\mathbb{R}^n$, i.e., every open set of $(\mathbb{R}^n, d)$ can be written as the union of some open balls.

Thus, any curve or any hyper-surface in $\mathbb{R}^n$ can be viewed as topological spaces with their respective relative topologies, acquired from their environment space. For the cases we are interested in, the environment spaces will be simply $\mathbb{R}^2$ or $\mathbb{R}^3$; thus, we can view these curves and surfaces as relative topological spaces on which we can apply topological tools that are introduced in what follows.

Now, let us recall a fundamental concept in topology, namely that of *continuity*. Let (X, τ_X) and (Y, τ_Y) be two topological spaces, with τ_X and τ_Y their respective topologies. A function $f : X \longrightarrow Y$ is *continuous* if for any open set $O \subset Y$ it holds that $f^{-1}(O) \subset X$ is open in X. Here, the notation $f^{-1}(U)$ denotes the inverse image, in X, of $U \subset Y$.

When f is bijection, then it has an inverse $f^{-1} : Y \longrightarrow X$ defined by: for any $y \in Y$, $f^{-1}(y) = x \in X$ if and only if $f(x) = y$. We say that f is a *homeomorphism* if it is a continuous bijection and with f^{-1} also being continuous. In this situation, X and Y are said to be *homeomorphic* topological spaces.

Homeomorphisms between topological spaces are also called *continuous transformations*, or *topological transformations*. Intuitively, these transformations can be thought of as functions that transforms points of one space, that are arbitrarily close to each other, onto points of another space that are also arbitrarily close. Spaces that are related in this way are said to be *topologically equivalent*. Proving that two spaces are homeomorphic requires one to find a homeomorphism between them. Now proving that two spaces are *not* homeomorphic is much harder since one must establish that no such homeomorphism between them exists. A common technique to achieve this non-existence crucially relies on the concept of connectedness, which we introduce now. Let X be a topological space. Suppose that $X = U \cup V$ for some open sets U and V with $U \cap V = \emptyset$, we say that X is *disconnected*. Otherwise, X is *connected*. This definition of connectedness is not intuitive and hence harder to use. A more natural version of connectedness is to require that any two points in the space be connected with a curve. Let X be a topological space. A continuous function $f : [0, 1] \longrightarrow X$ is called a *path*. We say that

X is *path-connected* if for any two points $x,\ y \in X$, there exists a path $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. An important and well-known result is that every path-connected space is connected.

Returning to the task of establishing certain two spaces are not homeomorphic, we use two well-known topological spaces of the figure-8, denoted by E, and the circle C as illustration of this topological proof technique. Let p be the point on E where it crosses itself. Suppose there is a homeomorphism $f : E \longrightarrow C$. Then the restriction map $\widehat{f} : E - \{p\} \to C - \{f(p)\}$ is again a homeomorphism; whence the spaces $E - \{p\}$ and $C - \{f(p)\}$ would be topologically equivalent. Now, $E - \{p\}$ is disconnected as its two disjoint components are the circles. But $C - \{f(p)\}$ is path-connected and hence connected, thus we arrived at the desired contradiction.

It is intuitively evident that all simple closed curves in the plane and all closed polygons are topologically equivalent to a circle. Similarly, all ellipsoids, closed cylinders, closed cones, convex polyhedra, and other simple closed surfaces are equivalent to a sphere. We assume all these aforementioned topologically equivalences in the ensuing development.

Recall that any surface is a topological space whose topological structure is inherited from the ambient space $\mathbb{R}^3$ it lives in. Also, two surfaces are topologically equivalent if there exists a homeomorphism f between them, where these surfaces are viewed as two topological spaces. Such a homeomorphism f is a bijection with the property that both f and its inverse f^{-1} preserve open sets. If a certain geometric object is transformed into another topologically equivalent by bending, stretching, etc., this process entails a special type of topological transformation called a *continuous deformation*.

In fact, we could refer to the surfaces we are working with in this paper as *topological manifolds* since, as subsets of $\mathbb{R}^3$, they comply with all the conditions for that kind of manifolds: being Hausdorff topological spaces, with a covering of open sets, each one homeomorphic to an open set in $\mathbb{R}^3$ and in the intersections of the sets of the covering, the *transition functions* are also homeomorphisms from a subset of the surface onto another subset of the surface. Thus, in what follows we shall be dealing with surfaces that are topological manifolds and we will refer to them with such a terminology.

1.1 Surfaces of perturbing the longitude and latitude affect the topological structures

Suppose S is a given parametrized surface, say

$$S = \left\{ \big(f(u,v), g(u,v), h(u,v)\big) \mid u, v \in R \right\}, \tag{3}$$

for a rectangular domain $R \subset \mathbb{R}^2$, and real-valued continuous functions f, g, and h defined on R. Usually the values of the parameters will be $0 \le u \le 2\pi$, and $0 \le v \le \pi$, hence we will refer to them as *longitude* and *latitude*, respectively.

Now, with the same parametrization (3) for S we can consider some other surfaces, namely

$$S^1 = \left\{ \big(f(u+\alpha_1, v), g(u+\alpha_2, v), h(u+\alpha_3, v)\big) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}, \tag{4}$$

or

$$S^2 = \left\{ \big(f(u, v+\beta_1), g(u, v+\beta_2), h(u, v+\beta_3)\big) \mid \beta_1, \beta_2, \beta_3 \in \mathbb{R} \right\}, \tag{5}$$

Recall that Cartesian coordinates (x, y, z) are related with spherical coordinates (ρ, u, v) by the rule

$$\begin{bmatrix} \rho \\ u \\ v \end{bmatrix} \longrightarrow \begin{bmatrix} x(\rho, u, v) \\ y(\rho, u, v) \\ z(\rho, u, v) \end{bmatrix} = \begin{bmatrix} \rho \cos u \sin v \\ \rho \sin u \sin v \\ \rho \cos v \end{bmatrix} \tag{6}$$

Throughout the paper, points in $\mathbb{R}^3$ are often expressed in column vector form. We first note how the new parameters α_i and β_i change the longitude u and latitude v. This *perturbation* of the parameters turns out to be quite noticeable.

Exercises:

We consider the ellipsoid S_0 , which is represented by $\begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} = \begin{bmatrix} 5\cos u \sin v \\ 4 \sin u \sin v \\ 3 \cos v \end{bmatrix}$, where $u \in [0, 2\pi]$ and $v \in [0, \pi]$, with parameters of $\alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{\pi}{3}, \alpha_3 = \frac{\pi}{6}, \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{3}$, and $\beta_3 = \frac{\pi}{4}$. We invite readers to verify the followings:

1. The surface $\begin{bmatrix} f(u+\alpha_1, v) \\ g(u+\alpha_2, v) \\ h(u+\alpha_3, v) \end{bmatrix}$ is topologically equivalent to the ellipsoid S_0.
2. The surface $\begin{bmatrix} f(u, v+\beta_1) \\ g(u, v+\beta_2) \\ h(u, v+\beta_3) \end{bmatrix}$ is not topologically equivalent to the ellipsoid S_0.
3. We include a Maple worksheet for readers to explore in [S1].

2 Topologically Equivalent or Non-equivalent Surfaces

We recall that a surface with a boundary is not topologically equivalent to a sphere, e.g., a cylinder with an open top, which may be stretched into a disk (a circle plus its interior). In addition, a closed surface such as a torus (a doughnut) is not equivalent to a sphere, since no amount of bending or stretching can get rid of the hole in the torus to make it into a sphere. We now explore a separate affine transformation that is inspired by (2) and applied on a surface. We ask if the image of the transformation is topologically equivalent to the original surface. We remind readers that the discussion here is from layman's point of view, we invite interested readers to establish concrete homeomorphism between two spaces. In view of (2), we replace the antipodal points C and D by two respective surfaces. Considering s as an scaling factor, we can ask for the properties of a new surface defined as the locus, when spherical coordinates are used. Consider the surface S given by (3). Let the locus surface $\{(F(u, v, \alpha_1, \beta_1, s), G(u, v, \alpha_2, \beta_2, s), H(u, v, \alpha_3, \beta_3, s))\}$, generated by the points in $\mathbb{R}^3$ defined by:

$$\begin{bmatrix} F(u, v, \alpha_1, \beta_1, s) \\ G(u, v, \alpha_2, \beta_2, s) \\ H(u, v, \alpha_3, \beta_3, s) \end{bmatrix} = s \begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} + (1-s) \begin{bmatrix} f(u+\alpha_1, v+\beta_1) \\ g(u+\alpha_2, v+\beta_2) \\ h(u+\alpha_3, v+\beta_3) \end{bmatrix}, \tag{7}$$

where $u, v \in \mathbb{R}$, α_1 and, β_1 are given angles, and $s \in \mathbb{R}$. If we start with a surface S, which is transformed, according to the transformation rule (7) when s is specified, into the surface $\widetilde{S} = \{(F(u, v, \alpha_1, \beta_1, s), G(u, v, \alpha_2, \beta_2, s), H(u, v, \alpha_3, \beta_3, s))\}$.

2.1 Our ellipsoid

If $\begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} = \begin{bmatrix} 5\sin(v)\cos(u) \\ 4\sin(v)\sin(u) \\ 3\cos(v) \end{bmatrix}$, then we consider the ellipsoid $\begin{bmatrix} f(u,\frac{v}{2}) \\ g(u,\frac{v}{2}) \\ h(u,\frac{v}{2}) \end{bmatrix} = \begin{bmatrix} 5\sin(\frac{v}{2})\cos(u) \\ 4\sin(\frac{v}{2})\sin(u) \\ 3\cos(\frac{v}{2}) \end{bmatrix}$ and denote it as S^*, where $u, v \in [0, 2\pi]$. We see from Figure 1(b) that S^* is a union of two top and bottom halves of the ellipsoid S^*, which are denoted by S_t^* and S_b^* and depicted in black and red respectively.

Next, we set $\alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{\pi}{3}, \alpha_3 = \frac{\pi}{6}, \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{3}$, $\beta_3 = \frac{\pi}{4}$, and $s = \frac{1}{6}$, we obtain the surface of our interest

$$S_1 = \{[F(u,v,\alpha_1,\beta_1,s), G(u,v,\alpha_2,\beta_2,s), H(u,v,\alpha_3,\beta_3,s)] : u, v \in [0, 2\pi]\}. \tag{8}$$

More specifically, we have

$$\begin{aligned} F(u,v,\alpha_1,\beta_1,\frac{1}{6}) &= \frac{5\cos u\sin v + 25\cos(u+\frac{\pi}{4})\cos v}{6}, \\ G(u,v,\alpha_2,\beta_2,\frac{1}{6}) &= \frac{2\sin u\sin v + 10\sin(u+\frac{\pi}{3})\sin(v+\frac{\pi}{3})}{3}, \\ H(u,v,\alpha_3,\beta_3,\frac{1}{6}) &= \frac{\cos v + 5\cos(v+\frac{\pi}{4})}{2}. \end{aligned} \tag{9}$$

We see S_1 is a *closed surface* from Figure 1(a). On the other hand, when $s = \frac{1}{3}$, the surface

$$[[F(u,v,\alpha_1,\beta_1,s), G(u,v,\alpha_2,\beta_2,s), H(u,v,\alpha_3,\beta_3,s)]]$$

is an *open surface* (see Figure 1(c)) and hence is not topologically equivalent to a sphere or an ellipsoid. *Since S_1 and S^* are both closed surfaces,* we naturally pose the following that was asked in [8].

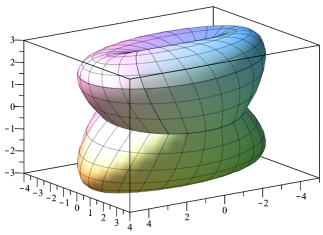


Figure 1(a). The closed surface S_1

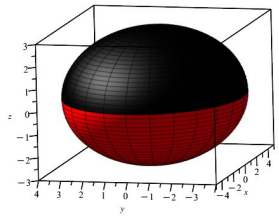


Figure 1(b). Ellipsoid S^*

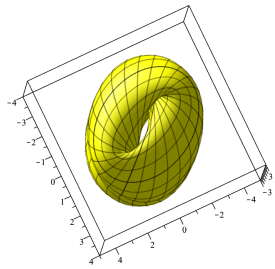


Figure 1(c). Not topologically equivalent to S^* for $s = \frac{1}{3}$.

2.2 Are the closed surfaces S_1 and S^* topologically equivalent?

Before we answer the question, we need to understand how the surface S_1 is formed, we thus proceed further as follows:

1. If we restrict the domain of S_1 to $u \in [0, 2\pi]\,, v \in [0, \pi]\,$, we obtain the surface

$$S_1^t = \left\{ \left[\left[F(u, v, \alpha_1, \beta_1, \frac{1}{6}), G(u, v, \alpha_2, \beta_2, \frac{1}{6}), H(u, v, \alpha_3, \beta_3, \frac{1}{6}) \right] \right] : u \in [0, 2\pi]\,, v \in [0, \pi] \right\}, \tag{10}$$

which is shown in Figure 2(a). On the other hand, if we restrict the domain of S_1 to $u \in [0, 2\pi]\,, v \in [\pi, 2\pi]\,$, we obtain

$$S_1^b = \left\{ \left[F(u, v, \alpha_1, \beta_1, \frac{1}{6}), G(u, v, \alpha_2, \beta_2, \frac{1}{6}), H(u, v, \alpha_3, \beta_3, \frac{1}{6}) \right] : u \in [0, 2\pi]\,, v \in [\pi, 2\pi] \right\}, \tag{11}$$

which is shown on Figure 2(b).

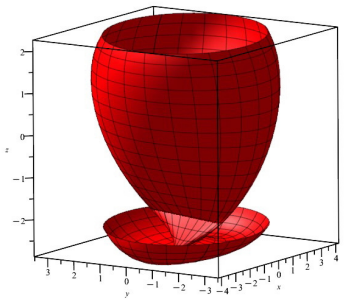


Figure 2(a). Graph of S_1^t

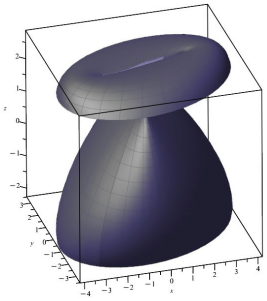


Figure 2(b). Graph of S_1^b

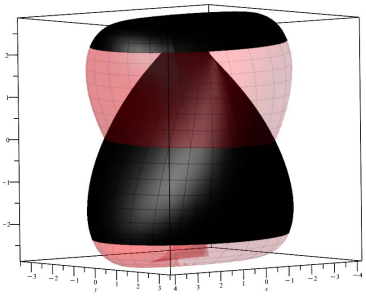


Figure 2(c). Graph of $S_1^t \cup S_1^b$

2. We included a Maple worksheet for readers to explore in [S2] on the closed surface shown in Figure 2(c) and the ellipsoid shown in Figure 1(b) .

3. With $\alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{\pi}{3}, \alpha_3 = \frac{\pi}{6}, \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{3}$, $\beta_3 = \frac{\pi}{4}$, and $s = \frac{1}{6}$, the animations of the space curves, using ([1]) for the longitudes (when $u \in [0, 2\pi]$), and latitudes for S_1 (when $v \in [0, 2\pi]$) respectively provide us some interesting graphical aspects of how S_1 is generated:

   (a) `animate(plot3d, [[F(u, v, 1/6), G(u, v, 1/6), H(u, v, 1/6)], u = 0 .. 2*Pi, v = 0 .. t], t = 0 .. 2*Pi, thickness = 3, color = yellow);`

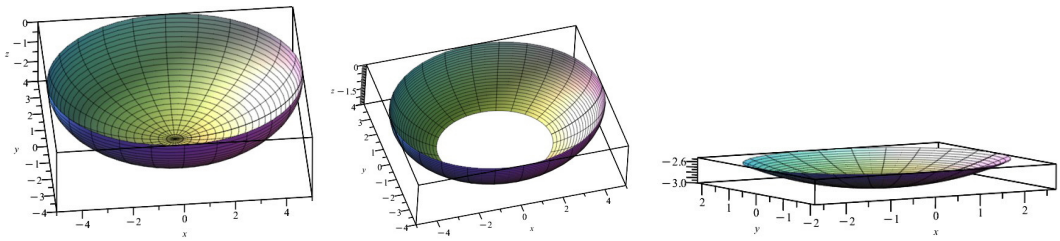
   (b) `animate(plot3d, [[F(u, v, 1/6), G(u, v, 1/6), H(u, v, 1/6)], u = 0 .. t, v = 0 .. 2*Pi], t = 0 .. 2*Pi, thickness = 3, color = yellow);`

   (c) See [S3] and [S4] respectively for the animations.

4. We remind readers that $S_1 = S_1^t \cup S_1^b$, shown in either Figures 1(a) or 2(c), resembles the union of two open tea-cups as seen in Figure 2(c) or its transparent 3D printing of Figure 5(b). If we look at the 3D printings for S_1^t (red) and S_1^b (black) in Figure 5(a), and if S_1 were topologically equivalent to the ellipsoid S^*, it is equivalent of asking the following

Sub-question: Is S_1^t or S_1^b is topologically equivalent to an open hemisphere with a boundary of a flat circular base? We proceed as follows:

1. First we set E_L to be the lower half of the ellipsoid S^* (see Figure 3(a)). In other words, E_L is $\begin{bmatrix} f(u,\frac{v}{2}) \\ g(u,\frac{v}{2}) \\ h(u,\frac{v}{2}) \end{bmatrix} = \begin{bmatrix} 5\sin(\frac{v}{2})\cos(u) \\ 4\sin(\frac{v}{2})\sin(u) \\ 3\cos(\frac{v}{2}) \end{bmatrix}$, where $u \in [0, 2\pi]$ and $v \in [\pi, 2\pi]$. We can decompose E_L into two sub-surfaces by cutting it with a horizontal plane $z = \pi+1$. We see $E_L = E_{L_1} \cup E_{L_2}$, with $E_{L_1} = \left[f(u,\frac{v}{2}), g(u,\frac{v}{2}), h(u,\frac{v}{2})\right]$, $u \in [0, 2\pi]$ and $v \in [\pi, \pi+1]$ (see Figure 3(b)), and $E_{L_2} = \left[f(u,\frac{v}{2}), g(u,\frac{v}{2}), h(u,\frac{v}{2})\right]$, where $u \in [0, 2\pi]$ and $v \in [\pi+1, 2\pi]$ (see Figure 3(c)). We see that E_{L_1} is an open surface with two boundaries, and E_{L_2} is another open surface with only one boundary, which is topologically equivalent to E_L or a hemisphere. In fact, any horizontal plane $z = v_0 \in (\pi, 2\pi)$ will result in similar topological decompositions.



Figures 3(a),(b) and (c). Plots of E_L, E_{L_1} and E_{L_2} respectively

2. We now decompose S_1^t into two sub-surfaces using the horizontal plan $z = \frac{\pi}{2}$. We see $S_1^t = S_{11}^t \cup S_{12}^t$, where S_{11}^t if $u \in [0, 2\pi]$, $v \in [0, \frac{\pi}{2}]$, shown in Figure 4(a) below. On the other hands, if $u \in [0, 2\pi]$, $v \in [\frac{\pi}{2}, \pi]$, we obtain the surface S_{12}^t, which is shown in Figure 4(b). We see S_{11}^t is topologically equivalent to E_{L_1}. However, since S_{12}^t is a surface which has two disjoint elliptical rims (indicated in red in Figure 4(b)), it cannot be topologically equivalent to E_{L_2} which only has one (circular) rim.

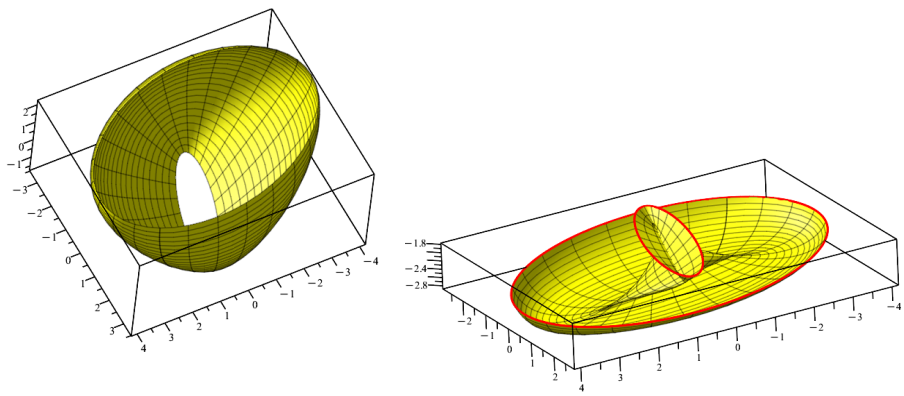


Figure 4(a). The plot of S_{11}^t. Figure 4(b). The plot of S_{12}^t.

3. We encourage readers to find the precise value v so that $(S_{11}^t)^*$ resembles the Figure 4(c), and $(S_{12}^t)^*$ is shown in Figure 4(d) , which both contain a *pinched* red line. For example,

one may consider picking a proper value for v in the equation for S_1 (9) (when $\alpha_1 = \frac{\pi}{4}$ and $\beta_1 = \frac{\pi}{2}$).

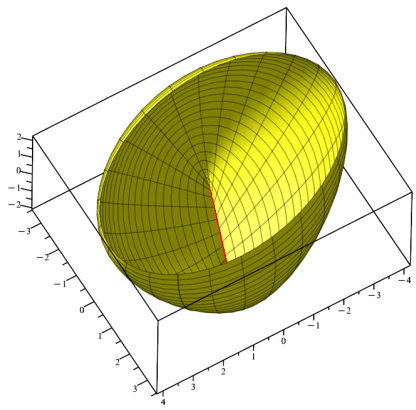


Figure 4(c). The red pinch-line for $(S_{11}^t)^*$, when $v \approx \frac{\pi}{2} + 0.15167$

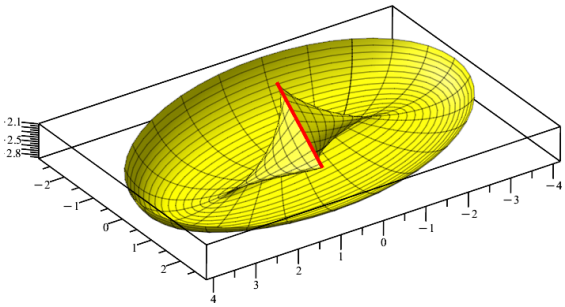


Figure 4(d). The red pinch-line for $(S_{12}^t)^*$, when $v \approx \frac{\pi}{2} + 0.15167$

Alternatively, we can find the places where we have pinch-lines as follows:

(a) For fixed t and $u \in [0, 2\pi]$, the graphs of the function

$$F^* : (u,t) \to \left[F(u,t,\alpha_1,\beta_1,\frac{1}{6}), G(u,t,\alpha_2,\beta_2,\frac{1}{6}), H(u,t,\alpha_3,\beta_3,\frac{1}{6})\right]$$

represents set of ellipses, and we conjecture through the animation in [S5] that there exist t such that ellipses degenerate to a pinched line. We shall show graphically how such t can be found such that the ellipses $F^*(u,t)$ degenerate to a line segment.

(b) It is clear that we may project the ellipses onto $z = 0$ without altering the respective areas. We set up the area function for the ellipses as

$$J(t) = \int_0^{2\pi} G(u,t,\alpha_2,\beta_2,\frac{1}{6}) \left(\frac{d}{dt} F(u,t,\alpha_1,\beta_1,\frac{1}{6})\right) du.$$

(c) We plot $y = J(t)$, $t \in [0, 2\pi]$ using ([1]) as in Figure 4(e).

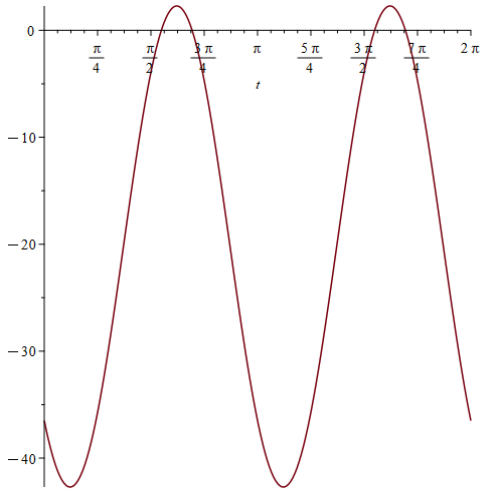


Figure 4(e). Plot of $y = J(t)$

(d) Now we set the area of ellipses $J(t) = 0$, and find the first

$$t = t_1 = 1.724344291. \tag{12}$$

We invite the readers to find the remaining places where ellipses are degenerated to pinch-lines, they are

$$t_2 = 2.178274393, t_3 = 4.865936945, t_4 = 5.319867047. \tag{13}$$

These pinch-lines will subsequently prove useful as we present the formal proofs that the surface S_1 is not homeomorphic to the ellipsoid or the sphere (see Section 3).

4. Therefore, we see that $(S_{11}^t)^*$ can not be topologically equivalent to E_{L_1} since there does not exist a homeomorphism between $(S_{11}^t)^*$ and E_{L_1}. Similar reasons can be drawn to show that $(S_{12}^t)^*$ cannot be topologically equivalent to E_{L_2}.

Hence, S_1^t cannot be topologically equivalent to a hemisphere with a boundary of a flat circular base. We conclude that neither S_1^t nor S_1^b is topologically equivalent to a hemisphere

with a boundary. Therefore, S_1 cannot be topologically equivalent to the ellipsoid S^*.

Figure 5(a) 3D printings of S_1^b (black) and S_1^t (red).

Figure 5(b) 3D printing of 2(c).

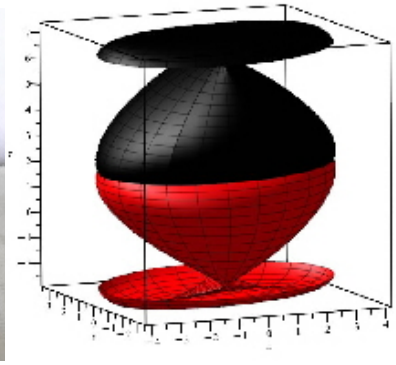


Figure 5(c). Mathematical plot of the 3D printing 5(a)

Remarks:

1. Since both S_1 and S^* are both closed surfaces, it is non-trivial at a glance, if two are topological equivalent. It is only after exploration with technological tools and 3D printing by knowing the 3D printing of Figure 5(b) is exactly the copy of $S_1 = S_1^t \cup S_1^b$ (or Figure 2(c)), do we realize that S_1 is a union of two open tea cups of S_1^t and S_1^b.

2. We also remark that we printed the 3D surfaces S_1^t (in red) and S_1^b (in black) individually, and stack S_1^b on top of S_1^t to get the graph of Figure 5(a). By using simple vertical shifting along the $z-$axis, we leave it to readers to explore how we can *generate the equation for the surface* that look like Figure 5(c).

3. Intuitively, if we start with the mathematical equation for S_2 (or Figure 5(c)), then it is much easier to visualize that S_2 is not topological to the ellipsoid S^* or a sphere.

3 Proofs: S_1 Is Not Homeomorphic To A Sphere

We recall a manifold in mathematics is a topological space that locally resembles Euclidean space near each point, see [S6]. We use the following notation throughout this section:

$$S_1(u,v) := (F(u,v), G(u,v), H(u,v)),$$

where

$$\begin{aligned} F(u,v) &= \frac{5\cos u \sin v + 25\cos(u+\frac{\pi}{4})\cos v}{6}, \\ G(u,v) &= \frac{2\sin u \sin v + 10\sin(u+\frac{\pi}{3})\sin(v+\frac{\pi}{3})}{3}, \\ H(u,v) &= \frac{\cos v + 5\cos(v+\frac{\pi}{4})}{2}. \end{aligned} \tag{14}$$

We shall use 3D Desmos to help us visualize the manifold and the accompanying details that we will disclose subsequently. Readers are encouraged to visit the graphical animation available at [S7] or go to`https://www.desmos.com/3d/lcr28pthtr`. We remind readers that the following numerical values are simply approximations for demonstration purposes only.

We decompose the manifold S_1 into four parts:

- f_1 represents the part where $0 \leq v \leq 2.18$. (See Figure 6). We observe that there are two pinch-lines when $v \in [0, 2.18]$. We have discussed the first pinched line from Equation (12). Here $v \approx 2.18$ tallies with the location of the second pinch-line (see below).

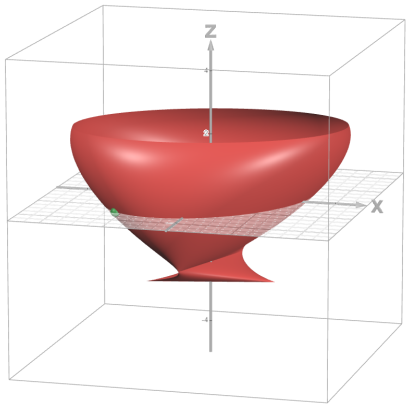


Figure 6. Surface for f_1

- f_2 represents the part where $2.18 \leq v \leq 4.0503$ which is the part ensuing from the second pinch-line and that lies in the lower plane $z \leq 0$.
- f_3 represents the part where $4.0503 \leq v \leq 4.87$. The value $v = 4.87$ tallies with the third pinch-line.
- f_4 represents the part where $4.78 \leq v \leq 2\pi$, which is the remaining portion ensuing from the third pinch-line and that lies in the upper plane, i.e., $z \geq 0$.

The reader can check on the corresponding circular icon next to the various f_i's to make the corresponding surface appear or disappear. One may also use the mouse-click to hold and rotate the manifold S_1 so as to view it from different angles.

3.1 Pinch-lines make S_1 not being topologically equivalent to a sphere

In this section, we shall supply the readers with the graphical details that lead to the following two major observations:

1. There are exactly *four* distinct so-called pinch-lines on the manifold S_1; the removal of which would result in a disconnected space with disjoint components.

2. The manifold S_1 is *self-intersecting* in the sense that there exists two distinct pairs of parameters (u, v) and (u', v') such that

$$S_1(u, v) = S_1(u'v')$$

but the tangential planes to the curve (that passes through this point) corresponding to these two pairs of parameters are non-parallel.

Using these results obtained graphically, we observe two different arguments that justify the result that S_1 is not homeomorphic to the sphere.

We begin by establishing the locations of the four pinch-lines that lie on S_1. To do this, we make use of the latitudes, given by

$$L_t = S_1(u, t), \qquad u \in [0, 2\pi].$$

From the graphical display, it can be seen that the latitude L_t is an ellipse centred at $(0, 0, H(u, t))$. The latitude L_t is given by the equation labelled by f_6 in the 3D Desmos display.

Because the analytical approach is way too tedious, we rely on a graphical, whence qualitative, approach.

Since $H(u, t)$ is independent of u it follows that this ellipse is parallel to the xy plane. As t increases over the interval $[0, 2\pi]$, the latitude L_t goes through the following vertical motion:

1. It begins its descent from the initial position at the rim of the open cup S_1^t (at $t = 0$) to the first pinch-line (at approximately $t = 1.722\,5$, see (12)).

2. It then continues its descent to the second pinch-line (at approximately $t = t_2 = 2.178274393$, see (13).

3. It continues its descent till it reaches it lowest height at approximately $t = 2.48$, and then starts its journey of ascent. Note that this is the value of t that minimizes $H(u, t)$ over the domain $[0, 2\pi]$.

4. It ascends till it reaches the third pinch-line (at approximately $t = t_3 = 4.865936945$, see (13), which is exactly π after the first pinch-line).

5. It then continues its ascent and reaches the fourth pinch-line (at approximately $t = t_4 = 5.319867047$ see (13), which is exactly π after the second pinch-line).

We now turn to the our second observation. This is inspired by examining the movement of the longitude and animation from [S4] that the surface S_1 is self-intersecting at several places. We observe the first place of such self-intersecting occurs at the junction where two cups meet (see Figures 2(c) and 5(b)), whose equation is given by

$$L^t = S_1(t, v), \qquad v \in [0, 2\pi].$$

This longitude is given the label f_5 in the 3D Desmos display. As t increases over the interval $[0, 2\pi]$, the longitude L^t gyrates counterclockwise (when viewed from above) relative to the xy-plane. It is worth-noting that at any value of $t \in [0, 2\pi]$ the longitude L^t cuts each of the pinch-lines at exactly one point. In particular, the direction of motion of these intersection

points with the two pinch-lines in Figure 6, for instance, are in *different* directions – hence, indicating the presence of a twist from the surface only up to the first pinch-line to the other surface shown in Figure 6, say f_1.

From the 3D Desmos display of the manifold S_1, one sees that the manifold self-intersects at the plane $z = 0$. More precisely, this happens when $H(t, v) = 0$, i.e.,

$$\cos(v) + 5\cos\left(v + \frac{\pi}{4}\right) = 0$$
$$\sqrt{5\sqrt{2} + 26}\arctan\left(\frac{5}{23}(5 - \sqrt{2})\right) = 0$$
$$v \approx 0.9087,\ 4.0503.$$

This then yields the two points of intersection of the longitude L^t with the plane $z = 0$:

$$P(t) = S_1(t, 0.9087), \qquad t \in [0, 2\pi],$$

and

$$Q(t) = S_1(t, 0.4.0503), \qquad t \in [0, 2\pi].$$

The equation for $P(t)$ is labelled f_8, while $Q(t)$ has been labelled f_7 in the 3D Desmos display.

In particular, for any $t \in [0, 2\pi]$ it always holds that

$$Q(t) = P(t + \pi).$$

Consequently, at any instant t, the tangential plane passing through $P(t)$ is always tangential to the lower half of the manifold, i.e., where $z \leq 0$, and the tangential plane passing through $Q(t)$ is always tangential to the upper half of the manifold, i.e., where $z \geq 0$. Hence at any instant t, the gradient vectors ∇S_1 at the points, $P(t)$ and $Q(t) = P(t + \pi)$ respectively, share the same x, y-components but have opposite signs for the z-component. We encourage the keen readers to calculate the explicit expression for ∇S_1 at the points $P(t)$ and $Q(t)$.

Note. We invite readers to explore that the surface S_1 is also self-intersecting at those four pinch-lines.

It follows from those two observations mentioned above, we are now ready to reap the fruits of our hard labor.

Proof 1.

We now argue that S_1 is not homeomorphic to the ellipsoid S^*. For the sake of contradiction, let us suppose that S_1 is homeomorphic to the ellipsoid S^*, say via the homeomorphism $f : S_1 \longrightarrow S^*$. Now, we remove the four pinch-lines $\ell_1, \cdots, \ell_4$ from the manifold S_1. By so doing, the resulting manifold $S_1 - \bigcup_{k=1}^4 \ell_k$ becomes a disconnected space that comprises of four disjoint components. Since f is a homeomorphism, $f(\ell_1), \cdots, f(\ell_4)$ represents four disjoint simple paths on the S^*. Even after removing these four simple paths, the resulting manifold still remains path-connected and hence connected (just like S^*). Now the restriction of f on the space $S_1 - \bigcup_{k=1}^4 \ell_k$ is still a homeomorphism between the two resultant spaces. But the space $S_1 - \bigcup_{k=1}^4 \ell_k$ is disconnected while the space $S^* - \bigcup_{k=1}^4 f(\ell_k)$ is connected, which is the desired contradiction we hope to arrive at. In conclusion, S_1 is not homeomorphic to the ellipsoid S^* and hence S_1 is not homeomorphic to a sphere.

We now give the second proof that S_1 is not homeomorphic to S^*.

Proof 2.
Our second observation informs us that S_1 is self-intersecting, i.e., the manifolds cuts itself at the plane $z = v_k, k = 1, 2, 3, 4$, where v_k's are places for those four pinch-lines. Had it been that S_1 is homeomorphic to the sphere S^*, then the S^* would also be self-intersecting, which we know is not for a fact! Thus, we may conclude that S_1 is not homeomorphic to S^*or a sphere, as desired.

4 Acknowledgements

The authors would like to express their gratitude to Petra SURYNKOVA and Laura MOSS for creating the 3D printing for Figures 11(d), which will aid students in understanding topological structures for surfaces.

5 Conclusion

In this paper, we employ the affordance of technological tools – three-dimensional visualization – to provide us crucial intuitions as we work towards more rigorous solutions to the topological problem at hand. More specifically, we have gained graphical insights by relying on graphing tools like *[1]* and *[2]*. Dynamic geometry software, together with computer algebra, points us to many valid conjectures, which otherwise would have evaded us. While it has been frequently reported that technological tools make learning mathematics fun and accessible, our partnership with computer technology shows that such tools also facilitate explorations into more challenging and theoretical mathematics, such as lower dimension topology in this case. It is hoped that this present technology-aided mathematical exploration convinces our readers that mathematical investigations – no matter simple or advanced – are accessible to students at all levels. Notably, 3D printing makes abstract and complicated objects *concrete*; for instance, students can now physically touch and feel the four pinch-lines, hence reinforcing their topological understanding.

Ultimately, inculcating a greater interest in mathematics for students by providing them accessible technological tools and concrete manipulatives goes a long way towards encouraging students to take on more challenging and intricate problems, which otherwise would remain beyond the reach of traditional pencil-and-paper approach. This we believe forms an important step towards cultivating mathematical creativity and innovation in students.

6 Supplementary Electronic Materials

[S1] Maple worksheet for Exercises:

https://atcm.mathandtech.org/EP2024/invited/22174/Exercise.mw.

[S2] Maple worksheet for two closed surfaces:

https://atcm.mathandtech.org/EP2024/invited/22174/two_closed_surfaces.mw.

[S3] Longitude animation for S_1: https://atcm.mathandtech.org/images/S1_longitude.gif.

[S4] Latitude animation for S_1: https://atcm.mathandtech.org/images/S1_latitude.gif.

[S5] The pinched line: https://atcm.mathandtech.org/EP2024/invited/22174/pinchline.gif.

[S6] Manifold from Wikipedia: https://en.wikipedia.org/wiki/Manifold.

[S7] Desmos Approach: https://www.desmos.com/3d/lcr28pthtr.

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[1] Maple, A product of Maplesoft, see http://Maplesoft.com/.

[2] A free too: https://www.desmos.com/.

[3] Shi Gi Gin Bang. "Strategies for High School Mathematics Complete Review". In: ed. by Cuiun Guan Zhiming. Century Gold. Yanbian University Press, 2015.

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[8] Yang, W.-C, DÁVILA, D., and HO, W. K. 'Another Topological View of Curves and Surfaces Inspired by 2D and 3D Locus Problems', (June 2023). The Electronic Journal of Mathematics and Technology, 17(2), ISSN 1933-2823, pp78-180.

Polynomials associated with bicentric polygons

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Abstract

A bicentric polygon *is one all of whose vertices lie on a circle (this makes it a* cyclic polygon*), and all of whose edges are tangential to another circle (this makes it a* tangential polygon*). All triangles are bicentric—every triangle has a circumcircle and an incircle—but not all quadrilaterals are bicentric. Non-square rectangles, for example, are cyclic but not tangential, and any non-square rhombus is tangential without being cyclic. Bicentric polygons are also important to a result called* Poncelet's Porism*, or* Poncelet's Closure Theorem*, which says in effect that the circles corresponding to any bicentric polygon also correspond to an infinite number of other polygons: indeed every point on the outer circle will be a vertex of some such polygon. Much work has gone into determining formulas connecting the radii of the two circles and the distance between their centres; this goes back to Euler and is still being actively investigated. As the number of sides of the polygon increases, the corresponding formulas grow in complexity. In this article we explore these formulas as polynomials in one variable, using dynamic geometry as our means of exploration.*

1 Introduction

Any plane triangle has both an incircle (tangential to all sides) and a circumcircle on which all vertices lie. In his *Elements*, Euclid showed how to construct both the incircle (Book IV Proposition 4) and a circumcircle (Book IV Proposition 5). However, Euler was the first to demonstrate a relation between the radii R and r of the circumcircle and incircle respectively, and of d, the distance between their centres:

$$R^2 - 2Rr - d^2 = 0. \tag{1}$$

However, given R, r, and d satisfying the above equation, there are an infinite number of associated triangles. In particular, given R and d, then

$$r = \frac{1}{2}\left(R - \frac{d^2}{R}\right).$$

For example, if $R = 1$ and $d = 0.5$ then we obtain $r = (1 - d^2)/2 = 0.375$. Figure 1 shows some examples of different triangles with the same circles.

Euler's equation can be written also as

$$\frac{1}{R+d} + \frac{1}{R-d} = \frac{1}{r}.$$

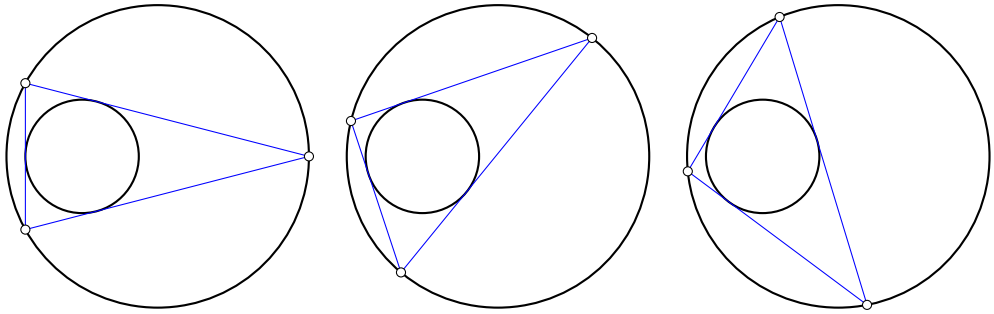


Figure 1: Triangles with the same incircle and circumcircle

Not long after Euler, Nicolas Fuss, another mathematician and Euler's assistant[1], extended this result to cyclic quadrilaterals [6]:

$$\frac{1}{(R+d)^2} + \frac{1}{(R-d)^2} = \frac{1}{r^2}. \tag{2}$$

From this, given R and d, a corresponding r can be obtained as:

$$r = \frac{R^2 - d^2}{\sqrt{2(R^2+d^2)}}.$$

Again, given appropriate values, there are an infinite number of different bicentric quadrilaterals with the same circles, as shown in figure 2.

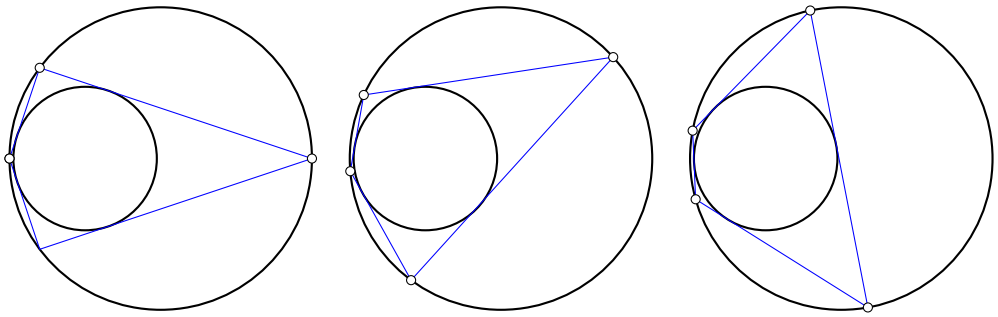


Figure 2: Bicentric quadrilaterals with the same incircle and circumcircle

The results shown in figures 1 and 2 are special cases of a deep and impressive result called *Poncelet's Porism* or *Poncelet's Closure Theorem*, one form of which can be stated as:

> Suppose C and D are two circles, with D lying entirely inside C. From any point p_0 on C, find the tangent to D, and let p_1 be the other point of its intersection with C. Similarly, define p_2 on C so that the line $[p_1, p_2]$ is tangential to D, and so on. This is illustrated in figure 3(a). If this series of points eventually "joins up"; that is, if there is a value k for which $p_k = p_0$, then this series will close independently of the initial point.
>
> In other words, if the path is closed for one starting point, it is closed for all possible starting points.

The result is generalizable to conic sections, and without the requirement of one being "inside" the other, and is considered one of the greatest theorems of plane geometry.

[1] He also married one of Euler's grand-daughters.

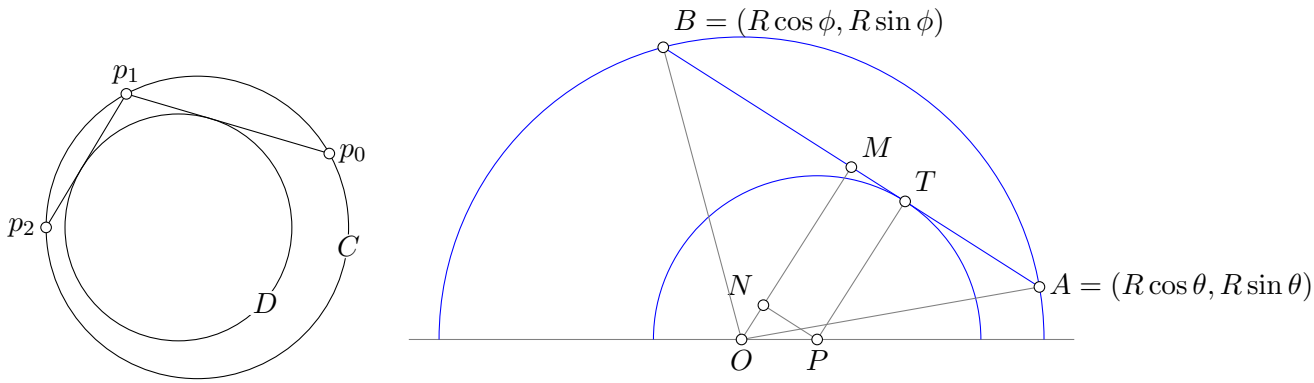
2 Bicentric relations and polynomials

We might hope, from the similarity of equations (1) and (2), that elegant and simple expressions, maybe with the same sort of structure, would be true for bicentric polynomials with more sides. Unfortunately this is not the case, and the relations quickly grow in complexity:

$$\begin{aligned}
\text{pentagon: } & (2Rr + d^2 - R^2)^3 = 8Rr(R^2 - d^2)((R-r)^2 - d^2) \\
\text{hexagon: } & 3(R^2 - d^2)^4 = 4r^2(R^2 + d^2)(R^2 - d^2)^2 + 16R^2d^2r^4 \\
\text{heptagon: } & (-R^2 + 4Rr + d^2)(R^2 - d^2)^5 - 4R(R + 6r)(R^2 - d^2)^4 \\
& \quad = 16Rr^3(2R^2 + Rr - 2r^2)(R^2 - d^2)^3 - 16(R - 6r)R^3r^4(R^2 - d^2)^2 \\
& \qquad - 64(R + r)R^4r^5(R^2 - d^2) + 64r^6R^6
\end{aligned}$$

Relations for polygons with a prime number of sides are particularly complicated.

We show now how to obtain these relations. First of all we need a relationship between the two points on the outer circle, for which the line between them is tangential to the inner circle. This is shown in figure 3(b) which is taken from McAndrew [8], following Kerawala [7].



(a) Poncelet traverse

(b) Relationships between chord and circles

Figure 3: The Poncelet traverse and its definition

From this diagram, we note that:

1. Angle AOM is $(\phi - \theta)/2$ and so the length of OM is $R\cos(\phi/2 - \theta/2)$.
2. By chasing angles, we find that angle PON is $(\phi + \theta)/2$ and since the length of OP is d, so the length of ON is $d\cos(\phi/2 + \theta/2)$.
3. Since $ON + PT = OM$, and since $PT = r$, we have $ON - OM = r$.

Thus we have the following expression relating all the variables:

$$R\cos\left(\frac{\phi}{2} - \frac{\theta}{2}\right) - d\cos\left(\frac{\phi}{2} + \frac{\theta}{2}\right) = r.$$

Using the standard expansion formulas, the above expression can be written as

$$R\left(\cos\left(\frac{\phi}{2}\right)\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right) - d\left(\cos\left(\frac{\phi}{2}\right)\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right) = r.$$

Collecting the cosine and sine terms produces the following:

$$(R + d)\sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}\right) + (R - d)\cos\left(\frac{\phi}{2}\right)\cos\left(\frac{\theta}{2}\right) = r. \tag{3}$$

In order to bring this expression into the theory of polynomial rings, we use the half angle substitutions

$$s = \tan\left(\frac{\theta}{4}\right), \quad t = \tan\left(\frac{\phi}{4}\right)$$

so that

$$\cos\left(\frac{\theta}{2}\right) = \frac{1-s^2}{1+s^2}, \quad \sin\left(\frac{\theta}{2}\right) = \frac{2s}{1+s^2},$$

and similarly for t and ϕ. Substituting these into equation (3) produces

$$4(R-d)st + (R+d)(1-s^2)(1-t^2) = r(1+s^2)(1+t^2). \tag{4}$$

If the points A and B are connected by a vertical tangent to the inner circle (as for example the vertical line in the left-most triangle of figure 1), then $t = -s$ and equation (4) simplifies to

$$4(R-d)s^2 + (R+d)(1-s^2)^2 = r(1+s^2)^2$$

and this can be rewritten as

$$R(1-6s^2+s^4) = (r-d)(1+s^2)^2. \tag{5}$$

Equations (4) and (5) were first obtained by Kerawala [7], although presented in a slightly different form.

Obtaining polynomial relations

Note that in fact this problem has been satisfactorily solved in many different ways over the years, often involving elliptic functions. See for example Mirman [9], Radić et al [10], and the historical review by del Centina [4, 5]. An original 1861 paper by Cayley is available through the Royal Society Publishing portal [1]. However, theoretically important as these solutions are, none of them help in actually drawing the polygons, or distinguishing one type of polygon (convex or star) from another. Our approach provides a greater geometric intuition, and is closer in spirit to the Euclidean nature of the porism.

We now show, as an example, how to turn the equations into relations for bicentric polygons; starting with a pentagon. We can, without loss of generality, place and rotate our figure so that the outer circle is centred at the origin, and the inner circle has its centre on the x axis. By Poncelet's Porism, we can start anywhere, so let the first point be $p_0 = (R, 0)$; here $s = 0$. By symmetry, the other four points will be mirrored about the x-axis. If points p_2 and p_3 correspond to parameters t and u, we have tangents from p_0 to p_1, from p_1 to p_2, and a vertical tangent between p_3 and is mirror image. Thus a bicentric pentagon could be defined by the three equations:

$$\begin{aligned}
&4(R-d)st + (R+d)(1-s^2)(1-t^2) - r(1+s^2)(1+t^2) = 0\\
&4(R-d)tu + (R+d)(1-t^2)(1-u^2) - r(1+t^2)(1+u^2) = 0\\
&R(1-6u^2+u^4) - (r-d)(1+u^2)^2 = 0.
\end{aligned}$$

However, given that we will be using a computer algebra system to simplify the polynomials, we can in fact use the simpler formulation given by equation (3) in the form:

$$\begin{aligned}
&(R+d)y_1y_2 + (R-d)x_1x_2 = r\\
&x_1^2 + y_1^2 = 1\\
&x_2^2 + y_2^2 = 1
\end{aligned} \tag{6}$$

Eliminating s, t, u from these equations will produce the relation we want. Here is where a computer algebra system which supports Gröbner bases will be useful. Here's how we might do this in Sagemath:

```
sage: tg(x1,y1,x2,y2) = (R-d)*y1*y2+(R+d)*x1*x2-r

sage: PR.<x1,x2,y1,y2,R,r,d> = PolynomialRing(QQ)
sage: I = PR.ideal([tg(1,0,x1,y1), tg(x1,y1,x2,y2), tg(x2,y2,x2,-y2), x1^2 +
y1^2 -1, x2^2 + y2^2 - 1])
sage: ie = I.elimination_ideal([x1,x2,y1,y2])
sage: pent = ie.gens().factor()[-1][0]
```

The result is

$$-4r^2R^4 - 2rR^5 + R^6 + 8r^3Rd^2 + 4r^2R^2d^2 + 4rR^3d^2 - 3R^4d^2 - 2rRd^4 + 3R^2d^4 - d^6 \quad (7)$$

and some fiddling by collecting terms produces a slightly neater expression:

$$(R^2 - d^2 + 2Rr)(R^2 - d^2 - 2Rr)^2 - 8r^2R(R^2 - d^2).$$

The relation for the hexagon is similar; by symmetry it must have points at $(R, 0)$ and $(-R, 0)$ plus two others. There's a slight subtlety here, in that the (x_i, y_i) values are the cosine and sine of the half angles. Thus at $(-R, 0)$, the half angle is $\pi/2$ and so $(x_i, y_i) = (0, 1)$. Again in Sagemath:

```
sage: tg(x1,y1,x2,y2) = (R-d)*y1*y2+(R+d)*x1*x2-r

sage: PR.<x1,x2,y1,y2,R,r,d> = PolynomialRing(QQ)
sage: I = PR.ideal([tg(1,0,x1,y1), tg(x1,y1,x2,y2), tg(x2,y2,0,1), x1^2 +
y1^2 -1, x2^2 + y2^2 - 1])
sage: ie = I.elimination_ideal([x1,x2,y1,y2])
sage: hex = ie.gens()[0].factor()[-1][0]
```

with output

$$-3R^8 + 4R^6r^2 + 12R^6d^2 - 4R^4r^2d^2 + 16R^2r^4d^2 - 18R^4d^4 - 4R^2r^2d^4 + 12R^2d^6 + 4r^2d^6 - 3d^8$$

By repeatedly dividing and keeping the remainder with different powers of $R^2 - d^2$, this expression can be written as

$$\begin{aligned} &-3(R^2 - d^2)^4 + 4r^2(R^6 - R^4d^2 + 4R^2r^2d^2 - R^2d^4 + d^6) \\ &\qquad - 3(R^2 - d^2)^4 + 4r^2((R^2 + d^2)(R^2 - d^2)^2 + 4R^2r^2d^2) \end{aligned}$$

or as

$$-3(R^2 - d^2)^4 + 4r^2(R^2 + d^2)(R^2 - d^2)^2 + 16R^2r^4d^2.$$

All these relations, of course, are to be set equal to zero for the values of R, r and d to form bicentric polygons.

A new set of polynomials

It will be seen that the relations just found increase in complexity as the number of sizes increases, and the relations because very quickly too unwieldy for any reasonable use. Many substitutions have been used to simplify these relations; Kerawala [7] recommended many, starting with

$$a = \frac{1}{R+d}, \quad b = \frac{1}{R-d}, \quad c = \frac{1}{r}$$

and

$$p = \frac{R+d}{r}, \quad q = \frac{R-d}{r}.$$

Using these, for example, Fuss's relation for bicentric quadrilaterals becomes

$$a^2 + b^2 = c^2$$

and the relation for the pentagon becomes

$$4p^2q^2(p-1)(q-1) = (p^2 + q^2 - p^2q^2)^2.$$

Our approach is different.

We note that any diagram can be scaled so that there are only two independent parameters (this was the case for the p and q substitution). We shall form polynomial relations by first setting $R = 1$, and writing the relation as a polynomial in r, whose coefficients are polynomials in d.

Quadrilateral. We start with $(R^2 - d^2)^2 = 2r^2(R^2 + d^2)$, which is a polynomial form of Fuss's relation given in equation (2). Setting $R = 1$ produces

$$(1 - d^2)^2 = 2r^2(1 + d^2) \quad \Rightarrow \quad 2(1 + d^2)r^2 - (1 - d^2)^2.$$

This means that given any d with $0 \leq d < 1$, we can find an appropriate r for bicentric quadrilaterals.

Pentagons. With $R = 1$, equation (7) reduces to

$$-d^6 + 8r^3d^2 - 2rd^4 + 4r^2d^2 + 3d^4 + 4rd^2 - 4r^2 - 3d^2 - 2r + 1$$

which as a polynomial equation in r becomes:

$$8d^2r^3 + 4(d^2 - 1)r^2 - 2(d^2 - 1)^2r - (d^2 - 1)^3 = 0. \tag{8}$$

Hexagons. The polynomial equation is

$$16d^2r^4 + 4(d^2 + 1)(d^2 - 1)^2r^2 - 3(d^2 - 1)^4 = 0. \tag{9}$$

Note that this is a quadratic in r^2, and so can be solved analytically.

Heptagons. The Sagemath expressions are similar to those for the pentagon, except that we have an extra point:

```
sage: tg1(x1,y1,x2,y2) = (R+d)*y1*y2+(R-d)*x1*x2-r
sage: PR.<x1,y1,x2,y2,x3,y3,R,r,d> = PolynomialRing(QQ)
sage: eqs1 = [tg1(1,0,x1,y1), tg1(x1,y1,x2,y2), tg1(x2,y2,x3,y3),tg1(x3,y3,x3,-y3)]
sage: eqs2 = [x1^2 + y1^2 - 1, x2^2 + y2^2 - 1, x3^2 + y3^2 - 1]
sage: I = PR.ideal(eqs1 + eqs2)
sage: ie = I.elimination_ideal([x1,x2,y1,y2,x3,y3])
sage: e7rd = ie.gens()[0].factor()[-1][0]
sage: e7rd
```

$$R^{12} + 4R^{11}r - 4R^{10}r^2 + [22 \text{ terms not shown}] - 6R^2d^{10} - 4Rrd^{10} + d^{12}$$

This expression could perhaps be simplified, but our polynomial form can be found to be:

$$\begin{aligned} 64d^2r^6 - 32(d^2+1)(d^4-1)r^5 - 16d^2(d^2-1)^2r^4 \\ + 8(d^2-1)^3(3d^2+1)r^3 - 4(d^2-1)^4r^2 - 4(d^2-1)^5r + (d^2-1)^6 = 0. \end{aligned} \tag{10}$$

Note that this polynomial can be simplified by the substitutions $x = 2r$, $b = d^2 - 1$ to obtain:

$$(b+1)x^6 - b(b+2)^2x^5 - b^2(b+1)x^4 + b^3(3b+4)x^3 - b^4x^2 - 2b^5x + b^6 = 0.$$

We shall find that these polynomials provide an elegant method of not only drawing bicentric polygons, but classifying them into different forms.

3 Bicentric pentagons

As opposed to bicentric triangles and quadrilaterals, which only exist in convex forms, bicentric pentagons exist in two forms: a convex pentagon, and a pentagonal star, sometimes known as a pentagram.

Figure 4 shows both types.

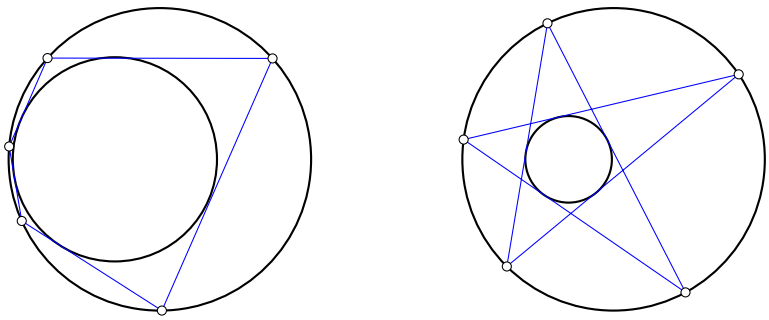


Figure 4: Bicentric pentagons

With $R = 1$, our aim is to solve the equation $8d^2r^3 + 4(d^2-1)r^2 - 2(d^2-1)^2r - (d^2-1)^3 = 0$ from equation (8) and for different values of d. Note that when $d = 0$ (so that the two circles are concentric), the equation is

$$-4r^2 - 2r + 1 = 0$$

which has solutions

$$r = \frac{-1-\sqrt{5}}{4}, \quad r = \frac{-1+\sqrt{5}}{4}.$$

These values are both negative, but since they are being taken to be the radii of circles, we can use their absolute values:

$$|r| = \frac{1+\sqrt{5}}{4}, \quad |r| = \frac{-1+\sqrt{5}}{4}.$$

Now if the circles are concentric, the radii of the inner circle can be easily found using standard trigonometry as shown in figure 5.

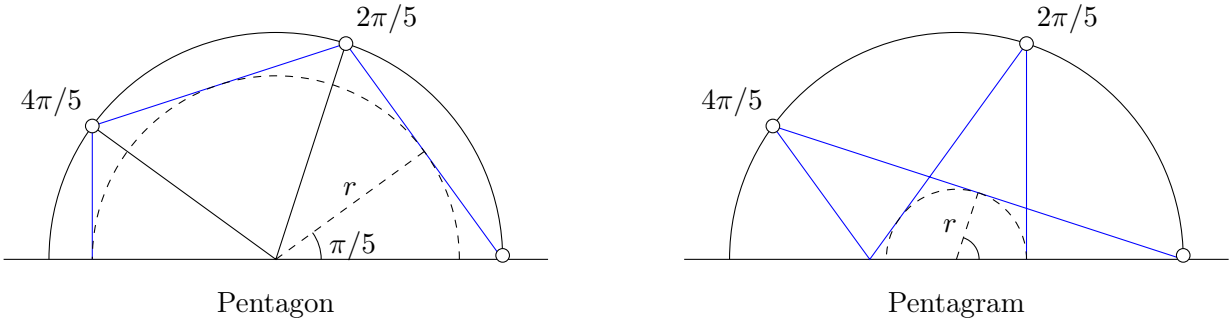


Figure 5: Radii of concentric circles for pentagons

It can immediately be seen that for the pentagon, $r = \cos(\pi/5)$ and for the pentagram, $r = \cos(2\pi/5)$. However, standard arguments with an isosceles triangle with angles $2\pi/5$, $2\pi/5$ and $\pi/5$ can be used to show that

$$\cos(\pi/5) = \frac{1+\sqrt{5}}{4}, \quad \cos(2\pi/5) = \frac{-1+\sqrt{5}}{4}$$

which are indeed the radii provided by our polynomial above.

The polynomial's coefficients are all themselves functions of d^2, which makes sense, in that when considering the symmetry of a bicentric polygon, the only relevant metric is the absolute distance between the centres. We shall choose $d < 0$, which puts the inner circle towards the left, as in figures 1, 2 and 4. Figure 6 shows some examples of the polynomial for different values of d.

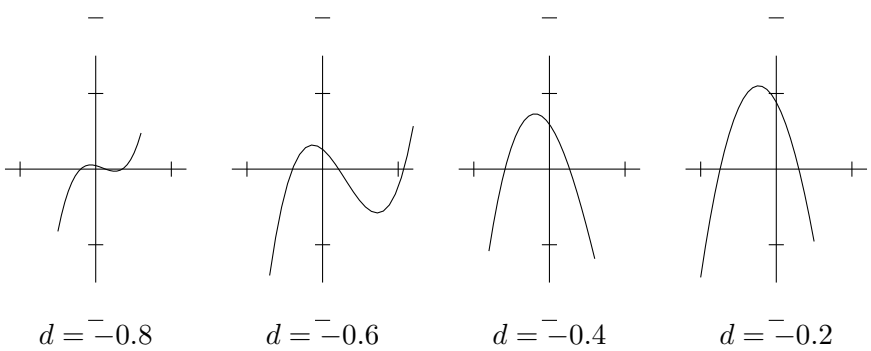


Figure 6: Cubic polynomials for bicentric pentagons

From the polynomial plots, it seems that the cubic has three values between -1 and 1 for d values less than about -0.6, and only two roots above that value. To determine the exact value, substitute $r = 1$ into the polynomial equation to obtain

$$-d^6 + d^4 + 13d^2 - 5 = 0.$$

This is a cubic in d^2, and the approximate roots can be found to be

$$d^2 \approx -3.340173, \quad d^2 \approx 0.377784, \quad d^2 \approx 3.962389$$

Of these, the only value for which $-1 \leq d \leq 1$ is the second root, for which

$$d \approx 0.614642$$

Note that the cubic can be solved analytically; it has three unequal real roots and so the trigonometric method can be used, which produces for the second root:

$$d^2 = \frac{2\sqrt{38}}{3} \cos\left(\frac{4\pi}{3} + \frac{1}{3}\arccos\left(-\frac{2\sqrt{38}}{361}\right)\right) + \frac{1}{3}.$$

What we have found is that when $-1 \leq d < -0.614642$ the cubic polynomial for r has three roots, and when $-0.614642 < d \leq 0$ there are only two roots.

Figure 7 shows plots of all the roots for different values of d. However, as we know there

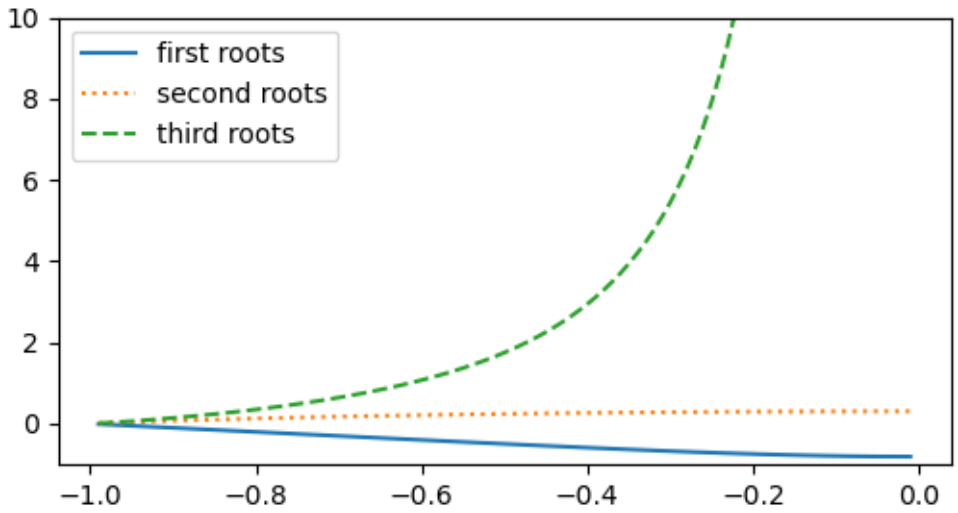


Figure 7: Plots of all the roots

are only sorts of polynomials: the convex pentagon and the pentagonal star. And we also know that for the inner circle to be entirely within the outer circle, we must have $|d| + |r| < 1$. The plots of $|d| + |r|$ are shown in figure 8. From this figure we see immediately that the third roots are not relevant to us; they may be considered "junk solutions" in being legitimate solutions of an equation, but not physically relevant to bicentric polynomials. Returning to the values of the roots, we note that as d approaches 0, the absolute values of the first and second roots approach the values 0.808841 and 0.308991 respectively; these being approximations to $(1 + \sqrt{5})/4$ and $(-1 + \sqrt{5})/4$. This means that the first roots will be the centres of circles for bicentric convex pentagons, and the second roots the centres of pentagonal stars.

4 Bicentric heptagons

We have seen the polynomial for heptagons; it was given as equation (10) earlier. Again the coefficients are polynomials in d^2, but this time we have a *sextic* polynomial in r—a polynomial of degree six. Some experimenting will show that when $0 < |d| < 1$, the polynomial has four real roots. However, there are three different forms for the pentagon, as shown in figure 9.

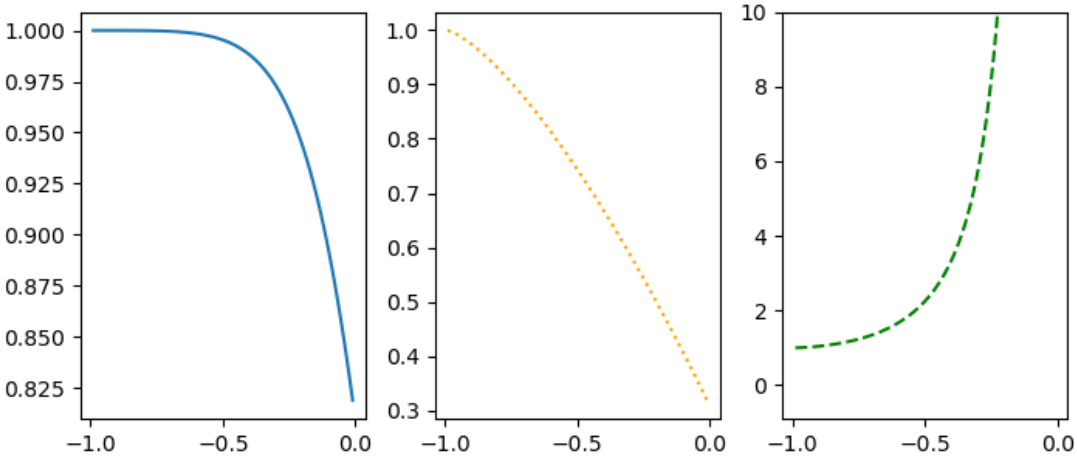
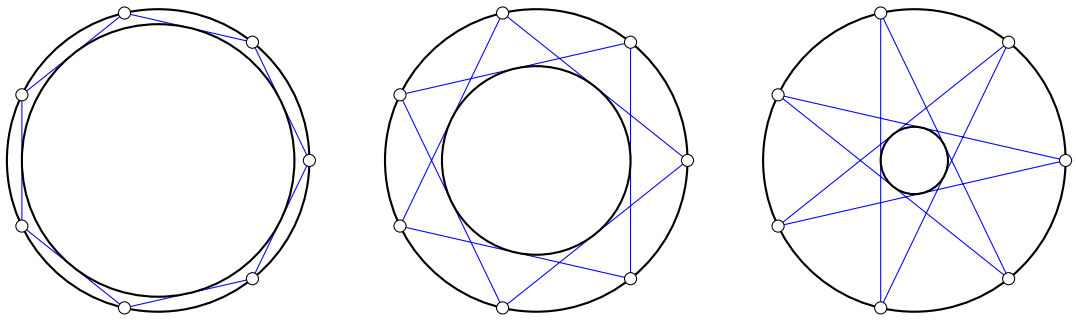


Figure 8: Plots of all the sums of roots and centres

The symbol in braces is the *Schläfli symbol*, which for a regular polygon is just the number of edges, and for a polygonal star lists the number of edges and the number of vertices from one to the next. That is, if the vertices are at $v_i = (\cos(2\pi i/n), \sin(2\pi i/n))$, then the symbol $\{n/k\}$ indicates that the vertices would be joined as $v_0, v_k, v_{2k} \ldots$, where the subscripts are evaluated modulo n. It will be seen that for a Poncelet traverse to join up, and to use all the points v_i, then k must be relatively prime to n, and so the number of different forms is the number of integers between 1 and $\lfloor n/2 \rfloor$ which are relatively prime to n. This means that if n is prime, there are $(n-1)/2$ different polygonal forms.



$\{7\}$ heptagon $\{7/2\}$ heptagonal star $\{7/3\}$ heptagonal star

Figure 9: Bicentric heptagonal forms

Similarly to the pentagonal radii shown in figure 5, the radii of the inner circles in figure 9 are $\cos(\pi/7)$, $\cos(2\pi/7)$ and $\cos(3\pi/7)$.

As before, we can plot the various roots of our function for different values of $0 < d < 1$. We find that for this range, the polynomial has four real roots and two complex roots, and that one of the roots can't be used. Figure 10 shows polynomials for $d = -0.5$, $d = -0.2$ and $d = -0.1$.

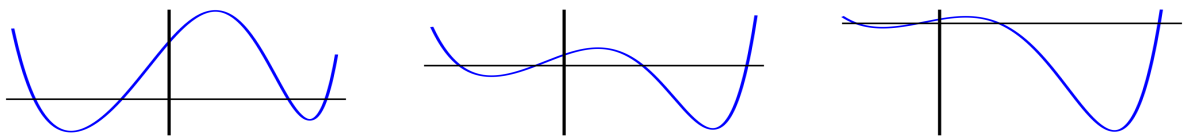


Figure 10: Polynomials associated with heptagons

5 Octagons and up

The polynomial for the octagon is:

$$128d^2(d^4+1)r^8 - 128d^2(d^2-1)^2(d^2+1)r^6 + 8(d^2-1)^4 \\ + (d^2+10d^2+1)r^4 - 8(d^2-1)^6(d^2+1)r^2 + (d^2-1)^8. \tag{11}$$

The polynomial for the nonagon is:

$$512d^6r^9 + 256(d^2+d+1)(d^2-d+1)(d^2-1)d^2r^8 + 128(d^2-1)^3d^4r^7 \\ - 256(d^2+1)(d^2-1)^3d^2r^6 - 32(5d^2+1)(d^2-1)^4d^2r^5 + 96(d^2-1)^5d^2r^4 \\ + 8(7d^2+1)(d^2-1)^6r^3 - 6(d^2-1)^8r - (d^2-1)^9 \tag{12}$$

For the decagon:

$$8dr^3 + 4d^2(d^2-1)r^2 - 2d(d^2-1)^2r - (d^2-1)^3. \tag{13}$$

For the enneagon (11 sided polygon), the amount of algebra, even for a computer system, is considerable[2]. However, Kerawala [7] gives this result:

$$\frac{(R+d)\sqrt{R+r-d}}{R^2-d^2+2rd} + \frac{(R-d)\sqrt{R+r+d}}{R^2-d^2-2rd} + \frac{r\sqrt{2R}}{d^2-R^2+2Rr} \\ + \frac{\sqrt{2R(R+r+d)(R+r-d)}}{R^r-d^2+2Rr} = 0 \tag{14}$$

This follows from a previous and similar result by Chaundy [2, 3] which in fact (as Kerewala pointed out) contained errors. Both Chaundy's result and Kerawala's correction follow from many pages of tangled algebra.

Start by setting $R = 1$, and so the above equation reduces to

$$\frac{(1+d)\sqrt{1+r-d}}{1-d^2+2rd} + \frac{(1-d)\sqrt{1+r+d}}{1-d^2-2rd} + \frac{r\sqrt{2}}{d^2-1+2Rr} + \frac{\sqrt{2(1+r+d)(1+r-d)}}{1-d^2+2r} = 0 \tag{15}$$

By repeated squaring we can eliminate all the square roots, and obtain one large fraction, of which only the numerator is significant. As a polynomial in r, this is:

$$-512d^8r^9 - 768(d^2-1)d^6r^8 + 128(7d^2-3)(d^2-1)^2d^4r^7 + 64(13d^2-1)(d^2-1)^3d^2r^6 \\ - 64(8d^2-3)(d^2-1)^4d^2r^5 - 224(d^2-1)^5d^2r^4 + 8(13d^2-1)(d^2-1)^6r^3 \\ + 16(d^2-1)^7r^2 - 10(d^2-1)^8r + (d^2-1)^9 \tag{16}$$

This polynomial can be slightly simplified by the substitutions $x = 2r$ and $b = d^2 - 1$:

$$-(b+1)^4x^9 - 3(b+1)^3bx^8 + (7b+4)(b+1)^2b^2x^7 + (13b+12)(b+1)b^3x^6 \\ - 2(8b+5)(b+1)b^4x^5 - 14(b+1)b^5x^4 + (13b+12)b^6x^3 + 4b^7x^2 - 5b^8x + b^9$$

For the dodecagon, we can simplify things by producing first an expression collecting two non-consecutive points. If s, t, u are the parameters corresponding to consecutive vertices on the

[2] I've had Sage running for several days trying to compute the appropriate Gröbner bases, without success.

circumcircle, we can eliminate the t from the two expressions connecting s and t, and t and u, to obtain a variant of equation (4), and putting $R = 1$:

$$((d-1)^2 - r^2)(d+1)^2(su+1)^2 - (d-1)^2 r^2 (s-u)^2 + 16r^2 dus \tag{17}$$

This means we have only three expressions, connecting every second vertex in the upper half circle, and the polynomial can be derived as previously; it is given in equation (18).

$$\begin{aligned}
&65536(d^4 - d^2 + 1)d^6 r^{16} - 65536(d^2+1)(d^2-1)^2 d^6 r^{14} \\
&\quad + 4096((d^4 + 6d^2 + 1)^2 - 9d^2(d^2+1)^2)(d^2-1)^4 d^2 r^{12} \\
&\quad - 4096(2d^4 + 3d^2 + 2)(d^2+1)(d^2-1)^6 d^2 r^{10} + 256(23d^4 + 24d^2 + 23)(d^2-1)^8 d^2 r^8 \\
&\quad - 1792(d^2+1)(d^2-1)^{10} d^2 r^6 16(d^4 + 26d^2 + 1)(d^2-1)^{12} r^4 \\
&\quad - 16(d^2+1)(d^2-1)^{14} r^2 + (d^2-1)^{16}
\end{aligned} \tag{18}$$

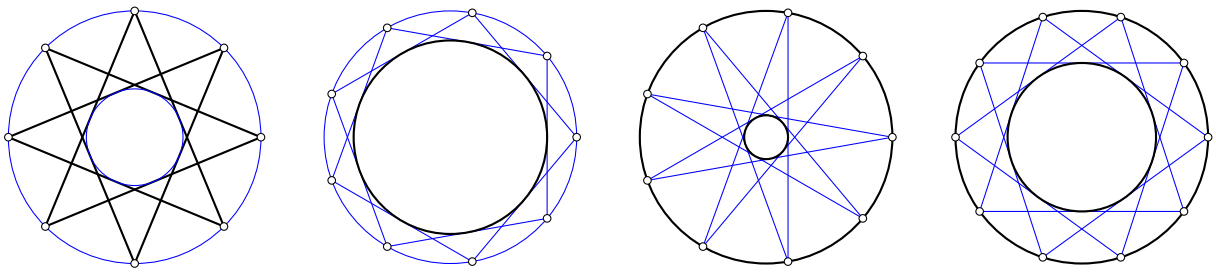
Note that this is an eighth degree polynomial in r^2, and all the coefficients are functions of d^2. Then equation (18) can be slightly simplified with the substitutions $x = 2r^2$ and $b = d^2 - 1$:

$$\begin{aligned}
&(b^2 + b + 1)(b+1)^3 x^8 - 4(b+2)(b+1)^3 b^2 x^7 + (b^4 + 7b^3 + 35b^2 + 56b + 28)(b+1)b^4 x^6 \\
&\quad - 4(2b^2 + 7b + 7)(b+2)(b+1)b^6 x^5 + (23b^2 + 70b + 70)(b+1)b^8 x^4 \\
&\quad - 28(b+2)(b+1)b^{10} x^3 + (b^2 + 28b + 28)b^{12} x^2 - 4(b+2)b^{14} x + b^{16}
\end{aligned} \tag{19}$$

6 Using the polynomials

Having obtained these recondite polynomials, what do we do with them? Aside from being, in general, much simpler expressions than the usual expressions connecting R, r and d (or any of the substitutions), they enable us to distinguish easily between the different forms of a polygon: convex or starred; and thus are powerful tools for drawing such polygons with dynamic geometry software.

For example, the stars corresponding to polygons of 8, 9 and 10 sides have Schläfli symbols $\{8/3\}$, $\{9/2\}$, $\{9/4\}$ and $\{10/3\}$; they are shown in figure 11.



$\{8;3\}$ $\{9;2\}$ $\{9;4\}$ $\{10;3\}$

Figure 11: Polygonal stars of orders 8, 9 and 10

We note that the central circle for a polynomial $\{n/k\}$ has radius $\cos(k\pi/n)$. We can then draw any polygon by relating roots of its associated polynomial to this central radius.

For the nonagon, as we see, there is a polynomial of degree 9. For any $0 < d < 1$ there will be five real roots and two sets of complex conjugate pairs. Two of the roots can be discounted, as the value of $|d| + |r| > 1$, which would indicate an "inner circle" which is not fully contained in the outer circle. So of the nine roots, only three correspond to bicentric polygons. For example, take $d = 0.5$.

Then:

$\lvert r\rvert$	$\lvert r\rvert + \lvert d\rvert$
0.138807	0.638807
0.488954	0.988954
0.499996	0.999996
0.511979	1.011979
7.862043	8.362043

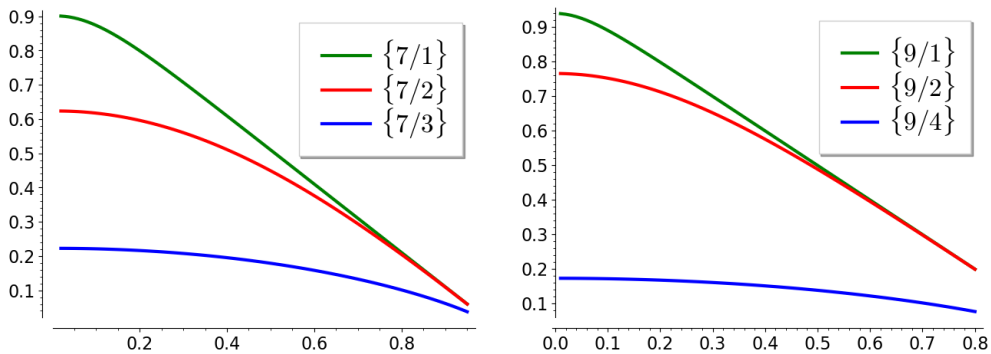
We see that only the first three of these roots are valid for us. Plots of the roots are shown in figure 12. The values on the $y$axis corresponding to the left hand ends of the curves for the heptagon are

$$\cos(\pi/7) = 0.900969, \quad \cos(2\pi/7) = 0.623490, \quad \cos(3\pi/7) = 0.222521$$

and for the nonagon are

$$\cos(\pi/9) = 0.939693, \quad \cos(2\pi/9) = 0.766044, \quad \cos(4\pi/9) = 0.173648$$

Similar plots can be drawn for any roots of such a polynomial, and they can be used to give a sort of "map" as to the nature of the polygon produced with the corresponding value of d and the chosen root.



Roots for the heptagon Roots for the nonagon

Figure 12: Roots of the polynomials associated with the heptagon and nonagon

A note on implementation

Given a positive integer n, and the associated polynomial $p_n[d](r)$ for a bicentric n-sided polygon, we can draw a bicentric polygon as follows:

1. Choose a value of d with $d^2 < 1$.
2. Find the real roots of the polynomial for that chosen value of d. To save doing this on the fly, it may be more efficient to precompute all roots for d in steps of 0.01 from 0.01 to 0.99. Then given any d, appropriate roots can be found by a simple interpolation.
3. Draw two circles: the unit circle centred at the origin, and the circle with centre at either $(d, 0)$ or $(-d, 0)$ and with radius r.
4. For any root r with $|r| + |d| < 1$, pick any point p_0 on the unit circle and draw a Poncelet traverse starting at p_0.

The definition of the polynomials mean that this Poncelet traverse must join up and so create a bicentric polygon. And we can determine which of the polygons $\{n/k\}$ will be drawn by our choice of root. Space forbids details, but we have found that the JavaScript library JSXGraph[3] is perfect for our explorations, combining as it does dynamic geometry with a full-featured programming language.

7 Conclusions

Bicentric polygons have shown themselves to be irresistible, both theoretically thanks to Poncelet's Porism, and the delight in determining conditions for them to exist. Increasingly complex expressions have been found joining the radius r of the incircle, the radius R of the circumcircle, and the distance d between their centres. This work predates Euler, and is still continuing. As well, the existence now of dynamic geometry software has allowed the exploration of bicentric polygons of all orders, with some very inventive programs to demonstrate them. In this article, we showed how for each order n, a particular polynomial can be found whose roots are the values of r, and as well allow us to distinguish whether the resulting figure will be a convex polygon, or an n-pointed polygonal star. We note that there has been little exploration involving non-circular conic sections (for which Poncelet's Porism also applies), as the complexity increases hugely from circles.

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The Tangram: A Timeless Puzzle in Modern Education

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Abstract

The Tangram, a traditional seven-piece puzzle originating from China, has long captivated educators with its simplicity and complexity. Beyond its historical roots as a form of entertainment, the Tangram has found a significant place in modern education, leveraging its potential to enhance cognitive and spatial reasoning skills. This paper explores the integration of Tangrams in educational settings, highlighting their application in teaching mathematics, geometry, and fostering creativity. Additionally, it examines innovative approaches such as digital Tangram applications with Scratch. These advancements have redefined how Tangrams are utilized, making them more accessible and versatile for learners. The discussion underscores the Tangram's enduring relevance as a tool that bridges traditional learning methods with contemporary technological innovations, ultimately promoting a more engaging and effective educational experience.

1 Introduction

The Tangram, a seven-piece puzzle originating from China, has fascinated and challenged minds for centuries [23]. Comprised of five triangles, one square, and one parallelogram, these pieces can be rearranged into countless shapes and patterns. The objective is to rearrange these seven pieces using rigid body transformations to fit them into a given pattern. An arrangement is considered a valid solution only if it includes all the pieces without any overlaps [12].

Beyond its initial use as an intriguing puzzle, the Tangram has made a significant impact in various fields. Artists and designers have incorporated the geometric elements of Tangram into their works of art, while mathematicians have explored its properties and connections with various mathematical concepts [25].

In the realm of education, Tangram puzzles have proven to be invaluable as manipulative learning tools and teaching aids. They help young students develop geometric thinking and reasoning skills by providing hands-on experiences that promote active learning. By physically manipulating the pieces, students can better understand geometric shapes, relationships, and

transformations [13]. The Tangram's simplicity and versatility make it an effective resource for teaching fundamental mathematical concepts in an engaging and interactive manner.

While traditionally a tool for entertainment, the Tangram has evolved into an educational resource that bridges the gap between traditional learning and modern innovation and technology. In contemporary classrooms, Tangram puzzles are used not only to teach geometry but also to foster problem-solving skills, critical thinking, and creativity [20]. Educators leverage digital platforms to bring Tangram puzzles to life, allowing students to interact with virtual Tangrams through educational software and applications. These digital tools enhance the learning experience by offering instant feedback, diverse challenges, and opportunities for collaborative learning [15].

Furthermore, the Tangram's integration into modern technology extends its relevance and appeal. Digital Tangram applications provide students with a dynamic and interactive environment where they can experiment with shapes and patterns, receive guidance, and track their progress. This integration of technology not only makes learning more engaging, but also equips students with valuable skills in using digital tools for problem-solving and creative expression [15].

2 Tangram in Education

The Tangram is more than a mere puzzle, it's a versatile educational tool that enhances cognitive and spatial reasoning skills [10]. Teachers worldwide have incorporated Tangrams into their curricula to teach various subjects, from mathematics and geometry to art and design (Figure 1). By manipulating the pieces to form different shapes, students learn critical concepts such as symmetry, fractions, and problem-solving strategies [19, 24].

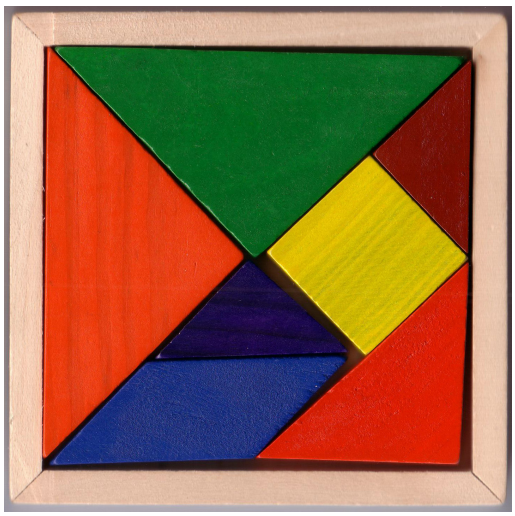

Figure 1: Wooden Tangram (initial square configuration)

According to Miller et al. [16] Tangram stimulates the imagination and creativity, it develops skills and abilities. The seven-pieces can be used in mathematic concepts, in art developing the creativity, such:

Mathematic Tangrams provide a hands-on approach to understanding geometric principles. Students can explore concepts like congruence, area, and perimeter by arranging the pieces into different shapes. This tactile learning experience helps solidify abstract mathematical ideas in a concrete manner.

Art Beyond mathematics, Tangrams foster creativity and artistic expression. Students can create intricate designs and patterns, encouraging them to think outside the box. This blend of creativity and logical thinking is essential in developing well-rounded problem-solving skills.

Tangram can be used to explore and solve challenging mathematical problems, especially in areas such as geometry, combinatorics and spatial reasoning. For example, given a set of tangram pieces, determine all the possible figures that have the same area but different perimeters. The difficulty is that this problem involves understanding how the different shapes can be combined to create new figures, keeping the same area but changing the perimeter. This requires a good understanding of geometry and the properties of shapes. According, to Yamada et, al. [26] point out that complex tangram puzzles have at least one of the following characteristics "(1) are composed of multiple connected components; (2) contain holes within the puzzle area; (3) require unconstrained rotations for the pieces; (4) require the reflection transformation of the parallelogram" (p.16). Figure 2 reflects what the authors say about the difficulty of constructing the puzzle using the seven pieces without overlapping.

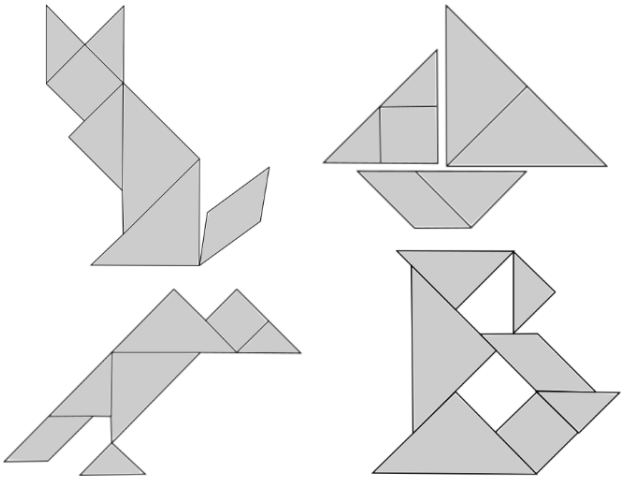

Figure 2: Complex Tangram Puzzle [26]

It is possible to build many shapes with the Tangram, but they all share a common characteristic: the area. Since each figure uses all the non-overlapping Tangram pieces, the areas of these figures are equal.

Decomposing figures is one of the simplest and most intuitive way for comparing the area of polygonal shapes. If two flat figures can be constructed using the same pieces without overlapping, they must have the same area. This principle often leads to intriguing pseudo-paradoxes when working with Tangrams.

The Dudeney's Paradox is a geometric puzzle where, for example, an equilateral triangle is cut into four pieces that can be reassembled to form a square. The paradox arises because the

triangle and square have different shapes, yet their areas are identical (Figure 3). The challenge lies in finding the precise way to dissect the triangle for this transformation. It highlights how geometry can create surprising links between different shapes.

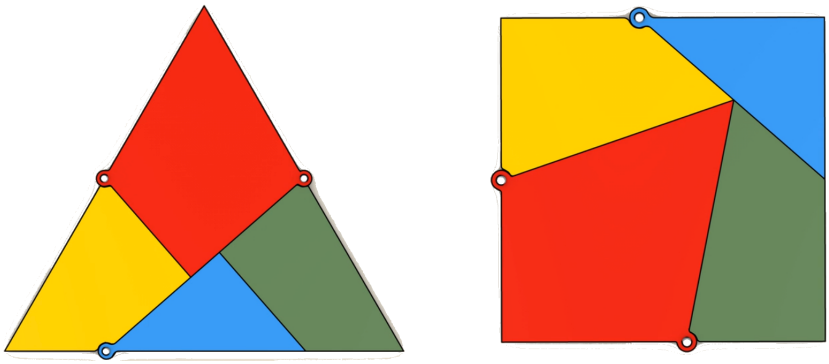

Figure 3: Equilateral triangle and Square (`https://en.etudes.ru/models/dudeney-dissection/`)

Consider the Figures 4 below:

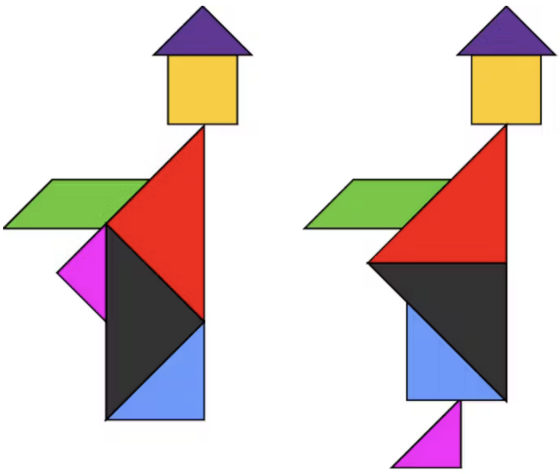

Figure 4: The two monks Pseudo-Paradox Tangram Puzzle (`https://tinyurl.com/pparadox`)

The images are on the same scale, and both can be constructed using the Tangram pieces [4]. However, upon closer inspection, the second figure appears to have a larger area than the first. Where did the extra area, such as the second person's feet, 'come from'?

Dudeney's Paradox is not the most subtle in terms of differences, with careful observation, one can see that one monk is slightly shorter than the other. However, other paradoxes are more difficult to spot.

The surface area of the Tangram pieces always remains constant. The areas have been transferred or shifted. The monks are not truly identical, everything has been moved slightly. This principle applies to all such paradoxes.

Even though all the pieces are used, the paradox appears longer, making it difficult to compare since they are identical in height.

The key to this effect is presenting these puzzles side by side. They appear to be the same height, but their widths differ slightly, which results in a significant part of the puzzle seeming to disappear [22].

3 Innovation in Tangram Usage

The effective use of information and communication technologies (ICT) as a teaching tool aims to encourage students to consolidate and update their technological and scientific knowledge. According to Prensky [17], students process information in a very different way to previous generations. This information-seeking potential associated with new technologies makes it essential to understand the contributions made to enriching learning contexts. The use of ICT by students and teachers contributes to educational development. In this way, appropriate use of ICT makes it possible to enrich, differentiate, individualise and implement all the curricular objectives.

Innovation in education often involves reimagining traditional tools for modern applications. The Tangram is no exception. Educators and developers have integrated Tangrams into digital platforms, creating interactive and engaging learning experiences. For examples:

- Digital Tangrams - With the advent of educational technology, digital Tangram applications have emerged, offering interactive puzzles that can be manipulated on screens. These apps provide instant feedback and step-by-step solutions, making it easier for students to grasp complex concepts [3].
- Gamification - Incorporating Tangrams into educational games has proven to be an effective way to maintain student engagement. Gamified learning experiences motivate students to solve puzzles and progress through levels, reinforcing their learning through repetition and challenge [11].

3.1 Tangram and Technology

Technology has transformed how Tangrams are used in educational settings, making them more accessible and versatile. In Augmented Reality (AR), technology has taken Tangram puzzles to a new level by overlaying digital information onto the real world [5, 21]. Students can use AR apps to see how Tangram pieces fit together in real-time, enhancing their spatial reasoning skills. Also in Virtual Reality (VR), provides an immersive environment where students can interact with Tangram puzzles in a three-dimensional space [2, 5]. This immersive experience helps students understand geometric concepts from multiple perspectives. Or with 3D Printing technology allows for the creation of custom Tangram sets, providing tactile learning tools that can be tailored to individual needs [14]. Teachers can design unique puzzles that align with their lesson plans, offering a personalised learning experience.

3.2 Tangram and Scratch

Scratch is a programming environment developed by a group of researchers from the Kindergarten and Lifelong Learning Group at the MIT Massachusetts Media Lab, under the direction of Resnick. The program allows for simple but efficient work on simple animations in which

scenes, characters and sound elements are integrated, whether created by the user in the work environment, developed in other programs or taken from the resource gallery [18].

The integration of Tangram with Scratch represents an innovative approach to education that bridges traditional and modern learning tools. By combining the tactile, hands-on nature of Tangram puzzles with the creative, digital environment of Scratch, educators can provide a rich, engaging learning experience that fosters a wide range of skills.

Studies by Felleisen and Krishnamurthi [9] show that using Scratch in the classroom makes it possible to learn basic mathematical concepts in a fun way and arouses students' interest and motivation in this area of knowledge. The idea is to stimulate students in areas such as maths so that they themselves develop a joy of learning. Felleisen and Krishnamurthi [9] emphasised the concept of 'imaginative programming', considering its importance as an element of programming, due to its close relationship between mathematics and computing.

4 Tangram as Activity

The study presented here takes place in the context of initial teacher training for early years schooling. It involves a class of 53 third-year Basic Education degree students attending the Didactics and Technology of Maths course. The predominant teaching model in this unit is based on carrying out exploratory/investigative tasks, solving problems, and presenting activities, often using manipulatives or technological resources. In this case, the focus is on geometry, and the aim is to introduce the Tangram as a classroom resource. This study examines the use of the Tangram by prospective teachers and its role in deepening their understanding of geometry.

The students, who are prospective teachers divided into small groups (4/5 each group), were asked to carry out an activity using the Tangram. The first part of this activity involved using physical materials, while the second part required implementing the proposed activity in Scratch. They were asked to explain the educational level, educational objectives, specific objectives, transversal skills, and provide a description of the activity they were proposing.

The aim of this activity is to mobilize and deepen knowledge and skills related to elementary mathematics, while mastering computer tools suitable for exploring the mathematical topics covered. Additionally, it seeks to develop autonomy in carrying out training activities, promote teamwork in group projects, and encourage continuous self-training in mathematics education. Students are expected to develop mathematical, didactic, and technological skills that will enable them to plan meaningful mathematics learning experiences in an informed manner, taking advantage of computer technologies.

4.1 Proposed Tasks

The task proposals enable the observation of mathematical concepts emerging in contexts that promote understanding and the (re)construction of meanings. Therefore, designing mathematical tasks is a crucial aspect of initial teacher training.

4.1.1 Proposed Tasks - Part I

The 12 groups of students presented their proposals for activities, which are divided into educational levels. The students proposed activities for the 1st, 2nd, 4th, 5th and 6th grades. The groups worked on the theme of geometry and measurement. Only a few of them will be mentioned here.

The group A develop an activity for the 1st grade, worked with the essential learning [6] in the topic plane figures. This topic the student has to:

- Recognise triangles, squares, rectangles, pentagons, hexagons and circles and using appropriate representations;
- Recognise congruent figures, using different strategies and resources to explain their ideas.

This group worked the topic operations with figures. This topic the student has to:

- Constructing, representing and comparing composite plane figures;
- Composing and decomposing a given plane figure, using physical or virtual manipulatives.

This group proposed the objectives of the activity, such as, to identify and recognise basic geometric shapes, like triangles, squares, and parallelograms; to develop skills in classifying and categorising shapes; to explore and understand spatial relationships between geometric shapes; to stimulate logical thinking and problem-solving; to promote creativity and artistic expression through manipulating the pieces. The transversal skills involved are critical thinking, problem-solving, communication and collaboration, autonomy and initiative.

The description of the activity is the following:

- First, distribute a uniformly coloured Tangram to each student (Figure 5). Individually, the student should be able to recognise the congruent pieces and paint them in the same way; the different pieces should be painted differently. The students should also compare the different triangles, checking that although the size is different, the properties remain the same (number of sides and angles). After this, design figures to be built with the Tangram, starting with easier figures and progressively increasing the degree of difficulty (Figure 6). At this stage. the students can work collaboratively. Once they've finished the figures they've asked for, they should create new ones and present them to the rest of the class.

Group B established relationships between the Tangram pieces (Figure 7):

- The two pairs of triangles establish a proportional relationship by the AA criterion (angle-angle);
- The square, the trapezoid and the medium triangle can be obtained by joining the 2 smaller triangles, depending on how they are arranged;

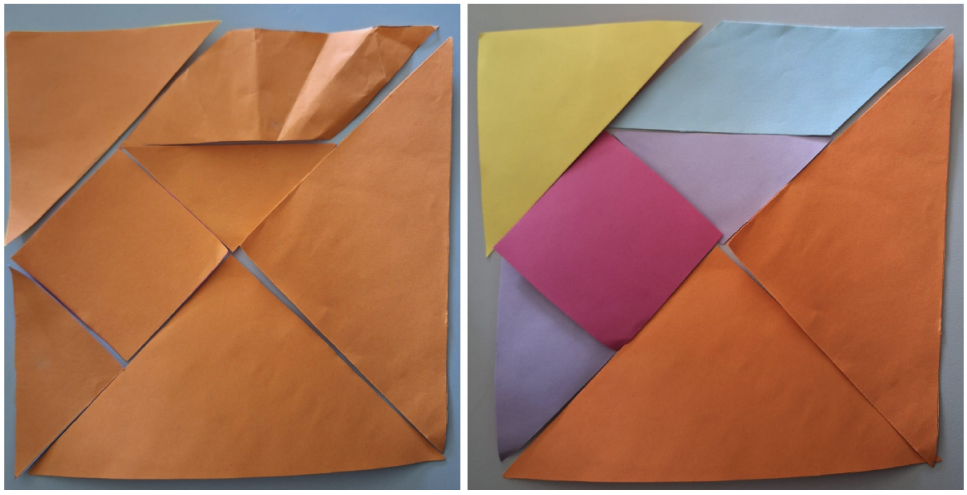

Figure 5: Activity Proposed - Part I

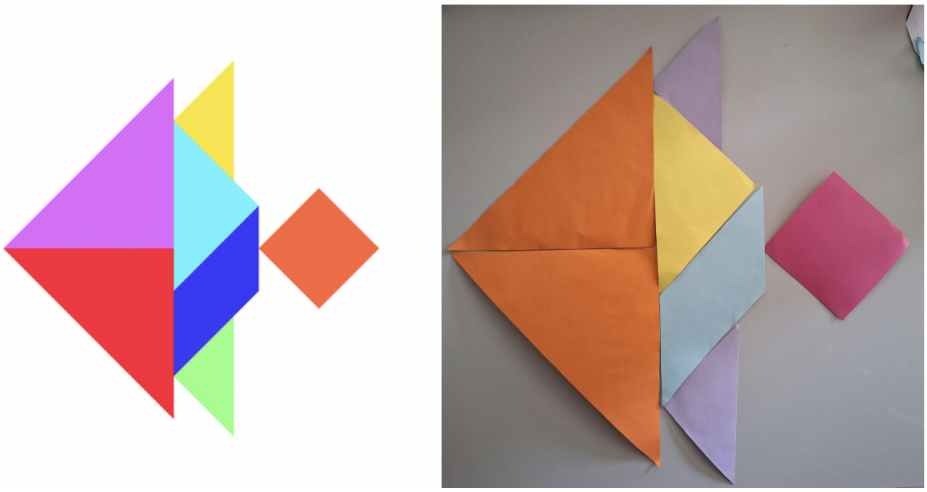

Figure 6: Activity Proposed - Part II (students image)

- The large triangle is made up of the square and the 2 small triangles;
- The sum of the hypotenuse of the two smaller triangles is equal to the hypotenuse of the larger triangle;
- The large triangle makes up 1/4 of the area of the Tangram.

The group B develop an activity for the 6th grade, worked with the essential learning [8] in the topic regular and irregular polygons. This topic the student has to:

- Distinguish between regular and irregular polygons (in the case of this activity, identify regular polygons);
- Solve problems involving regular and irregular polygons.

This group worked the topic sum of the measures of the internal and external angles of a triangle. This topic the student has to:

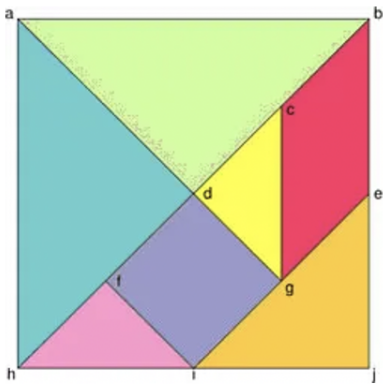

Figure 7: Activity Proposed - Tangram (students image)

- Conjecture about the sum of the internal and external angles of a triangle and explain the relationship found;
- Solve problems involving the properties of triangles.

This group proposed the objectives of the activity, such as, to develop problem-solving skills and logical reasoning; develop basic concepts of geometry; develop spatial perception; realising relationships between geometric figures. The transversal skills involved are patience and perseverance; creativity.

The description of the activity is the following:

- Divide the students into groups and give each group a Tangram. Ask them to find out the area of each Tangram figure, knowing that the measure of the long side is 6 cm. Ask them to justify how they arrived at each result.

4.1.2 Proposed Tasks - Part II

In this second part, the students were asked to implement an activity in Scratch. Two groups developed the activity proposed earlier, on the Tangram, which is presented below.

The group C prepare the activity with a duration of 45 minutes, with the objectives, to develop problem-solving skills and logical reasoning; to develop basic concepts of geometry; to develop spatial perception. The transversal skill to develop are patience and perseverance; creativity and to promote contact with ICT. The topics are analysing and working with geometric shapes and constructing patterns.

The description of the activity is the following:
After reading the book Tangram Cat (Figure 8) by Maranke Rinck and Martin van der Linden, divide the students into groups of 3 and let them explore the game 'Tangram remix copy' (Figure 9).

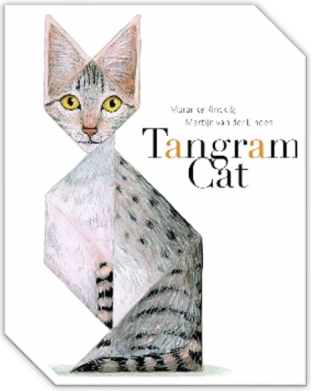

Figure 8: Tangram Cat (students image)

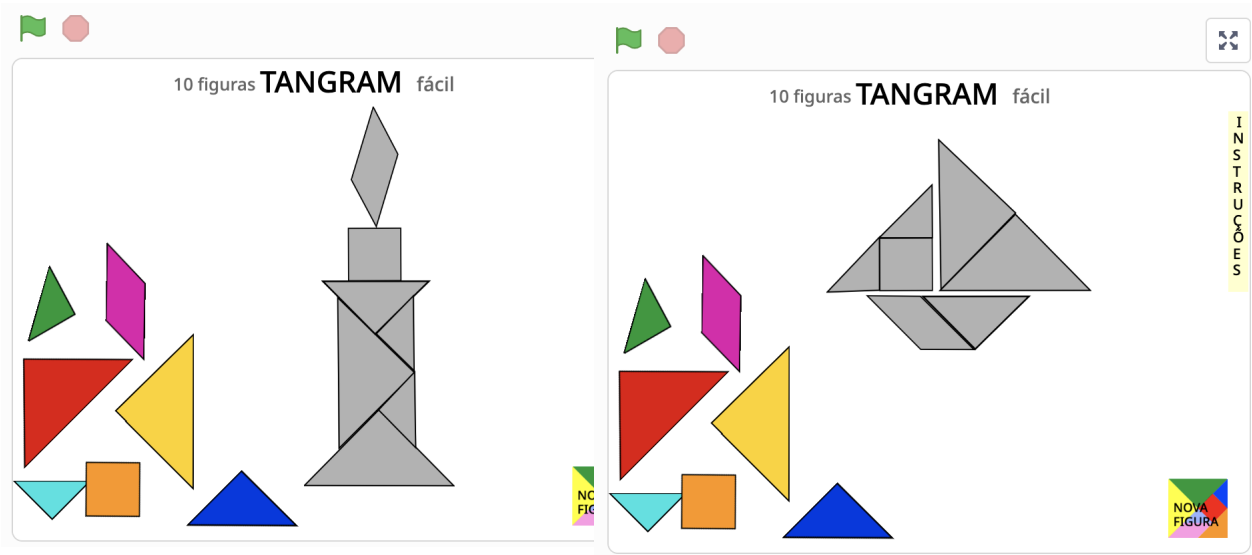

Figure 9: Scratch Tangram, two diferent constructions to do

Group D, prepare an activity (Figures 10) for the 2nd grade, that aims to classify plane figures based on their characteristics (such as straight lines, the number of sides, the number of vertices, and the equality of sides) [7]. The activity involves presenting and explaining their ideas, as well as justifying the congruence between plane figures using sliding, rotating, and turning movements. Additionally, the activity encourages students to present and explain their reasoning.

The transversal skills to be developed include problem-solving, where students apply their mathematical knowledge in various contexts and develop appropriate strategies to obtain valid solutions. Computational thinking is also emphasized, involving practices such as abstraction, decomposition, pattern recognition, analysis, and the definition of algorithms, along with developing habits of debugging and process optimization. Finally, the activity fosters creativity, encouraging the originality of creating new figures.

Notably, the activity plan did not specify its duration, leaving flexibility for its implementation.

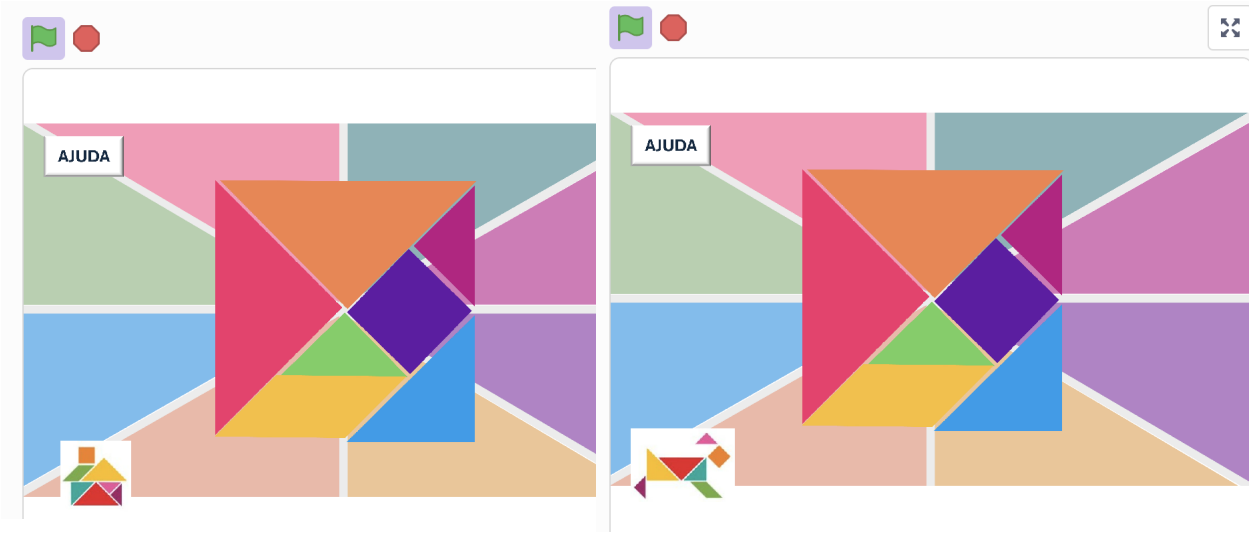

Figure 10: Scratch Tangram, two diferent constructions to do

4.2 Analysis and Discussion

Students are encouraged to reflect on the content, objectives, and target age group when designing activities, incorporating both physical and digital components. Creating an activity goes beyond simply aligning it with the curriculum, it involves understanding the underlying goals of the task, requiring students not only to design but also to solve the activity themselves. This process deepens their engagement with the material and fosters a more comprehensive understanding of how the task fits into broader educational objectives.

In terms of practical implementation, students found the first activity relatively easy and encountered few difficulties. However, the second activity, which required the use of Scratch, demanded more effort and posed greater challenges. Many students struggled to effectively utilise the digital tool to achieve the desired outcomes, revealing the complexities of incorporating programming into their educational toolkit.

The mastery of computer tools that are appropriated for exploring mathematical concepts, as well as the ability to demonstrate autonomy in completing these tasks, proved to be a significant challenge. Students found that developing proficiency in digital tools requires more than just basic familiarity, it demands sustained practice and a deeper understanding of the connection between technology and mathematics instruction.

This advancement will equip them with the capability to craft impactful and engaging mathematics learning experiences that are both well-informed and enriched through the strategic application of technology. The aim is to empower them to seamlessly integrate computer technologies into their instructional practices, thereby fostering an environment where learning is not only more interactive and dynamic but also tailored to the needs of contemporary education. By leveraging these tools effectively, they will be able to enhance student comprehension and engagement, making the learning process more responsive to the evolving landscape of educational demands and technological innovations. Ultimately, this approach seeks to prepare students to meet the challenges of modern teaching, where the integration of digital resources is essential for creating meaningful and effective learning experiences.

5 Conclusion

The Tangram puzzle, with its rich history and educational value, continues to be a relevant and powerful tool in modern education. By embracing innovation and technology, educators can harness the full potential of Tangrams to enhance learning outcomes. Whether through traditional methods or cutting-edge digital platforms, the Tangram remains a timeless resource that bridges the past and the future, fostering creativity, critical thinking, and a joy for learning.

Throughout the task, manipulating this resource provided opportunities to develop and articulate various geometry concepts. Notably, two key aspects emerged: the way students integrated Tangrams into their activity proposals and the incorporation of Tangrams into a digital context. Instead of starting with pre-made sets, having students create their own activities encouraged them to focus on the mathematical relationships embedded in the pieces. This approach effectively redirected their attention from the game itself to the mathematical concepts they aimed to study [1].

In class, the challenges of using a programming tool, even a basic one like Scratch, were evident among students with no prior programming knowledge. Despite the difficulties, students reported enjoying the tool and feeling motivated throughout the process. This enthusiasm highlights the potential of Scratch to engage learners, even those with no background in programming, and suggests that incorporating such tools in higher education contexts can enhance the learning experience.

The use of Scratch, specifically, not only makes the teaching-learning process more interactive but also serves as an accessible introduction to computational thinking. According to Resnick [18], programming significantly broadens the scope of what students can learn, extending beyond mere technical skills to foster problem-solving strategies and creative design across various domains. Engaging with programming encourages students to think critically and develop solutions in ways that transcend traditional subject boundaries. Resnick further emphasizes that programming also offers a unique opportunity for self-reflection. As students work through challenges, they begin to reflect on their own processes, gaining insights into their problem-solving approaches and overall learning strategies [18].

The Tangram remains a timeless resource that continues to inspire and educate. Its journey from a simple Chinese puzzle to a multifaceted educational tool illustrates its enduring appeal and versatility. By embracing both traditional methods and cutting-edge digital platforms, educators can harness the full potential of Tangrams to enhance learning outcomes.

6 Acknowledgements

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Professional Development in Coding and Computational Thinking for Mathematics Teachers

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Abstract: *Since 2006, computational thinking (CT) has been popularised as a critical interdisciplinary skill and linked to mathematical thinking, solidifying its applicability in mathematics education. Singapore has actively introduced CT in its mathematics curriculum and provided professional development (PD) opportunities for mathematics teachers to develop their competencies in incorporating CT in mathematics classrooms. However, research examining how and whether such PD prepares educators to teach mathematics using CT is scarce. The study fills this gap by examining how a PD course that introduces the VBA coding language in Microsoft Excel for computational problem solving develops mathematics educators' coding skills and CT, and how participating in-service teachers perceive the course with regards to learning coding. Qualitative analysis of the course design revealed that the course materials are capable of helping learners develop CT through instilling in them certain coding habits, while qualitative and quantitative analysis of Likert-scale and open-ended responses in the course feedback highlighted many strengths and suggested areas for improvement in various aspects of the course, like course structure, course materials, level of course difficulty, and perceived usefulness and applicability of the course. These findings reveal the benefits of computational approaches adopted in this study for developing CT and coding skills, the relevance of such approaches in mathematics education, areas that can potentially be improved for more effective PD, as well as how rich insights generated by feedback and course design analysis can contribute to assessing the impact of PD and tailoring PD courses to specific teacher needs and concerns.*

1. Introduction

One of the earliest mentions of computational thinking (CT) was made by Papert [23, p. 17] in 1980 when he suggested that "computational technology and computational ideas" could create "new possibilities for learning, thinking, and growing". This term gained prominence when Wing [31] re-introduced it in 2006 as an essential transdisciplinary skill central to problem solving. While there is no consensus on what constitutes CT, the literature generally recognises CT as a mental procedure of pinpointing problems and devising solutions that can be executed via computational means [18], foregrounding abstraction (extracting details relevant for problem solving), algorithmic thinking (approaching problems in steps), pattern recognition (identifying patterns in problems for understanding other problems), and decomposition (dividing problems into smaller, manageable parts) as core CT skills [10].

Numerous overlaps between CT and mathematical thinking suggest the relevance of computational approaches in mathematical problem solving. CT's computer science origins are founded on mathematical thinking [31], and scholars have recognised pedagogical benefits of CT in learning mathematical concepts and skills [30]. Frameworks linking CT to mathematics-specific skills include Sneider et al. [28] and Weintrop et al. [30], who respectively suggest areas where mathematical thinking and CT overlap as well as shared CT practices in mathematics and science.

There is also a close link between CT and mathematical modelling – both approaches share a common goal of problem solving as mathematical models are an "abstraction" of problems "in a mathematical form", and furthermore, modelling and computational tools are associated with enhanced mathematical skills [1, p. 1, 16, 27]. Pertaining to computational tools, coding has been identified as highly effectual for cultivating CT [20]. However, knowledge alone of using computational tools is not identical to developing CT [3, 26] – while such tools can execute solutions, a crucial step lies in generating solutions using a computational approach, which Ang [2, p. 127] terms *habits of mind*, or "mental habits" of those who utilise computational "tools… constructs… methods… to solve problems". He argues that coding creates opportunities to cultivate these CT habits, such as extracting relevant information in problems and transforming them into variables (i.e., abstraction).

Since Wing's article, there has been a marked increase in the educational integration of CT. Singapore for example has actively promoted CT since 2014, focusing the curricular integration of CT on mathematics. CT and computational tools are at the heart of Singapore's mathematics syllabus, with both featuring prominently in the secondary and junior college (i.e., Grades 7 to 12) levels. However, not many educators in Singapore are well acquainted with CT and how to use it to teach mathematics [7], with most having no prior exposure to coding in university or pre-service education [13]. Given the increasing pressure to prepare students for a workforce that favours computational proficiency, teachers need more opportunities to upgrade their CT capabilities.

This study examines the course design and feedback of a professional development (PD) course on computational approaches for mathematical problem solving. It is targeted at in-service mathematics teachers in Singapore and introduces the use of Visual Basic for Applications (VBA) in Excel as a tool for mathematical problem solving. It aims to answer the following research questions:

RQ1: How does the course cultivate mathematics educators' coding skills and CT?

RQ2: How do participants perceive the course with regards to learning and/or coding?

Excel is chosen as the computational tool for this course because of the pervasive use of Excel spreadsheets, with the number of users of spreadsheets as a coding tool estimated to be 11 million in the U.S. alone despite many users having minimal or no coding knowledge [9]. Many teachers and students have ready access to Excel at little to no extra cost and are more familiar with it compared to specialised coding software, as Excel is available on most computers and resembles the pen and paper format students are accustomed to [25]. Teachers can therefore implement Excel applications in class as a "natural and logical extension" of their existing proficiencies [13, p. 82]. Excel is also versatile as it can solve most school-level mathematical problems, contains numerous features that students can experiment with independently, and potentially enables prolonged information retention when mathematics is taught through it [9, 25].

2. Developing CT competencies through coding

There are numerous studies involving PD courses for developing mathematics teachers' capabilities in teaching using CT. These mainly examine program outcomes like changes in teachers' CT knowledge, skills, and perceptions. Gadanidis et al. [12] examine shifts in attitudes towards and knowledge of CT amongst pre-service primary level teachers in Canada who attended a course on integrating CT in mathematics. Analyses of assignments and discussions showed that the teachers shifted from having little to no knowledge of CT to being able to identify its integration in

mathematics and other subjects and its societal relevance. They also reported decreased anxiety towards CT and a growth mindset towards learning and teaching CT. Using participant observation and interviews, Martins [21] examines how primary level Portuguese pre-service teachers attending a mathematics pedagogy course perceive the link between CT and mathematics education, and ways in which they create learning activities that bolster skills in both areas. The teachers initially expressed unfamiliarity with CT aspects such as whether CT and mathematical skills are equivalent. However, as they worked on a mathematical task aimed at cultivating CT competencies, they gained familiarity in how to implement such tasks and honed their understanding of CT.

With regards to studies on PD courses on coding, Humble and Mozelius [15] analyse Grade 7 to 12 Swedish mathematics and technology educators' perceptions of CT during a PD session on coding. Essays submitted describe opportunities and challenges that teachers attributed to each aspect of CT and the general application of CT in school. They perceived different aspects as useful for their pedagogy and students' learning, albeit noting that CT may not always be relevant to the subject at hand and could be challenging to implement for reasons such as the perceived time-consuming nature of CT integration. Araujo et al. [4] on the other hand, examine knowledge related to CT acquired by Grade 7 to 12 pre-service teachers in Canada during a teacher education course on CT in mathematics and science education which covers coding. Analysis of discussion and reflection posts showed that their experiences of acquiring CT comprise four themes: pedagogical approaches (e.g., incorporation of CT in subjects), affordances (e.g., potential of CT in enhancing mathematics education), tools (e.g., affordances of CT tools for learning mathematics), and definitions (e.g., gaining a deeper understanding of CT).

These studies suggest the effectiveness of PD in improving mathematics teachers' perceptions, knowledge, and integration of CT. However, despite close links between mathematics learning and coding [22], there is little research on (and availability of) PD that introduces mathematics educators to coding [19] and virtually no studies evaluating the use of Excel in PD for mathematics teachers. Given the lack of coding proficiency amongst mathematics educators [13], this should be addressed so that appropriate PD can be introduced. This study therefore is an attempt to address this urgent need.

Studies on pedagogy involving CT typically examine CT concepts/skills/practices developed, mathematics topics taught through the use of computational tools, and how these tools augment the understanding of mathematics and CT. Chan et al. [8] identify as many as 64 tools in mathematics, with coding languages being the most common, digital tangibles and formative/summative assessments in second place, followed by apps/games and other digital tools (e.g., spreadsheets). Top CT skills and mathematics topics assessed using these tools include algorithmic thinking, abstraction, debugging, loops, and sequences for skills, and geometry and measurement, number and operations, algebra, statistics and probability, and calculus for topics. Kong [17] on the other hand observes that primary school students in Hong Kong develop CT concepts like conditionals, CT practices like debugging, and CT perspectives like expressing knowledge of mathematical concepts, as they learn to code an app for determining composite and prime numbers.

Several studies examine how Excel develops CT and mathematical thinking. Ang [3] discusses how simulating mathematical models in Excel helps to develop CT as mental habits, which include step-by-step coding and making assumptions to simplify problems into a form that can be expressed mathematically for computation. Ho et al. [14] and Sanford [25] introduce coding instructions for mathematical problems that are solvable on Excel, while van Borkulo et al. [29] demonstrate how CT can be developed through coding lessons on statistics in Excel. Combining data like questionnaires, interviews, and spreadsheets, the latter analysed how coding in Excel enables students to develop CT in the areas of computational practices, perspectives, and concepts (see [6]).

These studies suggest that there are several CT tools which mathematics teachers can be exposed to, and many mathematical topics where these tools may be applied in. However, research on the general use of tools like Excel in mathematics instruction remains scarce. Given the advantages of Excel in mathematics teaching and learning, there is much value in research which provides concrete, targeted examples of how Excel can be harnessed to teach mathematical concepts while developing one's competencies and habits in CT.

3. The present study

3.1 Data collection and selection

This study analyses two datasets: the course design of a PD course on computational approaches for mathematical problem solving and the participants' post-course feedback. The latter comes from a standard feedback form, which consists of a 5-point Likert-scale and open-ended questions. The selected dataset contains 125 open-ended responses and 364 Likert-scale responses collected from 42 participants across three runs of the course.

3.2 Data analysis

A mixed methods analysis was conducted where the open-ended responses and course design were analysed qualitatively using thematic analysis, and the Likert-scale responses quantitatively using percentages. Thematic analysis involves "identifying, analysing and reporting patterns (themes) within data" [5, p. 79]. The course design was analysed first, by outlining the topics taught and linking instructional instances in the course materials to attempts to develop CT habits through coding. This answers RQ1 on how this course develops coding and CT skills.

The analysis of the feedback focuses on strengths and areas for improvement in the course as suggested by participating educators, as well as their perceptions of the usefulness and applicability of the course in relation to mathematics teaching and other contexts. The findings of this analysis are presented in tables and bar charts with a thematic breakdown.

The Likert-scale responses were assigned to the following categories: 1) instructor's teaching, 2) course learning outcomes, 3) course structure, 4) course materials, and 5) likeliness to recommend the course, while the open-ended responses were assigned to the following themes: 1) course delivery, 2) course difficulty, and 3) usefulness and applicability of course content. Themes for the open-ended responses are decomposed into subthemes whose associated feedback are further categorised into strengths and areas for improvement in the course, summarised, and presented in a table, while categories for the Likert-scale responses are presented with associated statements in a table and the percentages of participants' level of agreement with each statement are shown in a bar chart. The analysis of these responses answers RQ2 on discerning participants' views about the PD.

The next two sections present the analysis of the course design followed by a quantitative and qualitative examination of the feedback.

4. Design of the PD course

This section analyses the design of the PD course entitled "*Computational Approaches for Mathematical Problem Solving*", which was conducted by the first author over three runs in 2022 and 2024 and comprises six sessions per run. The course was open to mathematics teachers in Singapore's junior colleges (Grade 11 to 12 schools) who have a basic understanding of Excel. It

aims to raise their confidence in using easily accessible computational tools like Excel, help them realise the potential of such tools in learning mathematics, and develop their CT habits via exposure to Excel with VBA in mathematical problem solving.

The course is guided by the following learning objectives: 1) Learning fundamentals of VBA coding in Excel for mathematical problem solving; 2) Utilising Excel and VBA to perform mathematical computations and problem solving; 3) Designing computational solutions using concepts and tools taught in this course; 4) Gaining exposure to aspects of CT in problem solving. More importantly, the course addresses the gap in coding experiences among mathematics teachers in Singapore.

The course adopts a self-directed learning pedagogy, comprising 50% online asynchronous lessons where participants watch videos related to the week's topic(s) and attempt problem exercises (8 hours), and 50% online synchronous sessions where the class convenes to discuss the topic(s) and exercises (8 hours, for a total of 16 hours). The course also implements spiral learning, where key concepts in VBA and Excel are revisited in increasing difficulty as the course progresses [11]. Table 1 outlines the course topics, which include Excel formulas and VBA concepts and functions for solving/simulating specific mathematical problems. Course materials include slides, videos where the instructor demonstrates the use of Excel formulas and VBA functions, and problem exercises that require participants to apply these formulas and/or functions to solve/simulate mathematical problems. Only the videos and slides (featured in the videos) are discussed in this paper.

Table 1. Course content

Session	Topic(s) for that session
1	• Introduction to computational thinking in mathematical problem solving • Introduction to Excel layout, data types, and cell formatting • Basic Excel formulas (e.g., SUM( ), AVERAGE( ), SIN( ) etc.) • Using SOLVER tool to produce mathematical models
2	• Form controls (i.e., scroll bars) • Performing simulations for random number generation using RAND( ) and RANDBETWEEN( ) formulas • Condition checking using IF and COUNTIF formulas
3	• Creating more form controls (buttons) for running VBA code (assigning macro) • Introduction to VBA concepts (subroutines) and syntax (Sub, MsgBox, If…Then…ElseIf statement, logical and conditional operators (=, And, etc.)) • Declaring variables (Dim), assigning values to variables, variable name constraints, and data types (numeric and non-numeric) in VBA
4	• Performing iteration using For-Next loop statement in VBA • Creating user-defined functions via Function… End function statement in VBA • Condition checking using While-Wend loop statement in VBA
5	• Performing simulations for random number generation in VBA using Randomize and Rnd functions
6	

4.1 Developing CT habits through coding

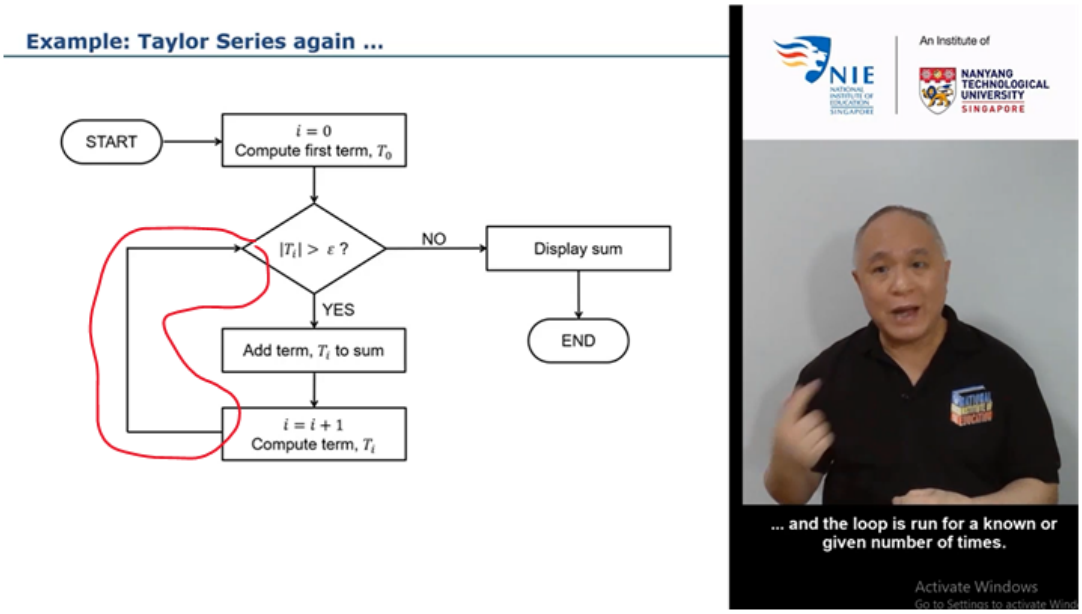
This section examines implicit attempts in the course at developing CT habits through solving problems using computational tools including coding and flowcharts. An instructional video from Session 4 is presented to illustrate this point. As shown below in Figure 1, this video guides participants through the process of coding a "While loop" to compute the approximation of e^x using Taylor's series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!} \ .$$

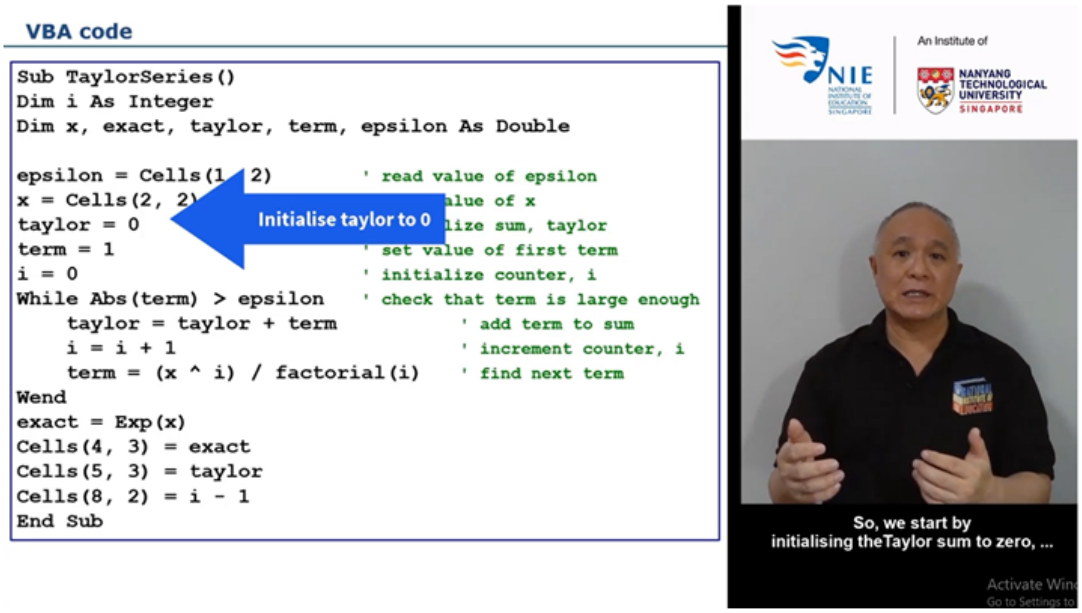
Mathematically speaking, the process of approximation is as follows (see above formula): beginning with a first term of 1, each successive term in the Taylor's series expansion is computed and if the term is sufficiently significant (that is, large enough in magnitude to make a difference to the approximation), it is added to the partial sum. In coding, this process may be achieved computationally using a "While loop". The following paragraphs explain how each step of coding a "While loop", from a flowchart of simplified coding procedures to the actual VBA code, develops CT habits through encouraging certain coding practices/habits.

In the instructional video on the "While loop", the instructor first introduces the basic syntax of the While loop statement (which checks for conditions and stops the (infinite) repetition/loop of a code when these conditions are met), before presenting a flowchart of the steps for computing the approximation of e^x using Taylor's series, as described above. In the flowchart, the first term (T_0) is computed and compared with epsilon (ε) to determine if the condition of $|T_0| > \varepsilon$ is fulfilled. If No, then the sum of terms (which contains only the value of T_0) will be displayed. If Yes, then T_0 is added to the sum of terms and the successive term (T_1) is computed and reevaluated against the condition of $|T_i| > \varepsilon$ (where i represents the increase in degree for each successive term) to determine if the current sum of terms (containing only T_0) should be displayed or if T_1 should be added to it. Terms will be continuously evaluated and added until the condition becomes "No". The flowchart therefore incorporates the loop via branching conditions (YES / NO) and repetition/looping using arrows (see red highlight in (a) in Figure 1). By encouraging the habit of creating flowcharts to represent simplified coding procedures for solving a mathematical problem, this lesson helps participants to develop the CT habits of (a) algorithmic thinking through understanding the stepwise procedure of formulating solutions, (b) abstraction through extracting relevant information about Taylor's series approximation for presentation in a flowchart, and (c) pattern recognition through understanding how the While loop can be applied not only in a general situation, but also in specific mathematical contexts (i.e., Taylor's series). The course's spiral learning approach is also demonstrated here, as the basic syntax and general usage of the loop is presented first, which scaffolds understanding of its application in more complex mathematical contexts later in the course such as in this example.

Next, the instructor shows how this flowchart can be converted into a full VBA code. Converting between different representations (i.e., flowchart to code) can be seen as helping participants develop the CT habit of abstraction, by identifying essential flowchart elements which need to be incorporated into a code so that it can perform the processes indicated in the flowchart. That is, by converting pertinent parts of the flowchart's written procedures into code. One way this is done is through the coding habit of declaring variables to represent mathematical symbols in the flowchart. ε is declared as a variable called `epsilon` and the expansion of Taylor series as it increases in degree (T_i) is represented by variables like `taylor`, `term`, and `i`.



(a)



(b)

Figure 1. The "While loop" instructional video in Session 4

The instructor then sequentially examines each line/sequence of lines in the code and explains their purpose with respect to solving the problem (see (b) in Figure 1). Participants are encouraged to develop the habits of dealing with code sequentially and seeing code as a combination of chunks/modules with different functions that work together to solve an overarching problem. For example, the instructor first explains how the line `taylor = 0` and `i = 0` initialises the variables

`taylor` (sum of terms) and `i` (increase in degree for each successive term) respectively to 0, and how `term = 1` sets the value of the first term. He then moves down the code to clarify that the use of the While loop in `While Abs(term) > epsilon` checks for the condition of whether $|T_i|$ is more than ε. If $|T_i| > \varepsilon$, then the term is added to the sum (`taylor = taylor + term`), and the degree (`i = i + 1`) and the value (`term = (x ^ i) / factorial(i)`) of the successive term is computed and compared with ε in a new loop. This helps participants to cultivate the CT habits of (a) algorithmic thinking by understanding the sequential execution of codes, and (b) decomposition by breaking down codes into modules and understanding how each module works.

5. Analysis of course feedback

5.1 Quantitative findings

Participants' feedback was sought through a form that contains statements about the course. Table 2 presents these statements, which are grouped into five categories, and participants had to indicate their level of agreement (in a typical 5-point Likert-scale) with these statements. For easier reference, these statements were coded (e.g. IT1, CLO2 etc.) based on the category names.

Table 2. Statements in the Likert-scale-based feedback

Category	**Statement**
Instructor's teaching (IT)	IT1: The facilitator is skillful at facilitating the participants' learning.
	IT2: The presentation was clear.
Course learning outcomes (CLO)	CLO1: The objectives were achieved.
	CLO2: The course helped me to reflect on my teaching practices.
	CLO3: The course met my learning needs.
	CLO4: I can apply the ideas/knowledge/skills learnt from the course.
	CLO5: The questions were adequately addressed at the course.
Course materials (CM)	CM1: The learning resources supported me in my learning.
Likeness to recommend this course (LRC)	LRC1: I would recommend the course/workshop to my colleagues.
Course structure (CS)	CS1: The duration of the course was sufficient to meet its objectives.*

*CS1 is excluded from Figure 2 as it uses a different rating scale ("Just right" and "Too short")

Figure 2 below shows participants' ratings for statements across the five categories. Generally, more than 90% of participants expressed agreement ("Strongly Agree" or "Agree") with the statements. Certain statements received 100% agreement, and these are related to: the instructor's skillfulness at teaching (IT1), addressing of participants' questions (CLO5), and learning support provided by the course's learning resources (CM1). Only a minority (4 to 18%) disagreed ("Disagree" or "Strongly Disagree") with a few statements concerning: the clarity of instruction (IT2), attainment of course objectives (CLO1), impact on participants' teaching practices (CLO2), addressing of participants' learning needs (CLO3), applicability of course content (CLO4), course duration (CS1), and likeliness to recommend the course to others (LRC1). All ten statements reflect the strengths of the course due to high agreement ratings, although the presence of negative ratings in some of them suggests that these are aspects of the course that can be improved on.

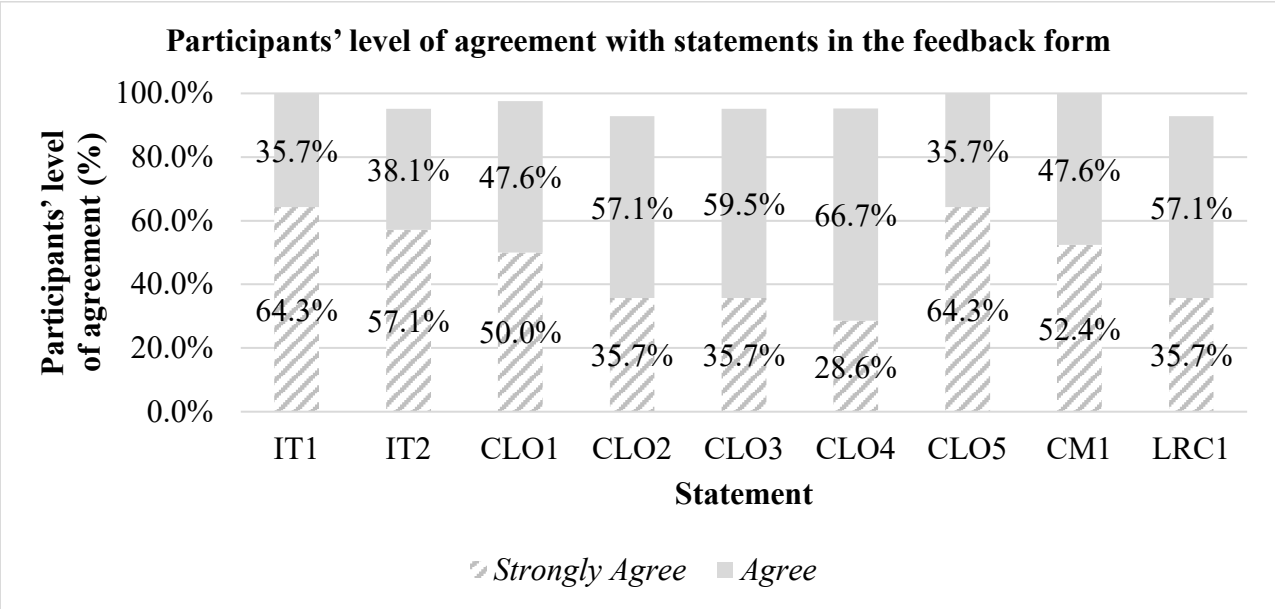


Figure 2. Participants' level of agreement with statements in the Likert-scale-based feedback

5.2 Qualitative findings

Participants' responses to open-ended questions were analysed and grouped into three main themes, namely, (1) course delivery, (2) course difficulty, and (3) usefulness and applicability of the course content. Their responses to these questions form the basis of the qualitative findings for this study. A total of 21 subthemes were uncovered. However, due to space constraints, only the nine most prominent subthemes are discussed here. These nine are the top three to four subthemes per theme except for "course difficulty" which has only two subthemes (see Table 3). The content of these responses, categorised as strengths and areas for improvement, are summarised in Table 3, while notable responses are cited in the analysis below.

The theme of usefulness and applicability of the course content dominates, followed by course delivery and course difficulty. For the theme of usefulness and applicability of course content, prevalent subthemes are related to whether the course addresses the curricular/pedagogical needs of mathematics educators, how it exposes participating educators to various uses of VBA, and the educators' general learning gains. Next, the allocation of learning/teaching activities, the course's/instructor's level of engagement, clarity of the instructor's teaching, and pacing of the course are prominent subthemes associated with the theme of course delivery. Lastly, the theme of course difficulty comprises two subthemes: the course content and instructor's teaching. These subthemes suggest the main strengths of the course as well as pertinent areas of concern that can be addressed in future course runs or other PD courses on coding.

Next, the specifics of strengths and areas for improvement in each theme are presented. In relation to the theme of usefulness and applicability of the course content, many participants described useful ideas from the course that they could apply to their mathematics lessons, such as the use of simulations to help students visualise concepts in topics like statistics as well as creating activities that develop students' CT. Some described how they have acquired the skills to implement

these ideas, and others anticipated some learning benefits for students such as deepening metacognition, simplifying concepts, and developing CT.

Table 3. Themes and subthemes in the open-ended feedback

Theme	Subtheme	Summary of strengths	Summary of areas for improvement
Usefulness and applicability of the course content	*Addressing mathematics curricular/ pedagogical needs*	• Promotes awareness of the pedagogical potential of VBA in mathematics and its benefits • Shows how to create pedagogical activities for mathematics using these tools	• Provide examples tailored to participating educators' curricular needs • Provide examples of how VBA is used in classrooms globally
	Exposure to different uses of VBA	• Promotes awareness of diverse uses of VBA	• Introduce more complex applications of VBA
	General learning gains	• Perceived gains in pedagogical knowledge and learning of VBA/programming	
Course delivery	*Allocation of course activities*		• More hands-on coding practice • More time to discuss solutions and do exercises • Focus more on covering content in synchronous lessons, less on flipped classroom and asynchronous lessons
	Level of engagement	• The instructor is passionate, approachable, addresses questions, open to sharing his expertise, and creates fun exercises	• Engage more with participants through questions and coding demonstrations
	Clarity of instruction	• The instructor is clear and precise at explaining concepts	
	Course pacing	• The instructor is patient • Pace of the course is ideal and tailored to address queries and participants' differentiated speed of coding	• Course is slow paced and draggy, especially for synchronous lessons
Course difficulty	*Course content*	• The difficulty level of exercises is ideal	• Provide hints for the exercises • Set less exercises as homework • Exercises are difficult, especially for those with no coding experience • Exercises should be simplified and/or more scaffolded/guided

			• Exercises can be more difficult • More time for exercises
	Instructor's teaching	• The instructor makes learning easy	• In the synchronous lessons, account for those with no coding experience

One participant reported gains in coding skills through the course, being able to now "*create some simple codes and students can interact [with these codes] to understand some concepts better*", while another became aware of how Excel could positively impact students' mathematics learning by "*facilitat[ing] learning of mathematics to deepen students' metacognition*". Participating teachers also expressed that they gained an understanding of how VBA can be used in diverse contexts like administrative work, general simulations, and results analysis. One of them was "*impressed by what excel VBA can achieve*", while another reflected on his/her already ongoing use of VBA in "*daily productivity work… with Excel by automation*".

Several respondents also discussed their overall learning gains, such as how they received pedagogical guidance and the enjoyment of learning coding in the course. They described the instructor as "*great at guiding [them] in [their] thinking and pedagogy*" and an exemplary figure for "*teachers in designing lessons for blended learning*", adding that the process of learning VBA was "*meaningful to do*" and became a stimulus "*to explore a bit more of programming*".

Nevertheless, participating teachers also noted some aspects of the course which could be improved regarding its utility and applicability. There were a few suggestions for more problem exercises and examples that address the educators' own curriculum/syllabus (i.e., A-level mathematics) as well as real-life instances of VBA usage in mathematics instructional contexts to "*seed more ideas for [teachers] to see the possibilities*" of how they can apply it to their own classes, because mathematics teachers in Singapore currently have, as one educator perceives, "*insufficient triggers as [they] still lack the exposure*". One teacher also suggested a further exploration of more complex VBA applications such as in animations.

In terms of course delivery, participants generally felt that the instructor was able to explain concepts with clarity and adjust his pace to adapt to their questions and differing speeds of learning. They also found that they were able to clarify their doubts, and some enjoyed the exercises. However, participants also highlighted several areas that could be improved. They pointed out that more time could be allocated for course activities, especially for attempting the exercises and for discussion of the solutions to these exercises. Quite a few thought that coding could be practiced during lessons, rather than as part of the asynchronous activities. In fact, some expressed that the focus on the flipped classroom approach could be reduced and instead more content could be covered in synchronous lessons.

A small number of teachers also proposed that apart from encouraging queries, live coding demonstrations would be "*most useful instead of just videos*" in boosting engagement while a few others remarked that the course, especially in its synchronous sessions, was too slow-paced and draggy, with one recommending that the course summary "*could outline in more detail what the course will cover*" prior to participants registering.

Lastly, in terms of course difficulty, a few participants commented that the exercises were of "*the right level of difficulty*" while some underscored the instructor's teaching skills in making the course "*easy to follow*" and "*comfortable*" especially for the asynchronous sessions. On the flip side, there were numerous suggestions to reassess the difficulty level of the course content. Many teachers voiced concerns about the difficulty level of the exercises, mentioning that these were "*quite daunting*

for newcomers who are not familiar with coding". They provided several suggestions to alleviate this, such as more hints or guidance/scaffolding from the instructor, simpler exercises, reducing the number of exercises, and increasing the time given for solving these exercises. Comments that proposed more hints and/or scaffolding appeared several times, remarking that due to the lack of these aids, such as the intentional omission of certain VBA functions, it "*was quite difficult to figure out what needs to be used sometimes for the exercises*" and "*the math involved [in VBA/coding] can be unfamiliar*". On the contrary, one respondent proposed that the difficulty level could be increased. Pertaining to the instructor's teaching, one participant expressed that during synchronous sessions, some teachers "*without coding background may struggle to understand*" immediately what is being taught in the class itself.

6. Discussion and implications

In general, the above findings show that the course presented in this study can potentially help learners, including practicing mathematics teachers, develop some level of coding skills through designing and implementing computational solutions to mathematical problems. Secondly, while the course displays strengths in many aspects such as course materials and instructor's teaching, there are some areas for improvement, especially in aspects like course learning outcomes and level of course difficulty. Nevertheless, these findings suggest the role of coding activities in developing a computational mindset and the applicability of these activities in mathematics learning.

Furthermore, findings from this study also suggest implications for pedagogical principles and practices. One should bear in mind the importance of accommodating variations in coding aptitude and experience amongst mathematics teachers, and perhaps a course with differentiated instruction could be designed. Coding examples could be better prepared with attention paid to tailor them to teachers' curricular/pedagogical needs. The feedback analysis revealed that a number of teachers expressed difficulty in handling the exercises, especially for those without coding experience, and several suggested exercises that were more relevant to their curricular requirements.

Some were also not very comfortable with the self-directed/flipped classroom approach and instead preferred more guided learning. Offering differentiated guidance and targeted examples helps educators see the value and direct applicability of CT and coding in their teaching and ensures that educators with different coding abilities can receive customised guidance, rather than being demotivated to adopt coding and CT due to perceived difficulty and/or lack of relevance.

In terms of policies, analysing intended learning outcomes in the course materials and perceived learning gains in the feedback reveals potential mismatches between mathematics teachers' beliefs and educational policies regarding CT. Although the findings suggest that the course materials could be effective in cultivating CT and coding skills and that the course exhibits strengths in doing so, there are numerous areas of concerns that still need to be addressed. These include the relevance of the course content to the educators' pedagogical/curricular needs and their perceptions of the feasibility of acquiring coding skills with regards to factors like the difficulty of coding, amount of guidance received, and preferences for certain pedagogical approaches. Educational policies concerning PD that involves coding or CT should be informed by studies on what mathematics educators need, prefer, and are comfortable with in their teaching and learning as well as what has been done well and what needs to be improved in existing PD courses and teaching resources.

Lastly, in terms of research, this study serves as a starting point for piloting and evaluating PD courses or resources related to coding and CT for mathematics teachers, given the lack of such programmes and information about their impact, strengths, and areas for improvement especially in Singapore.

7. Conclusion

This study evaluated the potential impact of a PD course that focusses on computational approaches in mathematical problem solving, with a view towards developing coding competencies and CT habits in the context of mathematics learning. This was done by analysing the course's strengths and areas for improvement in various aspects of the course, and examining how an instructional segment helps learners develop some aspects of CT by solving a problem computationally.

There are some limitations that may affect the generalisability of the findings. Firstly, the course has a short duration of only six sessions. It has been suggested that a short course duration may not be very effective for developing teachers' CT in the long run [24, 32]. Secondly, the course was only taught thrice and had a relatively small participant size (average of 14 per run). More course runs with larger participant numbers may be necessary to verify the effect and usefulness of such a PD course. Lastly, the analysis only reflects intended learning objectives (in the materials) and participants' perceived learning outcomes. Classroom footage or CT scores would provide more concrete evidence of learning gains.

Nevertheless, despite these limitations, insights gained from this study will serve as a stepping stone to developing more effective and better designed PD courses and teaching resources on coding and CT. In fact, the present study has provided some very valuable lessons and directions for the authors' next research project on mathematics teachers' CT knowledge and their views on CT integration in mathematics instruction, its challenges, and CT pedagogical tools.

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Automated generation of solved problems for self-study

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Abstract

We study the possibility of generating step-by-step solutions to a certain class of problems, and its practical implementation with a CAS.

1 Introduction

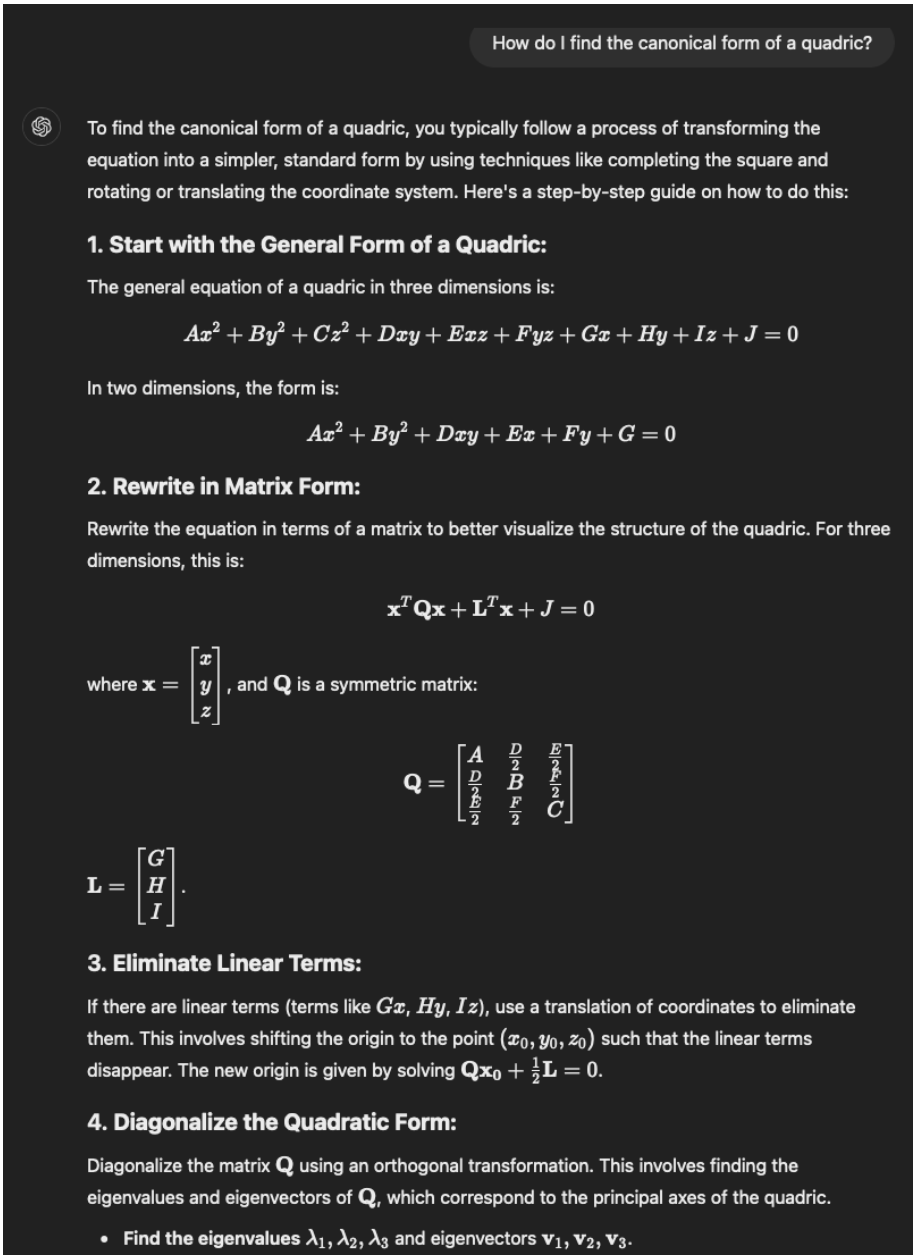
Every mathematics teacher faces the same question at the beginning of a course: 'Where can we find examples and/or solved problems on that topic?' Some students are just lazy when it comes to solving problems, others try hard to work on them, but are unsure about the correctness of their solutions. For the vast majority, the absence of problems solved in a step-by-step manner is a serious difficulty that very often leads to abandoning the course. Another source of frustration for the students is the fact that sometimes they find those sought-after problems in the page of some other professor in a different university, but they are using a different technique than the one required in their own courses. Moreover, the differences in notation, terminology and choice of topics, make the translation of these solutions a task as difficult as solving the original problems.

Nowadays, the irruption of ChatGPT in every aspect of teaching solves many of these problems, provided that the student knows how to formulate the question in proper terms. The point is that more often that not, the student lacks the capacity of stating precise questions in mathematical terms. Thus, the first answer provided by ChatGPT can be confusing or even misleading.

To illustrate what we are talking about, consider the case of a class of mathematics for first-year Engineering students. They know some single variable calculus, and some linear algebra, including computing determinants but not yet diagonalizing matrices and the like (they are studying an Algebra course alongside). They are studying minimization methods for two variable real functions, and they are discussing the simple case of quadratic functions arising from the Hessian determinant. It is very important to know if the function is positive or negative definite (or indefinite), and the teacher mentions that this can be readily seen

if the corresponding quadratic form is put into its canonical form, having in mind a very simple computational method such as completing squares, which involves nothing but basic manipulations. The teacher considers that given the elementary nature of the method, a quick example suffices to illustrate and writes it on the blackboard without too many details.

A clueless student takes some notes and, once at home, checks them. As he does not grasp what the teacher wrote (when he was not paying attention), he goes to ChatGPT and asks how is this done. Figure 1 shows the output of an actual ChatGPT session.



How do I find the canonical form of a quadric?

To find the canonical form of a quadric, you typically follow a process of transforming the equation into a simpler, standard form by using techniques like completing the square and rotating or translating the coordinate system. Here's a step-by-step guide on how to do this:

1. Start with the General Form of a Quadric:

The general equation of a quadric in three dimensions is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

In two dimensions, the form is:

$$Ax^2 + By^2 + Dxy + Ex + Fy + G = 0$$

2. Rewrite in Matrix Form:

Rewrite the equation in terms of a matrix to better visualize the structure of the quadric. For three dimensions, this is:

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{L}^T\mathbf{x} + J = 0$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\mathbf{Q}$ is a symmetric matrix:

$$\mathbf{Q} = \begin{bmatrix} A & \frac{D}{2} & \frac{E}{2} \\ \frac{D}{2} & B & \frac{F}{2} \\ \frac{E}{2} & \frac{F}{2} & C \end{bmatrix}$$

$\mathbf{L} = \begin{bmatrix} G \\ H \\ I \end{bmatrix}$.

3. Eliminate Linear Terms:

If there are linear terms (terms like Gx, Hy, Iz), use a translation of coordinates to eliminate them. This involves shifting the origin to the point (x_0, y_0, z_0) such that the linear terms disappear. The new origin is given by solving $\mathbf{Q}\mathbf{x}_0 + \frac{1}{2}\mathbf{L} = 0$.

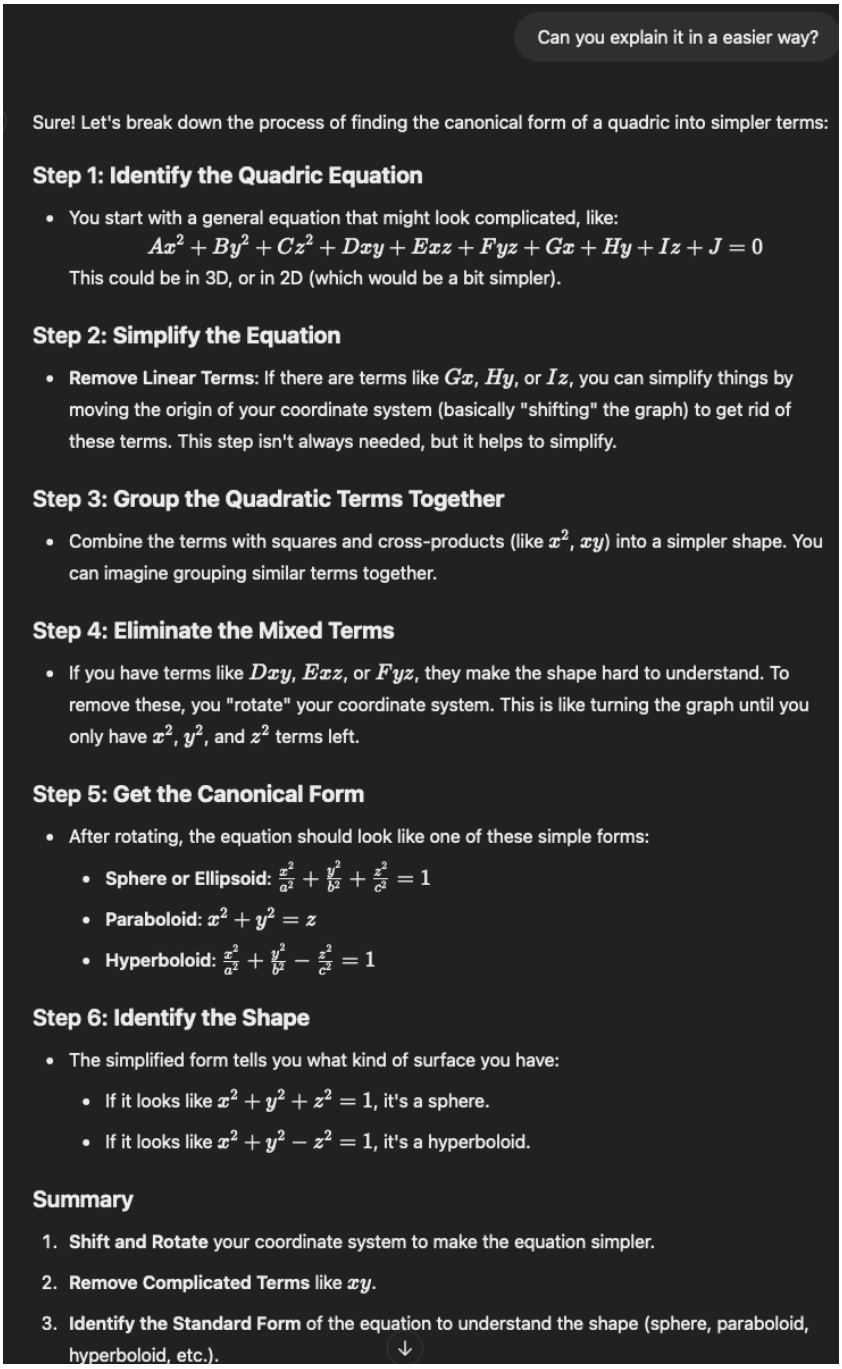
4. Diagonalize the Quadratic Form:

Diagonalize the matrix $\mathbf{Q}$ using an orthogonal transformation. This involves finding the eigenvalues and eigenvectors of $\mathbf{Q}$, which correspond to the principal axes of the quadric.

- **Find the eigenvalues** $\lambda_1, \lambda_2, \lambda_3$ and eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Figure 1: Asking ChatGPT how to reduce a quadric.

The student gets overwhelmed when looking at the answer: He knows nothing about orthogonal transformations or eigenvalues! So he asks if there is a simpler explanation. Figure 2 shows the answer.



Can you explain it in a easier way?

Sure! Let's break down the process of finding the canonical form of a quadric into simpler terms:

Step 1: Identify the Quadric Equation

- You start with a general equation that might look complicated, like:
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$
This could be in 3D, or in 2D (which would be a bit simpler).

Step 2: Simplify the Equation

- **Remove Linear Terms**: If there are terms like Gx, Hy, or Iz, you can simplify things by moving the origin of your coordinate system (basically "shifting" the graph) to get rid of these terms. This step isn't always needed, but it helps to simplify.

Step 3: Group the Quadratic Terms Together

- Combine the terms with squares and cross-products (like x^2, xy) into a simpler shape. You can imagine grouping similar terms together.

Step 4: Eliminate the Mixed Terms

- If you have terms like Dxy, Exz, or Fyz, they make the shape hard to understand. To remove these, you "rotate" your coordinate system. This is like turning the graph until you only have x^2, y^2, and z^2 terms left.

Step 5: Get the Canonical Form

- After rotating, the equation should look like one of these simple forms:
  - **Sphere or Ellipsoid:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
  - **Paraboloid:** $x^2 + y^2 = z$
  - **Hyperboloid:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Step 6: Identify the Shape

- The simplified form tells you what kind of surface you have:
  - If it looks like $x^2 + y^2 + z^2 = 1$, it's a sphere.
  - If it looks like $x^2 + y^2 - z^2 = 1$, it's a hyperboloid.

Summary

1. **Shift and Rotate** your coordinate system to make the equation simpler.
2. **Remove Complicated Terms** like xy.
3. **Identify the Standard Form** of the equation to understand the shape (sphere, paraboloid, hyperboloid, etc.).

Figure 2: Asking ChatGPT if is there a simpler way.

We can see that it is not very helpful. However, the original answer contains the key: In figure 3 we notice that ChatGPT is well aware of the technique of completing squares, so maybe restating the question leads to a better answer. This is done in Figure 4.

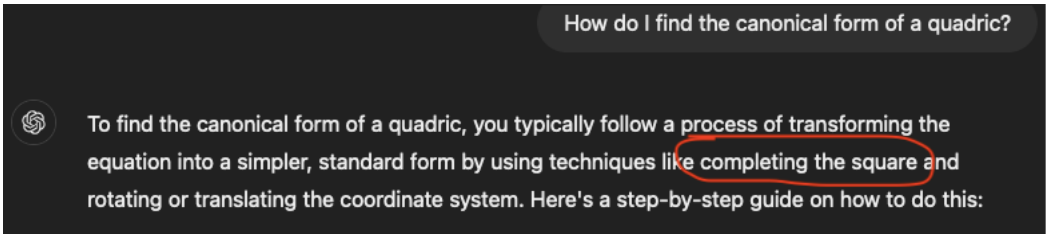


Figure 3: Completing squares!

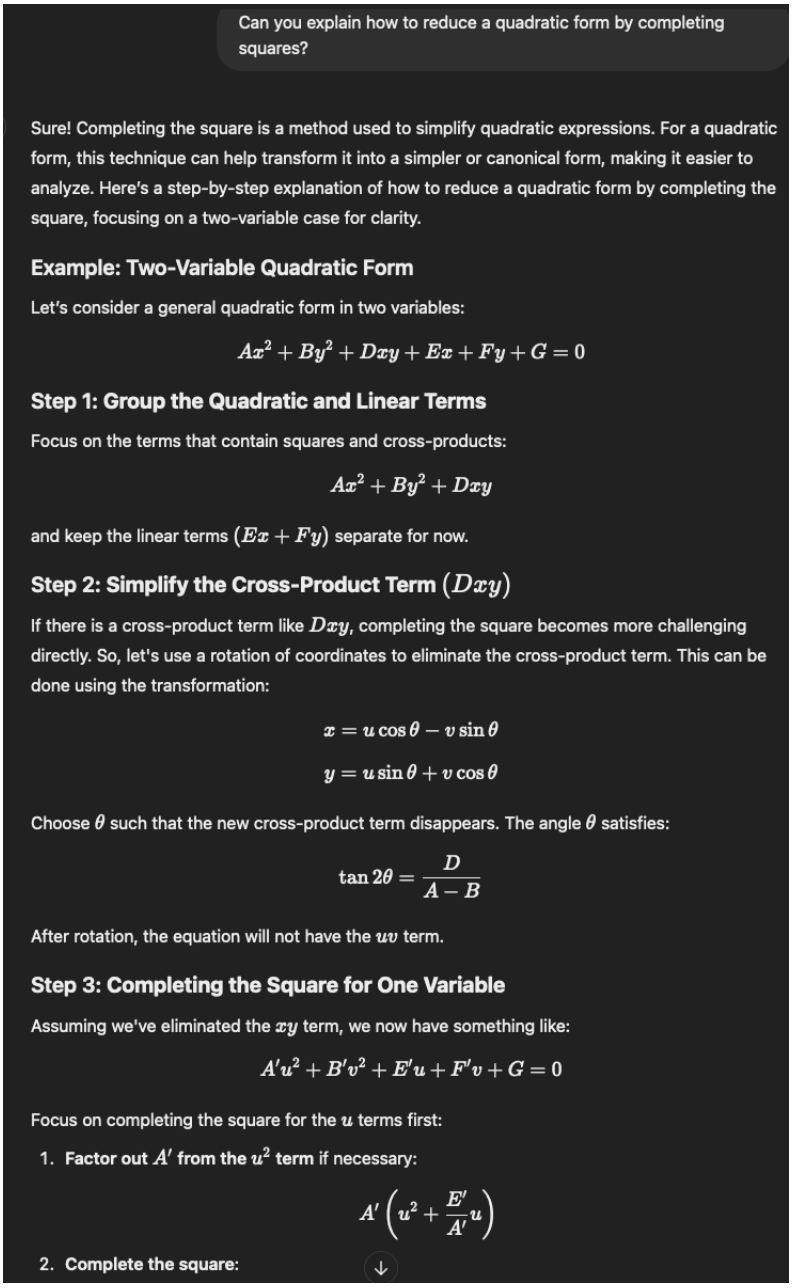


Figure 4: Explaining the technique of completing squares.

The result is much more clear for our student, but some annoying comments still remain. What are these rotations used with terms of the form Dxy? The teacher did not mention anything like that[1].

The discussion could go on and on, but the idea is more or less evident: ChatGPT is as useful as well-formed are the questions we ask. And the inescapable problem is that (many of) our students do not have the mathematical maturity required to write such questions. Thus, we see that there is still room for alternatives to ChatGPT, and in this paper we describe a way to generate solved problems using a Computer Algebra Software (CAS). Of course, this procedure only applies to those problems whose solutions have an algorithmic character, but it has the advantage that the teacher decides which method will be used, and allows for the insertion of tips in those steps that he considers more problematic for students.

2 The main idea

Any programming language, hence any CAS, offers the possibility of printing text at any stage of a program's execution. Whenever we have an algorithm for solving a problem, we can document what we are doing by printing some descriptive message, and if the text is carefully chosen, this will provide a complete report on the application of the algorithm to the particular problem at hand. In other words, printing these mid-execution messages is the equivalent of a verbal explanation at the blackboard.

Let us see that idea at work in a simple example. Consider the technique of completing squares. Suppose you have an expression which is quadratic in a variable x, so it has the form $ax^2 + bx + c$, where a, b, c can be expressions themselves, but free of x. Completing the square in x consists in manipulating the expression so the variable x only appears under the form $(Ax + B)^2$, for some expressions A, B (free of x). This can be achieved as follows: First, factor the coefficient $a \neq 0$ of x^2, to get the equivalent expression

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right).$$

Now, focus on the terms $x^2 + bx/a$. By comparing with the desired form $(Ax + B)^2 = A^2x^2 + 2ABx + B^2$, we see that it must be

$$A = 1 \quad \text{and } B = \frac{b}{2a},$$

but there is an additional term $B^2 = \frac{b^2}{4a^2}$, so, if we introduce A, B, we must extract that term. The final expression is

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$

[1] In fact, what the teacher used in the classroom example was the equality

$$Dxy = \frac{D}{4}((x + y)^2 - (x - y)^2)$$

The right hand side clearly shows that the square completed expression consists of two parts: the binomial term containing x, and the remainder $c - b^2/4a$, free of x. To implement this technique in Maxima, we start with an expression `expr` (which has the form $ax^2 + bx + c$) and get the coefficients a, b, c with the command `coeff`. For instance, a (the coefficient of x to the power of 2) is obtained as `a:coeff(expr,x,2)`, while c (the independent term, or the coefficient of x to the power of 0), is `c:coeff(expr,x,0)`. Once we have identified a, b, c, we can directly write the square completed expression

$$a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) . \tag{1}$$

This is a one-line function in Maxima[2]

```
(%i1) completesquare(expr,x):=block([a,b,c],
a:coeff(expr,x,2),b:coeff(expr,x,1),c:coeff(expr,x,0),
fullratsimp(a)*(x+ratsimp(b/(2*a)))^2+multthru(ratsimp(c-b^2/(4*a)))
)$
```

And here is the application to the expression $x_1^2 + 3x_1x_2 - x_2^2$, completed with respect to x_1:

```
(%i2) completesquare(x[1]^2+3*x[1]*x[2]-x[2]^2,x[1]);
```

$$\left(x_1 + \frac{3x_2}{2}\right)^2 - \left(\frac{13x_2^2}{4}\right) \tag{\% o2}$$

We can make all of this much more instructive if we print some messages explaining what we do at each one of the two steps of the algorithm, for instance:

```
(%i3) completesquareexplained(expr,x):=block([a,b,c],
print("First, we obtain the coefficients a,b,c in expression"),
print(expr),
a:coeff(expr,x,2),b:coeff(expr,x,1),c:coeff(expr,x,0),
print("These are"),
print(['a=a,'b=b,'c=c]),
print("Next, we write the square completed expression as in (1)"),
fullratsimp(a)*(x+ratsimp(b/(2*a)))^2+multthru(ratsimp(c-b^2/(4*a)))
)$
```

To create a visually not-so-appalling output, notice that each `print` command appears in a separate line. Also, it is worth remarking that the single quote operator `'` prevents evaluation, so in `['a=a,'b=b,'c=c]` we actually get a list with the values assigned to a, b, c on the right hand sides, and the names of the variables (not evaluated) on the left hand ones. The output now becomes a full explanation of the steps followed, with a clear reference to the formulas being applied, the notation, and anything that the teacher considers worth including:

[2]The commands `fullratsimp`, `ratsimp`, and `multthru` are included only to get simplified rational expressions.

```
(%i4) completesquareexplained(x[1]^2+3*x[1]*x[2]-x[2]^2,x[1]);
```

First, we extract the coefficients a, b, c in expression

$$-x_2^2 + 3x_1x_2 + x_1^2$$

These are

$$\left[a = 1\,, b = 3x_2\,, c = -x_2^2\right]$$

Next, we write the square completed expression as in (1)

$$\left(\frac{3x_2}{2} + x_1\right)^2 - \frac{13x_2^2}{4} \qquad (\%\ o4)$$

That is the whole idea. Of course, for more complicated algorithms the comments must be clear enough for the student to be able of following the line of reasoning, and it is the task of the teachers to create useful ones, based on their familiarity with the difficulties the students experience on a particular subject.

3 Practical implementation

We will examine the case of reducing quadrics to canonical form, which served for illustration purposes in the introduction. The Lagrange algorithm for reducing a quadratic form by completing squares is straightforward: Let

$$Q(x_1, \ldots, x_n) = \sum_{k=1}^{n} a_{kk}x_k^2 + \sum_{\substack{i,j=1\\ i<j}}^{n} a_{ij}x_{ij}$$

be a quadratic form in n variables $x_1, \ldots, x_n$.

1. Locate the lowest index i such that $a_{ii} \neq 0$. Then, we can complete the square using all the terms containing x_i to obtain a decomposition

$$Q(x_1, \ldots, x_n) = a_{ii}\left(\sum_{j=1}^{n} \frac{a_{ij}}{a_{ii}}x_j\right)^2 + Q(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\,.$$

Here, $Q(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is a quadric in $n-1$ variables, free of x_i, to which we apply the same procedure until no terms of the form $a_{ii}x_i^2$ are left. Notice that a change of variables of the form

$$y_i = \sum_{j=1}^{n} \frac{a_{ij}}{a_{ii}}x_j$$

allows us to rewrite the quadric as

$$Q(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) = y_i^2 + Q(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\,.$$

2. If $a_{ii} = 0$ for all i, then we choose an $a_{ij} \neq 0$ (there must be at least one, otherwise the quadric would be $Q = 0$, which is already reduced). Then we make the change of variables
$$\begin{cases} x_i = y_i + y_j \\ x_j = y_i - y_j \\ x_k = y_k \text{ if } k \neq i, j\,, \end{cases}$$
and proceed to step 1, as
$$a_{ij}x_ix_j = a_{ij}^2y_i^2 - a_{ij}^2y_j^2\,.$$

This is quite simple: No diagonalization of matrices, or orthogonal transformations are needed, just plain simple sums and products. That can be explained with an example in the blackboard, and everybody will grasp it. However, when doing particular examples students frequently commit mistakes; maybe it is a sign, maybe a forgotten coefficient. What is required in this case is a list of the intermediate steps for comparison, to detect the error in the student's calculations.

So, here is our proposal for this particular case, programmed in Maxima.

```
(%i1) load(fourie)$ /* To enable equalp */
```

```
(%i2) load(linearalgebra)$ /* To enable locate_matrix_entry */
```

```
(%i3) completesquare(r,x):=block([a,b,c],
a:coeff(r,x,2),b:coeff(r,x,1),c:coeff(r,x,0),
    if is(not(equalp(a,0))) then
[fullratsimp(a)*(x+ratsimp(b/(2*a)))^2,multthru(ratsimp(c-b^2/(4*a)))]
    elseif is(not(equalp(b,0))) then
((x+b)^ 2-(x-b)^ 2)/4
    else
r
 )$
```

Let us give an example of the use of `completesquare`, because it is a slightly modified version of the function we defined earlier with the same name. When applied to an expression r containing the variable x, its output is a pair where the first element is the square completed and the second is the remainder of the original expression, free from x^2:

```
(%i4) completesquare(x[1]^2+3*x[1]*x[2]-x[2]^2,x[1]);
```

$$\left[\left(\frac{3x_2}{2} + x_1\right)^2, -\frac{13{x_2}^2}{4}\right] \qquad (\%\ o4)$$

We prefer this form of `completesquare` because of the recursive nature of the algorithm, which at each step will be applied to the expression free of the variable with respect to which

we are completing the square, *plus* the second element of the list provided by `completesquare` in the previous step

Now we can write the main function. Notice how the printed output closely follows the coding comments, but written in a less terse style. It is here where teachers must resort to their knowledge about their students, and prepare brief but helpful sentences that explain the steps followed.

```
(% i5) reducequadric(exprsn,variabs):=block([lvariabs:length(variabs),coeffsmatrix, coeffssquares,
laux,expraux:exprsn,solaux:[],indaux,coeffsmatrixaux,tolaux:[],fxpraux],
/* We start by writing the symmetric real matrix assocaited to the quadric */
print("We start by writing the real symmetric matrix associated to the quadratic form:"),
coeffsmatrix:genmatrix(lambda([i,j],if equalp(i,j) then coeff(exprsn,variabs[i],2)
else coeff(coeff(exprsn,variabs[i],1),variabs[j],1)/2),lvariabs,lvariabs),
print(coeffsmatrix),
/* Now, we determine the first variable for which the coefficient of its square is non-null */
print("Next, by reading the variables following the ordering given in "),
print(variabs),
print("we determine the first one for which its square has a non-zero coefficient,"),
print("and factor with respect to it. If that is not the case (that is, there are no squares) we still"),
print("have to factor, but using sums/differences applied to the variables for which there are cross terms."),
coeffssquares:makelist(coeffsmatrix[i,i],i,1,lvariabs),
laux:sublist_indices(coeffssquares,lambda([unk],not(equalp(unk,0)))),
/* and factor with respect to it */
/* if there are no squares we still have to factor, but using sums/differences starting from the first two variables */
/* for which there is a cross term. To find them, we read the coefficients matrix row by row with locate_matrix_entry */
while (is(length(laux)≠0) and expraux≠0) do (
print("The expression "),print(expraux),print("contains a square with non-zero coefficient in "),print(variabs[first(laux)]),
print("and we complete it (if needed) resulting the term:"),
solaux:push(first(completesquare(expraux,variabs[first(laux)])),solaux),print(first(solaux)),
expraux:second(completesquare(expraux,variabs[first(laux)])),
laux:rest(laux,1)
),
print("All the possible squares are already complete. If needed, we will continue with cross terms."),
if is(equalp(expraux,0)) then (print("In this particular case, it is not needed and we have finished:"),
return(apply("+",solaux)))
elseif (is(equalp(length(laux),0)) and is(length(solaux) > 0)) then
(print("In this particular case, it is not needed and we have finished:"),return(apply("+",solaux)+expraux)) else
(coeffsmatrixaux:genmatrix(lambda([i,j],if equalp(i,j) then coeff(expraux,variabs[i],2) else
coeff(coeff(expraux,variabs[i],1),variabs[j],1)/2),lvariabs,lvariabs),
indaux:locate_matrix_entry(coeffsmatrixaux,1,1,lvariabs,lvariabs,lambda([u],u≠0),'bool),
fxpraux:expraux,print("We have to factor "),print(expraux),print("And we do it using a sum/difference scheme."),
while is(not(equal(indaux,false))) do
(tolaux:push(multthru(arrayapply(coeffsmatrixaux,indaux)/2*((variabs[first(indaux)]+variabs[second(indaux)])^ 2
-(variabs[first(indaux)]-variabs[second(indaux)])^ 2)),tolaux),
fxpraux:ratsimp(fxpraux-2*arrayapply(coeffsmatrixaux,indaux)*variabs[first(indaux)]*variabs[second(indaux)]),
coeffsmatrixaux:genmatrix(lambda([i,j],if equalp(i,j) then coeff(fxpraux,variabs[i],2)
else coeff(coeff(fxpraux,variabs[i],1),variabs[j],1)/2),lvariabs,lvariabs),
indaux:locate_matrix_entry(coeffsmatrixaux,1,1,lvariabs,lvariabs,lambda([u],u≠0),'bool)
),
print("We get the terms:"),print(apply("+",tolaux))
),print("Then, the final result is:"),
if (listp(solaux)) then (apply("+",solaux) + apply("+",tolaux)) else solaux+apply("+",tolaux)
)$
```

4 Some examples

Let us start with a very simple example: The quadric $Q(x_1, x_2) = x_1^2 - 4x_1x_2$ only needs completing the square with respect to x_1 to be in canonical form, as

$$x_1^2 - 4x_1x_2 = (x_1 - 2x_2)^2 - 4x_2^2\,.$$

In Maxima, we get:

```
(%i6) reducequadric(x[1]^2-4*x[1]*x[2],[x[1],x[2]]);
```

We start by writing the real symmetric matrix associated to the quadratic form:

$$\begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix}$$

Next, by reading the variables following the ordering given in

$$[x_1\,, x_2]$$

we determine the first one for which its square has a non-zero coefficient,
and factor with respect to it. If that is not the case (that is, there are no squares) we still have to factor, but using sums/differences applied to the variables for which there are cross terms.
The expression

$${x_1}^2 - 4x_1\,x_2$$

contains a square with non-zero coefficient in

$$x_1$$

and we complete it (if needed) resulting the term:

$$(x_1 - 2x_2)^2$$

All the possible squares are already complete. If needed, we will continue with cross terms.
In this particular case, it is not needed and we have finished:

$$(x_1 - 2x_2)^2 - 4{x_2}^2 \qquad (\%\ o6)$$

Another example, this time with the quadric $Q(x_1, x_2) = 4x_1x_2$, which does not contain squares.

```
(%i7) reducequadric(4*x[1]*x[2],[x[1],x[2]]);
```

We start by writing the real symmetric matrix associated to the quadratic form:

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Next, by reading the variables following the ordering given in

$$[x_1\,, x_2]$$

we determine the first one for which its square has a non-zero coefficient,
and factor with respect to it. If that is not the case (that is, there are no squares) we still have to factor, but using sums/differences applied to the variables for which there are cross terms.
All the possible squares are already complete. If needed, we will continue with cross terms.
We have to factor

$$4x_1x_2$$

And we do it using a sum/difference scheme.
We get the terms:

$$(x_2 + x_1)^2 - (x_1 - x_2)^2$$

Then, the final result is:

$$(x_2 + x_1)^2 - (x_1 - x_2)^2 \qquad (\%\ o7)$$

Let us finish with a more complete example. Given the quadratic form

$$Q(x_1, x_2, x_3) = x_1^2 + 4x_2^2 + 3x_3^2 - 4ax_1x_2\,,$$

it is required to classify it in terms of the values of $a \in \mathbb{R}$. We run the command `reducequadric` looking for some help, and here is what we get:

```
(% i8)  reducequadric(x[1]^ 2+4*x[2]^ 2+3*x[3]^ 2-4*a*x[1]*x[2],[x[1],x[2],x[3]]);
```

We start by writing the real symmetric matrix associated to the quadratic form:

$$\begin{pmatrix} 1 & -2a & 0 \\ -2a & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Next, by reading the variables following the ordering given in

$$[x_1\,, x_2\,, x_3]$$

we determine the first one for which its square has a non-zero coefficient,
and factor with respect to it. If that is not the case (that is, there are no squares) we still have to factor, but using sums/differences applied to the variables for which there are cross terms.
The expression

$$-4x_1x_2a + 3x_3^2 + 4x_2^2 + x_1^2$$

contains a square with non-zero coefficient in

$$x_1$$

and we complete it (if needed) resulting the term:

$$(x_1 - 2x_2a)^2$$

The expression

$$-4x_2^2a^2 + 3x_3^2 + 4x_2^2$$

contains a square with non-zero coefficient in

$$x_2$$

and we complete it (if needed) resulting the term:

$$x_2^2\left(4 - 4a^2\right)$$

The expression

$$3x_3^2$$

contains a square with non-zero coefficient in

$$x_3$$

and we complete it (if needed) resulting the term:

$$3x_3^2$$

All the possible squares are already complete. If needed, we will continue with cross terms. In this particular case, it is not needed and we have finished:

$$(x_1 - 2x_2a)^2 + x_2^2\left(4 - 4a^2\right) + 3x_3^2 \qquad (\%\ o8)$$

Thus, $Q(x_1, x_2, x_3)$ is degenerate (semidefinite positive, with signature $(+, 0, +)$) whenever $a \in \{-1, 1\}$. If $|a| > 1$ it will be indefinite, with signature $(+, -, +)$, and positive definite if $|a| < 1$.

5 Conclusion

For those problems that admit an algorithmic solution, we have shown how we can implement a Maxima function that solves them and outputs the intermediate steps of the procedure, with canned comments that help the student identify the precise block of the algorithm that is being executed. In this way, the students can compare the output with their own solutions and detect any mistake, thus gaining confidence in their work. This also alleviates the workload of teachers, as sometimes a great effort is needed to answer the 'where could I be wrong?' kind of questions, and even the more common 'where can I find solved examples?'

Contributed Papers

Bayesian Sequential D-Optimal Sampling Designs for Fractional Partial Differential Equations

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Abstract

Researchers are becoming increasingly interested in using Fractional Partial Differential Equation (FPDE) models for physical systems such as gas flows through porous materials. These models rely on the fraction of the differentiation α, which needs to be estimated from empirical data. Experimentation is needed to obtain empirical data where pressures need to be measured at various times, t, from the initial pressure and distances x from the pressure source which produces an output pressure $p(x,t)$. While sampling times are easy to choose when a sensor is in place. Typically the location of sensors from the pressure source are arbitrarily chosen. This work shows how to design experiments using a sequential design with a base design and sequentially adding sampling design points by finding the optimal sensor locations along x by minimizing D-optimality criteria which is essentially minimizing the of the volume of the variance-covariance matrix of all the parameters. For parameter estimation a Bayesian framework is utilized a sequential design is used to search through the possible locations for the next sensor in the follow up design. Two simple FPDE parameterisations are used to illustrate the method with a base design of six sensor locations and with five additional sensors locations determined sequentially. The simple examples suggest that the parameter values influence the location of the next best sensor location.

1 Introduction

Fractional Partial Differential Equations (FPDE) are growing in interest among researchers confronted with modeling flow through porous materials [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13]. Often they would like to be able to estimate various parameters associated with the model such as the fractional derivative parameter α as well as measurement error. Once empirical data is collected the parameters can be estimated using techniques discussed in [13]. However, generating this empirical data may require experimentation on samples of the porous materials under study. This work considers how to place sensors for observation along the distance from the pressure source axis x using a follow-up study approach. Figure 1 shows and example of how an experimental set up may look. Pressure is applied to one end of the porous material and at specified distances $x_1, x_2, ...x_k$ sensors are placed to measure the pressure output at those locations at various times t from the pressure being applied, denoted $p(x_i, t)$. Notice that the choice of locations of the sensors may or may not be optimal for learning about the parameters that govern the process. The question posed here is: How can a follow-up study framework allow for better estimation of the parameters, specifically the fraction parameter α.

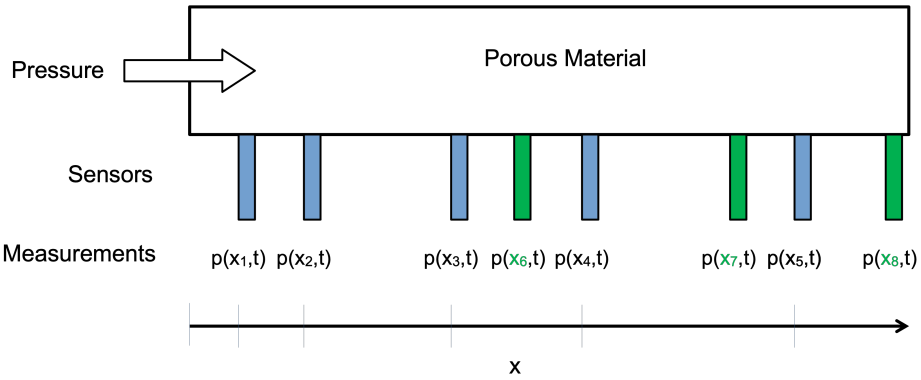

Figure 1: Example experimental device with pressure input, sensors at various locations and resulting $p(x, t)$ measurements. Blue sensors correspond to the *base* design and the green sensors are the additional sensor as suggested via the *follow-up* design.

The follow-up approach to design of experiments consist of two parts, first, a *base* design is conducted where sensors are placed and initial data is obtained and the information is used to help determine where the additional sensors are placed for the second stage of experimentation, the *follow-up* design. The advantage of follow-up studies is that researchers can place the additional sensors in locations that will allow for the most information about the parameters to be learned from the data. Experimental Design is a broad field in statistics which is covered well by [14] and [15]. For this study, a base design with six locations chosen arbitrarily, and follow-up designs with 1, 2, 3, 4 and 5 possible new locations are considered. In Figure 1, the blue sensors reflect the *base* design and the green sensors reflect the addition of three sensors once data has been collected from the base sensor. These three green sensors are the *follow-up* design, in which the locations are chosen to minimize the volume of the Variance-Covariance matrix of the model parameters.

The outline of the paper is as follows: In Section 2 we give the various definitions of fractional derivatives and some properties related to their Laplace transforms. We also consider the

mathematical model through a factional partial differential equation (FPDE) with boundary conditions. The solution to the FPDE is also explicitly presented. Section 3 defines the statistical model in terms of likelihood, prior distributions, parameter estimation and optimality criterion. Section 4 describes the search algorithm to find the new locations for sensors as well as computational details. This is followed by the results in Section 5 that shows the placement of sensors when done simultaneously for 1, 2, 3, 4, and 5 sensors. Finally, Section 6 gives a discussion of the work, it's limitations and future work.

2 Preliminaries and Mathematical Model

In this section we present some useful definitions and results that will help us throughout the paper.

2.1 Preliminaries

In this section we present some definitions and results which will be needed later in our model.

Definition 1 *The Riemann-Liouville fractional integral of order α of f is defined by*

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \;\; t > 0, \alpha > 0$$

when the right hand side exists.

Definition 2 *The Riemann-Liouville fractional derivative of order α of f is defined by*

$$({}^{RL}D_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds, \;\; t > 0, 0 < \alpha < 1$$

when the right hand side exists.

Theorem 3 *The relationship between the Riemann-Liouville fractional and the Riemann-Liouville fractional integral is given by*

$$({}^{RL}D_{0+}^{\alpha} f)(t) = \frac{d}{dt}(I^{1-\alpha} f)(t)$$

Definition 4 *The Caputo fractional derivative of order α is defined by*

$$({}^{C}D_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \;\; t > 0, 0 < \alpha < 1$$

when the right-hand side exists.

The following theorem gives the relationship between the Caputo and Riemann-Liouville fractional derivatives.

Theorem 5

$$({}^{RL}D_{0+}^{\alpha} f)(t) = ({}^{C}D_{0+}^{\alpha} f)(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0^+), \;\; t > 0, 0 < \alpha < 1.$$

Definition 6 *The Hilfer fractional derivative of f of order α and type β is defined by*

$$(D_{0^+}^{\alpha,\beta} f)(t) = \left(I^{\beta(1-\alpha)} \frac{d}{dt} I^{(1-\alpha)(1-\beta)} f \right)(t)$$

whenever the right-hand side exists

Proposition 7 *1. If $\beta = 0$ in Definition 6, then the Hilfer derivative becomes the Riemann-Liouville derivative,*

2. If $\beta = 1$ in Definition 6, then the Hilfer derivative becomes the Caputo derivative.

It is well known that the Caputo fractional derivative is often preferred over the Riemann-Liouville fractional derivative at least for two main reasons: The Caputo derivative allows us to use the usual initial data such as $p(x,0) = f(x)$, whereas the Riemann-Liouville derivative requires the prior knowledge of $D^{\alpha-1}p(x,0) = g(x)$ when the order of the equation is α. In addition to that, the Caputo derivative of a constant is zero which is not the case for the Riemann-Liouville derivative.

Finally the Laplace transforms for these fractional derivatives are given by

Theorem 8

$$\begin{aligned}
\mathcal{L}\left[({}^{RL}D_{0^+}^{\alpha} f)(t)\right](s) &= s^{\alpha}\mathcal{L}\left[f(t)\right](s) - (I^{1-\alpha} f)(0^+) \\
\mathcal{L}\left[({}^{C}D_{0^+}^{\alpha} f)(t)\right](s) &= s^{\alpha}\mathcal{L}\left[f(t)\right](s) - s^{\alpha-1} f(0^+) \\
\mathcal{L}\left[((D_{0^+}^{\alpha,\beta} f)(t)\right](s) &= s^{\alpha}\mathcal{L}\left[f(t)\right](s) - s^{\beta(\alpha-1)}(I_{0^+}^{(1-\alpha)(1-\beta)} f)(0^+)
\end{aligned}$$

where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.

2.2 Model

In this section we consider the model prersented by N. Malik et al. [10]. In [10], a mathematical model for transport through tight porous rocks which are characterized by nano-scale size porous networks with ultra-low permeability. The mathematical model was presented in terms of time-fractional advection-diffusion equation given by

$$\frac{\partial^{\alpha} p}{\partial t^{\alpha}} = \frac{\partial^2 p}{\partial x^2} - \frac{\partial p}{\partial x}, \qquad t > 0, x > 0. \tag{1}$$

with the initial condition is $p(x,0) = e^{-cx}$, $(c > 0)$, and the boundary condition is $p(x,t) \to 0$ as $x \to \infty$. $p(x,t)$ is the pressure in tight gas reservoir.

Malik et al [10] use the Variational Iteration Method (VIM) to solve the above equation. Ji-Huan He [11] proposed the VIM to obtain solutions of nonlinear differential equations. The method provides the solution in the form of a successive approximations that may converge to the exact solution if such a solution exists. In the case where a closed form of exact solution is not achieved achievable, we use a truncated series as an approximate solution. Using the VIM, Malik et al present the following solution to Equation 1

$$p(x,t) = e^{-x} \sum_{k=0}^{\infty} \frac{(2t^{\alpha})^k}{\Gamma(\alpha k + 1)}. \tag{2}$$

A closed form solution to Equation 1 can be written as

$$p(x,t) = e^{-x} E_\alpha(2\,t^\alpha) \tag{3}$$

where E_α is the one-parameter Mittag-Leffler Function given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{4}$$

3 Model and Optimality Criterion

The experimental error associated with any experimental studie should be incorporated into the model. In the case under consideration here, pressure measurements (outcome variable) will always be non-negative and hence any error structure needs to account for this. In many statistical models a Normal distribution is chosen for the error term, however this can produce negative pressures, and hence an alternative is needed. The log-Normal error term is used and to incorporate this into the model as it will never allow for a negative pressure. Combining the log-Normal error structure with the pressure model $p(x_i, t)$ the observed data model given by:

$$y_{x_i,t} = p(x_i, t)\epsilon_{i,t} \tag{5}$$

where $\epsilon_{i,t} \sim log - Normal(1, \sigma^2)$ and σ^2 is the variance of the measurement error. Note that he log-Normal error structure is multiplicative versus additive to ensure the positivity of the outcome. A Bayesian framework is employed here, which requires prior distributions for the parameters α and σ^2 be specified. As $0 < \alpha < 1$ the natural choice for the prior probability distribution for $\alpha \sim Beta(2, 2)$ and since $\sigma^2 > 0$ a natural choice the prior distribution for $\sigma^2 \sim Exp(1)$. Recall, for α, the $Beta(2, 2)$ distribution has likelihood of zero when $\alpha \leq 0$ and $\alpha \geq 1$ obtaining a maximum at 0.5 reflecting that the fractional parameter cannot be 0 and fractional derivatives are not needed when $\alpha = 1$ and that as the value of x becomes closer to 0 or 1 the less likely that value is. For parameter estimation, MCMC techniques such as the Metropolis-Hastings algorithm are utilized to obtain samples from the posterior distribution of α and σ^2. This is a similar approach to that used by [13]. All parameter inferences are based on these posterior samples, specifically, the posterior Variance-Covariance of the α and σ^2.

In order to determine the optimal location of new sensors an Information Matrix approach is used. Recall, given a data set D, parameter vector θ and probability density f (that is twice differentiable), the Fisher Information matrix is defined as [16]:

$$\mathcal{I}(\theta) = -E\left[\left(\frac{\partial^2}{\partial\theta^2} log f(D|\theta)\right)|\theta\right]. \tag{6}$$

Unfortunately, the likelihood here is not differentiable with respect to α which is problematic. Upon, closer inspection $\mathcal{I}(\theta)$ is the second moment of the parameters and is an approximation of the inverse of Variance-Covariance of the parameters, which allows us to not need the differentiability with respect to α requirement. Here, this work uses the estimated posterior distribution Variance-Covariance matrix of the parameters calculated from the MCMC samples, which will be denoted as $\hat{\mathcal{I}}^{-1}(\theta)$.

To find the optimal locations, one wishes to optimize the Information in some respect. The D-optimal criterion is [17]:

$$D = \min_x |\hat{\mathcal{I}}^{-1}(\theta)| \tag{7}$$

Here the minimization is over the candidate points x or the locations of the the follow up sensors and the criterion to be minimized is the volume the Information Matrix, which is essentially the sum of the marginal variances for α and σ.

4 Methods

A simulation study is used to illustrate how locations of additional sensors can be selected. The base design (test locations) were arbitrarily chosen to span the range of reasonable values and set tobe $x = 0.2,\ 1.0,\ 2.0,\ 5.0,\ 7.0,\ 10.0$ and measurements were taken at times $t = 0.5,\ 1.0,\ 1.5$. A data set of pressures at each distance x and time t, $p(x,t)$ was generated for this scenario which is used throughout the study to obtain the initial parameter estimates and variances. Let X_{base} denote the x locations of the base design and $y_{base,t}$ be the corresponding y values of $p(X_{base}, t)$ at time t. The follow up study design consisted of 44 candidate locations with x ranging from $x = 0.4$ to $x = 9.8$ in 0.2 increments and denote this as X_{cand}. Here additional points will be considered in a sequential manner with no duplicate sampling locations permitted as each location can physically have only one sensor. We will denote the set of x values that are accepted as X_{acc} and the set of values that are candidates to be added considered for inclusion as X_{cand}. To start off, we set $X_{acc} = X_{base}$ and $X_{cand} = X_{cand} \setminus X_{base}$. Algorithm 1 details the steps of the process to obtain the optimal x values.

Algorithm 1 Algorithm for Sequential D-optimal Sampling locations (5 new points).

for $j \in 1$ to 5 **do**
  for $i \in |X_{cand}|$ **do**
    Sample: $(\alpha^{(m)}, \sigma^{(m)}) \sim \pi(\alpha, \sigma | x_{acc}, x_i)$
    Calculate: $\hat{\mathcal{I}}^{-1}(\alpha, \sigma) = cov\left(\alpha^{(m)}, \sigma^{(m)}\right)$
    Store: $D_i = |\hat{\mathcal{I}}^{-1}(\alpha, \sigma)|$
  end for
Find: $x_{opt,j} = arg\min_x D_i$
Combine: $X_{acc} = X_{acc} \cup \{x_{opt,j}\}$
Remove: $X_{cand} = X_{cand} \setminus \{x_{opt,j}\}$
end for

Both the MCMC sampler and sequential design algorithms were programmed in MATLAB 2024a. As the design space is rather small at each stage (at most 44 points) all candidate points were evaluated, resulting in a total computation time of approximately 370 seconds for all five steps using an AMD Ryzen Threadripper Pro 5975WX 32-Core with 64GB of RAM. For quality assurance MCMC diagnostics such as ACF plots and trace-plots were examined during the entire process, which high acceptance probability and sampler convergence to the posterior distribution.

5 Results

Two simulated scenarios are shown here to illustrate the methodology. In the first scenario the true underlying parameter values are specified as $\alpha = 0.82$ and $\sigma = 0.01$. To determine the optimal placement of each of five new sensors is done sequentially using the process described above. Figure 2 gives the results of this scenario where the true underlyng solution for the fPDE is the blue curve, the base design sampling locations are presented as dashed blue vertical lines with red circles corresponding to the mean of process at that location. New sensors are placed at the locations corresponding to the vertical red lines which have the x value denoted. Notice that the the first sensor placed at $x = 3.2$, the second at $x = 2.6$, the third at $x = 2.2$, the fourth at $x = 8.8$ and the fifth at $x = 8$.

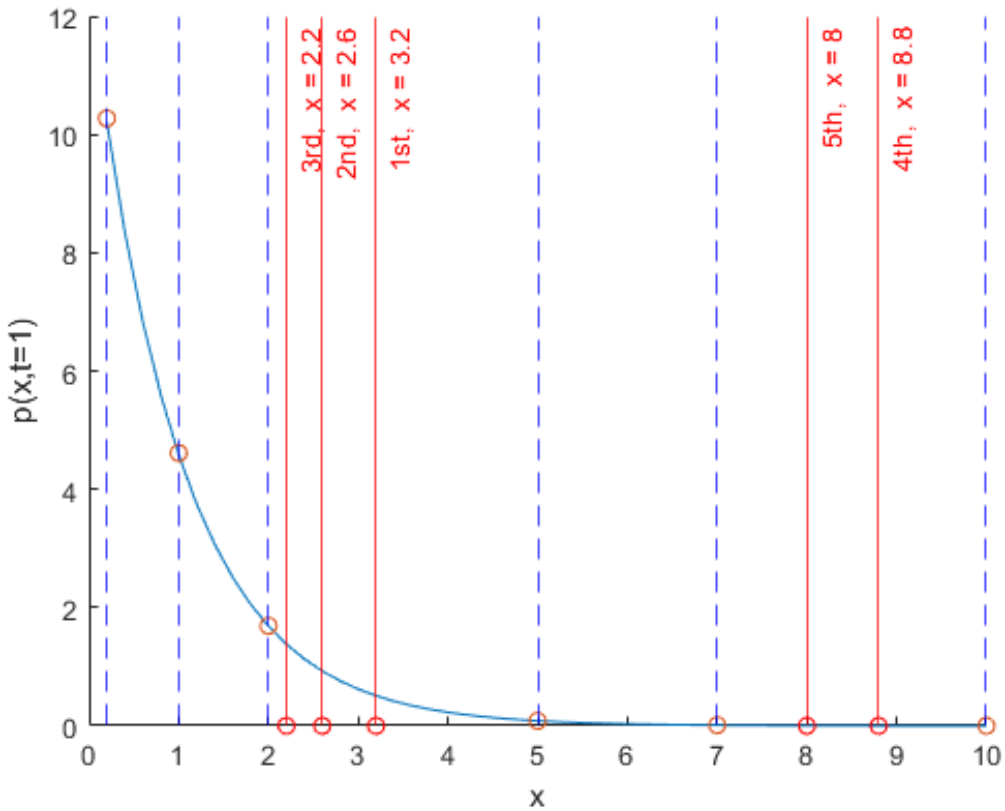

Figure 2: Plot of additional sensor locations when added sequentially. Simulation example where $t = 1$ and $\alpha = 0.82$. and $\sigma = 0.01$.

For the second simulated scenario, the true underlying $\alpha = 0.41$ and $\sigma = 0.01$. The process outlined above determines the optimal sequential placement of five new sensors. Figure 3 gives the results for this scenario with the true underlying fPDE solution being the blue curve, the base sampling design locations as dashed blue vertical lines with red circles at the predictive mean of the process at that location. Vertical red lines indicate the suggested optimal locations of the new sensors are annotated with the x corresponding x value. Notice that the the first sensor placed at $x = 3.6$, the second at $x = 7.4$, the third at $x = 2.8$, the fourth at $x = 6.6$ and the fifth at $x = 5.6$. By contrasting these two examples, one can see that there is variability

between additional sampling locations simply due to the true underlying fractional derivative value α. Another interesting item to notice is that all of the new sampling locations are above $x = 2$. This suggests that information collected for low values of x do not carry as much information as those with larger values. A larger scale simulation study should be conducted to determine if this phenomenon exists in a broader sense.

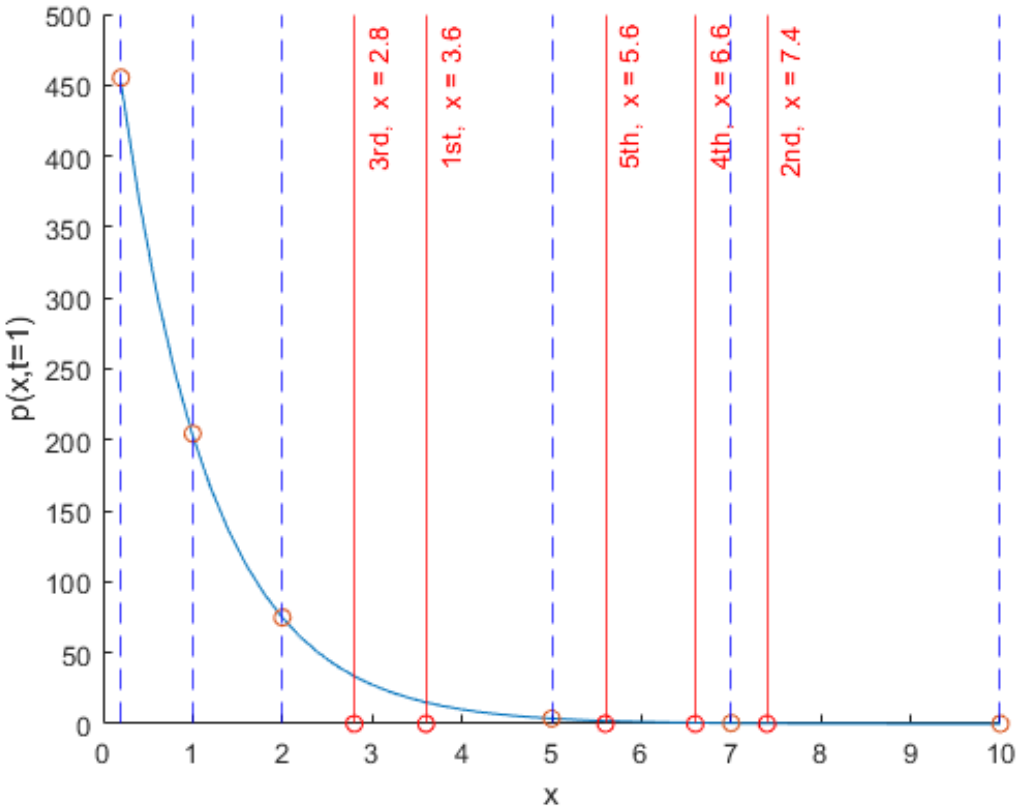

Figure 3: Plot of additional sensor locations when added sequentially. Simulation example where $t = 1$ and $\alpha = 0.41$. and $\sigma = 0.01$.

6 Acknowledgements

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7 Conclusion

Using the base design considered here, in all scenarios considered additional experimental sensor locations should to be placed in the asymptotic region of the FPDE curve in order to minimize the volume of the space generated by the variance-covariance matrix of the parameter estimates.

In addition, points where the curve approaches asymptotic behavior appear to be important as well. In this simple model there are two parameters of interest α and σ^2. One can easily determine the contribution to the total variance of each component as each is simply the diagonal value in the estimated Information matrix. In our case, both α and σ are on similar scales, both are in the interval $(0, 1)$, and hence one parameter should not dominate the results.

Presenting a large scale simulation study is not permissible due to page limitations. However, such a study is needed to determine if there are any generalizable patterns or rules-of-thumb that can be obtained across values of α and σ. In addition, there are other criteria such as E-optimal and A-optimal that could be considered as well. In addition, a study could examine where the best sensor locations are for optimal hypothesis testing. Both of these studies should be done across a wide spectrum of values of α and σ to determine any generalizeable patterns. This work considered a very simple FPDE scenario as to not complicate the design of experiments portion with other phenomenon that may be generated from a more complex FPDE. Studies should be conducted on more complex FPDEs with larger number of parameters for a variety of optimality criteria.

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Engaging Future Secondary Teachers in Technology Experiences

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Abstract: *Work with future secondary teachers should engage them in technology experiences for both teaching and learning mathematics. These tasks should address both content and pedagogical issues. Building on this foundation, this paper shares sample tasks that provide an intellectual need for someone to use technology to solve varying mathematical problems.*

1. Introduction

Future secondary teachers need focused experiences learning to use technology to both teach and learn mathematics. Further, these experiences should occur across multiple technologies (graphing calculators, dynamic graphers, Interactive Geometry Systems, spreadsheets, Computer Algebra Systems, etc.) using familiar mathematical content. This perspective is more than learning how to use a specific technology or tool for specific content, but instead it is focused on helping one to learn (through these experiences) how to effectively use technology with mathematics.

The examples presented here come from a required course for future secondary teachers at the college level that is focused on using technology in teaching and learning mathematics. To that end, all experiences are focused on how the mathematics of a task interacts with one's understanding and learning of mathematics while using a technological tool to engage various ideas [1]. While there are many ways one can interact with technology, these experiences highlight the ways these tools change (or have the potential to change) how we do what we do in mathematics. The outcome being that these future teachers learn how technology not only supports their learning but how it can support their future students' learning.

Work with technology should emphasizes using technology both as an amplifier and as a reorganizer [2]. Using a tool as an amplifier essentially means we are replacing what we could have done by hand with technology as a way to more efficiently get work done. In contrast, using technology as a reorganizer means we have in some way fundamentally changed what is being learned or thought about due to the use of the tool. Additionally, most tasks attempt to introduce an intellectual need for the use of the technology [3], thus providing a reason that results in reorganizing behavior. Further, the level of demand [4][5] of the tasks and how it is implemented should engage students in multiple ways to learn procedures with connections, as well as doing mathematics.

Finally, the mathematical fidelity of each tool should not be ignored. The fidelity of a tool is critically important to being able to teach effectively with the tool [6]. While most tools accurately represent mathematical ideas in a way congruent to what we would do without the tool, it is still important to know when it is not and even further, why it might not be correct. For example, a TI-83 calculator will show the graph of a circle as an ellipse when a standard window is used. The standard window provides axes that both run from -10 to 10 and thus should give a 1:1 aspect ratio.

Instead, a non-square (or non 1:1 ratio) window (from an x and y delimiter point of view) is needed. This fidelity issue is due to the nature of the screen of the device rather than any underlying mathematical issue.

2. Graphing and Solving 'unsolvable' Equations

Many future secondary teachers I work with generally feel proficient with symbolic reasoning and many algebraic procedures. The need for technology in a math class does not initially exist for these students because by hand procedures already exist. In fact, for some that is what mathematics is all about. This perception portrays the use of technology as a hindrance or roadblock to doing mathematics the "real" way.

To counter this perception and provide one with the intellectual need for technology, students are asked to solve the equation, $2^x = x + 3$. Most try to solve the equation using any algebraic tools they know, especially attempting to use logarithms and a variety of logarithmic bases. When taking the logarithm of both sides of the equation is unsuccessful, the resulting cognitive conflict causes a few students to try and use the erroneous "rule" $\log(x+3) = \log(x) + \log(3)$ in order to try and find a solution. This highlights how strong the desire and belief are that any equation should be able to be solved by hand, as many have never encountered one that could not be algebraically manipulated to a solution.

Truth is, this equation is not solvable by traditional algebraic methods. Before the advent of current graphing technology one effective approach to solving such an equation was a numerical based one. The bisection algorithm or Newton's Method are two such numerical iterative methods one could use to get an approximate solution to a certain degree of accuracy. A spreadsheet can nicely illustrate the iterative nature of both methods, and provides a decent exercise in elementary programming. However, in this case, a graphical method is the most efficient path to a solution. Using a graphing utility to graph each side of the equation as its own function yields a solution at the identified intersection point shown in Figure 1. This seemingly simple approach provides the opportunity to highlight many mathematical ideas and engage future teachers in deepening their mathematical and pedagogical content knowledge.

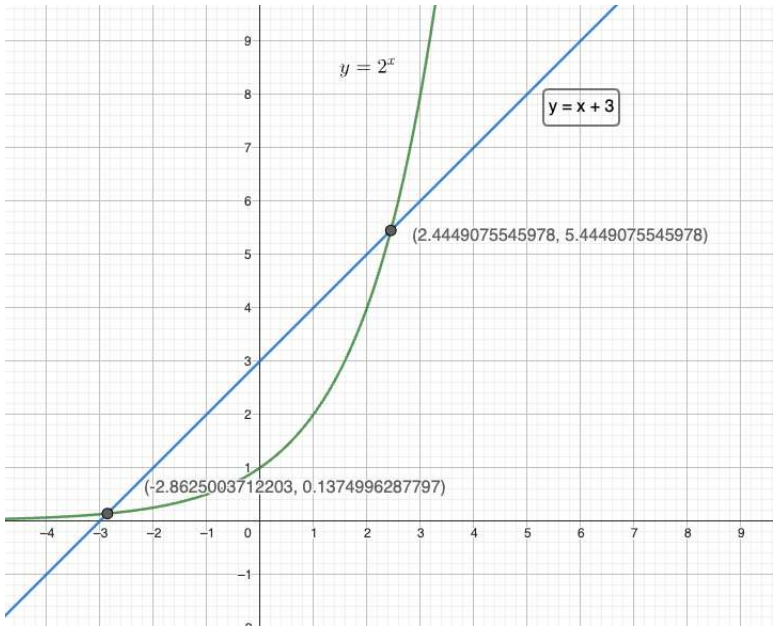

Figure 2.1 – Graphical solution to $2^x = x + 3$

Once an intersection between the two functions is found, another issue is knowing that all intersections (solutions) have been found. This question opens the discussion to how both linear and exponential functions behave with respect to rate of change. The problem provides the basis for the discussion and does not require knowledge of calculus. After this initial experience, the initial equation used can be varied in multiple ways. For example, if the equation were $2^x = x^{100}$ it is now not as easy to see all the graphical solutions on a small screen and thus we need to rely on our mathematical knowledge of rate of change for both functions to know how many solutions could exist.

Outside of these questions, it is worth nothing that the technology also provides the avenue to consider another idea that is often ignored or not discussed until later in mathematics. If one were to subtract one side of the equation from the other, we get the related equation $2^x - (x + 3) = 0$. Graphing the left hand side of the equation (shown in Figure 2) provides a different mathematical experience and it is one that is new for most of the future teachers I work with (at the very least it is not something they have thought about in this way). The solutions to the original equation are now the values where this new equation cross the x-axis (which is the right hand side of the equation). This change of perspective allows one to introduce ideas of the Intermediate Value Theorem well before any calculus experiences.

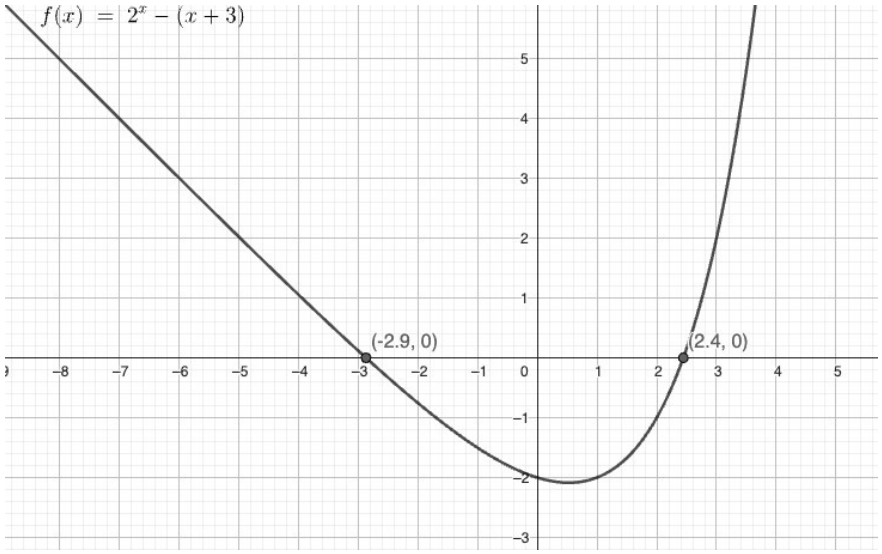

Figure 2.2 – A different graphical solution

The experience of solving unsolvable equations provides at least as much mathematical engagement and learning as any by hand procedure for solving. In this way getting the solution to the given equation is almost tangential to the mathematics of the situation. One must consider the types of functions involved, the fact that they are continuous and be able to be flexible in the approach to graphing.

Once experiences such as the one described have taken place, I ask my students to solve other related equations on homework. The purpose is to allow them to both use and extend what has been done in class to their own work. It also provides them the opportunity to use the technological tools in their own way. For example, solving $x^{50} = 2^x$ is no more challenging than solving the equation in the Figure 2.1, but depending on the approach, the method used in Figure 2.2 is more efficient and generally helps highlight the rate of change issues discussed previously.

3. Dynamic Graphs and Sliders

The slider is truly an innovation when looking at parameters of relations. The ability to consider how a parameter instantaneously influences the behavior of a relation both graphically and analytically at the same time in a dynamic way also allows one to collect data quickly and accurately. This tool allows one to focus on the family of graphs or solutions generated while simultaneously considering how the parameter is impacting those graphs and solutions.

One way to introduce the slider tool is to use a familiar function that (like in the previous example) both challenges what one thinks they know about the mathematics of a situation and shows how new ideas can be introduced because of the technology. Exploring the parameters for $y = mx + b$ with sliders for both the parameter m and the parameter b is just such an experience. Most future teachers likely have had dozens of experiences with linear functions and yet the dynamic nature of the slider allows ideas to come forth that have not been explicitly considered before.

For example, when graphing $y = 2x + b$, where the value for b is controlled by a slider, it is striking that instead of vertical movement, almost everyone sees horizontal movement of the graph. Given that the b value is the vertical intercept and changing it should result in a vertical shift, many are surprised to explicitly see that the change also impacts the horizontal intercept. Chances are

these students have probably never been asked to consider all the ways that parameter b impacts the function. Instead, the focus has probably been on having students learn how the parameter is the vertical intercept. Related to using a slider, we can plot dynamic points. This provides a way for students to connect their algebraic knowledge and procedures with the dynamic representation before them. Plotting either intercept with respect to the function is a new idea in this environment as it requires abstracting the parameters of the function and use them as generic input values to generate the needed information.

A second experience with sliders has students look at quadratic functions of the form $y = ax^2 + bx + c$. In this case, most have a working knowledge of how both a and c impact the function. The vertical movement of the function as the c-value changes more closely matches their notion of vertical translation (although the c-value here also impacts the x-intercepts). Similarly, the impact the a-parameter has on the concavity (width) of the function is familiar (once again this value also impacts the x-intercepts). What is not familiar, is the impact of the b-parameter. Some students know that b is involved in determining the vertex value but no one can initially explain the movement of the parabola as the b-value varies. This specific task has been highlighted a few times in a few different ways (for one example see [7]) and so it will not be detailed here, other than to reveal that the b-value traces a parabolic path. As an extension students can also be asked how the roots of the two quadratics (the original and the one created by the path of the b-value) are related. This extension provides an opportunity for a straightforward algebraic exercise that gives a surprising result and allows for even further exploration with complex numbers. The point of this example is that the slider tool makes possible the explicitness of us being able to see this behavior through movement of the graph. While the tool amplifies our ability to produce several graphs easily, the technology also reorganizes understanding of the impact of that parameter.

Another task asks students to compare parameter behavior across two different quadratic forms, $y = ax$^2 $+ bx + c$ and $y = a(x - h)^2 + k$. In the investigation, there is often surprise that while the a-values determine the same information regarding concavity between the forms, they do not behave exactly the same way by providing the same graph for the same a-values. This initial cognitive conflict is due to the a-value impacting both the x^2-term and the x-term in the second form (often called vertex form). Algebraically the two forms can be reconciled to determine the correspondence between parameters. Outside of this, students are also asked to consider how the control of the sliders varies in what control you have over the quadratic. The purpose here is to highlight how horizontal and vertical translations are clearly emphasized in one form over another.

Once the power and use of sliders is established, I ask my students to provide the 'biography' for a given family of functions. A biography assignment allows students time to engage in mathematical exploration of a familiar family functions (e.g. $y = kx^p$, where k and p are any real number) while becoming more proficient in using the given technological tool. The exploration has to be structured because most of my students have not ever thoughtfully or thoroughly explored a function family in this way before. Because of this, the assignment is allowed several revisions, letting students better organize and refine their work.

4. Dynamic Geometry and Diagrams

Interactive Geometry Systems (IGS) provide an opportunity to explore and analyze geometric concepts dynamically. This dynamic aspect potentially creates problems for what can be considered a proof while also giving students access to being able to prove something that they

might not have been able to prove without the technology. Another quandary is that an IGS provides multiple ways to efficiently create geometric objects and while the end product might look the same, the process to get the end product needs to be (and can be) highlighted and understood because it impacts the ways the object can be used. For example, the following three definitions are all for an isosceles triangle-

- An isosceles triangle is a triangle with at least two equal sides.
- An isosceles triangle is a triangle with at least two equal angles.
- An isosceles triangle is a triangle that is symmetric about at least one angle bisector of the triangle.

The process to create each end figure is clearly different, but the end result is always an isosceles triangle. Without the ability to see the steps taken to create each, we would have no idea what properties were used and how that impacts the dynamics of the final diagram. The IGS provides the tools to efficiently and effectively see how the definition impacts the creation of the figure.

One example of proof issues can be illustrated through constructing a dynamic triangle and demonstrating how each angle measure of the triangle can be taken from 0-degrees to 180-degrees all while the angle sum of the three interior angle of the triangle is shown on the screen. During this demonstration the angle sum never changes from the constant of 180-degrees. Once complete, students are asked if this demonstration proves that the interior angle sum of a triangle is always 180-degrees. The purpose here is to consider the pedagogical implications of an IGS and what makes something a proof. The demonstration claims to have tested all possible cases of the interior angles sum. Further, even though all students 'know' that this angle sum is true, many have trouble recalling a proof. Figure 3 shows just such a triangle in two stages of change.

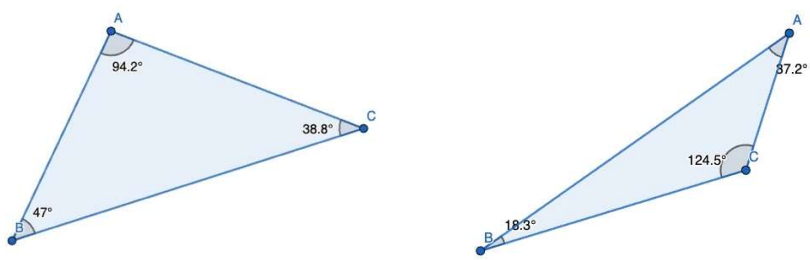

Figure 4.1 – Dynamic Triangle

An example using an IGS to provide access to proof ideas and links to other representations is provided by the square in a square problem. The version used here comes from a calendar problem in *Mathematics Teacher* [8, pp. 343]. This problem reveals the power technology has to assist students in reasoning and sense making. The problem is stated as-

> A square is inscribed in a larger square. (That is, the four vertices of the inscribed square lies on the four sides of the larger square.) What is the smallest possible value of the ratio of the area of the inscribed square to that of the larger square?

Students are asked to initially provide a proof for this problem. Both geometric and algebraic arguments can be made to prove that midpoint is the desired location. However, in my experience most students often provide a proof that the ratio at that point is ½ and not that the point gives the minimum ratio. Further, many students find it challenging to properly parameterize the figure for an algebraic approach. A dynamic diagram provides a chance for a student to access the problem

and to explore several measurements and ratios of the problem because it allows one to move a vertex of the interior square and then dynamically measure whatever one wants to measure as shown in Figure 4. The size of the square is arbitrary.

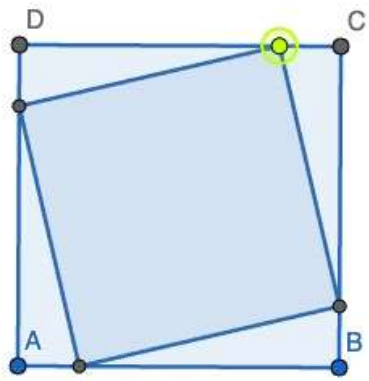

Figure 4.2 – Dynamic Diagram for a Square

Because the orientation of the interior square can change, a dynamic graph linked to specific changing lengths from the diagram could trace out a concave up parabola and show that the area ratio is at a minimum when the interior square is at the midpoint as shown in Figure 5. The trace plot here is the length of EC and the resulting area ratio of the two squares.

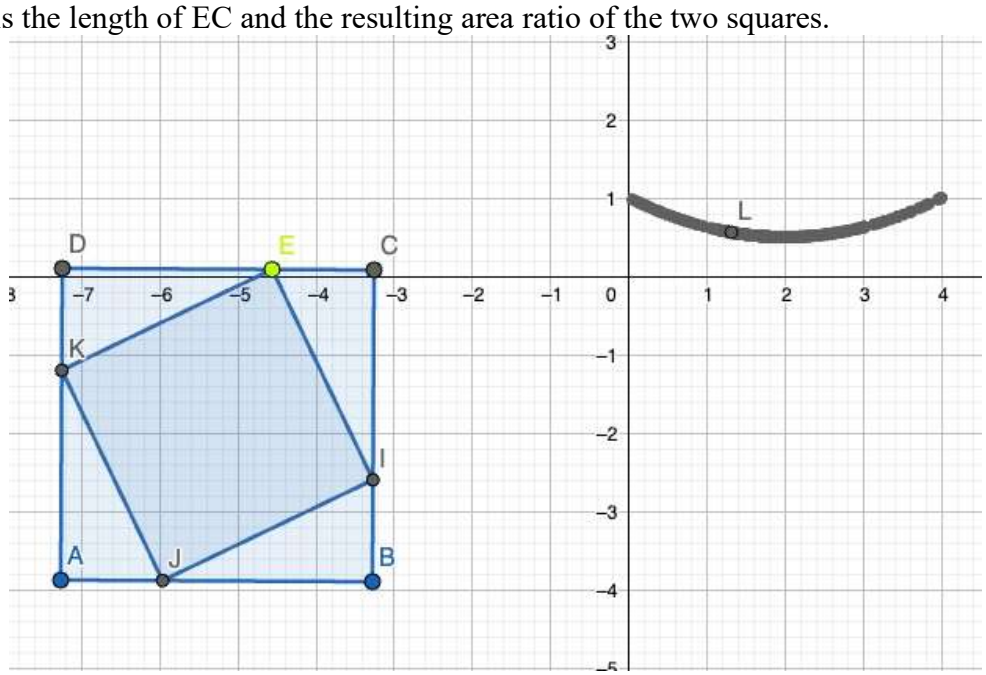

Figure 4.3 – Dynamic Diagram Plot

For students who are unable to provide an initial proof, the dynamic diagram provides a way to look for potentially any relationship to prove the needed claim without needing to engage in the symbolic representation. In [9] the stories of four students who could not initially provide a proof for this problem are shared. Through the use of dynamic diagrams, each was able to access the problem and see connections that they were unable to see before. Note that the students did not suddenly gain algebraic insight into the problem, instead they gained the ability to engage the problem in a way that made sense to them.

As with most problems, this problem provides the chance for further investigation by looking at area ratios for other regular polygons. This extension is a mathematically involved problem and provides an interplay between the technology and brute force calculations [10]. The technology enables one to engage such extensions that might otherwise be missed.

5. Closing Thoughts

The purpose of technology in mathematics should be to allow students access to mathematical ideas that they might not otherwise not access. Additionally, its use should provide mathematical insight in ways that might not be possible without it. The experiences shared here highlight how it can both support teaching and learning mathematics. The examples also show how initial questions and investigation lend themselves to further and more involved mathematical investigations. This work is significant work for future teachers as it addresses many mathematical ideas as well as several pedagogical ideas.

Without technology none of the learning experiences, problems, and learning shared here would have been possible. Technology use in each case was purposeful, useful, and essential to the task. Technology provides access to mathematical ideas and procedures that might not otherwise be accessible. This is clearly the ladder that [11] argued for so many years ago, At that time he used the graphing calculator and called for revising the curriculum because of the great advantages this tool could bring to mathematics. This is a call that is routinely continued but ultimately appears to be unanswered. Through working with future teachers and providing them the intellectual need for how and why technology enhances and deepens mathematical learning can help this vision to be realized.

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Learning guidance based on analysis of symbolic comprehension

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Abstract

When two concepts contain a common concept, overgeneralization (the phenomenon of overgeneralizing specific rules or semantic features) may occur in the process of learners gaining an understanding of the two concepts in relation to each other. We use a neural network to analyze the overlap singularity phenomenon and the elimination singularity phenomenon in singular regions and perform simulations on the loss surface in previous research [4], [5]. In this study, the analysis technique "semantic comprehension" is applied to "symbolic comprehension". We defined four learning stages in the process of understanding of "permutations"and "combinations" and considered the simulation results regarding the change in training loss and the dynamics on the loss surface. Findings from this analysis are used to formulate learning guidance for teachers of mathematics.

1 Introduction

Interrelationship problems can be thought of as problems that require consideration of the common concepts of permutations and combinations to derive an answer. For example, consider the following question: "How many ways are there to choose 3 people from the ten members of a committee?" The correct answer is ${}_{10}P_3$, whereas the answer ${}_{10}C_3$ is considered a semi-correct answer. In this study, we separate the conceptual understanding of "permutation" and "combination" into two parts: "symbolic understanding" and "semantic understanding." If we classify mathematics problems into categories of "semantic understanding", the test in this case is to correctly judge the meanings of "arrange" and "choose" in the question sentence. We attempted to apply the results of analyzing the overlap singularity phenomenon and the elimination singularity phenomenon in singular regions, as a method for visualizing the state of "semantic comprehension" ([4], [5]). Considering answering the interrelationship problems requires understanding semantics and then symbols. To consider a transformation of the concept of "arranging" in permutations, we need to analyze the state of "symbolic comprehension" in learning mathematics.

2 Preparation for analysis

More details on the basic definitions of learning theory, see [1], [2]. If we classify mathematics problems into categories of "symbolic comprehension", the test here is to correctly compute permutations and combinations of symbols. To test symbolic comprehension in the field of "permutations" and "combinations" in the unit "The Number of Cases and Probability" in high school mathematics, we performed three examinations(1st and 2nd and 3rd) and created a set of four questions as follows: (a) ${}_5P_3$ (1st and 2nd), ${}_8P_4$ (3rd), (b) 6! (1st and 2nd), ${}_7C_3$ (3rd), (c) ${}_6C_4$ (1st), ${}_8C_3$ (2nd), (d) ${}_{20}C_{18}$ (1st), ${}_{16}C_{14}$ (2nd). Questions (a) (1st to 3rd) and (b) (1st and 2nd) are permutation problems; questions (b) (3rd) and (c) (1st to 3rd) and (d) (1st to 3rd) are combinatorial problems . The total score for permutations was set to 0.5, as was the total score for combinations. A student was credited with a semi-correct answer if the student mistakenly treated questions (a) as combination problems, questions (c) and (d) as permutation problems . For the 1st and 2nd examinations , the scores for correct answers were (a) 0.4 points, (b) 0.1 points, (c) 0.3 points, and (d) 0.2 points; and the scores for semi-correct answers were (a) 0.2 points, (c) 0.15 points, and (d) 0.1 points. For the 3rd examination, the scores for correct answers were (a) 0.5 points and (b) 0.5 points; and the scores for semi-correct answers were (a) 0.25 points and (b) 0.25 points. We excluded students whose scores in both the permutation and combination areas were less than 0.5 and used the remaining data for analysis.

We defined four learning stages: Learning stage (1), in which correct answers are obtained by considering and progressing in understanding both permutations and combinations between the 1st and 2nd examinations. Learning stage (2), in which semi-correct answers are influenced by the study of combinations and are obtained despite little progress in understanding combinations between the 1st and 2nd examinations. Learning stage (3), in which semi-correct answers are influenced by the study of combinations between the 2nd and 3rd examinations. Learning stage (4), in which correct answers are obtained by considering the interrelationship with combinations between the 2nd and 3rd examinations.

Three student groups were defined based on questions (a) through (d): (i) Students with full points in the permutation problems and partial points in the combination problem. (ii) Students with partial points in the permutation problems and full points in the combination problem. (iii) Students with full points in both the permutation and combination problems. We compared student groups (i) and (iii) in learning stage (1), student groups (i) and (ii) in learning stages (2) and (3), and student groups (ii) and (iii) in learning stage (4).

Definition 1 *Given inputs x, θ_0, the output Y of the two-layer neural network is defined as follows:* $Y := f(x, \theta_0) = w_3 \tanh(w_1 x) + w_4 \tanh(w_2 x)$.

The coordinate transformation from the parameters $\theta = (w_1,\ w_2,\ w_3,\ w_4)$ to the new parameters $\xi = (a,\ b,\ v,\ w)$ can be defined as follows: $a = w_2 - w_1,\ b = \frac{w_3 - w_4}{w_3 + w_4},\ v = \frac{w_3 w_1 + w_4 w_2}{w_3 + w_4},\ w = w_3 + w_4$.

For the two student groups, let c be the average score for the combinatorial problem, d be the average score for the permutation problems, and e and f be the numbers of students in the two groups. The weights of the neural network can then be determined as follows: $w_1' = c$, $w_2' = d$, $w_3 = \frac{e}{e+f}$, $w_4 = \frac{f}{e+f}$. More details on correction, see [4], [5].

Figure1 shows the average score, number of students, and correction coefficients for students who gave correct answers (left two columns, center two columns) and semi-correct answers (right two columns) to the interrelationship question (d) in the three examinations.

	Students who answered correctly		Students who answered correctly		Students who answered semi-correctly	
1st to 3rd examinations	(i) (full, partial) points	(iii) (full, full) points	(ii) (partial, full) points	(iii) (full, full) points	(i) (full, partial) points	(ii) (partial, full) points
Average c, d	0.282352941	0.5	0.127777778	0.5	0.291071429	0.172727273
Number of people e, f	34	121	9	121	28	11
Overall average g	0.452258065		0.474230769		0.257692308	
Total number of people	155		130		39	
1st examination	(i) (full, partial) points	(iii) (full, full) points	(ii) (partial, full) points	(iii) (full, full) points	(i) (full, partial) points	(ii) (partial, full) points
Average c, d	0.282142857	0.5	0.15	0.5	0.3	0.425
Number of people e, f	28	50	6	50	7	2
Overall average h_1	0.421794872		0.4625		0.327777778	
Total number of people	78		56		9	
Correction factor $\frac{g}{h_1}$	1.072222767		1.025363825		0.786179922	
Correction w_1, w_2	0.302519995	0.536111383	0.153804574	0.512681913	0.235853977	0.334126467
2nd examination	(i) (full, partial) points	(iii) (full, full) points	(ii) (partial, full) points	(iii) (full, full) points	(i) (full, partial) points	(ii) (partial, full) points
Average c, d	0.283333333	0.5	0	0.5	0.305263158	0.4
Number of people e, f	6	28	0	28	19	2
Overall average h_2	0.461764706		0.5		0.314285714	
Total number of people	34		28		21	
Correction factor $\frac{g}{h_2}$	0.979412369		0.948461538		0.81993007	
Correction w_1, w_2	0.277500171	0.489706185	0	0.474230769	0.250294442	0.327972028
3rd examination	(i) (full, partial) points	(iii) (full, full) points	(ii) (partial, full) points	(iii) (full, full) points	(i) (full, partial) points	(ii) (partial, full) points
Average c, d	0	0.5	0.083333333	0.5	0.125	0.035714286
Number of people e, f	0	43	3	43	2	7
Overall average h_3	0.5		0.472826087		0.055555556	
Total number of people	43		46		9	
Correction factor $\frac{g}{h_3}$	0.904516129		1.002970822		4.638461538	
Correction w_1, w_2	0	0.452258065	0.083580902	0.501485411	0.579807692	0.165659342

Figure 1: Examination results

Parameter a is the difference between the average score of the permutation problems and the average score of the combinatorial problem. Parameter b is the proportion of students in the two student groups to be considered .

We investigated this proportion by selecting two groups from the three student groups (i), (ii), and (iii) . For the selected groups, we can visualize the understanding of permutations and combinations on the learning loss surface. More details on the construction of neural networks and learning loss surfaces, see [3], [4]. Here, the state of understanding is displayed as points for the 1st examination (blue), 2nd examination (red), 3rd examination (green), and 1st to 3rd examinations (yellow). Student groups (i) and (iii) are shown on the left side of Figure 2; student groups (i) and (ii) are shown in the center; and student groups (ii) and (iii) are shown on the right side.

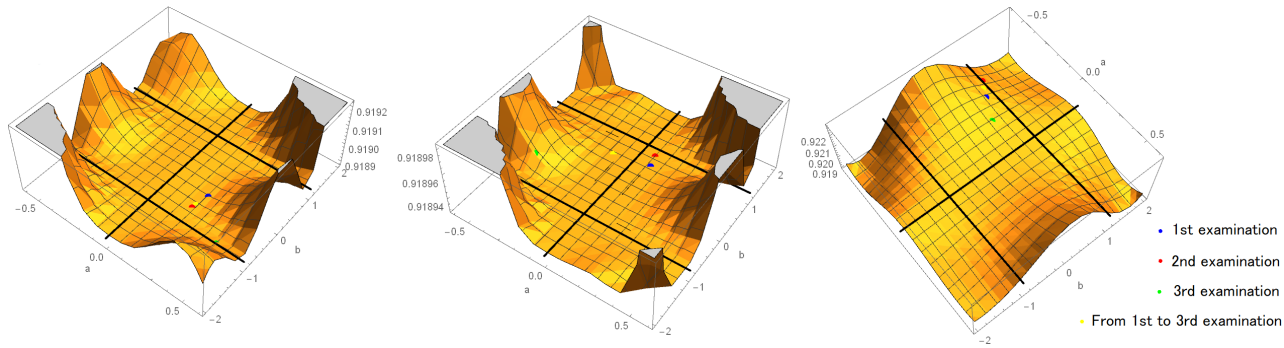

Figure 2: Learning loss surface of symbolic computation

More details on overlap singularity and cross-overlap singularity in information science, see [4], and more details on elimination singularity, near-elimination singularity, and fast convergence, see [5].

3 Simulations

In the following sections, the results obtained from real data are used to perform simulations in which the initial values are changed in 0.05 increments from -0.6 to 0.6 for a and from -1.1 to 1.1 for b ([4], [5]). In the figures, the initial values for the 1st (blue), 2nd (red), and 3rd (green) examinations are indicated by ◎, and the true distribution is indicated by $\times$. For the dynamics from the 1st to the 2nd examination, the initial value (simulation) is shown in blue, and the value after the simulation is shown in red. For the dynamics from the 2nd to the 3rd examination, the initial value (simulation) is shown in red, and the value after the move is shown in green.

3.1 Analysis of students: learning stage (1)

We analyzed the change from the 1st to the 2nd examination to target the students who gave correct answers. We can now consider the process of understanding the two concepts by comparing student groups (i) and (iii).

The dynamics of the simulation with a varied, the change in training loss, and the dynamics on the loss surface are shown in Figure 3. Learning begins from the state $b = -0.28$, where

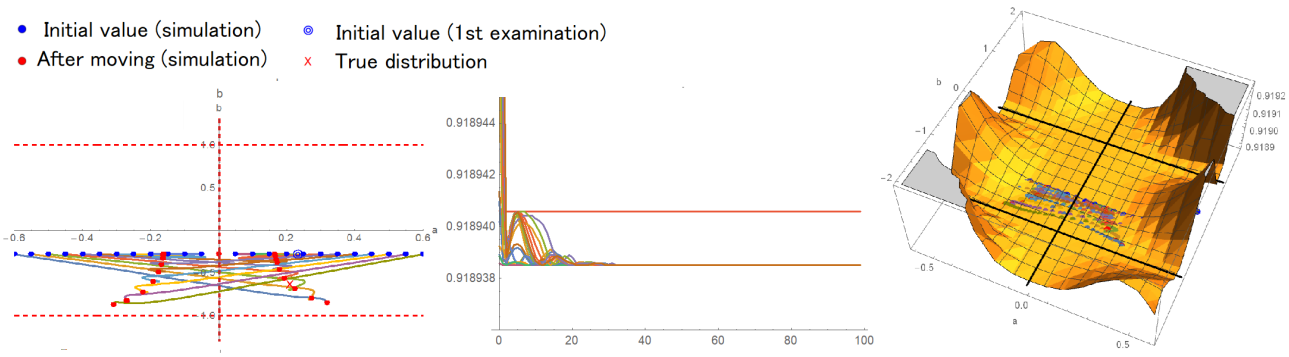

Figure 3: a varied: correct answer

the proportion of student group (iii) is slightly greater than that of student group (i). As the overgeneralization of combinations as permutations increases (a decreases), near-overlap singularity occurs due to the influence of the critical line $a = 0$, stagnating at $a = -0.16$. Furthermore, the difference in overgeneralization disappears from $v = 0.45$ and finally cross-overlap singularity occurs due to the influence of the critical line $a = 0$, stagnating at $a = -0.16$. Since the proportion of student group (iii) decreased from $b = -0.28$ to $b = -0.83$, the learning of combinations progresses and a transformation of the permutational schema takes place. This can be considered a natural way of understanding in the developmental process.

The dynamics of the simulation with b varied, the change in training loss, and the dynamics on the loss surface are shown in Figure 4.

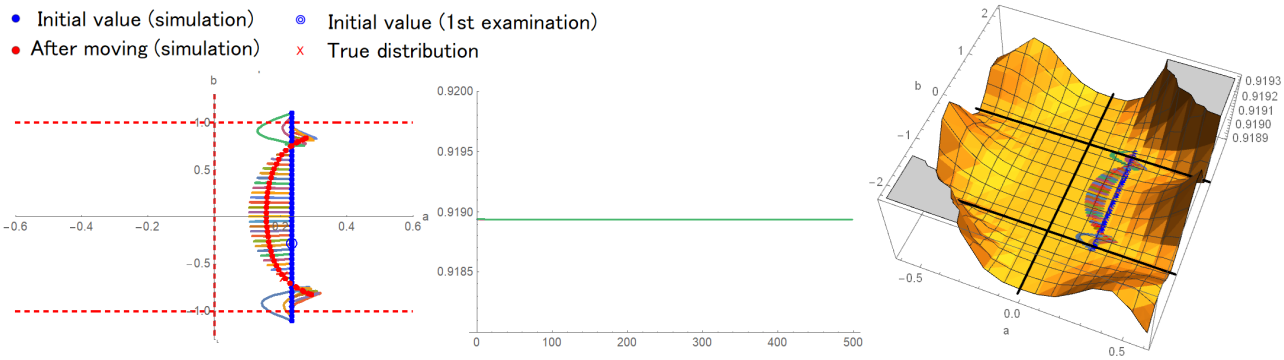

Figure 4: b varied: correct answer

Here, we start learning from $a = -0.233$, a state in which the overgeneralization of permutations as combinations is highly biased. As the percentage of student group (iii) increases (b decreases), a becomes small when $b < -0.74, b > 0.74$ and converges (fast convergence) without the influence of the critical line $a = 0$. From $b = -0.28$ to $b = -0.74$, the difference from the overgeneralization of permutations as combinations is smaller, so that a positive transition is happening. From $b = -0.74$ to $b = -0.83$, the difference from the overgeneralization of permutations as combinations is even smaller, so that a negative transition is happening. When the proportion of student group (i) increases (b increases), a becomes small within $-0.74 \leq b \leq 0.74$ and near overlap singularity occurs. A positive transition occurs as the difference from the overgeneralization that occurs becomes smaller.

3.2 Analysis of students: learning stage (2)

We analyzed the change from the 1st to the 2nd examination to target the students who gave semi-correct answers. We can now consider the process of understanding the two concepts by comparing student groups (i) and (ii).

The dynamics of the simulation with a varied, the change in training loss, and the dynamics on the loss surface are shown in Figure 5.

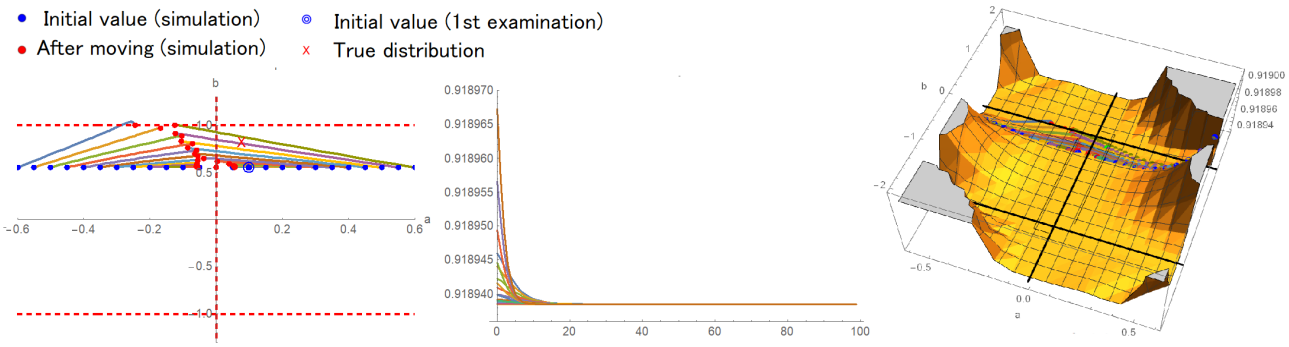

Figure 5: a varied: semi-correct answer

Learning begins from the state ($b = 0.55$), where the proportion of student group (i) is larger

and more unbalanced than the proportion of student group (ii). When the overgeneralization of combinations as permutations increases (a decreases), the critical line $b = 1$ is affected and elimination singularity occurs. Only student group (i) is included; there is no student group (ii). If the overgeneralization of combinations as permutations is smaller than $a < -0.5$, the learning of combinations does not progress and a transformation of the permutation schema takes place. As the overgeneralization of permutations as combinations increases (a increases), near overlap singularity occurs due to the influence of the critical line $a = 0$, the difference in overgeneralization disappears from $v = 0.25$ (overlap singularity), and cross-overlap singularity occurs beyond the critical line $a = 0$. Since the proportion of student group (i) increases slightly, the learning of combinations progresses and a transformation of the permutational schema takes place.

The dynamics of the simulation with b varied, the change in training loss, and the dynamics on the loss surface are shown in Figure 6.

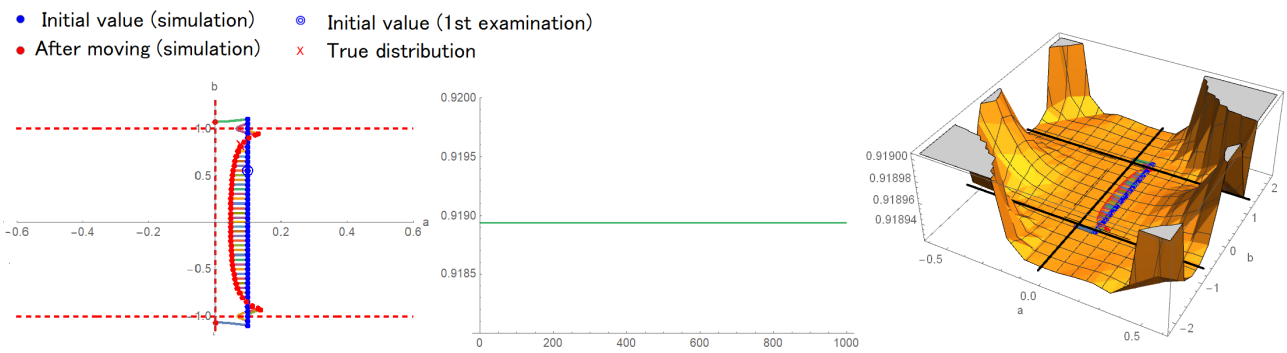

Figure 6: b varied: semi-correct answer

Learning starts from the state $a = 0.09$, where the overgeneralization of permutations as combinations is large. At first, as the proportion of student group (i) becomes larger (b increases), overlap singularity occurs under the influence of the critical line $a = 0$. When the overgeneralization of permutations as combinations is large, w_1, w_2 become 0.25 from $v = 0.25$, and there is no difference from the overgeneralization of combinations as permutations. Therefore, a positive transition occurs. When the proportion of student group (ii) becomes larger (b decreases) occurs the same transition as a larger proportion of student group (i) becomes larger (b increases).

3.3 Analysis of students: learning stage (3)

In order to target students who are able to answer semi-correctly, we analyzed changes between the 2nd and 3rd examinations. We can now consider the process of understanding the two concepts by comparing student groups (i) and (ii).

The dynamics of the simulation with a varied, the change in training loss, and the dynamics on the loss surface are shown in Figure 7.

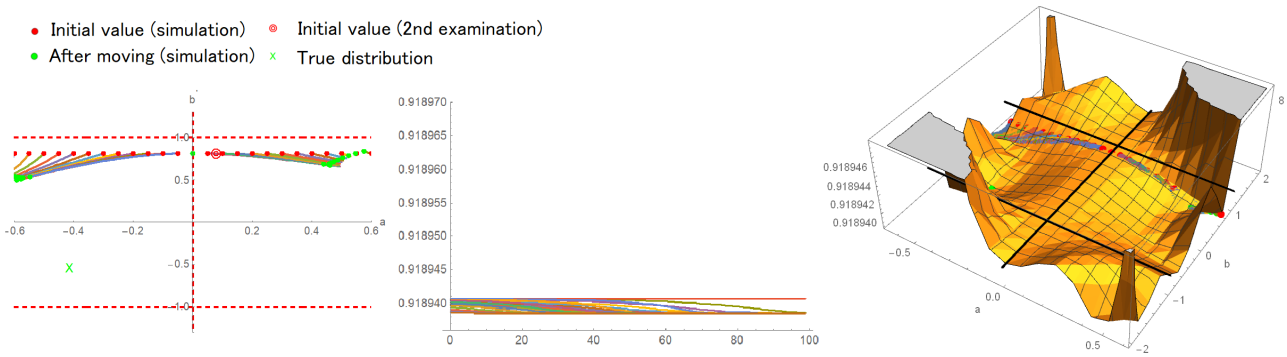

Figure 7: a varied: correct answer

Learning begins from the state ($b = 0.80$), where the proportion of student group (i) is larger and more unbalanced than the proportion of student group (ii). The overgeneralization of combinations as permutations increases (a decreases) and converges at $a = -0.57$ (fast convergence) without the influence of the critical line $a = 0$. Since the proportion of student group (i) decreased from $b = 0.80$ to $b = 0.49$, a transformation of the combinational schema takes place. As the overgeneralization of permutations as combinations increases (a increases) occurs the same transformation as the overgeneralization of combinations as permutations increases (a decreases) and fast convergence occurs. The proportion of student group (ii) increased slightly.

The dynamics of the simulation with b varied, the change in training loss, and the dynamics on the loss surface are shown in Figure 8.

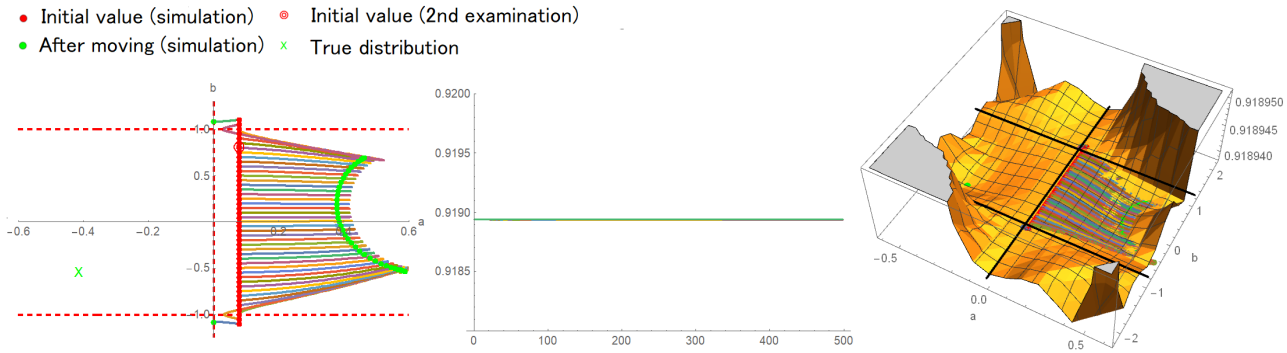

Figure 8: b varied: correct answer

Here, we start learning from $a = 0.03$, a state in which overgeneralization of permutations as combinations is unlikely to occur.As the percentage of student group (ii) increases (b decreases), fast convergence occurs without the influence of the critical line $a = 0$. The difference from the overgeneralization of permutations as combinations is even greater, so that a negative transition occurs. When the proportion of student group (ii) increases (b increases), fast convergence occurs without the influence of the critical line $a = 0$. The large overgeneralization of permutations as combinations is suppressed, so that a negative transition occurs. By learning combinations (relearning permutations), the overgeneralization of permutations as combinations can be suppressed from $a = 0.46(0.58)$ to $a = 0.37$.

3.4 Analysis of students: learning stage (4)

In order to target students who are able to answer correctly, we analyzed changes between the 2nd and 3rd examinations. We can now consider the process of understanding the two concepts by comparing student groups (ii) and (iii).

The dynamics of the simulation with a varied, the change in training loss, and the dynamics on the loss surface are shown in Figure 9.

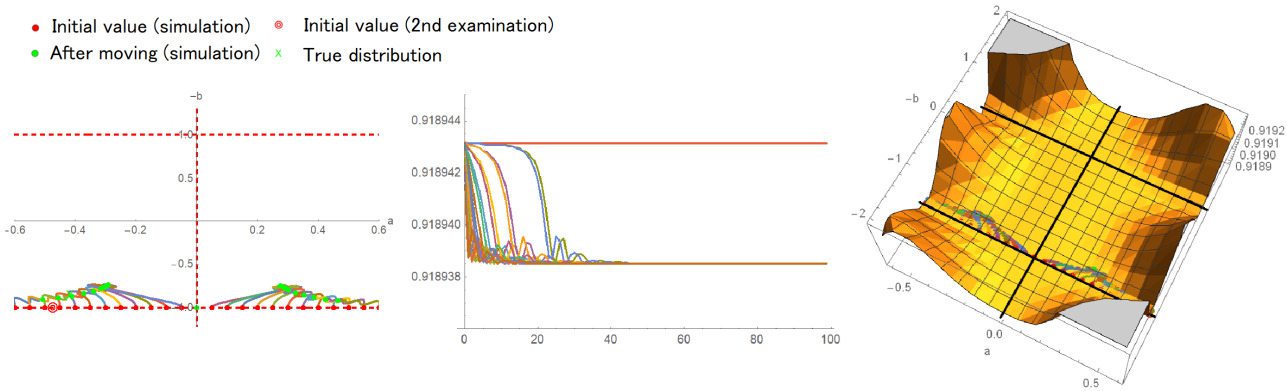

Figure 9: a varied: semi-correct answer

Learning starts from $b = 0.58$ when only student group (iii) is included. As the overgeneralization of combinations as permutations increases ($a > -0.3$ decreases), fast convergence occurs at $a = -0.29$ without the influence of the critical line $a = 0$. Moreover, the overgeneralization of combinations as permutations increases ($a < -0.3$ decreases) and near overlap singularity occurs due to the influence of the critical line $a = 0$. Since the proportion of student group (iii) decreased from $b = 1.0$ to $b = 0.73$, a transformation of the combinational schema takes place. As the overgeneralization of permutations as combinations increase (a increases) occurs the same transformation as the overgeneralization of combinations as permutations increases (a decreases) and fast convergence occurs at $a = -0.29$.

The dynamics of the simulation with b varied, the change in training loss, and the dynamics on the loss surface are shown in Figure 10.

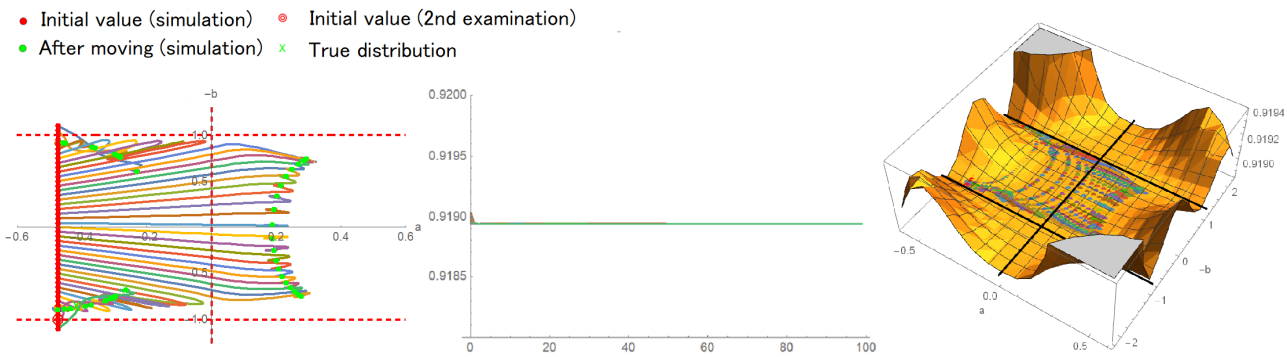

Figure 10: b varied: semi-correct answer

Learning starts from the state $a = -0.47$, where the overgeneralization of combinations as

permutations is large. At first, as the proportion of student group (ii) becomes larger (b decreases), a near-elimination singularity occurs due to an approach to the critical line $b = 1$ and a return to the original state. A positive transition occurs because a large overgeneralization of combinations as permutations is suppressed. As the proportion of student group (iii) becomes larger (b increases), a cross-overlap singularity occurs beyond the critical line $a = 0$. When the overgeneralization of permutations as combinations is large, w_1, w_2 become 0.47 from $v = 0.47$, and there is no difference from the overgeneralization of combinations as permutations. Furthermore, the overgeneralization of permutations as combinations becoming large leads to a negative transition.

3.5 Direction of teacher guidance

By simulating changes in a and b, the transformation and transition of the schema can be assumed, allowing the teacher to devise a teaching strategy.

3.5.1 Direction of teacher guidance: learning stage (1)

From varying a, by increasing the overgeneralization, a transformation of the schema of permutations is caused to occur, and student understanding of both permutations and combinations progresses. Students answered correctly and were able to identify the problem correctly as a permutation. It appears that understanding is achieved by considering the interrelationship with permutations in trying to understand combinations. From varying b, if we increase the proportion of student group (iii), the overgeneralization of permutations as combinations increases and negative transitions occur. If the proportion of student group (i) is increased, the overgeneralization of combinations as permutations is reduced, and positive transitions occur. Although the problem is answered correctly, the above relationship is affected by the overgeneralization of permutations as combinations. Therefore, even after learning combinations, it is necessary to proceed with the re-learning of permutations while still paying attention to the understanding of permutations .

3.5.2 Direction of teacher guidance: learning stage (2)

From varying a, by increasing the overgeneralization, a transformation of the schema of permutations is caused to occur, but student understanding of combinations progresses little. From varying b, if the proportion of student group (ii) is increased, the overgeneralization of permutations as combinations is reduced, and positive transitions occur when student understanding of combinations progresses. Although the problem is answered partially correctly, the above relationship is not affected by the overgeneralization of permutations as combinations. In symbolic comprehension, the proportion of student group (ii) is not increased in the process of understanding the concept of "combinations" when the students are given the opportunity to learn about combinations. Although the students answered partially correctly to the interrelationship question, it can be assumed that the reason for these partially correct answers is not symbolic comprehension. It is necessary to provide guidance for deepening understanding combinations by considering the interrelationships between permutations and combinations. Students need to be taught to make correct judgments in problems that require an understanding of the interrelationships between permutations and combinations.

3.5.3 Direction of teacher guidance: learning stage (3)

By increasing the overgeneralization of permutations as combinations by varying a , the schema of the combinations is changed. However, the proportion of student group (ii) is not increased, so the learning of combinations does not progress. From varying b, increasing the proportion of student group (i) increases the overgeneralization of permutations as combinations, so that a negative transition occurs even as the learning of combinations progresses, but the overgeneralization of permutations as combinations can be slightly suppressed by learning combinations (relearning permutations). Although the problem is answered partially correctly, the above relationship is greatly affected by the overgeneralization of permutations as combinations. This can be considered as result of a superficial understanding of the developmental process. In symbolic comprehension, however, learning combinations do not affect learning permutations, and thus it is considered necessary to learn combinations, paying attention to understanding permutations even after learning combinations.

3.5.4 Direction of teacher guidance: learning stage (4)

By increasing the overgeneralization of combinations as permutations by varying a , the schema of the combination is changed, but the learning of both permutations and combinations is improved. From varying b, increasing the proportion of student group (ii) increases the overgeneralization of combinations as permutations, so that a positive transition occurs even as the re-learning of combinations progresses. (The levels of understanding of both permutations and combinations are stable.) Additionally, decreasing the proportion of student group (iii) increases the overgeneralization of permutations as combinations, so that a negative transition occurs despite the learning of permutations progressing little. Although the problem is answered correctly, the above relationship may be affected by the overgeneralization of combinations as permutations due to an insufficient understanding of permutations. Students need to be re-learning permutations continuously to make correct judgments in problems that require understanding the interrelationships between permutations and combinations while still paying attention to the reduction in understanding permutations.

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Classroom Practices with RLA to Promote Inquiry-Based Learning

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Abstract: *In this paper, the author will discuss RLA (Researcher-Like Activity) as a practice that promotes inquiry-based learning, present its characteristics and examples of class practice, and describe its educational effects. The author found that this practice leads to independent, interactive, and deep learning by students. The interactive nature of RLA not only engages students but also involves educators in the learning process. The effective use of ICT in this practice will also be discussed, encouraging educators to participate actively in their students' learning.*

1. Introduction

In Japan, curriculum standards called "Courses of Study" are established by the Ministry of Education, Culture, Sports, Science and Technology (MEXT) to ensure that a particular standard is maintained nationwide. These standards stipulate general curriculum considerations, the number of instructional hours, and the general objectives, content, and treatment of content for each subject. Each school determines the textbooks and schedules for its students based on these standards. In high schools, education under the new Courses of Study started in 2022, and "inquiry" has been identified as a keyword for learning throughout the curriculum. The author, who has been a pioneer in introducing teaching through inquiry-based learning (Researcher-Like Activity) and actively incorporates mathematical activities, will share insights on this approach. RLA means researcher-like activity; its basic concept is the reduced activity of the researcher. It is an educational practice proposed by Shinichi Ichikawa (1996) and introduced into middle school mathematics by Satoshi Karimata (1996). In this class, mathematicians discover and solve problems. They then express them in papers and present them at conferences. Students will share their knowledge through mutual evaluation. The students will do these activities by incorporating mathematical activities. The students' activities replace the mathematician's "problem discovery" with "problem creation by changing conditions from the basic problem." The students set the problem, conduct exploratory activities, and hold a poster session as in a mock academic conference. The author presents a classroom practice using RLA in this paper and describes its characteristics and educational effects.

2. Characteristics of classes teaching with RLA

As shown in Fig. 2.1, teaching with RLA is a practice that uses the subject matter and replaces the mathematician's activities with those of the students (see [1]). In addition, this practice has the following two characterized points. The first point is that "finding a problem" in the mathematician's activity is to "create a problem by changing the conditions" of a basic problem presented by the teacher. Setting up a problem is an important factor in inquiry learning, but in mathematics, it is difficult for students to do so in some respects. Therefore, it is easy for students

to change the conditions of the basic problem, and by changing the rules and initial conditions, new facts and differences from the results of the revealed basic problem, making it easy for students to set their exploratory problems. This activity is possible even if the students have different learning backgrounds and abilities. Second, the students will be able to communicate and interact with each other mathematically by compiling their explorations into posters and holding a poster session. Expressing and sharing one's ideas deepens the presenter's understanding and leads to learning for the audience. In addition, based on past practices, using concrete objects, ICT, and appropriate teacher advice according to the situation effectively promotes inquiry activities. Furthermore, it would be better if it is possible to set aside a continuous or more extended period, such as two weeks. Or a month to conduct inquiry activities.

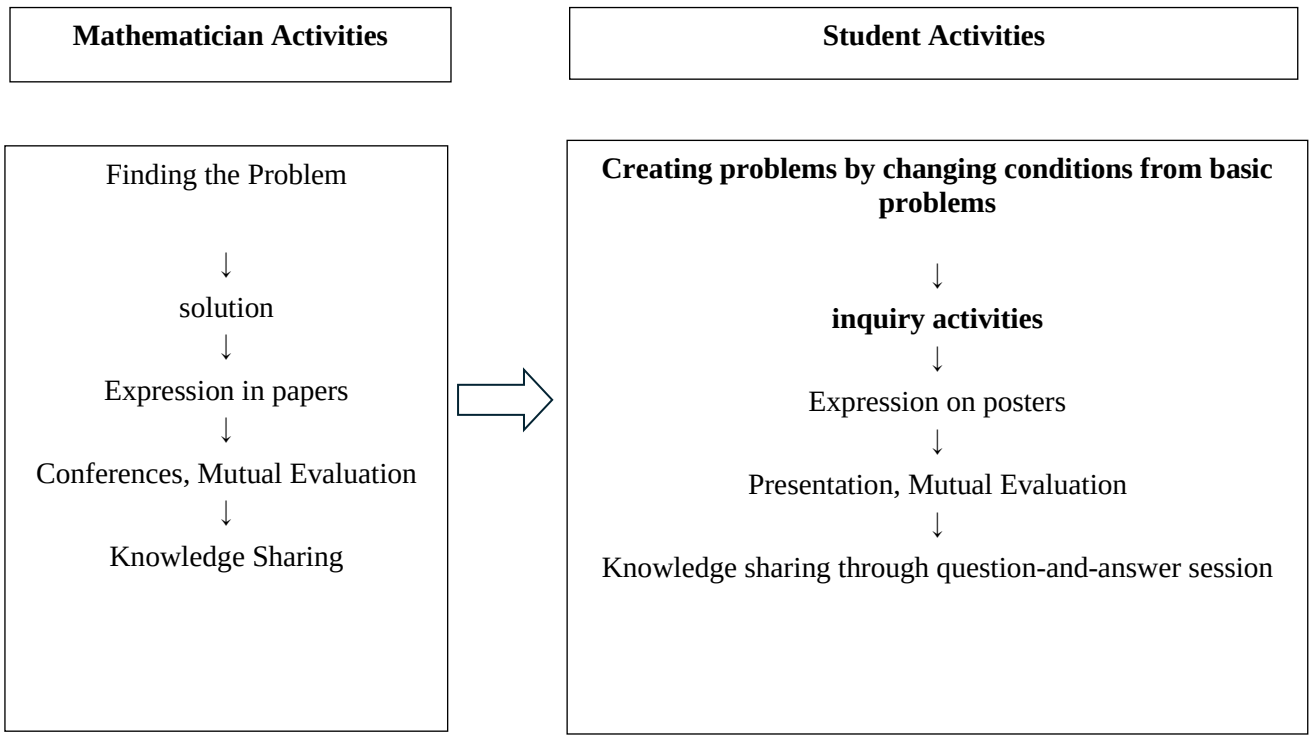

Figure 2.1: Activities of mathematicians and students in RLA

3. Examples of class practice

The author will describe two examples of RLA practice. For these two examples, we will describe the specific task setting and the characteristics of the learners. After setting up the tasks in both cases, the students conducted exploration activities, compiled their explorations into posters, and held a poster session.

3.1 Practice “Regular polyhedron”

After an activity to make regular polyhedrons using regular polygonal plastic parts such as equilateral triangles and squares, the students deepened their understanding of regular polyhedrons, including their characteristics and the basic theorem. After that, the students created problems: " Polyhedron Consisting of Two or More Regular Polygons," "Development of Regular Polyhedron," and "The Relation between Soccer Balls and Regular Icosahedrons.” etc. The author taught classes

to first-year high school students. The author taught lessons in an intensive course for university students in the mathematics department who wish to become teachers. The author found differences in problem creation according to the age of the learners that the first year of high school students focused on planar content, such as the difference of regular polygons. In contrast, the college students developed into spatial quantities, such as volume.

3.2 Practice in the field of "Properties of Integers."

The author taught classes to first-year high school students. Under the theme of "The Mysterious Calculation - The Wonder of the 17th Step," the students first grasped the calculation's characteristics and discovered their group's regularity and periodicity. The author described the activities of students in the next chapter. The author also taught classes for only college students and a mixed-age group of first- and second-year high and junior high school students. These practices found that creating problems and inquiry activities are possible for different age groups and that even junior high school students can naturally and easily change the conditions.

4. Class Outline

The practice in properties of integers will be discussed and explained. The subject matter of "Mysterious Calculations" is as follows.

Write a 5-digit number on the first line—all with different numbers.
On the second line, write 55555.
For each number of ranks, find the sum of the numbers in the first and second lines, and write the last digit number of the sum on the third line.
Similarly, find the sum of the numbers in lines 2 and 3 for each column, and write the last digit number of the sum in line 4.
Repeat this procedure until line 17.

The following Figure 4.1 shows one example of this calculation. (see [1])

Step					
1	1	2	3	4	5
2	5	5	5	5	5
3	6	7	8	9	0
4	1	2	3	4	5
5	7	9	1	3	5
6	8	1	4	7	0
7	5	0	5	0	5
8	3	1	9	7	5
9	8	1	4	7	0
10	1	2	3	4	5
11	9	3	7	1	5
12	0	5	0	5	0
13	9	8	7	6	5
14	9	3	7	1	5
15	8	1	4	7	0
16	7	4	1	8	5
17	5	5	5	5	5

Figure 4.1: Calculation example

The following are some of the activities of the first year of high school students. Using the worksheet, students performed the calculations, discussed their findings in groups, and presented them on paper. Students raised the finding facts: "The second and seventeenth lines are 5," "The first, fourth, and tenth lines are the same," "If the first line is even, the seventh line is 0, and the twelfth line is 5," and "If the first line is odd, the seventh line is 5, and the twelfth line is 0. Next, they proved these discovered properties. Here, writing the last digit number means that we are focusing on the remainder divided by 10, so the author explained that in such cases, it is best to use the idea of congruence expression ($a_2 \equiv a_{17} \equiv 5$ mod10). Then, students proved the property by putting the number by a letter. With this activity, the students understood the mathematical content of the basic problem. Next, each group thought about what would happen if they changed the number in the second line. While giving desk-to-desk guidance, students were encouraged to try to prove what they had noticed. They proved a property related to the relationship between the number of the second line and the number of the 17th line (the number of the 17th line is the number of the first place of the number of the second line multiplied by 7: $a_{17} \equiv 7\, a_2$ mod10).

The author added that the background of these calculations is the Fibonacci sequence. After understanding the basic problem, students created problems for each group. "Let's change mod10 to mod3" and "What happens if we try to multiply addition by multiplication?" Some groups noticed the periodicity from the activity by changing the number of the second line, and some groups focused on this point. These properties naturally emerged from the students' inquiry activities, and students actively worked together as a group to formulate hypotheses about them and verify them. Next, students summarized their explorations, created posters, and held a poster session. Students, university professors, mathematics teachers from other schools, and graduate students participated in the poster session. The author documented the posters created by students with images, distributed them to students as deliverables in a booklet, and instructed them to prepare individual reports to reflect this activity.

5. Class Considerations

The author believes this RLA-based class practice effectively promoted students' inquiry-based learning and realized independent, interactive, and deep understanding based on students' comments, self-evaluations, written impressions, and surveys on emotional changes about mathematics. In addition, since the "Mysterious Calculations" is a simple calculation, it is not affected much by the mathematical abilities that the students have acquired. Therefore, students can have their ideas and experiences and then explain them to other students in the group. Listening to others makes it easier for them to solve problems and realize different viewpoints from their own. Easy to compute but deep enough to be suitable for inquiry activities.

Furthermore, during these activities, "classify and summarize the properties found by everyone," "prove mathematically," and "share mathematical knowledge" in the poster session. These experiences lead to "the ability to identify and select carefully what is essentially the same," "the ability to understand the advantages of using letters and the characteristics of their structure and to explain them logically by generalizing," and "the opportunity to have a sufficient understanding of the presenter's facts, to make discoveries by listening, and to reconsider one's ideas. The presentation allows the students to make discoveries and reconsider their ideas.

Next, the author would like to analyze the abilities gained from this activity based on the individual reflection reports. This report describes what a student learned from this activity, citing keywords. In a student's report, the Keywords are: "Think about it from different perspectives,"

"Generalize," "Persuade," " Consider the reasons for success or failure," and "Doubt that it's true." From the contents of this report, the author could read the abilities that the author wants the students to acquire through the study of mathematics. Below are the keywords and impressions that the students listed.

"Think about it from different perspectives."
Even within our group, we discovered that different people focus on various points of view, and through repeated successes and failures, we came up with our answers.
"Persuade"
It was important to accept successes and failures and link them to ideas about what to do.
"Generalize"
This is one way to convince people, but by using letters, we can show that it is not a coincidence. Make it clear without omission. Generalization requires considering what satisfies the equation and why it satisfies it.
" Consider the reasons for success or failure."
Many times, when things did not go as expected and the research turned out to be a failure, I thought about the reasons but moved on to the next step without really understanding them. When I do my subsequent research, I would like to consider what went wrong.
"Doubt that it's true."
It was important to doubt the results and check whether they were true once they were obtained.

The content of this report allows the author to read from the students' expressions the skills that the author wants them to acquire through the study of mathematics. The author can also read the students' mathematical thinking, including logical and critical thinking.

6. Consideration from the viewpoint of using ICT

Next, the author will discuss how ICT can be used in RLA learning activities.
First, ICT can be used in inquiry activities. Students can use spreadsheet software such as Excel to repeat calculations. For example, using the MOD function command, entering “=MOD(B2+B3,10)” in cell B4 will output the remainder obtained by adding the numbers in cells B2 and B3 and dividing by 10. (See [1])

	A	B
1	Step	
2	1	2
3	2	5
4	3	=MOD(B2+B3,10)
5	4	=MOD(B3+B4,10)
6	5	=MOD(B4+B5,10)

Figure 6.1: Functions used in Excel

Figure 6.2 shows the results of this simulation using the MOD function command (See [2]). In this example, the numbers from 1 to 9, including 2 in the first column and 5 in the second column, are summarized because of the calculation. From this operation, students can examine the results when the numbers in the second line are varied; we can see that the numbers in the second line do not necessarily coincide with those in the 17th line.

Step									
1	2	2	2	2	2	2	2	2	2
2	5	4	3	2	1	6	7	8	9
3	7	6	5	4	3	8	9	0	1
4	2	0	8	6	4	4	6	8	0
5	9	6	3	0	7	2	5	8	1
6	1	6	1	6	1	6	1	6	1
7	0	2	4	6	8	8	6	4	2
8	1	8	5	2	9	4	7	0	3
9	1	0	9	8	7	2	3	4	5
10	2	8	4	0	6	6	0	4	8
11	3	8	3	8	3	8	3	8	3
12	5	6	7	8	9	4	3	2	1
13	8	4	0	6	2	2	6	0	4
14	3	0	7	4	1	6	9	2	5
15	1	4	7	0	3	8	5	2	9
16	4	4	4	4	4	4	4	4	4
17	5	8	1	4	7	2	9	6	3

Figure 6.2: Result of fixing the number in the first column and changing the number in the second column.

Step									
1	2	2	2	2	2	2	2	2	2
2	5	4	3	2	1	6	7	8	9
3	7	6	5	4	3	8	9	0	1
4	2	0	8	6	4	4	6	8	0
5	9	6	3	0	7	2	5	8	1
6	1	6	1	6	1	6	1	6	1
7	0	2	4	6	8	8	6	4	2
8	1	8	5	2	9	4	7	0	3
9	1	0	9	8	7	2	3	4	5
10	2	8	4	0	6	6	0	4	8
11	3	8	3	8	3	8	3	8	3
12	5	6	7	8	9	4	3	2	1
13	8	4	0	6	2	2	6	0	4
14	3	0	7	4	1	6	9	2	5
15	1	4	7	0	3	8	5	2	9
16	4	4	4	4	4	4	4	4	4
17	5	8	1	4	7	2	9	6	3
18	9	2	5	8	1	6	3	0	7
19	4	0	6	2	8	8	2	6	0
20	3	2	1	0	9	4	5	6	7
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
50	7	8	9	0	1	6	5	4	3
51	3	8	3	8	3	8	3	8	3
52	0	6	2	8	4	4	8	2	6
53	3	4	5	6	7	2	1	0	9
54	3	0	7	4	1	6	9	2	5
55	6	4	2	0	8	8	0	2	4
56	9	4	9	4	9	4	9	4	9
57	5	8	1	4	7	2	9	6	3
58	4	2	0	8	6	6	8	0	2
59	9	0	1	2	3	8	7	6	5
60	3	2	1	0	9	4	5	6	7
61	2	2	2	2	2	2	2	2	2

Figure 6.3: Results of repeated calculations

Students can also verify that the number changes after repeating this operation after the 17th line. It is easy to notice that the numbers in the first row reappear as the student repeats the calculation. Figure 6.3 shows the calculation results. (See [3].) Using Excel in this way, students can further investigate what might happen if the mod is changed.

In the case study described in Chapter 4, a student's poster presentation on mod changes and circulation was based on the report's content. Figures 6.4 and 6.5 are part of the poster. (See [4] and [5]) Figure 6.4 generalizes the results of the calculations in Figure 5.3. Figure 6.5 further generalizes that the numbers in the first row reappear. Students did the calculations by hand, got the facts, and generalized them to show them.

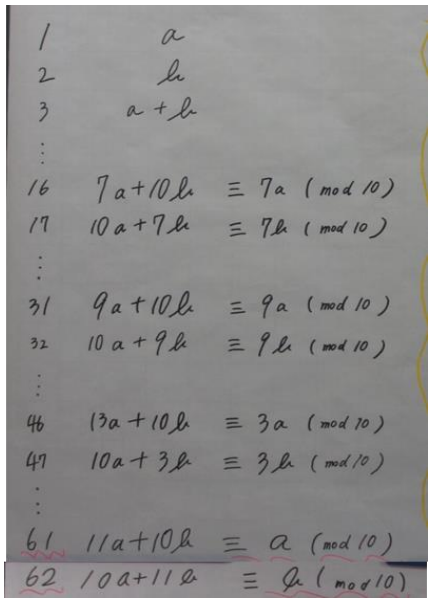

Figure 6.4: Student’s Poster

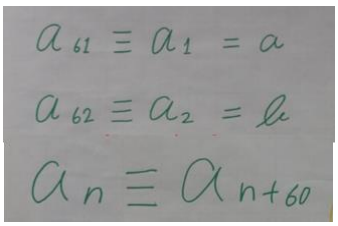

Figure 6.5: Student’s Poster

Using ICT in this student inquiry activity reduces the time spent repeating the same procedure manually and generates time for reflection and analysis of the facts and their mathematical background. Students can discover regularities by hand calculations and confirm their predictions using ICT, or they can discover regularities from the results of calculations made with ICT; using ICT will broaden the range of mathematical approaches, which we believe will allow students to engage in more profound inquiry activities.

Making and testing predictions is an effective use of ICT in inquiry activities. This approach encourages students to think critically and develop problem-solving skills. It can be repeated several times to reinforce learning. Thus, creating new time for reflection, factual analysis, and deepening mathematical background is possible.

Classroom practice with RLA can effectively engage students in mathematical inquiry activities. Deep learning can be achieved through problem-making and inquiry activities, such as changing the conditions of basic problems and poster sessions. Among these, we believe that inquiry activities are the pillar of learning and that using ICT is effective. We believe that using ICT can bring more profound inquiry activities to students.

The future task is to develop a class using ICT for inquiry activities on the same theme as the class practice at RLA and to clarify the differences from the previous practice based on the student's activities, impressions, reports, and posters.

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Connecting Trigonometry to Its Geometric Roots: An Introduction to Trigonometric Values

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Abstract: *This research explores two methods through which ten preservice math teachers develop an understanding of trigonometric values. Using the unit circle, preservice math teachers engage in knowledge-building activities such as paper folding and GeoGebra application. Grounded in Altman and Kidron's 2016 didactical design research, this study examines the cognitive processes ten preservice math teachers undergo during knowledge acquisition procedure. Employing the dynamically nested epistemic action model for abstraction, we analyze how different tasks facilitate ten preservice math teachers' comprehension of unit circle representations for trigonometric expressions and their associated values. Furthermore, we apply the abstraction in context framework to observe how ten preservice math teachers' knowledge progresses from traditional 'triangle' trigonometry to 'circle' trigonometry, aiding in determining trigonometric values.*

1. Introduction

Trigonometry is rooted in geometry, particularly in studying right-angled triangles and trigonometric functions. The fundamental trigonometric functions—sine, cosine, and tangent—are defined based on the ratio of the sides of a right triangle. So, trigonometric values and functions are the foundation of geometry, relating angles to sides of a right-angled triangle. The measurements of trigonometric ratios have captivated mathematicians from the inception of trigonometry and remain a topic of intense fascination today. Studies on the pedagogy of trigonometric functions have identified challenges faced by learners in grasping these mathematical concepts [1-3]. Learning trigonometry involves understanding these relationships through functions and/or ratios like sine, cosine, and tangent. These functions and/or ratios are essential for solving math, physics, and engineering problems. Teaching trigonometry effectively requires clear explanations, visual aids, and real-world examples. It is about guiding learners to see beyond numbers and equations to the patterns and principles that govern the physical world [4-5]. Whether through interactive tools such as GeoGebra or Desmos applications or hands-on activities such as paper folding, the goal is to make trigonometry understandable and connect to its root with relevant exploration [6-7].

With the gap identified above, in this study, we propose a didactical design to fill the gap by connecting trigonometry to its geometric roots using two methods (1) paper folding method and (2) visual representation using the GeoGebra application for finding the trigonometric values. This study is grounded on the previous research proposed by Altman and Kidron that describes how students' cognitive process through constructing knowledge [3]. Hence, this research presents a didactical approach to teaching trigonometry, transitioning from right-angle triangle principles to the unit circle, thereby facilitating a comprehensive finding of trigonometry values. The study aimed to explore effective instructional designs that teachers could adopt to connect the discovery of trigonometric values from right-angle triangles through hands-on activities like paper folding and digital visualization with the GeoGebra application. Specifically, the research sought to answer the question:

How can the proposed lesson design enable preservice math teachers to integrate the paper folding method and GeoGebra visual tools to elucidate trigonometric values derived from right-angle triangles and unit circles?

2. Abstraction in Context and Construction of Knowledge

Mathematical principles and even trigonometric concepts find their application across various domains such as engineering, astronomy, finance, economics, and statistics, making them instrumental in navigating real-world scenarios. These principles are often elucidated through tangible means—be it physical, digital, or graphical representations. This methodological approach, supported by educational psychology (as noted by Bruner in 1966 and Piaget in 1970), underscores the significance of concrete materials in fostering a deeper understanding of theoretical knowledge for the construction of knowledge. The advantage of employing physical objects such as paper folding lies in their ability to offer a hands-on context, which not only aids learners in grasping practical knowledge but also helps the exploration and construction of abstract mathematical ideas in trigonometry [6-7]. In this context, the integration of ICT tools like GeoGebra in mathematics while finding the value of trigonometry is noteworthy. GeoGebra bridges the gap between the abstract nature of geometric transformations and their tangible counterparts through visual aids and animations. This interactivity encourages the use of other software that allows for the manipulation of trigonometry values by altering variables, thereby simplifying the visualization process for students. Consequently, GeoGebra transforms the inherent abstractness of mathematics into something more concrete, enhancing the learning experience through improved visualization capabilities.

3. The Learning Experience and Methodological Issues

3.1 Research Context

For this purpose, the design is based on didactical design research. The study focuses on the learning experiences of ten preservice math teachers teaching at the secondary level, each boasting over a decade of teaching in the field. Our research, as outlined in the introductory section, was craft to eliminate the need for prior knowledge of trigonometry. The intent was to guide preservice math teachers through a novel method of knowledge construction by engaging in the geometric creation of the unit circle. This was achieved through two distinct approaches: traditional paper folding and the modern GeoGebra software. Emphasis was placed on observing the progression of knowledge construction rather than the ultimate learning outcomes of the students. Consequently, aligned with our research purpose, this investigation adopts a qualitative approach, incorporating a singular case study and an in-depth micro-analysis of the preservice math teachers' knowledge construction processes.

3.2 The Learning Setting and a-Priori Analysis

The journey is structured into two distinct stages. Initially, the preservice math teachers engage with activities centered around paper folding, which revisit the right-angle triangle concept. Subsequent exercises prompt the preservice math teachers to embark on crafting a unit circle, which in turn aids in discerning trigonometric values. This initial stage aims to acquaint the learner with the unit circle's framework. While no significant challenges were predicted for triangles within the first quadrant, the dynamics altered for other quadrants where triangles assume various orientations. It is

envisaged that grappling with these complexities shall fortify the preservice math teacher's understanding, paving the way for the subsequent stage that involves broadening the scope of trigonometric values within the GeoGebra application. The second stage's tasks are designed to draw a correlation between trigonometric values and the geometric representation of the unit circle established earlier, with a reflective analysis of these findings within the GeoGebra application. The purpose of these bifurcated stages is to facilitate the preservice math teacher's comprehension of the geometric significance of trigonometric values as they pertain to the unit circle. Our preliminary analysis posited that learners would draw parallels with prior knowledge from 'triangle' trigonometry, recognizing the interplay between the sides of a right triangle and its angles. The tasks are methodically crafted to promote a seamless shift from 'triangle' trigonometry to 'circle' trigonometry, with the goal of enabling learners to deduce the trigonometric values of the unit circle and show their relevance. While we observed several anticipated outcomes in our retrospective analysis, we also encountered a range of intriguing and unforeseen constructs.

4. Reporting and Analysis of Findings

4.1 Reporting and Analysis of Findings in the First Phase of the Preservice Math Teacher Experiences

Prior to starting the activity, we ensured items: a single sheet of unmarked A4 paper, an item with a round shape, like a bowl or a compact disc, and a bold marker or a pencil with an HB lead for clear markings. Drawn the circle. See Figure 1. Using a circular object, such as a bowl or CD, carefully draw a circle in the lower left portion of your A4 paper, as shown in Figure 1. The circle has a radius of about 6 cm and about 2 cm from the bottom and left edges of the paper. Throughout this activity, we thought that the circle as the unit circle $x^2 + y^2 = 1$.

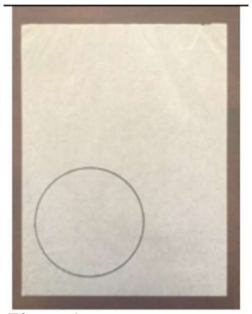
Figure 1

Fold 1. The x-axis was created as the horizontal diameter of the circle. See Figure 2. Holding the paper up to a light source so we can see through it, fold the circle onto itself, matching semicircle to semicircle. Be sure to line up the left edge of the paper. The two halves of the circle should line up perfectly, and the left edge of the paper should line up with itself. Ensured that the fold line is parallel to the paper's bottom edge. Creased the paper on the fold line (shown in red).

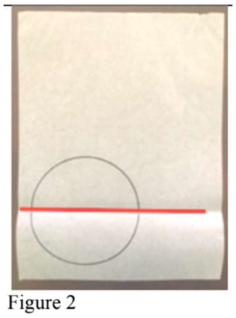
Figure 2

Located the first 2 of the 16 points. See Figure 3. We have marked the intersection points between the fold line and the circle.

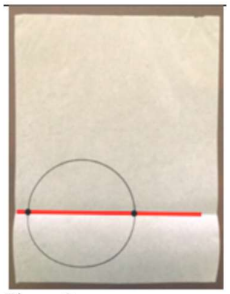
Figure 3

Fold 2. The y-axis was created as the vertical diameter of the circle. See Figure 4. Holding the paper up to a light source, fold the circle onto itself while lining up the bottom edge of the paper. The two halves of the circle should line up perfectly, and the bottom edge of the paper line up with itself. (The two marked points should fold exactly onto each other. Also, the fold line parallel the paper's left edge.) Creased the paper on the fold line.

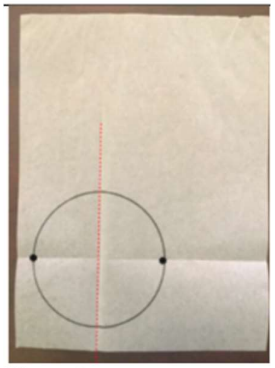
Figure 4

Locate the next 2 of the 16 points and the center of the circle. See Figure 5. We have marked the new intersection points between the fold line and the circle. It also, marked the intersection of the two folds, which is the center of the circle. (The circle's center is marked because it is a useful reference point, but it is NOT one of the 16 points we seek on the unit circle itself.)

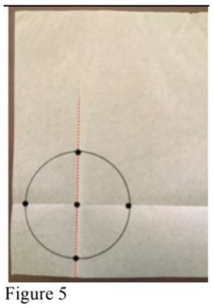
Figure 5

Fold 3. Create the line y = x.
See Figure 6. Folded the positive x-axis onto the positive y-axis. Make sure that the fold passes through the origin (0, 0), and that red point (1, 0) folds onto red point (0, 1). The red line illustrates the location of the fold line. Using light to see through the paper, fold the circle onto itself, matching semicircle to semicircle.

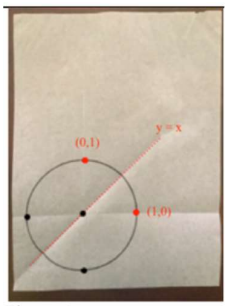

Figure 6

Locate the 2 new points on the line y = x.
See Figure 7. Creased the paper and marked the intersection points between the new fold line and the circle.

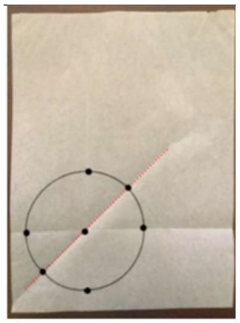
Figure 7

Fold 4. Create the line y = –x.
See Figure 8. Folded the negative x-axis onto the positive y-axis. Make sure that the fold passes through the origin (0, 0), and that red point (–1, 0) folds onto red point (0, 1). The red line illustrates the location of the fold line. Using light to see through the paper, fold the circle onto itself, matching semicircle to semicircle.

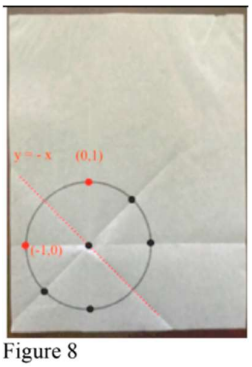

Figure 8

Locate the 2 new points on the line y = –x.
See Figure 9. Creased the paper and marked the intersection points between the new fold line and the circle.

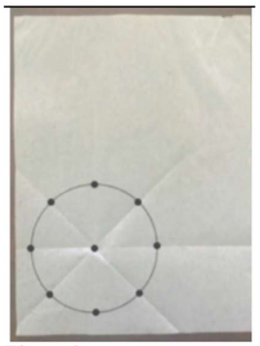
Figure 9

Fold 5. Create the vertical line x = 1/2.
See Figure 10. Folded the red point (1, 0) onto the red origin (0, 0). The red line showed the location of the fold line.

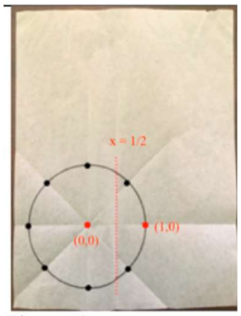

Figure 10

Locate the 2 new points on the line x = 1/2.
See Figure 11. Creased the paper and marked the intersection points between the new fold line and the circle.

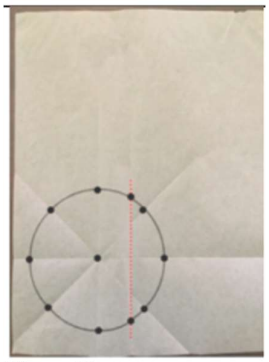
Figure 11

Fold 6. Create the horizontal line y = 1/2.
See Figure 12.
Folded the red point (0, 1) onto the red origin (0, 0). The red line showed the location of the fold line.

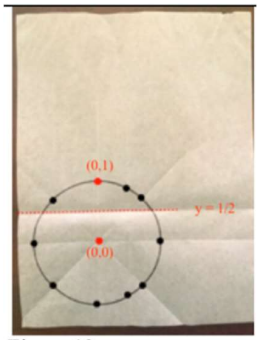

Figure 12

Locate the 2 new points on the line y = 1/2.
See Figure 13.

Creased the paper and marked the intersection points between the new fold line and the circle.

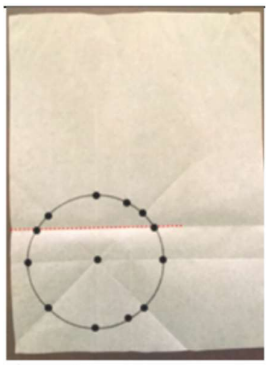

Figure 13

Fold 7. Create the vertical line x = –1/2.
See Figure 14.
Folded the red point (–1, 0) onto the red origin (0, 0). The red line showed the location of the fold line.

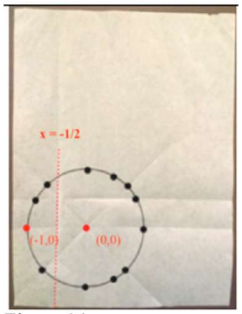

Figure 14

Locate the 2 new points on the line x = –1/2.
See Figure 15.
Creased the paper and marked the intersection points between the new fold line and the circle.

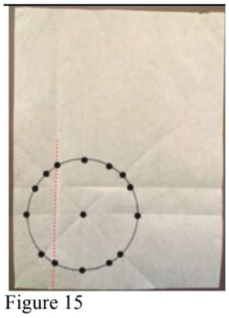

Figure 15

Fold 8. Create the horizontal line y = –1/2.
See Figure 16.
Folded the red point (0, –1) onto the red origin (0, 0). The red line showed the location of the fold line.

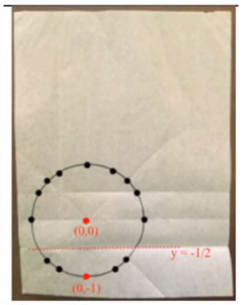

Figure 16

Locate the 2 new points on the line y = –1/2.
See Figure 17.

Creased the paper and marked the intersection points between the new fold line and the circle. Your 16-point circle was completed!

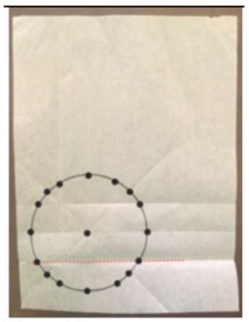

Figure 17

Label the 16 points on the unit circle.
Ensure that your paper is oriented so that the circle is in the bottom left corner, as shown in Figure 18. Begin labeling at the 0° position on the unit circle and label the degrees for each of the 16 points. The first 5 points were labeled in Figure 18; continue labeling the other 11 points.

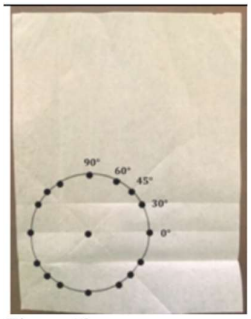

Figure 18

Bonus. Two additional folds...
See Figure 19.
Folded to create the line x = 1, tangent to the unit circle at 0°. Also, folded to create the line that passes through both the origin (0, 0) and the point on the unit circle at the 60° position. Drawn lines along these new creases. Compute the exact values of the following trig functions by any method you wish:

- sin (60°)
- cos (60°)
- tan (60°)
- sec (60°)

Find line segments in Figure 19 that match these values.

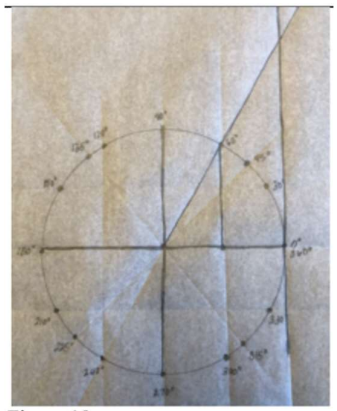

Figure 19

The discussion was initiated based on the questions (1) are all of the angle measures shown in Figure 18 correct?, how can you be sure? Explain your thinking. For example, how do you know that the point labeled 60° actually forms a 60° angle with the positive x-axis? (2) what are the coordinates of each of the 16 points? (3) what are the sine (sin) and cosine (cos) values associated with each point? And (4) how can you be sure about your answers to Questions 2 and 3?

4.2 Reporting and Analysis of Findings in the Second Phase of the Learning Experience

While exploring a dynamic point on a unit circle, Trigonometric situation 1 (GeoGebra Configuration 1, click to open the following link: https://www.geogebra.org/m/vqzuvuyy, See figure 20 in the next page).

The directions and discussion questions for configuration 1 were as follows:

1. Grab point P and move it around the Unit Circle. Consider all four quadrants as you move point P. Notice how the numerical values change. Does any value stay fixed?
2. Describe the line segments that make up the right-angle triangle OBP. How do the side lengths of triangle OBP relate to the coordinates of point P?
3. How is the angle a related to the x-coordinate of point P? How is the angle a related to the y-coordinate of point P?
4. Describe the trigonometric relationships within the dynamic configuration. Can you locate the 16 points that you found during the Paper-Folding Activity?
5. For any particular location of point P, how would you calculate sin θ, cos θ, and tan θ?

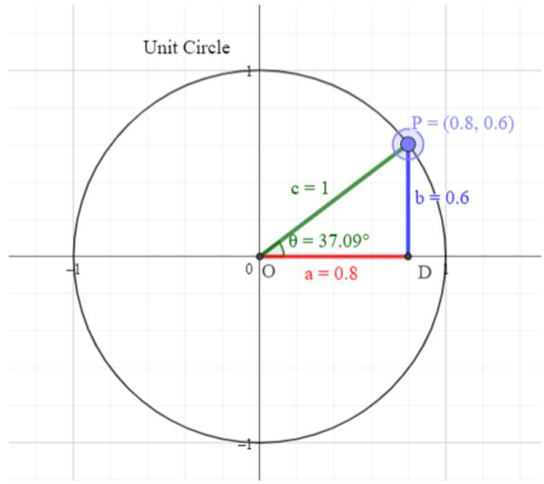

Figure 20

Next is trigonometric situation 2 (GeoGebra Configuration 2, click to open the link: https://www.geogebra.org/m/jh4qrzdg, See figure 21). The directions and discussion questions were for configuration 2 as following:

1. Grab point P and move it around the Unit Circle. Consider all four quadrants as you move point P. Notice how the numerical values change. Does any value stay fixed?
2. Describe the line segments that make up the right-angle triangle OBA. How do the side lengths of triangle OBA relate to the measure of angle θ?
3. Describe the trigonometric relationships within the dynamic configuration. Can you locate the 16 points that you found during the Paper-Folding Activity?
4. For any particular location of point P, how would you determine the values of tan θ and sec θ?

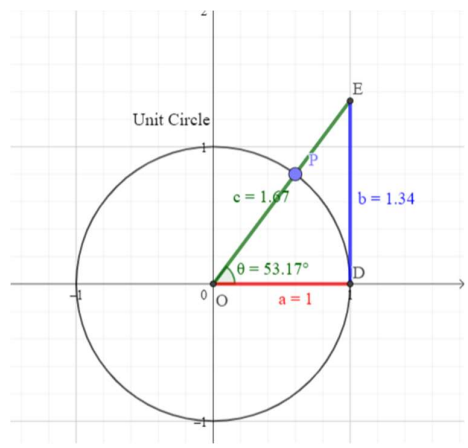

Figure 21

Finally, in trigonometric situation 3 (GeoGebra Configuration 3, see Figure 22). The directions and discussion questions for configuration 3 as follows:

1. Grab point P and move it around the Unit Circle. Consider all four quadrants as you move point P. Notice how the numerical values change. Does any value stay fixed?
2. Describe the line segments that make up the right-angle triangles OPA and OPB. How do the side lengths of these two triangles relate to the measure of angle θ?
3. Describe the trigonometric relationships within the dynamic configuration. Can you locate the 16 points that you found during the Paper-Folding Activity?
4. For any particular location of point P, how would you determine the values of

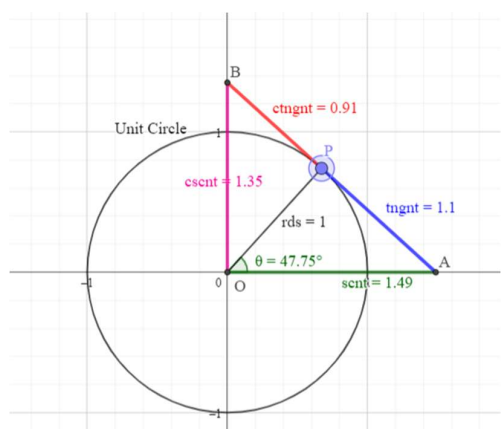

Figure 22

- tan θ ?
- sec θ ?
- cot θ ?
- csc θ ?

4.3. Reflection on Analysis of Findings in the two Phases

In the first phase of the learning experience, preservice math teachers are involved in a hands-on paper-folding activity designed to illustrate the geometric and trigonometric properties of the unit circle. Preservice math teachers began by preparing their materials, ensuring each had an A4 paper, a round object for tracing, and a bold marker or HB pencil for clear markings. The activity meticulously guided participants through multiple folds, each aimed at establishing creases of the lines, such as the x-axis and y-axis, and various angular bisectors represented by lines y = x and y = -x. This meticulous process helped locate 16 points on the unit circle, each corresponding to specific angle degrees. The folding activity served as a practical introduction to circle geometry and set the stage for understanding more complex trigonometric relationships. The completion of the task was marked by labeling the angles, reinforcing the angular concepts, and ensuring comprehension of the geometric placements of the points on the circle. In the second phase, the experience of preservice math teachers transitioned to a digital exploration using the GeoGebra software, which allowed for a dynamic engagement of preservice math teachers with the trigonometric concepts previously approached through paper folding. Preservice math teachers manipulated a point on a digital unit circle, observing the changes in trigonometric values as the point moved through various quadrants in the GeoGebra application. This interactive module was divided into multiple scenarios, prompting preservice math teachers to explore different trigonometric relationships and calculate values such as sine, cosine, tangent, and their reciprocals. This phase significantly enriched the understanding of preservice math teachers by linking the tangible insights from the paper-folding exercise with the abstract mathematical functions, providing a comprehensive learning experience emphasizing trigonometry and geometry's practical and theoretical dimensions. The discussion points and questions raised during this phase aimed at deepening the analytical skills of the preservice math teachers, ensuring a thorough grasp of the trigonometric identities and their real-world applications.

5. Discussion and Concluding Remarks

Our strategy was designed to integrate two methodologies: the time-honored practice of paper folding and the innovative GeoGebra software, both essential for preservice math teachers and learners to grasp trigonometric concepts and their corresponding values. We incorporated hands-on paper folding activities to engage preservice math teachers in connecting these methods, followed by a reflective analysis of their work. This approach guided preservice math teachers to a new understanding of trigonometry values within different frameworks, such as the unit circle, using both traditional and modern techniques. Our study validated this approach, demonstrating its effectiveness in comprehensively understanding trigonometry values. Preservice math teachers were able to locate points on a circle's coordinates with fewer difficulties and misconceptions than previously reported in academic studies [8-17, 24]. Additionally, it cultivated a holistic grasp of trigonometry [7], both as ratios and within the context of the unit circle. We identified errors in preservice math teachers' responses related to the coordinate system, though it was uncertain whether these were due to oversight or underlying misconceptions. Further in-depth research is necessary to clarify this issue. Upon reviewing our didactical design, we recognized the critical factor of time management, particularly concerning technical aspects [18-21]. To optimize the design, we have highlighted some

areas for revisions to minimize time expenditure: 1) Ensuring all preservice math teachers or groups have sufficient resources throughout the learning process, 2) Providing clear, detailed instructions, and 3) Confirming preservice math teachers' foundational knowledge, such as understanding the coordinate system, paper-folding skills and handling GeoGebra application. In terms of group work evaluation, we recommended that educators assign specific tasks to each group member of preservice math teachers to enhance collaboration and performance. Lastly, these adjustments aim to facilitate smoother cooperation among preservice math teachers and learners.

6. Acknowledgements and Funding

We extend our heartfelt gratitude to the ten secondary-level preservice math teachers from different schools in the Kathmandu Valley, Nepal, for their invaluable participation in this journey. We sincerely appreciate their voluntary involvement. Likewise, we extend our special thanks to Prof. Gregory Foley for sharing his insights on trigonometry and its geometric roots during one of the webinars during COVID-19 via the Zoom platform. We greatly appreciate his valuable contribution as the foundation of this work. Additionally, we acknowledge the partial funding support from the Kathmandu University School of Education (KUSOED), Department of STEAM Education, Hattiban, Lalitpur, Nepal.

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The Role of Educational Robotics in Integrated STEM Learning Towards the Formation of 21st-Century Skills

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Abstract: *As a crucial element in Integrated STEM Learning, Robotics plays a vital role in developing 21st-century skills such as communication, collaboration, critical thinking, and creativity. This research aims to (1) analyse how 21st-century skills emerge in integrated STEM learning and (2) analyse how Educational Robotics supports the formation of 21st-century skills in Integrated STEM Learning. The study was conducted at a non-formal school in Central Java, Indonesia, involving primary and junior school students. The research adopted a qualitative approach, with learning conducted using Lego Mindstorm EV3 Educational Robotics by creating a door security alarm.. Data were collected through interviews and questionnaires to evaluate the role of robots in knowledge construction. Data analysis was performed using Qualitative Content Analysis in Atlas. ti. The results showed that (1) 21st-century skills emerge in integrated STEM learning by encouraging students to communicate, collaborate, think critically, and create, and (2) Educational Robotics supports the development of 21st-century skills by providing challenges that require cooperation, collaboration, and communication to solve problems. However, students need to further hone their skills, especially in communication and critical thinking, to achieve a higher level of 21st-century skill development.*

1. Introduction

Education is a crucial element in the development of a country, and to advance it, new solutions that keep up with the times are required. There are four main aspects to consider in improving the quality of education: policy, leadership, infrastructure, and the learning process [1]. One solution that can address all four aspects is robotics. The use of robotics in education has been increasing in various countries. However, it is still minimal in Indonesia due to the high costs of the necessary facilities and infrastructure.

As a developing country, the current industrial revolution demands that Indonesia adopt new technologies like robotics as a learning medium to prepare for the future. Therefore, skilled teachers who can help students learn robotics are needed. Teachers proficient in teaching robotics will be more effective in delivering the material to students, thus enhancing the quality of education [2].

Robotics education incorporates integrated STEM (Science, Technology, Engineering, and Mathematics) elements [3], which connect real-world STEM problems through concepts and skills from various disciplines [4]. Integrated STEM provides a structured learning environment and space for collaboration, making it easier for students to understand STEM concepts.

As a part of STEM, robotics is drawn from real-world applications, allowing it to be studied and applied through simple principles and used as teaching material in schools [5]. Even though students may not fully understand how robotics works, its use can attract their interest in learning Integrated STEM. Robotics also serves as a practical step in engaging students with Integrated STEM [6].

However, there are still some issues in its implementation, mainly related to the curriculum, which is not fully aligned yet. A strong discipline is needed to apply Integrated STEM within the educational curriculum, using robotics as a technological platform in teaching materials[7]. This is crucial to ensure that STEM-based learning operates effectively and aligns with educational objectives.

The use of robotics in STEM-based learning is also influenced by 21st-century skills, which emphasize the integration of various disciplines with the goals of effective communication, collaboration, problem-solving, and critical and creative thinking [8]. These 21st-century skills are highly relevant in supporting the implementation of Integrated STEM and using robotics as a learning medium.

Based on the background above, the problem statement of this research is, How do 21st-century skills emerge in Integrated STEM learning? How does educational robotics support the development of 21st-century skills in Integrated STEM learning?

2. Literature Review

A. Implementation of Integrated STEM Learning in School Education

Integrated STEM learning offers a holistic educational approach by combining multiple disciplines to address real-world problems and prepare students for modern challenges. [9, 10].This pedagogy aims to develop critical 21st-century skills, such as critical thinking, creativity, collaboration, and problem-solving, by engaging students in interdisciplinary projects that connect academic concepts with practical applications [11]. Teachers facilitate this learning process by creating dynamic environments where students explore, discover, and collaborate, fostering intrinsic motivation and active engagement. [12].

While Integrated STEM learning presents significant educational benefits, it also faces challenges, including the need for adequate resources, policy support, and professional development for educators. [13]. Overcoming these challenges requires a strong commitment from stakeholders and an awareness of the long-term advantages of this

approach. Integrated STEM can enhance educational quality and prepare students for a complex and evolving future when effectively implemented.

Critical attributes of Integrated STEM education include transdisciplinary integration, authentic context, and problem-solving design. [9]. Transdisciplinary integration combines various disciplines to address real-world issues, while authentic context involves students actively exploring and relating problems to their daily lives. Problem-solving design consists of engaging with complex, poorly structured problems to meet success criteria through innovative design and production [14].

B. The Role of Robotics in Education

Creativity is essential in the 21st century for global competitiveness and personal development, mainly as it enhances metacognitive abilities and prepares individuals for future career challenges [15]. Robotics in education provides a dynamic platform for integrating multidisciplinary skills, which can significantly improve students' academic performance and creativity by engaging them in tasks like building and programming [16] [17]. As technological advancements continue to evolve, the integration of robotics into educational settings allows students to interact with scientific and technological concepts in a practical and stimulating environment [18].

Robotics education supports the development of key 21st-century skills, such as collaboration, communication, critical thinking, and problem-solving. By involving students in programming and operating robots, they gain hands-on experience with STEM principles and enhance their readiness for the modern workforce [19]. Additionally, the engaging nature of robotics activities helps boost students' motivation and interest in science and technology, leading to greater satisfaction and a positive attitude toward learning and innovation [20].

Despite the positive impacts of robotics on education, including its application across various scientific fields and interdisciplinary STEAM activities [21], there is a need for further research to explore its role in developing 21st-century skills. Previous studies have primarily focused on elementary education and the use of LEGO robots [22], but a comprehensive understanding of how robotics contributes to skill development remains limited. This research aims to address these gaps and assess the outcomes of robotics in fostering essential modern skills.

C. The Relationship Between STEM Learning and 21st-Century Skills

STEM education integrates Science, Technology, Engineering, and Mathematics to develop students' critical thinking, problem-solving, and analytical skills. By engaging in interdisciplinary and project-based learning, students collaborate, communicate, and think creatively, which enhances their ability to evaluate information and make informed decisions. This approach also fosters essential 21st-century skills like creativity, collaboration, and adaptability, preparing students for future careers and innovation.

Through real-world problem-solving and innovative design, STEM education encourages students to think flexibly and creatively, broadening their perspectives and enhancing their motivation and self-efficacy [23]. While STEM education offers numerous benefits, including improved teamwork and communication skills[24], there is still a need for more research on students' perspectives across different levels and subjects to understand its impact fully [25].

A well-structured STEM learning approach should effectively support the development of 21st-century skills such as leadership, critical thinking, and problem-solving. Ensuring that STEM education is accessible and fosters creativity will help students face future challenges and contribute meaningfully to society and the global workforce [6].

D. Educational Robotics Facilitates 21st-Century Skills

In recent years, educational robotics platforms and programs have rapidly developed to suit various academic levels, from postgraduate and undergraduate students to school-age children [26]. The use of robotics in education has proven to be an effective tool in facilitating the development of 21st-century skills [27]. Students gain an understanding of technological concepts and develop critical, creative, and collaborative skills essential in the modern era. Through hands-on experiences in designing, programming, and operating robots, students engage in practical and deep learning, preparing them to face future challenges with confidence [28].

Robotics in education significantly contributes to developing students' 21st-century skills [29]. Using robots as learning tools introduces students to the latest technological concepts and trains them to work in teams, collaborate, and communicate effectively in completing tasks. This process enhances problem-solving and critical thinking skills as students face challenges requiring creative solutions and adaptability [1]. Robotics education not only provides engaging and enjoyable learning experiences but also equips students with relevant skills needed for future challenges [30].

Moreover, educational robotics encourages students to think critically and innovatively [31]. The process of designing and programming robots involves complex challenges that require creative and innovative problem-solving, fostering the development of lateral thinking and out-of-the-box thinking. Additionally, robotics education enhances collaborative and communicative skills as students often work in teams on robotic projects [32]. This teamwork requires effective cooperation, clear communication, and appreciation of each team member's contributions. Thus, robotics education helps students acquire technical skills and shapes them into individuals skilled in teamwork, communication, and collaboration. Studies have shown that educational robotics supports conceptual understanding in STEM and fosters a positive attitude toward these subjects, reinforcing 21st-century skills [33].

3. Methodology

This study used a qualitative approach. The participants of this study were eight students from a nature-based elementary school consisting of 6 early-grade students (grades 1-3) and 2 middle-grade students (grades 4-6), with a primary focus on the role of educational robotics in integrated STEM learning and its impact on 21st-century skills development. The researchers surveyed this school to establish a baseline understanding of students' knowledge of educational robotics. Data collection was conducted using questionnaires and interviews. The process involved giving interviews to all early and middle-grade students and then confirming the results of the interviews with questionnaires. In this study, the researchers used the Qualitative Content Analysis data analysis method.

Qualitative Content Analysis is a systematic method for explaining the meaning of qualitative data [34]. This approach involves classifying data into pre-determined categories within a coding framework. Qualitative Content Analysis is suitable for describing material that requires interpretation. When engaged in qualitative research, the data often requires interpretation to be fully understood.

4. Result and Discussion

A nature-based school was chosen as a research site to continue STEM-related activities, with the research problem focusing on analyzing the role of educational robotics in integrated STEM learning on 21st-century skills. Robotics will be used to solve daily life problems, specifically on home security systems, by integrating the system through robot assembly, programming, and ultrasonic sensors to detect motion. This approach aims to improve students' understanding of integrated STEM learning. The participant data consisted of 8 elementary school students from early and intermediate levels in grades I-IV.

The interview results will be processed using the Atlas.ti application and define it with codes according to the discussion. Then the codes will be triangulated into the questionnaire so that the data can be concluded to support or not. Overall, the results of the interviews show that they are indicators of 21st-century skills [35] and STEM integration [36]. Then, the results of the questionnaire also show that the practices that have been carried out have an increasing effect on the 21st-century skills of these students and contain

elements of STEM that are seen implicitly or explicitly.

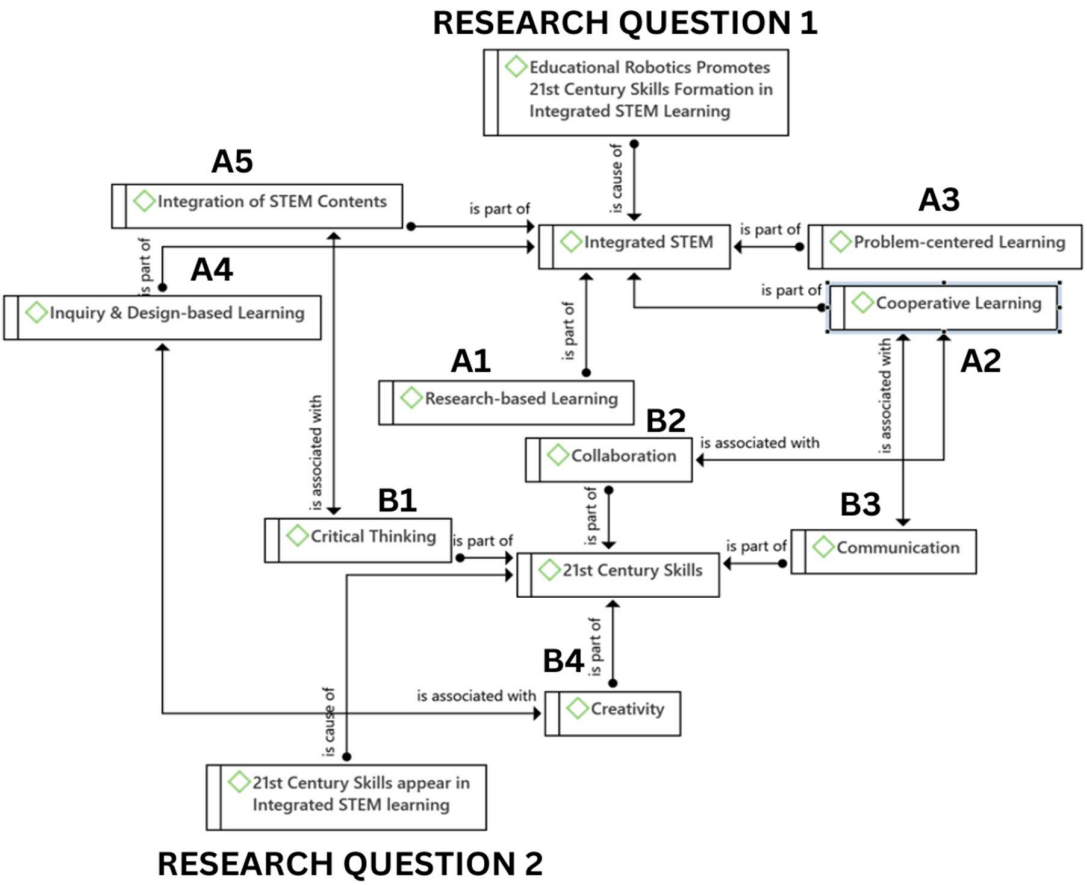

Figure 1. The flow of answers to the research questions

Figure 1 shows the flow of answers to the research questions created with the ATLAS.ti application, based on the existing codes. Figure 1 can be explained as follows:

1. The first research question, "How do 21st-century skills emerge in integrated STEM learning?" is answered through the aspect of 21st-century skills, referring to codes B1, B2, B3, and B4.
2. The second research question, "How does educational robotics support the formation of 21st-century skills in integrated STEM learning?" is answered through the aspect of Integrated STEM, referring to codes A1, A2, A3, A4, and A5.

The discussion about the codes are presented below

B1 (Critical Thinking)

The analysis results show that most students are already thinking critically by mentioning aspects related to STEM, and some students also mention the need to define specific distances for the robot to detect accurately. For instance, Participant 4 noted several aspects of STEM that had already been learned:

"learning the program and assembling Lego robots" (Participant 4).

B2 (Collaborative)

The results for code B2 indicate that all participants reported engaging in various forms of collaboration. For example, Participant 4 collaborated by adding sensors and asking other friends to fetch the necessary items:

"helping to choose Lego components and sensors..." (Participant 4).

This analysis aligns with the study by Saad et al. (2024), which found that integrated STEM learning can develop 21st-century skills, including students' collaboration abilities. It shows the ability to break down problems into smaller tasks, an indicator of collaborative traits and teamwork [37].

B3 (Communication)

The results for code B3 indicate that all participants responded positively, with the majority stating that the material delivery was good. The material covered basic issues such as safety concerns, robot parts, and basic application methods, from assembly to ensuring the robot functions correctly. The students reported understanding and enjoying the learning experience, as noted by Participant 1:

"yes, and it was fun" (Participant 1).

From the above analysis, it can be stated that STEM learning, in the form of solving security issues by creating door alarms through educational robotics, meets the 21st-century skills aspect of communication. This is consistent with the description by [37], which states that an indicator of communication is the presence of group discussions that prompt members to solve problems by analyzing existing issues and communicating effectively, both orally and in writing, to understand and convey messages in various situations and for different purposes.

B4 (Creativity)

The results for code B4 indicate that some students were able to propose and provide creative solutions for designing the robot to be more effective and useful according to the existing problems. Participant 1 described a creative solution by adding accessories to the robot and increasing detection speed by adjusting the distance between the robot and the object to facilitate easier detection:

"yes, adding accessories and speed as well as distance" (Participant 1).

This demonstrates the development of students' thinking, which is an indication of creative thinking [38].

The following is the explanation of the codes from the second Research Question,

A1 (Research-Based Learning)

The analysis of code A1 from the interviews indicates that the majority of students stated that the allotted time was sufficient. However, some students felt that the time was not enough because they wanted to explore educational robotics further. As described by Participant 1:

"The time was insufficient, but for the program, 90% was good enough." (Participant 1)

A2 (Cooperative Learning)

The results for code A2 indicate that most students reported collaborating with their peers by helping direct and provide the necessary items, as stated by Participant 8:
"cooperating by fetching the required items (components)" (Participant 8).

A3 (Problem-Centered Learning)

The analysis of code A3 shows that most students described the steps they took during the previous practice sessions, which included building the robot, creating the program, and installing sensors to secure the house door, as mentioned by Participant 4:
"Building the robot, creating the program, and installing sensors to secure the house door" (Participant 4).

A4 (Inquiry & Design-Based Learning)

The analysis of code A4 reveals a variety of opinions from the students, with most having their own ideas for modifying the robot, such as adding a remote control, as stated by Participant 2:
"adding a remote control" (Participant 2),
and adding speed and distance adjustments, as stated by Participant 1:
"…adding speed and distance" (Participant 1).

A5 (Integration of STEM Content)

The analysis of code A5 indicates that the majority of students were able to explain the robot they built based on its function and operation, such as motion detection, as stated by Participant 5:
"detecting motion…" (Participant 5),
then starting the workflow by assembling the robot and programming it, as mentioned by Participant 4:
"learning the program and assembling the Lego robot" (Participant 4).

5. Conclusion

Based on the results and discussion above, we can conclude the findings of the research as follows:

1. Through the analysis and fulfillment of aspects of 21st-century skills from codes B1, B2, B3, and B4, it is seen that 21st-century skills appear in Integrated STEM learning through various aspects inherent in these skills. This is shown in the integrated and continuous STEM learning as in code B2. Integrated STEM learning facilitates the emergence of 21st-century skills by creating an environment that allows students to practice critical thinking in code B1, collaborate in code B2, and communicate effectively in code B3, as well as solve real-world problems and design innovative solutions in codes B1 and B4. Students are trained to think creatively.
2. Through the analysis and fulfillment of STEM integration aspects of codes A1, A2, A3, A4, and A5, it can be concluded that educational robotics supports aspects related to 21st-century skills through integrated STEM learning. Integrated STEM learning supports students in honing analytical skills (critical thinking) as evidenced in code A5 and generating new ideas (creativity) in codes A4 and A1, as well as collaboration and communication in code A2 as a solution to existing problems, so as to develop

students' 21st-century skills as in code A3. Overall, this STEM learning is well delivered and fulfills students' improvement in 21st-century skills.

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Some Poncelet invariants for bicentric hexagons

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Abstract

Tangential polygons are (convex) polygons for which every side is tangent to an inscribed circle. Cyclic polygons are those for which every vertex lies on a circle, the circumcircle. Bicentric n-gons are those which are both tangential and cyclic. Every triangle is bicentric. Bicentric quadrilaterals are those for which the sum of the lengths of opposite sides is the semiperimeter and for which opposite angles sum to π. Here we give some results pertaining to invariants of (convex) bicentric hexagons.

A remarkable result of Poncelet is that if one has a pair of circles admitting a bicentric n-gon, then for every point on the circumcircle can be a vertex for a bicentric n-gon. This is illustrated in the animation at
`https://mathworld.wolfram.com/PonceletsPorism.html`

The animation indicates that, along with the incentre and circumcentre, the point of intersection of the principal diagonals of a $2m$-gon is invariant under the motion. Such invariants – here called Poncelet invariants – have been studied for two centuries, in particular for triangles and bicentric quadrilaterals. We present results, for bicentric hexagons, that various combinations of distances between vertices - lengths of diagonals and of sides - are invariant. CAS supplements are available for checking the results.

1 Introduction

Several years ago there was an ATCM Invited Talk on bicentric polygons and Poncelet's Porism [2]. The second author of this paper has presented on them too: [4, 5]. The area - Poncelet invariants, bicentric polgons - has a colourful history. It and adjacent problem areas, e.g. elliptic billiards, remain active areas of research: there is a huge and growing literature. References in this paper post-dating 2020 are [11, 6, 15, 14]. Some of the literature is very technical (e.g. [6, 3], the latter of which has a good historical account but in later parts is technical). One function of this paper is to illustrate, yet again, that classical polynomial algebra continues to yield results which, once found, could easily be explained to undergraduate mathematics students. Theorem 1 in 1.3 is an example.

Poncelet was an officer in Napoleon's Grande Arm'ee and was taken prisoner during the invasion of Russia in 1812. While in prison, he discovered "invariants of bicentric polygons" (and made really major contributions to projective geometry). See [1, 3].

The history, and the MathWorld animations referenced in the abstract, make some of the area appropriate in various undergraduate mathematics courses. For example, it might be an aside in a history of mathematics course when Poncelet is being treated.

1.1 Notation

Label the vertices of the (convex) polygon, n-gon, by numbers from 1 to n, increasing in the counter-clockwise direction. The interior angle at vertex j is denoted α_j: the distance between vertices i and j by d_{ij}. When we use Cartesian coordinates we will take the origin of coordinates at the incentre of the tangential polygon and denote the coordinates of the vertices by z_j. We have $d_{ij} = |z_i - z_j|$. From §1.4 onwards we take the inradius ρ to be 1 (with occasional reminders on this).

1.2 Symmetric polynomials

Given n angles α_j with $0 < \alpha_j < \pi$ there exists a tangential n-gon with these angles. Denote the inradius by ρ. If one fixes the incircle and permutes the 'ears', vertex angles, the n-gon remains tangential. The perimeter L and area A satisfy

$$L = 2\rho \sum_{j=1}^{n} T_j, \qquad A = \frac{1}{2}\rho L.$$

Amongst n-gons with sides $d_{j,j+1}$ there are cyclic n-gons each of which has the maximum area. Different cyclic n-gons, with the same circumradius R and area A are formed as one permutes the sides.

Elementary symmetric polynomials are appropriate in studies of these: polynomials $e_j(\mathbf{T})$ for tangential n-gons, with the obvious first example that e_1 is the semi-perimeter. For an example of the use of symmetric polynomials in connection with cyclic n-gons see the `mathworld.com` page on 'Cyclic Hexagon' and §3.2.

Bicentric n-gons are tangential n-gons which are also cyclic. Some of the invariants, e.g. that of equation (22) (and related results associated with $\cos(\alpha_j)$) are special cases of results first derived for tangential n-gons, e.g. equation (21). The key steps in moving from this result for tangential hexagons to bicentric hexagons is due to Radic, equations (3,4) and implications for the $e_j(\mathbf{T})$ given at equations (9). All invariants are functions of the single variable e_2.

1.3 Sides and the circumradius

As stated in the abstract, much is popularly known for triangles and bicentric quadrilaterals. The wikipedia entry for bicentric quadrilaterals notes that the product of the diagonals ($d_{13}\, d_{24}$) is a Poncelet invariant, specifically,

$$\frac{d_{13}\, d_{24}}{\rho^2} - \frac{4R^2}{d_{13}\, d_{24}} = 1, \qquad \text{for a bicentric quadrilateral ,}$$

where ρ is the inradius and R the circumradius. This re-writes to

$$4\, R^2 = d_{13}\, d_{24} \left(\frac{d_{13}\, d_{24}}{\rho^2} - 1 \right) \qquad \text{for a bicentric quadrilateral .} \tag{1}$$

From these one sees that the sum of products of opposite sides is also a Poncelet invariant as Ptolomy's Theorem for cyclic quadrilaterals states

$$S_4 = d_{12}d_{34} + d_{23}d_{14} = d_{13}\, d_{24} \qquad \text{for a cyclic quadrilateral .}$$

Pitot's Theorem gives

$$(d_{12} + d_{34}) - (d_{23} + d_{14}) = 0 \qquad \text{for a tangential quadrilateral .}$$

There are many invariants already discovered, some of which will be reviewed later in this paper. We prescribe the inradius ρ and the circumradius R. For a bicentric n-gon to exist the distance d between the incentre and the circumcentre has to satisfy a polynomial equation, Fuss's equation: see §1.6. So ρ, R and d are all fixed, invariant, as the bicentric n-gon traverses in its different positions. As noted in the preceding paragraph, there are other quantities which are invariant. Results on bicentric hexagons seem less well known than results for triangles and quadrilaterals. An example of the various items we discovered (and which we have not seen elsewhere - but might well be in nineteenth century literature) is as follows.

Theorem 1 *For bicentric hexagons the sum of the products of the opposite sides, that is* $S = d_{12}d_{45} + d_{23}d_{56} + d_{34}d_{16}$*, is a Poncelet invariant.*

Our (re?-)discoveries include relatively simple formulae for the circumradius of a bicentric hexagon. Perhaps the simplest is equation (2). This, like equation (1), just depends on products of the side-lengths of opposite sides and, when as always henceforth $\rho = 1$, is

$$R^2 = \frac{(8-S)S(4+S)^2}{192(S-2)^2}. \tag{2}$$

The formulae are more elaborate than the quadrilateral case of equation (1) but very much simpler than the elaborate formula for the circumradius of a general cyclic hexagon.

1.4 Radic's formulation and notation

Radic's formulation [8, 10] is here very slightly modified as we take the inradius $\rho = 1$. Let T_j be the tangent length at vertex j. With the angles at the vertices denoted by α_j we have $T_j = \cot(\alpha_j/2)$. For bicentric hexagons Radic gives the vector of tangent lengths as

$$\mathbf{T} = \left[T_1, \frac{(T_1+T_3)t_s}{1-T_1T_3}, T_3, \frac{t_s}{T_1}, \frac{1-T_1T_3}{T_1+T_3}, \frac{t_s}{T_3}\right], \tag{3}$$

$$t_s^2 = \frac{T_1T_3 - T_1^2T_3^2}{1+T_1^2+T_1T_3+T_3^2}, \qquad t_s > 0 \text{ (i.e. } 0 < T_1T_3 < 1). \tag{4}$$

(Radic's lower case t_j corresponds to my ρT_j.) The regular hexagon has $T_1 = T_3 = 1/\sqrt{3}$ and $t_s = 1/3$ and hence all entries in $\mathbf{T}$ have $T_j = 1/\sqrt{3}$. The constraints on T_1 and T_3 include that they are positive and $T_1\,T_3 < 1$, and this yields that $0 < t_s \leq 1/3$.

We will not repeat Radic's derivation of (3,4). We observe, as did Radic, that $T_j\,T_{j+3} = t_s$ and this has the immediate consequence that the product of the 6 entries T_j being t_s^3, a fact we record in equation (8). There are several other consequences of equations (3,4), e.g.

$$T_1T_3 + T_3T_5 + T_5T_1 = 1 = T_2T_4 + T_4T_6 + T_6T_2, \quad \frac{T_1T_3T_5}{T_1+T_3+T_5} = t_s^2 = \frac{T_2T_4T_6}{T_2+T_4+T_6},$$

and, perhaps less obviously, equation (7).

If one regards t_s as given, equation (4) is a biquadratic for the pair (T_1, T_3). The biquadratic can be written as $e_2 = 2 + 1/t_s$ with e_k the k-th elementary symmetric polynomial in the T_j.

Radic's calculations lead to a simple relation between t_s and the circumradius R (having set $\rho = 1$),

$$R^2 = \frac{(1-t_s^2)(1+t_s)(1+3t_s)}{16t_s^2}, \tag{5}$$

$$= \frac{(e_2-3)(e_2-1)^2(e_2+1)}{16(e_2-2)^2}. \tag{6}$$

Another quantity treated by Radic is the distance $R_j = |z_j|$ of vertex j from the origin, the incentre. We have $R_j = \operatorname{cosec}(\alpha_j/2)$ and $R_j^2 = 1 + T_j^2$. On using this last equation with (3,4) we have

$$\frac{1}{R_1^2} + \frac{1}{R_3^2} + \frac{1}{R_5^2} = \frac{1}{R_2^2} + \frac{1}{R_4^2} + \frac{1}{R_6^2}. \tag{7}$$

1.5 Elementary symmetric polynomials

More can be said about the various e_k. The semiperimeter, or equivalently the area as $\rho = 1$, is e_1. That the sum of the angles of the hexagon is 4π accords with

$$e_1 - e_3 + e_5 = 0.$$

The elementary symmetric polynomials $e_j(\mathbf{T})$, besides satisfying the angle constraint $e_1 - e_3 + e_5 = 0$, satisfy

$$e_6 = t_s^3, \quad e_4 = 1 + 2\,t_s, \quad e_2 = 2 + \frac{1}{t_s}, \text{ and } e_5 = t_s^2 e_1, \quad e_3 = (1 + t_s^2)e_1. \tag{8}$$

This enables all the e_j to be expressed in terms of e_1 and e_2:

$$e_3 = \left(1 + \frac{1}{(e_2 - 2)^2}\right) e_1,\ e_4 = \frac{e_2}{e_2 - 2},\ e_5 = \frac{e_1}{(e_2 - 2)^2},\ e_6 = \frac{1}{(e_2 - 2)^3}. \tag{9}$$

We now have as invariants t_s, e_2, e_4, e_6 and e_3/e_1, e_5/e_1.

1.5.1 Palindromic polynomials

As e_j have been, at least since Newton, inextricably linked with questions associated with polynomials, we briefly note here – with the application to our problems being that given e_1 and e_2 (area and circumradius along with $\rho = 1$) one can find the T_j from a palindromic equation. This is very obvious for bicentric quadrilaterals as one has $e_1 = e_3$ and $e_4 = 1$ so the equation satisfied by the T_j is

$$T^4 - e_1 T^3 + e_2 T^2 - e_3 T + 1 = 0 \ \text{ for bicentric quadrilaterals.}$$

For bicentric hexagons the equation satisfied by the T_j is

$$T^6 - e_1 T^5 + e_2 T^4 - \frac{(5 - 4e_2 + e_2^2)e_1}{(e_2 - 2)^2} T^3 + \frac{e_2}{e_2 - 2} T^2 - \frac{e_1}{(e_2 - 2)^2} T + \frac{1}{(e_2 - 2)^3} = 0.$$

From T define $\hat{T} = T\sqrt{e_2 - 2}$. Then $\hat{T}$ satisfies a palindromic equation, which one solves (reducing the problem to solving a cubic in u) with the substitution $u = \hat{T} + 1/\hat{T}$. This accords with that, if T_j is a tangent length, then so is $T_{j+3} = t_s/T_j$. In any event, specifying the bicentric hexagon with $\rho = 1$ through e_2 giving the circumradius and other invariants, and the area/semi-perimeter e_1 to give particular instances, seems natural with the additional reassurance that (e_1, e_2) gives a solvable sextic for the tangent lengths.

1.5.2 Biquadratic equations

This is an aside from our main aim of presenting invariants. However the readership of ATCM have diverse backgrounds so some mention of the widely applied biquadratic equations is perhaps justified. From equations (4) and (8) we have the symmetric biquadratic equation

$$1 + T_1^2 + T_3^2 - (-3 + e_2)(-1 + e_2)T_1 T_3 + (-2 + e_2)^2 T_1^2 T_3^2 = 0. \tag{10}$$

The Poncelet traverses, geometrically visualised as in the MathWorld animation, map 1-1 and onto to points (T_1, T_3) moving around the level curve (10). There are Jacobian elliptic function parametrisations of the level curve, due to Jacobi and Cayley. (Amongst the many applications of biquadratic equations we mention a couple. They are used in connection with QRT-maps in the theory of discrete integrable systems. They are also used to provide solutions of the Yang-Baxter equation.)

1.6 Distance d between the incentre and circumcentre

With the inradius ρ set to one, Fuss's equation is

$$3(R^2 - d^2)^4 - 4(R^2 + d^2)\,(R^2 - d^2)^2 - 16R^2d^2 = 0.$$

See, for example, [1, 4, 5]. A short calculation verifies that with R^2 given by (5) or by (6)

$$d^2 = \frac{(-5+e_2)(-3+e_2)^2(-1+e_2)}{16(-2+e_2)^2} = \frac{(1-t_s)^2(1+t_s)(1-3t_s)}{16t_s^2},$$

satisfies Fuss's equation.

1.7 First invariants involving the sides

Our main results later in this paper concern invariants related to geometric quantities, distances and angles associated with the bicentric hexagon. Those in this subsection are well-known.

The distance matrix DM is the matrix with entries $d_{i,j}$ and for any cyclic n-gon, at least with $3 \le n \le 6$, the determinant of DM is a numerical factor times the product of the side lengths $d_{j,j+1}$. See Theorem 2. With the inradius $\rho = 1$ we have $d_{j,j+1} = T_j + T_{j+1}$ and using (3,4) we have, concerning the sides,

$$d_{12} + d_{34} + d_{56} = e_1 = d_{23} + d_{45} + d_{16} \quad \text{and} \quad d_{12}\,d_{34}\,d_{56} = t_s\,(1+t_s)e_1 = d_{23}\,d_{45}\,d_{16}. \tag{11}$$

Thus the quantity $d_{12}\,d_{34}\,d_{56}/(d_{12} + d_{34} + d_{56})$ is invariant. In other words, the product of the sides divided by the square of the perimeter is invariant.

A classical result concerning the principal diagonals of cyclic hexagons is Fuhrmann's Theorem which we have used to check some calculations.

$$d_{12}\,d_{45}\,d_{36} + d_{23}\,d_{56}\,d_{14} + d_{34}\,d_{16}\,d_{25} + d_{12}\,d_{34}\,d_{56} + d_{23}\,d_{45}\,d_{16} - d_{14}\,d_{25}\,d_{36} = 0. \tag{12}$$

We will revisit this in §3.1:

1.8 Examples of bicentric hexagons

Once one has fixed ($\rho = 1$ and) R, equivalently e_2, if one wishes to find numeric values of some invariant, it suffices to evaluate it at some particular bicentric hexagon with that e_2. Favourite examples for this are shown in Figure 1.

For these bicentric hexagons at given $e_2 \ge 5$ (i.e. given R), as shown by Radic [9] (see his Figures 3.1a and 3.1b), the area (semiperimeter) e_1 takes extreme values for the hexagons shown in Figure 1. We can also plot how these bounds on e_1 change as e_2 changes, and note, in particular, as $e_2 \to 5 = e_{2*}$ that the allowed interval gets smaller and both bounds tend to $2\sqrt{3} = e_{1*}$. Thus, for bicentric hexagons,

$$12 + 4(e_2 - 5) = 4e_2 - 8 \le e_1^2 \le \frac{(e_2-3)(e_2+1)^3}{4(e_2-2)^2} = 12 + (e_2 - 5) + \frac{(e_2-5)^3(e_2-1)}{4(e_2-2)^2}. \tag{13}$$

The two formulae for the areas of these two bicentric hexagons, presented as functions of R, ρ and d, are given in [7].

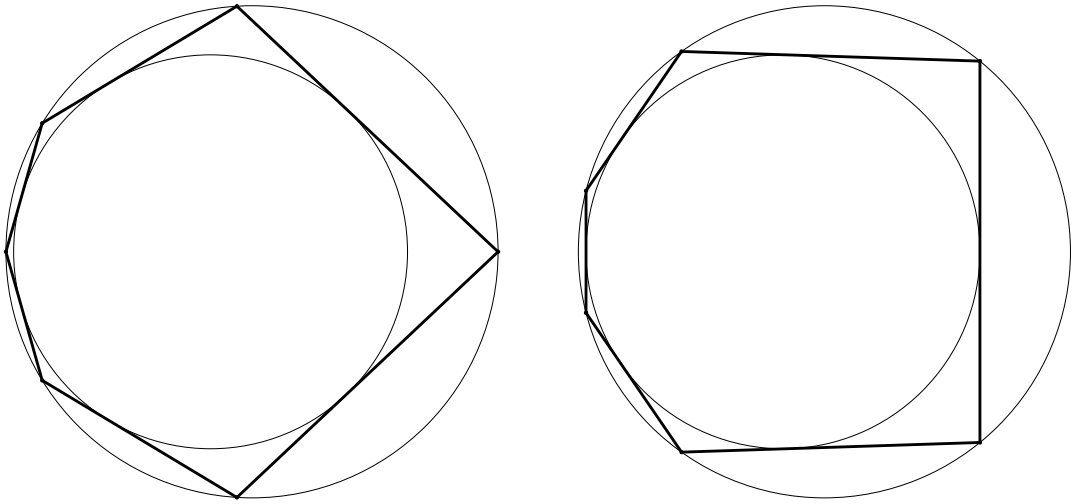

Figure 1: Bicentric hexagons at left with maximum area $e_1 = 3.65198$; at right with minimum area, 3.65148. Here $\rho = 1$ and $t_s = 3/10$, $e_2 = 16/3$.

2 Distance geometry and distance matrices

We do not need any of the various properties of distance matrices, but have used them, via Theorem 2, as checks in our code. As regards work in this paper, consider the distance matrices as convenient record keeping for distances used in the various applications.

2.1 An observation concerning cyclic polygons

Theorem 2 *(i) The determinant of the distance matrix $DM(n)$ for a set of n points on a circle with $3 \leq n \leq 6$ satisfies*

$$\det(DM(n)) = -(-2)^{n-2} \prod_{j=1}^{n} d_{j,j+1}. \tag{14}$$

For a cyclic n-gon the product on the right hand side is the product of the side lengths of the n-gon.
(ii) The Euclidean distance matrix $EDM(n)$, the matrix with entries the squares of those in $DM(n)$, for a cyclic n-gon with $3 \leq n \leq 6$ has rank 3.

At $n = 4$, items (i) and (ii) are equivalent to Ptolomy's Theorem. We have no reason to believe that the restriction $n \leq 6$ is necessary. However the proof we have requires n to be bounded (and within the memory and time limits of our computer and algebra packages). Our proof starts from the z_j on a circle, and continues with brute calculation.

2.2 The distance matrix of a bicentric hexagon

The side lengths are given (when $\rho = 1$) by $d_{j,j+1} = T_j + T_{j+1}$. The "2-diagonals" for a bicentric n-gon with circumradius R are given by $d_{j-1,j+1} = 4RT_j/(1 + T_j^2)$. The principal diagonal d_{14} is found by applying Ptolomy's Theorem to the cyclic quadrilateral with vertices 1, 2, 3 and 4. The

other principal diagonals are found in the same way. We have

$$\begin{aligned} d_{14} &= \frac{d_{13}d_{24} - d_{12}d_{34}}{d_{23}} = \frac{d_{15}d_{46} - d_{16}d_{45}}{d_{56}}, \\ d_{25} &= \frac{d_{15}d_{26} - d_{12}d_{56}}{d_{16}} = \frac{d_{24}d_{35} - d_{45}d_{23}}{d_{34}}, \\ d_{36} &= \frac{d_{13}d_{26} - d_{23}d_{16}}{d_{12}} = \frac{d_{35}d_{46} - d_{34}d_{56}}{d_{45}}. \end{aligned}$$

$$DMbi(6) = \begin{bmatrix} 0 & T_1+T_2 & \frac{4RT_2}{1+T_2^2} & d_{14} & \frac{4RT_6}{1+T_6^2} & T_1+T_6 \\ T_1+T_2 & 0 & T_2+T_3 & \frac{4RT_3}{1+T_3^2} & d_{25} & \frac{4RT_1}{1+T_1^2} \\ \frac{4RT_2}{1+T_2^2} & T_2+T_3 & 0 & T_3+T_4 & \frac{4RT_4}{1+T_4^2} & d_{36} \\ d_{14} & \frac{4RT_3}{1+T_2^3} & T_3+T_4 & 0 & T_4+T_5 & \frac{4RT_5}{1+T_2^5} \\ \frac{4RT_6}{1+T_6^2} & d_{25} & \frac{4RT_4}{1+T_4^2} & T_4+T_5 & 0 & T_5+T_6 \\ T_1+T_6 & \frac{4RT_1}{1+T_1^2} & d_{36} & \frac{4RT_5}{1+T_2^5} & T_5+T_6 & 0 \end{bmatrix}.$$

One check we have performed is substituting the values we have, on using (3,4) in $d_{j,j+1}$ and $d_{j-1,j+1}$ in the above, and then using these values, and (5), in Fuhrmann's equation (12) we find the Fuhrmann equation is satisfied.

3 Invariants

3.1 Preliminary notes on invariants

We begin with a brief review of the setting. The inradius $\rho = 1$ and circumradius $R \geq 2/\sqrt{3}$ are given, and we suppose the incentre is at the origin of coordinates and the circumcentre is at $(d, 0)$ where the invariant d is given by Fuss's equation. The circumradius R of a bicentric hexagon is related to e_2 or t_s by equation (5).

We already have e_4 and e_6 and e_3/e_1 and e_5/e_1 as invariants.

Points with stay fixed during a Poncelet traverse are of interest. Results, obtained by others, include the following.

For an even-sided [bicentric] polygon, the diagonals are concurrent at the limiting point of the two circles. This is treated at the following pages.
`https://math.stackexchange.com/questions/3604700/the-concurrency-property-of-the-diagonals-sem`
`https://mathworld.wolfram.com/LimitingPoint.html`
The concurrency point of the diagonals is invariant, and is on the line joining the incentre and circumcentre. The stackexchange page gives references for proofs:
Lachlan, Modern Pure Geometry, 1893, pg 217, Ex. 4.
Halbeisen and Norbert, A Simple Proof of Poncelet's Theorem, 2014, Theorems 4.1,4.2.

The mathworld page gives the distance x' of the limiting point from the circumcentre as

$$x' = \frac{d^2 - \rho^2 + R^2 \pm \sqrt{(d^2 - \rho^2 + R^2)^2 - 4d^2R^2}}{2d}.$$

Special case of Weill's Theorem. *Let ζ_j denote the tangency points. Then the centre of mass $(\sum \zeta_j)/n$ is fixed during a Poncelet traverse.*

The centres of mass of the vertices $(\sum z_j)/n$ traverse a circle, and its centre remains fixed during a Poncelet traverse. See [12]

3.2 Invariants involving $d_{i,j}$ and $\cos(\alpha_j)$

We remark that angle information can be found from distances. With θ_{ikj} denoting the angle between (i,j) and (j,k),

$$\cos(\theta_{ikj}) = \frac{d_{ik}^2 + d_{kj}^2 - d_{ij}^2}{2\, d_{ik}\, d_{kj}}.$$

In particular

$$\cos(\alpha_k) = \frac{d_{k-1,k}^2 + d_{k,k+1}^2 - d_{k-1,k+1}^2}{2\, d_{k-1,k}\, d_{k,k+1}}.$$

3.2.1 Involving just the sides $d_{j,j+1}$

Several results have been stated before. See equation (11) for results in Radic's papers. Here we add the observation that

$$\frac{d_{12}d_{34}d_{56}}{d_{12}+d_{34}+d_{56}} = t_s(1+t_s), \tag{15}$$

is a Poncelet invariant. The proof of the invariance stated in Theorem 1 is just a calculation verifying

$$S = d_{12}d_{45} + d_{23}d_{56} + d_{34}d_{16} = 2(1+3t_s). \tag{16}$$

Let $\mathbf{d} = [d_{j,j+1}]$ and also write $\mathbf{d}^2 = [d_{j,j+1}^2]/$ Consider first elementary symmetric functions of $\mathbf{d} = [d_{j,j+1}]$. We have, from equation (15),

$$e_1(\mathbf{d}) = 2e_1, \qquad \frac{e_6(\mathbf{d})}{e_1^2} = t_s^2(1+t_s)^2.$$

Some other invariants are $e_3(\mathbf{d})/e_1$ and $e_5(\mathbf{d})/e_1$. There are polynomial relations between the $e_j(\mathbf{d})$ but these seem more complicated than those between the $e_j(\mathbf{T})$ given at equations (9).

The quantities $\sigma_j(\mathbf{d}) = e_j(\mathbf{d}^2)$ are used in connection with formulae for the area of cyclic hexagons. See `https://mathworld.wolfram.com/CyclicHexagon.html`
Clearly $\sigma_6(\mathbf{d}) = e_6(\mathbf{d})^2$.

There are easier results coming from not restricting attention to just sides, but considering diagonals too.

3.2.2 Involving also the diagonals $d_{j,j+2}$ and $d_{j,j+3}$

Theorem 3 *The 2-diagonals of a bicentric hexagon satisfy:*

$$\frac{\sum d_{j-1,j+1}}{\sum d_{j,j+1}} = \frac{\sum d_{j-1,j+1}}{2e_1} = \frac{4t_sR}{1-t_s^2}, \tag{17}$$

$$\prod d_{j-1,j+1} = (4R)^6 \frac{t_s^5}{(1-t_s^2)^4}. \tag{18}$$

Theorem 4 *The principal diagonals of a bicentric hexagon satisfy*

$$\frac{\sum_{j=1}^3 d_{j,j+3}}{\sum_{j=1}^6 d_{j,j+1}} = \frac{\sum_{j=1}^3 d_{j,j+3}}{2e_1} = \frac{1+t_s}{2(1-t_s)}, \tag{19}$$

$$X = \frac{d_{14}\, d_{25}\, d_{36}}{e_1} = \frac{(1+t_s)^3}{1-t_s}. \tag{20}$$

Use Fuhrmann's Theorem (12) and then equation (15) and (20) to give

$$\frac{d_{12}\, d_{45}\, d_{36} + d_{23}\, d_{56}\, d_{14} + d_{34}\, d_{16}\, d_{25}}{e_1} = X - 2t_s(1+t_s) = \frac{(1+t_s)(1+3t_s^2)}{1-t_s}.$$

3.3 Invariants involving R_j and $\cos(\alpha_j)$

We have, for any tangential n-gon,

$$\cos(\alpha_j) = 1 - \frac{2}{R_j^2}, \qquad R_j^2 = \frac{2}{1-\cos(\alpha_j)} = \frac{1}{\sin(\alpha_j/2)^2} = 1 + T_j^2.$$

Specializing to $n = 6$,

$$\sum_{j=1}^{6} \frac{1}{R_j^2} = \frac{e_5(\mathbf{R}^2)}{e_6(\mathbf{R}^2)} = \frac{2(-3+2e_2-e_4)}{-1+e_2-e_4+e_6}. \tag{21}$$

From this and equations (7) and (9) we have, for a bicentric hexagon,

$$\frac{1}{R_1^2} + \frac{1}{R_3^2} + \frac{1}{R_5^2} = \frac{2}{1-t_s^2} = \frac{1}{R_2^2} + \frac{1}{R_4^2} + \frac{1}{R_6^2}. \tag{22}$$

3.4 The circumradius in terms of $d_{i,j}$

Just as in the case of the bicentric quadrilateral, the circumradius-squared can be expressed in terms of the sum of products of opposite sides, S as in equation (16). This equation gives $t_s = (S-2)/6$ and this in equation (5) gives, with $2 < S \leq 4$, the equation (2) stated in the Introduction.

One can find other formulae for R^2 using different combinations of the d_{ij}, e.g. from equations (5,15,18).

3.5 More invariants involving vertex angles α_j

$$\begin{aligned} \sum \cos(\alpha_j) &= 6 - \frac{8}{1-t_s^2} = -\frac{2(1+3t_s^2)}{1-t_s^2}, \\ &= -3 + \frac{(e_2-5)(e_2+1)}{(e_2-3)(e_2-1)}. \end{aligned}$$

$$\sum \cos(\alpha_j + \alpha_{j+1}) = 6 - \frac{4(1+3t_s)}{1-t_s^2}.$$

$$\sum \cos(\alpha_j + \alpha_{j+1} + \alpha_{j+2}) = 6 + \frac{8((-1+5t_s^2) + t_s(1+3t_s^2))}{(1-t_s^2)^2}.$$

More general results, for bicentric n-gons, not merely hexagons, are given in [11].

4 Conclusion

Our conclusion is completely consistent with one of ATCMs themes: that Computer Algebra is immensely useful in both checking the impressive algebraic results of past generations and also for discovering, possibly new, results, for example, our theorems. Questions involving elliptic billiards, and rays, sometimes are similar to, or related to, the bicentric polygon work, e.g. [15, 14] and [6, 11, 3].

There are, of course, many open questions. There are results for triangles and bicentric quadrilaterals aligning with those presented here for bicentric hexagons. However we do not have them for bicentric pentagons, geometries presented, at least in special cases, at earlier ATCMs, see [13].

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Mathematica for checking the results is currently available via links at
`https://sites.google.com/site/keadyperthunis/home/papers`

Exploring Conic Sections through Desmos Artworks: Where Math, Art, and Technology Combine

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Abstract: *This study explores the use of Desmos in creating artworks out of the four conic sections discussed in Pre-calculus, namely: circle, parabola, ellipse, and hyperbola. This performance task was designed to give senior high school students an opportunity to integrate their creativity in arts and their knowledge of conic sections to produce their own artworks using a graphing calculator. Description of the performance task, concepts underlying its design, and examples of Desmos artworks produced by Grade 11 students are presented in this paper. Moving forward, mathematics educators could use this information as a springboard for designing performance tasks that integrate mathematics, art, and technology.*

1. Introduction

It has been widely recognized that the disciplines of mathematics and arts are interrelated, despite them seemingly being opposites. This relationship was argued to exist long ago, as evidenced by the incorporation of symmetry and patterns in the design of ancient structures and figures [1]. To this day, this connection between mathematics and arts persists even with the emergence of new technologies, such as computers. Due to the fast-expanding potential and use of relevant technological tools, adding this third dimension to the equation has therefore gained traction among researchers.

Various computer software is now being utilized as a medium for creating art and as an exploratory tool for facilitating the understanding of mathematical concepts [2]. For instance, the use of graphing software to support classroom activities in mathematics has grown interest among mathematics educators because of its capability to generate graphs of functions quickly and accurately, thereby improving the pace at which related topics can be explored [3]. Considering the advantages of graphing software in facilitating the exploration of certain mathematical concepts, this study therefore aims to investigate the use of a graphing software called Desmos in understanding conic sections. In particular, students were given the opportunity to explore the equations and graphs of the four conic sections, namely: circle, parabola, ellipse, and hyperbola, by creating artworks out of these mathematical objects on Desmos.

2. Creativity and Collaboration in the Four-C Model

The Framework for 21st Century Learning identifies creativity, critical thinking, communication, and collaboration as essential skills for the 21st century [4]. In order to prepare students for the future, it is important that these skills, which are collectively referred to as the Four Cs, are integrated into the teaching of core academic subjects, such as mathematics. The Four-C model therefore encourages mathematics educators to design learning activities that promote creative problem-solving among students and develop their communication and collaboration skills.

The discussion presented in this paper focuses mainly on creativity and collaboration. Creativity was described in terms of three distinct features [5]: First, creativity must be something new, novel, or original. Second, creativity must involve a combination of two or more concepts that have never been combined before by the individual. Third, creativity must be expressed in a certain way in the world. Aligned with these features, the Four-C Model describes creativity as a skill that entails the creation and refining of new and worthwhile ideas using a wide array of idea creation strategies, such as brainstorming [4]. It also highlights the importance of working creatively with others to give students the opportunity to communicate and develop these ideas with their peers, pool their knowledge together for a common goal, and incorporate new and diverse ideas into the work. Furthermore, collaboration allows students to value the contributions made by each individual and assume shared responsibility for their work, thereby increasing their engagement in the learning process. As similarly described by [6], when students collaborate, they exercise a shared agency, repair their shared understanding, and build their output together by suggesting ideas, listening to one another, accepting similar interpretations, and resolving conflicting ones. Such collaboration is said to be important in fostering students' learning of mathematics.

Interest in creativity in mathematics has been reignited in the past years due to the rapidly growing use of relevant technologies, such as computers, which have made it possible for the invisible to become visible [7, 8]. That is, abstract concepts, which were once inaccessible, can now be easily visualized through the use of computers and various software. Moreover, the emergence of collaborative learning technologies, such as online learning spaces that allow students to work on tasks remotely and in real time, has increased the ways through which collaboration can be facilitated and integrated into instructional practices [9]. There is therefore a great potential for incorporating these technologies into the teaching and learning of mathematics, as they can serve as an outlet for creative expression of ideas and as a platform for collaboration by providing the necessary resources that allow students to discuss and build creatively and jointly.

To explore the potential of such technologies, this study therefore looked into the integration of the use of a graphing calculator called Desmos into the discussion of conic sections. This provided the students an opportunity to collaborate and create their own artworks out of the four conic sections discussed in Pre-calculus, namely: circle, parabola, ellipse, and hyperbola. Combining mathematics and arts could balance the learning of students with the combination of creativity and logic [10]. Moreover, the integration of arts enables students to engage in a creative process and demonstrate their understanding of a concept through an art form [11]. The aforementioned task therefore aims to enrich the students' learning of the graphs and equations of the four conic sections by allowing them to express their creativity, collaborate with their peers, and demonstrate their understanding of the lesson through a Desmos artwork.

3. Desmos Artwork as a Performance Task

In the Philippines, the K-12 Basic Education Program follows a standards- and competency-based grading system in which students from Grades 1 to 12 are graded based on their written work, quarterly assessment, and performance task [12]. Of these components, performance task bears the highest weight, which is 40 to 50% of a student's grade. Performance tasks are concrete products or performance-based activities that serve as evidence of learning. They are given by teachers every academic quarter to enable the students to demonstrate and integrate their knowledge and understanding of the topics discussed during the quarter in various forms, either individually or in collaboration with their peers. For instance, performance tasks may be given in the form of portfolios, hands-on exercises, multimedia presentations, and research projects, among others. Since the form in which performance tasks may be given is flexible, teachers could tap the students' knowledge and skills within or across subject areas and maximize the use of available resources to support them in demonstrating what they have learned so far.

In this performance task, the students were instructed to work in pairs and create their own artworks on Desmos using the four conic sections discussed in Pre-calculus, namely: circle, parabola, ellipse, and hyperbola. They were also allowed to include graphs of other functions that they were already familiar with to enable them to integrate their previous learning into the current task. The students were therefore expected to work hand in hand with their partners by sharing, developing, and implementing their ideas together on how they should incorporate the graphs of the four conic sections and other functions, if any, into the artworks that they intended to create. The final output should include their artwork and the equations used, which must be categorized into the four conic sections.

In order to help the students proceed with the performance task, the teacher first allotted a portion of the class period to introduce the basic features of Desmos, such as its ability to plot points, graph functions, and add sliders, among others. The students were then encouraged to explore Desmos on their own asynchronously so that they could familiarize themselves with the aforementioned features. The students who participated in an interview claimed that it took them at least two hours to a day of exploring and practicing before they became accustomed to using Desmos.

It is also important to highlight that this performance task was given during the COVID-19 pandemic. Since face-to-face classes in the Philippines were suspended for an indefinite time to prevent the spread of the COVID-19 virus, schools were forced to migrate to online platforms to support learning continuity in the midst of the pandemic. It was then recommended that technological tools be maximized and integrated into lessons and tasks to reinforce student learning despite the fully online setup. This performance task, which involved the use of Desmos, was therefore given to the students so that they could work together on the artwork with their partners in real time, albeit virtually.

4. Assessment Criteria

The Desmos artworks submitted by the students were assessed using the criteria shown in Table 1.

Table 1
Assessment Criteria for the Performance Task

Features	Percentage (%)
Originality and creativity of artwork	40
Completeness of graphs	25
Accuracy of equations	25
Punctuality	10
Total	100

For the first component, the artworks were assessed based on their originality and creativity. That is, the artwork must be an original creation by the students and must showcase their creative or imaginative ideas. This bears the highest weight of all the components, which is 40% of the students' final score. As for the second component and third component, all four conic sections must be present in each artwork, and the equations representing these conic sections must be accurate. Each of these two components constitutes 25% of the final score. Lastly, the students must be able to submit the performance task on or before the deadline set by the teacher. The students were given four (4) weeks to work on this performance task. This constitutes 10% of the final score.

5. Students' Desmos Artworks

This section presents two examples of Desmos artworks created by Grade 11 students as part of their performance task in Pre-calculus. The students were from a private senior high school situated in the National Capital Region (NCR) of the Philippines and were enrolled under the Science, Technology, Engineering, and Mathematics (STEM) strand.

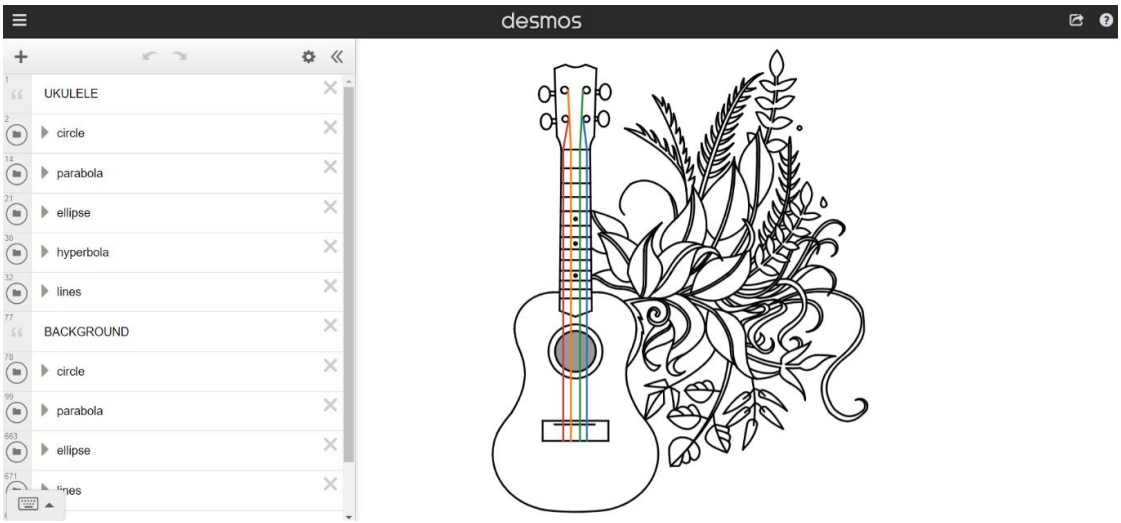

Figure 4.1 Screenshot of the first Desmos artwork

Figure 4.1 shows a screenshot of the Desmos artwork created by one pair of Grade 11 students from the aforementioned school. It features an image of a ukulele with various types of leaves as the

background. On the left side of the screen, equations of the graphs of the four conic sections and linear functions used to create the artwork were listed. Examples of the graphs present in the artwork are the following: a circle was used to create the sound hole of the ukulele; ellipses were used to create the turning pegs; and lines were used to create the strings.

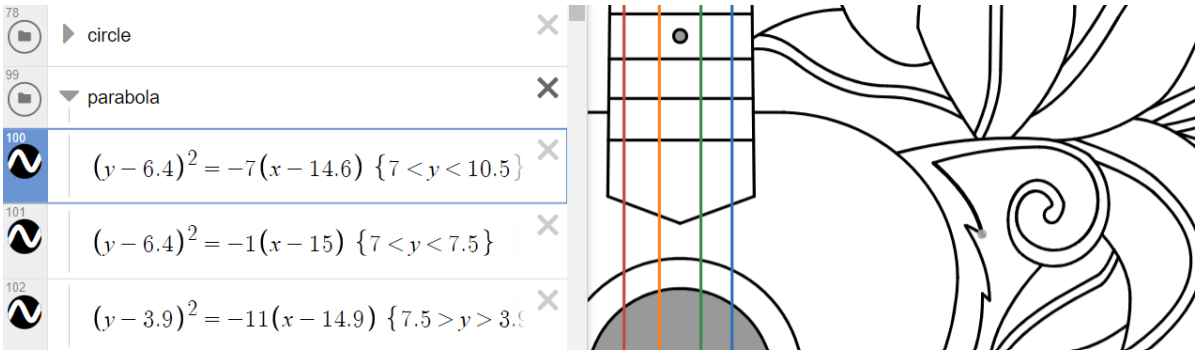

Figure 4.2 Screenshot of some equations and inequalities used in the first Desmos artwork

The concept of inequalities was also used in defining the domain and range of the graphs in order to create bounded graphs and form the students' desired images. An example of such a case is shown in Figure 4.2, where some of the equations used by the students to create the leaves in the background were listed. As shown, the range of the graph of the selected parabola was restricted, which is why only a segment of the parabola appeared in the image on the right. It is interesting to note that the concept of inequalities was not part of the teaching content in Pre-calculus. It is therefore assumed that it was a lesson that they had previously learned and effectively integrated into the current task.

Figure 4.3 Screenshot of the second Desmos artwork

Figure 4.3 shows a screenshot of another Desmos artwork, which is inspired by the animated film "Up." Similar to the first artwork, the equations of the graphs of the four conic sections were

listed on the left side of the screen. This time, the students used linear functions and sine functions to complete their artwork. Some of the graphs present in the artwork are the following: circles were used to form the balloons; ellipses were used to form the leaves; and sine functions were used to form the strings of the balloons. The concept of inequalities was also used in defining the domain and range of the graphs in order to create the students' desired images and to add colour to certain areas. For instance, the bounds of the graphs of the parabola were defined by adding domain and range restrictions to each equation in order to form the roof tiles. The area inside the biggest circle to which blue colour was added was likewise defined using inequalities.

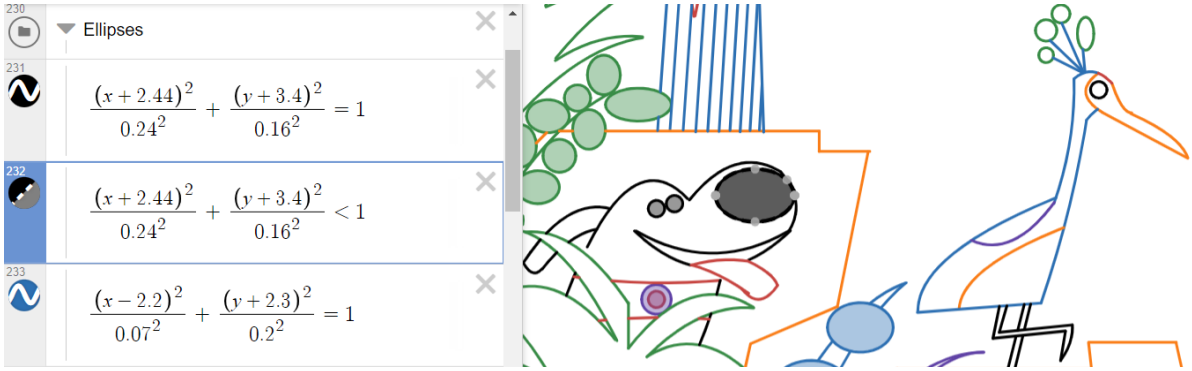

Figure 4.4 Screenshot of some equations and inequalities used in the second Desmos artwork

Figure 4.4 shows some of the equations and inequalities used by the students to complete the artwork. The first equation shown is an equation of an ellipse, which corresponds to the nose of the dog in the artwork. The inequality below this equation corresponds to the colour added to the area inside the ellipse.

An interview with select students was subsequently carried out to gather their insights about having the Desmos artwork as their performance task. Among the difficulties encountered by the students who were interviewed include the need to modify equations on a trial-and-error basis to achieve their desired graphs and restrictions on the design due to their limited knowledge of other graphs, which were not covered in the current lesson but could have been a helpful addition to their artwork. Despite these difficulties, the students also had positive responses towards the task. According to them, understanding the graphs of the conic sections through the equations alone was really challenging, but the task helped them visualize the graphs of the conic sections and their transformations more clearly. They also realized through the task that many objects from their everyday lives can be modelled using conic sections.

Overall, this performance task opened up an opportunity for the students to explore the graphs and equations of the four conic sections in an online collaborative environment and find solutions among themselves to the difficulties they encountered while accomplishing the task. It also helped the students visualize and establish the relationships between the equations and their graphs, specifically how modifying the equations could transform the parent graphs, whether by translating, resizing, reflecting, or a combination of these transformations, so that the transformed graphs would suit their artwork. This process where the students had to manipulate the equations on a trial-and-error basis can be considered a productive learning experience as it allowed the students to observe, reflect, and make changes reiteratively until their desired artwork was achieved. Moreover, as

claimed by [13], utilizing technological tools, such as Desmos, in mathematics provides students with unique ways of engaging in the learning process by allowing them to make observations and conduct experiments. Lastly, the performance task allowed the students to make connections between the four conic sections and various objects found in out-of-the-classroom contexts, thereby enriching their learning of the current lesson. According to [14], enabling students to visualize mathematical concepts and their application using technology can increase student motivation and accessibility to understanding these concepts.

6. Conclusion

This study presented a description of a performance task that involves the creation of artworks on Desmos using conic sections, as well as concepts underlying its design and sample artworks created by the students. The design of this performance task illustrates how technological tools, such as Desmos, can be integrated into the teaching and learning of mathematics to facilitate creativity and collaboration among students. Both creativity and collaboration constitute the Four-C model, which encourages teachers to design learning activities that aim to develop essential skills for the 21st century among the students. In addition, considering the fast-expanding use of technological tools and the implementation of hybrid learning, it is wise to design activities that maximize the potential of such resources in facilitating students' understanding of mathematical concepts, whether classes are conducted in person or virtually. This study could therefore serve as a springboard for designing other performance tasks that incorporate mathematics, arts, and technology.

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Impact on Student Learning Outcomes in Mathematics Using GeoGebra

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Abstract: *In recent years, dynamic geometry software tools for teaching and learning in mathematics have been an integral part of a technology-based learning environment and become ubiquitous in educational institutions. The learning strategy assisted by these mathematics software tools allows students to better understand abstract contents with symbolic expressions through computer visualization with interactive and dynamic graphical representation. This paper presents the study of the effect of using dynamic geometry software, GeoGebra, on student performance and learning skills of geometrical concepts in mathematics teaching. The study was conducted for two groups with the same learning contents but different teaching styles for lectures with and without using GeoGebra software. The collected data from the students' learning records, including post-test scores, were analyzed, focusing on two categories of symbolic calculations: graphs and inequalities. The performance of two teaching styles was evaluated based on the student achievements on paper-based quizzes, considering the number of retrials and the success rates. This study also investigates the impact of using GeoGebra on student learning skills by comparing score distributions for symbolic calculations and graphs and inequalities categories among three different academic years. The results show that integrating mathematics visualization software in teaching algebra can significantly improve student learning outcomes, providing higher post-test average scores when compared with the conventional teaching style. The results also reveal that using GeoGebra as a teaching tool considerably enhances students' graphing skills and understanding of geometrical concepts in mathematics, particularly in graphing functions and solving inequalities. In addition, the effectiveness of using GeoGebra for improving mathematics learning outcomes for low-performing students has been confirmed with a higher success rate and lower number of retrials for paper-based quizzes. Thus, early intervention in using GeoGebra to enhance students' visualization and graphing skills might be helpful, particularly for graphing functions and solving inequalities.*

1. Introduction

Engineers use mathematics as their everyday tool and many engineering subjects, especially in Electrical engineering, heavily depend on the learners' understanding and manipulating skills of symbolic expressions to proceed with the lessons. Naturally, the students must master mathematics before their journey of engineering education. However, there are not few engineering students who are not ready in actual situations, and their performance is poor. The poor performance was suspected to be the direct result of a 'black box' approach [1]. It suggests that the students tended to memorize a series of formulas and manipulate procedures without examining their features or considering the connections between the symbolic representations and the features of graphic objects. Their learning is likely to be dedicated towards memorizing shallow procedural knowledge and possibly abandoning conceptual understanding.

The target of this work is the engineering students in KOSEN*. The Electrical engineering department in a Japanese KOSEN had introduced engineering mathematics courses in 1995 academic year (AY) and enforced students' solving skills of algebra and calculus as their preparation for learning electrical theories such as Electrical Circuits. In the courses, students solved quizzes and wrote the solutions in detail on paper, so the others could read and understand their solving processes. Writing their solutions on blank paper also helped the learners to clarify their thinking processes including the logics they used and the questions they had on the way and helped them find mistakes and misconceptions.

In the courses, the quizzes were first given to all the students in a lesson, and if the score didn't reach the acceptable level (75% in this course), the students were allowed to challenge similar questions (retrials) in the following weeks until they succeeded in solving them. The lower the initial score, the more retrials there are to achieve mastery. The department decided that intense effort was necessary to build the foundation for learning engineering subjects, even though the workload of the teachers and students was reasonably high compared to the other courses. The improved students' performance in the two-year courses was confirmed in the standard mathematics examination of the institution in 1999 AY [2]. The department also reported fewer low-performing students in the following engineering subjects and fewer dropouts after 1998 AY [3], and the courses kept the combination of quizzes and interval retrials for 20 years.

However, paper-based quizzes were not popular among some students especially when the given questions didn't match with the students' knowledge and skills. The mismatched questions gave the unprepared students a hard time and hopelessness. The extent of mismatch widened when the students' content knowledge was shallow, such as several formulas as black boxes, which tended to evaporate in a few weeks after the examinations. Effective teaching and learning approaches were expected to guide students to be ready to stand at the start line before the paper-based quizzes.

According to the analysis [2], the average students improved their knowledge and skills well in symbolic calculations but more slowly in functions and geometry. For example, relating a symbolic expression of a function to its graph took more work for average students than the instructor expected. Also, the explanations in lectures and students' practice in quizzes in these fields still had a gap.

To deepen the student's understanding of mathematics, the link between conceptual understanding and procedural knowledge is necessary, and the MODEM (Model construction for didactic and empirical problems of mathematics education) framework [4] proposed to include concept-building steps (concept orientation, definition, identification, production, and reinforcement) in learning activities. In the concept-building steps of the MODEM approach between concrete applications and abstract mathematical ideas, visualization [5], visual thinking [6, p.89], or visual reasoning [7] plays an important role. Visual reasoning is based on visual interpretations and the construction of logical inference chains. Visualization makes it possible to perceive abstract mathematical objects through the senses. Visual representations can be considered more concrete than analytic ones because they are based on external objects. Analytic reasoning is often exact and detailed, but visual reasoning is needed to reveal wider trends of the whole problem-solving process and holistic features of the problem situation. The inclusion of visualization into the teaching process by increasing geometrical rather than algebraic presentation enhanced university students' conceptual learning of vector spaces [8, 9]. Kllogjeri [10] argues mathematical software GeoGebra's role is 'teaching the "feel" to the students (teach them having an eye or sense how to get from "here" to "there")'. The research suggests that geometrical structures helped the students' meaningful and conceptual learning.

However, many students had difficulties analyzing visual representations, so they could not utilize them in problem-solving [11]. Zazkis [5] observed the value of the interrelationship between the visual and analytic approaches from their interviews with 32 university students in their first abstract algebra course. In their proposed model of students' development, novice students' acts of visualization and analysis were separate and quite different, but both kinds of thinking became more and more deeply interiorized within the mind of the individual until finally, analysis and visual understandings were synthesized so that it could become very hard for the individual to distinguish between them.

Nishizawa [12] incorporated a visualization method using GeoGebra activities to teach 3D vector equations to engineering students. They found that when students had the opportunity to manipulate 3D graphic objects in virtual space and observe how symbolic equations change their parameters

interactively, they recognized the relations between symbolic and graphic representations more easily provided the graphic effects of symbolic functions, such as addition or scalar product, were familiar to them. However, the impact of the interactive change diminished when the graphic effect of symbolic function was not familiar to them in such a case as an inner product.

In this paper, we seek to extend the visualization method to the learning of algebra and precalculus using graphic objects more familiar to the students: 2D graphs of functions.

* KOSEN is a unique Engineering educational institution based in Japan, and 50 colleges have a history of over 60 years. In KOSENs, the students start their Engineering education at fifteen, the same age as senior high school students, along with general education such as mathematics, sciences, social sciences, and languages. Japanese manufacturing companies evaluate KOSEN graduates' engineering knowledge and skills equally or more highly than average bachelors from Japanese engineering universities. We now have Japanese-style KOSENs in Thailand.

2. Methods

2.1 The effect of students' individual works with GeoGebra

We compared the performance of student groups that had received different teaching styles (Table 1) in the first semester of the first year of KOSEN in a post-test conducted in the first month of the second semester. The post-test consisted of several review questions from the first semester.

Table 1 Teaching styles

Academic year	Group A (2011, 13, 15-19 AYs)	Group B (2020 AY)
Number of students	$39-44\ (42.1 \pm 1.6)$	46
Teaching style	Face-to-face lectures and paper-based quizzes with retrial opportunities throughout the semester	Remote lecture and independent work with GeoGebra (4 months) + paper-based quizzes without retrials (one week)

Group A had seven sub-groups, each consisting of the first-year students in each academic year, as shown in Table 1. They did paper-based quizzes every week throughout the first semester. Each quiz was given to the students during a face-to-face lesson, and if the score didn't reach the 75% level, the students were allowed to challenge similar questions (retrials) in the after-school sessions in the following weeks. Through these quizzes in lessons and after-school sessions, students had plentiful opportunities to solve questions and to explain their thoughts on paper.

Group B was the experimental group, where the students were forced to join remote lessons for four months in the first semester because of the COVID-19 pandemic. Instead of the paper-based quizzes, Group B students were assigned individual work using GeoGebra during the remote lessons (Table 2). The instructor posed the lecture during the personal work and gave students time to try the tasks individually. After school, teaching assistants tutored students on how to use GeoGebra in remote group sessions. The students only joined the paper-based quizzes in the last week of the first semester when face-to-face lessons restarted, but they didn't have time to sit on retrials.

The learning content in the first semester was the same for Group A and Group B, although the learning styles were different. In the second semester, the content was also the same, and students in all the groups did paper-based quizzes with retrials.

The post-test questions are shown in Table 3. They asked for the skills of symbolic calculations, graphing functions, and solving inequalities, which the students had learned in the first semester. They were review questions from the first semester. The students knew that they were to solve review

quizzes in the first month of the second semester from the course syllabus; however, they didn't know the exact questions prior. The post-test was also a part of paper-based quizzes in the second semester.

Table 2 Example of students' individual work with GeoGebra (Group B)

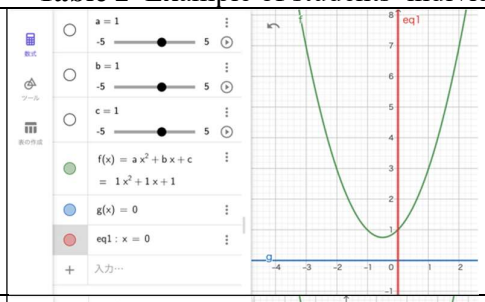
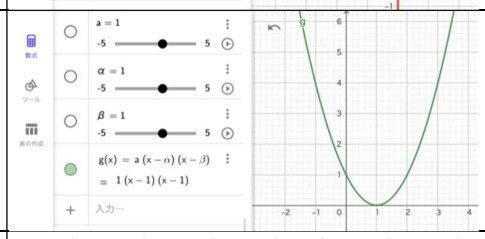
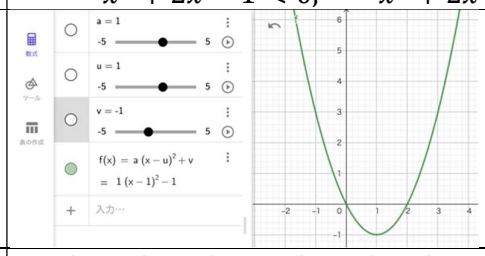
Week 1		Students observe the effect of three parameters a, b, and c in $y = ax^2 + bx + c$. They also confirm the equations of x-axis and y-axis, which they sometimes mistakenly memorize.	15 min.
Week 2		Students observe that the parameters α and β in $y = a(x-\alpha)(x-\beta)$ are closely related to the x-intercepts of the graph.	5 min.
Week 2	Students draw the pair of graphs and solve the inequalities visually $x^2 + 2x - 1 < 0, \quad x^2 + 2x + 1 < 0, \quad x^2 + 2x + 2 < 0$		20 min.
Week 3		Students observe that the parameters u and v in $y = a(x-u)^2 + v$ show the coordinates of the vertex.	10 min.
Week 4	Students draw the graphs and analyze the effect of the parameters $y = \frac{1}{x^3}, \frac{1}{x^4}, \frac{1}{x^5}, \frac{1}{2-x} + 3$		10 min.
Week 5	Students observe the graph of fractional function $y = \frac{x+2}{(x+1)(x-3)}$ and how the curve crosses the x-axis.		10 min.

Table 3 Questions used in the post-test

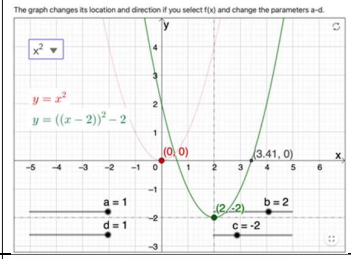
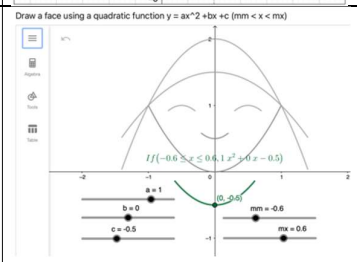
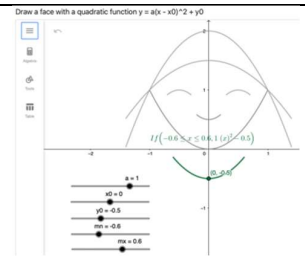
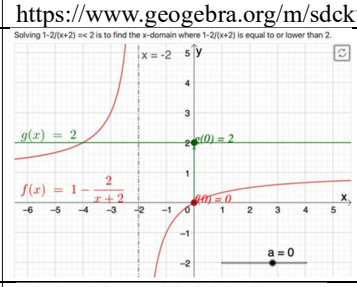
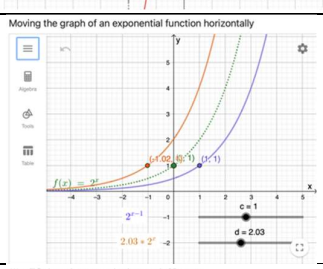
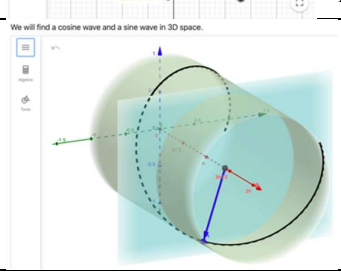
#	Category	Content	Question
1	Graphs and inequalities	Polynomial inequality	Solve $x^4 - 4x^3 + 3x^2 + 2x - 2 \geq 0$
2		Irrational inequality	Solve $8 - x \geq \sqrt{x+4}$
3		Fractional inequality	Solve $\frac{1}{x-2} + 2 \geq x^2$
4	Symbolic calculations	Calculation of powers	Simplify $2^{1/2} \times 5^{-1/2} \times 10^{1/2}$, $\left(a^{1/2} + a^{-1/2+1}\right)\left(a^{1/2} + a^{-1/2+1}\right) \quad (0 < a)$

Then, we examined the learning records of the students and analyzed the influence of the post-test scores on the student's quiz scores in the following semester over time, monthly scores for all the

students and weekly scores for the three lowest-performed students (7%) in Week 8, and the average workload in the second semester. As this course provided more retrial opportunities to the student group with lower initial quiz scores, the student's average workload during the second semester varied from year to year. The average workload was measured by the total number of retrials per student and their success rates through the second semester.

2.2 The effect of GeoGebra activities in face-to-face lessons and assignments

Table 4 Examples of GeoGebra Activities

Week 3		Students observe the effects of four parameters a, b, c, and d in $y = a \cdot f\{d(x-b)\} + c$ with several functions; while b and c move the graph horizontally and vertically, a and d flip the graph vertically and horizontally. https://www.geogebra.org/m/u25bhncu	10 min.
Week 4	 	Students compare the two formulas of a quadratic function and find that one makes it easier to adjust the parameters to draw eyes than the other. https://www.geogebra.org/m/sdckvcww https://www.geogebra.org/m/hr3gaefr	10 min.
Week 5		Students solve an inequality $1 - \frac{2}{x+2} \leq 2$ visually. They confirm that solving means finding the horizontal area where the heights of two graphs satisfy the condition. https://www.geogebra.org/m/aubxeyx6	10 min.
Week 6		Students compare the graphs of exponential functions $y = 2^{x-c}$ (c shifts the graph horizontally) and $y = d \cdot 2^x$ (d expands/shrinks the graph vertically) and confirm both equations can express the same graph. https://www.geogebra.org/m/r2ndd4mf	15 min.
Week 8		Students observe a helix in 3D space and confirm that when they change the viewpoint, the object looks like the graphs of $y = \cos x$, $z = \sin x$, or a unit circle in $y - z$ plane. https://www.geogebra.org/m/anemzjby	15 min.

The effect of tailored GeoGebra activities was examined in the first semester of 2024 AY at the Electrical and Electronics department of a KOSEN in Thailand. The institution started 'Electrical and Information Mathematics 1' course in the first semester of the second year and followed the practice of the Japanese KOSEN, which included paper-based quizzes with retrials, and 31 students joined the course. As the course was held in the second year, it must review the content taught in 'Mathematics 1' one year earlier in the students' first year.

The teaching team also prepared tailored GeoGebra Activities (Table 4) and let the students work with them individually during the lessons. This was to follow the example of Group B described in the previous section (Table 1). The GeoGebra Activities were meant to improve students' graphing skills and encourage them to think visually.

Then, the teaching team gave extra assignments using GeoGebra activities for two low-performing students who failed to increase their graphing quizzes in several retrials during the first month of the semester. Their hand-written solutions showed they still needed help correlating the symbolic expressions of functions with graphs. The extra assignments used GeoGebra activities of Week 3 and 5 in Table 4 and had more detailed instructions and questions. The assignments were meant to give them more time to try the approach at their own pace, which was obviously not enough for them in the limited time during the lessons. The team tracked their weekly quiz score and compared it with the average score of the other 29 students.

3. Results

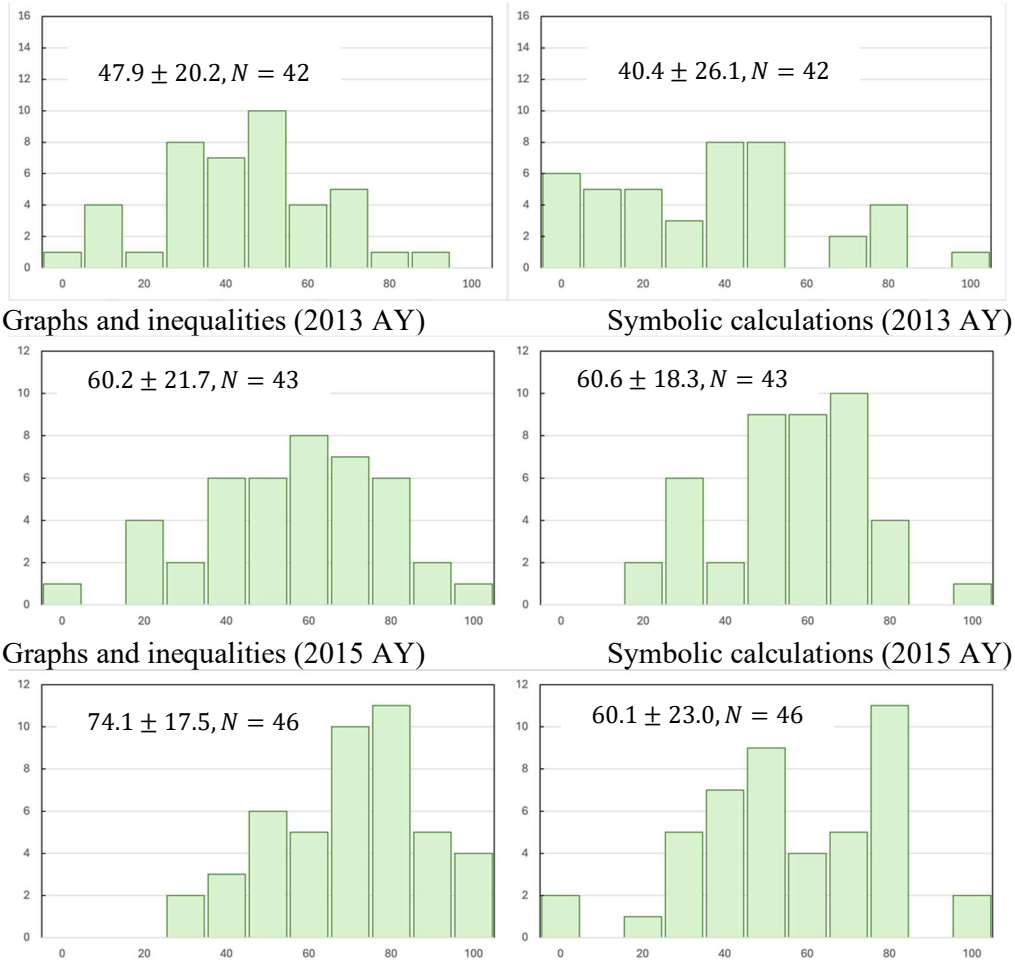
3.1 The effect of students' individual works with GeoGebra

Table 5 shows the average scores of student groups in the post-test. The average scores of Group A varied widely from 45.3% to 60.4%, with the standard deviation of group averages 5.75%. The average score of Group B (2020 AY) was higher than the highest subgroup (2015 AY) among Group A, and the difference between Group B and the average of Group A is 2.85 of the standard deviation of group averages (Group A).

Table 5 Average scores of student groups in the post-test

AY	Group	N	Score		Statistics
			Average	deviation	
2011	A	43	47.7	22.7	Average of the groups: $AV_A = 52.1$ The standard deviation of group averages: $DV_A = 5.75$
2013		42	45.3	19.4	
2015		43	60.4	16.3	
2016		43	53.2	19.6	
2017		44	56.7	19.8	
2018		39	55.2	19.4	
2019		42	46.4	19.4	
2020	B	46	68.5	16.7	$AV_B = 68.5, (AV_B - AV_A)/DV_A = 2.85$

Figure 1 shows the comparison of score distributions in the post-test among Group B (2020 AY), the highest subgroup (2015 AY) in Group A, and the lowest subgroup (2013 AY) in Group A. Scores from two categories: graphs and inequalities, and symbolic calculations are compared. The scores of two subgroups, 2013 AY and 2015 AY in Group A, differed in both categories (or the difference was more extensive in symbolic calculations). However, the score difference between Group B and the 2015 AY subgroup in Group A was mainly the difference in the graphs and inequalities category.

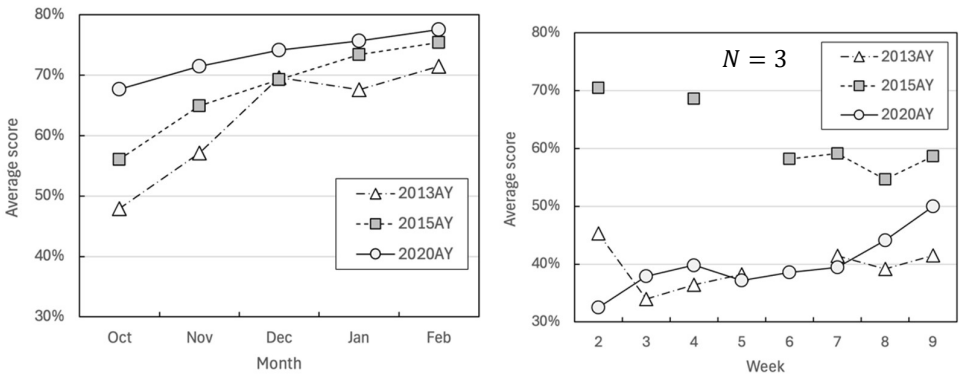


Graphs and inequalities (2013 AY) Symbolic calculations (2013 AY)

Graphs and inequalities (2015 AY) Symbolic calculations (2015 AY)

Graphs and inequalities (2020 AY) Symbolic calculations (2020 AY)

Figure 1 Score distributions in the post-tests



All the students Lowest-performed students in Week 8

Figure 2 Average scores over time in the second semester

Figure 2 shows the average quiz scores over time in the second semester for three groups: the lowest-score group (2013 AY) and the highest-score group (2015 AY) among Group A in the post-test and Group B (2020 AY). For all the students, the 11.7% difference between Group B (2020 AY)

and the 2015 AY subgroup in the first month (October) shrank to 4.8% in the middle of the semester (December) and 2.1% at the end of the semester (February) possibly because of more intense students' effort in after-school retrials through the semester. The 2013 AY subgroup made the hardest effort especially in December to increase the average score. However, the scores of the three lowest-performed students (7% of the students) in Week 8 tend to stagger during the first half (seven to nine weeks) of the semester in every AY-group. They obviously failed many times in the retrials in the former part of the semester (until seventh to nineth week or later) until finally started to increase their total scores in the later part of the semester.

Table 6 shows the intensity of the after-school retrials in the second semester. It shows the average number of retrials per student and the average success rate of the retrials, where success in a retrial required a 75% or higher score. The 2015 AY subgroup needed six more retrials with a 7% lower success rate to reduce the score gap from 11.7% to 2.1% with Group B. The 2013 AY subgroup did 3.5 more trials with a 3% lower success rate than the 2015 AY subgroup in average.

Table 6 Average number of retrials and success rate in the second semester

Group	A (2013 AY)	A (2015 AY)	B (2020 AY)
Retrials / student (success rate)	37.0 (49%)	33.5 (52%)	27.5 (59%)

3.2 The effect of GeoGebra activities in face-to-face lessons and assignments

Figure 3 shows the effect of tailored GeoGebra activities in the first semester of 2024 AY. Their average score in Week 2 is relatively higher than the average scores of Japanese KOSEN in the first month (Figure 2), although comparing their scores with those of former attempts in Japanese KOSEN is difficult because the students' years and educational histories in the former school are not the same.

Figure 3 also shows the weekly average score of the two lowest-performing students who tackled additional assignments using GeoGebra activities after Week 5 and the average score of the other 29 students. The former decreased their scores during the early three weeks, from Week 3 to 5, and the score difference with the others increased gradually up to 27% in Week 5. Additional assignments were given to the two students at the end of Week 5, and then they finally started to increase their average score and narrow the score gap with the other students to 14% in Week 9.

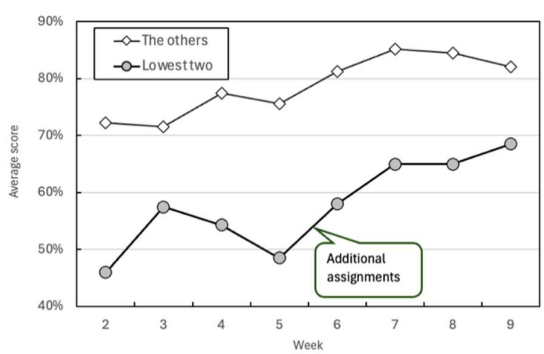


Figure 3 Weekly average scores of low-performing students and the others

4. Discussion

The higher performance of Group B compared to Group A in the post-test (Table 5) showed the potential effectiveness of students' independent work with GeoGebra. Group B's higher performance was at an unachievable level by Group A's weekly paper-based quizzes with retrials. Group B's

students might deepen conceptual understanding by visual thinking through their works with GeoGebra. It was also better as Group B's students could perform better with fewer retrials in the following semester (Table 6). However, it was a one-time practice under exceptional circumstances that fresh students started their remote school life without friendly communication with their classmates because of the pandemic. As the effect of GeoGebra activities is expected to depend heavily on the actual participation of the students, we still need more practice in the normal teaching environment to confirm the effect.

On the other hand, the comparison of score distributions between the highest subgroup (2015 AY) in Group A and Group B (Figure 1) reveals the nature of GeoGebra activities' effect. The score distributions were the same for symbolic calculations between the 2015 AY subgroup and Group B, but Group B had higher scores in graphs and inequalities. The selective effect of GeoGebra activities suggests that they helped the students realize the connection between symbolic expressions and graphs and think visually. GeoGebra activities can significantly decrease the burden of low-performing students' repeated trials of paper-based quizzes. We could use tailored GeoGebra activities to support struggling students in recognizing the connection between symbolic and graphic representation of various functions, have a bird-eyes-view in this field, and start solving questions confidently.

Further, Group B maintained higher performance through the second semester (Figure 2) with a lighter student workload (Table 6). As Group B's average success rate in retrials was higher, GeoGebra activities in the first semester probably improved students' graphing or visual thinking skills and eased the learning in the second semester. We expect that a 3D graphic object such as a helix (Week 8 of Table 4) further assists students in learning trigonometric functions.

Lastly, Figure 3 suggests that the combination of paper-based quizzes with retrials and GeoGebra activities during the lessons in 2024 AY worked well for average students. GeoGebra activities might encourage the students to think visually and increase scores in weekly paper-based quizzes with retrials. Figure 3 also suggests the possible effectiveness of early intervention for low-performing students to learn graphing functions and solving inequalities. As the low-performing students recognize the necessity of doing something after failing in retrials, we can expect their active participation in GeoGebra activities, while the other students may not feel the necessity. However, the effect of early intervention needs more time to be examined.

5. Conclusions

This paper presents the study of the effect of using dynamic geometry software GeoGebra on student performance and visual thinking skills in learning mathematics. The study was conducted on two occasions, the first in Japan and the second in Thailand.

On the first occasion, the two student groups with the same learning contents but different teaching styles, face-to-face lessons with paper-based quizzes, remote lessons with activities using GeoGebra software and paper-based quizzes, are compared. The collected data from the students' learning records, including post-test score distribution, were analyzed, focusing on two categories of symbolic calculations: graphs and inequalities. The student group using GeoGebra activities had a significantly higher average score in the post-test, especially in the graphs and inequalities section. They also needed fewer retrials and higher quiz success rates during the following semester. The students' visual thinking skills might have improved their performance in the post-test and decreased their learning load in the subsequent semester.

On the second occasion, GeoGebra activities are used regularly throughout the lessons and also used for additional assignments for low-performing students. This student group had relatively higher

average scores during the first half of the semester. The low-performing students started to increase their scores earlier after the additional assignments. The GeoGebra activities may familiarize the students with visual thinking, which is not an automatic learning style, and encourage them to use visual thinking in their learning. Thus, it helped the students learn graphing functions and solving inequalities deeply.

We also want to share simple and effective GeoGebra activities to promote visual thinking in learning mathematics deeply and enjoyably.

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Technology Integration Frameworks in Geometry Education: A Synthesis of Recent Research

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Abstract: *This synthesis examines technology integration frameworks from empirical research published from 2016-2022 on integration of computer technology (CT) into geometry education across grade levels. We conducted article searches through article database, i.e., Google Scholar and ERIC with relevant keywords. Inclusion criteria focused on peer-reviewed empirical studies combining geometry education, computer technology integration, as well as the availability of technology integration framework. Out of 338 initial search results, 14 articles met the criteria. The results show that there are 10 frameworks and the predominant technological frameworks used in these time periods is Technological Pedagogical Content Knowledge (TPACK). The technology integration frameworks from selected articles were categorized based on framework categorizations: how students learn with technology, design and evaluation of tools/tasks, how teachers use technology, and how teachers learn to use technology. The majority of frameworks addressed how teachers learn to integrate technology, with nine out of fourteen selected articles. Most studies focused on in-service or pre-service teachers rather than K-12 students. This synthesis reveals framework usage mostly focused on in-service and preservice teachers and highlights gaps in research on frameworks for student learning with technology in geometry.*

1. Introduction

Geometry is one of the topics in mathematics that has the potential to provide students with opportunities to pose their own questions, investigate concepts, develop logical arguments, construct proofs, and convey mathematical ideas. However, historically in the U.S. curriculum, geometry has not been valued as highly as algebra [2]. Moreover, research shows that many students struggle with learning geometry concepts and developing proficiency in geometric thinking [3]. It is important to create excellent and meaningful mathematics classes that integrate mathematical technology as the main resource for students to understand and make sense of mathematical concepts, to reason mathematically, and to communicate their mathematical thinking [4]. Specifically, in geometry classes, the use of dynamic geometry software (DGS) or computer technology (CT) has been proposed as an effective approach to enhance geometry teaching and learning since it offers interactive diagrams and visualization features that provide students with the opportunity to dynamically construct, manipulate, and analyze geometric shapes and measurements. These features facilitate students' exploration of geometry concepts, and the creation, testing, and revision of their conjectures and constructions [5, 6, 7]. This geometric conjecture exploration is possible in DGS since the ability to drag geometrical objects "preserves the underlying relationships among them in well-constructed diagrams" [4, p. 78].

Frameworks have the potential to affect teachers and educators in various ways. A framework serves as a shared lens for teachers or educators to work collectively towards enhancing instructional practices and supporting one another in developing effective teaching methods that are important for ensuring all students succeed in mathematics learning [4]. In teaching practice, research-based frameworks, along with tools, technologies, and strategies, are important to connect theory and practice and guide candidates in their work with authentic artifacts of teaching [8]. Particularly, technology integration frameworks can help them prepare to teach mathematics with technology, specifically on how students use and learn with technology, how to design and evaluate technology

tools and tasks, and how teachers use and learn to use technology [1]. Additionally, these frameworks discuss the importance of teaching for equity and engagement, which can be both rewarding and complex for teachers [9]. This suggests that the frameworks aim not only to improve current teaching practices but also to build a pipeline of talented educators for the future.

As part of our previous study [16], investigating the trend of the integration of computer technology in geometry classes that my advisor, Dr. Jane-Jane Lo, and I submitted and which was accepted for the North American chapter of the International Group for the Psychology of Mathematics Education (PME-NA) conference this Fall 2024, this study focuses on synthesizing the technology integration frameworks that were used in peer-reviewed research on the integration of computer technology in geometry education during the years 2016-2022. The findings will provide an overview of the research-based frameworks on technology integration in geometry classes that will be useful for educators or teachers to teach geometry classes with technology.

2. Theoretical framework

Computer Technology

In this study, we restrict computer technology (CT) to software rather than hardware. We categorize CT into four primary types: (a) tutors or tutorials, (b) communication media, (c) exploratory environments, and (d) tools [10, 11, 12]. Tutorial software provides direct instruction through information delivery, demonstrations, and practice. Communication media enable collaboration and information sharing between students and teachers through email, e-learning platforms, videoconferencing, and collaborative systems. Exploratory environments promote active learning by allowing students to explore and discover concepts independently through platforms like Logo programming, GeoGebra, Geometer's Sketchpad, simulations, and multimedia/hypermedia. Tools leverage technologies for general purposes like word processing, spreadsheets, and data analysis. This includes instructional management platforms.

In this synthesis study, we focus our computer technology on the category of exploratory environments. The rationale for choosing this category is, as noted in Lou et al. [11], that exploratory environments encourage active student exploration and discovery-based learning. This aligns with modern constructivist learning theories, which emphasize students constructing their own knowledge. Using exploratory CT allows for open-ended mathematical investigation and may promote deeper conceptual understanding compared to direct instruction.

Framework

Generally, a theoretical framework serves several purposes, such as supporting predictions, guiding the crafting of appropriate methods to test predictions, helping interpret findings, and showing how a study contributes to the field's knowledge base [13]. In this study, we particularly focus on the framework, using the term "technology integration framework" precisely to show the phenomena related to the use of technology or the integration of technology in teaching and learning mathematics, especially in the topics of geometry.

A study on the technology integration framework was conducted by McCulloch and colleagues [1]. While McCulloch et al.'s [1] findings provide a helpful snapshot of the ways MTEs are framing the work related to teaching PSMTs to teach with technology, their study has limitations. The sample of frameworks they examined was offered by participants who completed a particular item on the survey. In our study, instead, we draw frameworks from article searches on Google Scholar and ERIC through a screening process of peer-reviewed articles on the integration of technology in geometry teaching and learning.

We adopt four categories of frameworks "what is being framed" from McCulloch et al.'s [1] study to categorize the frameworks we found from the screening process. The first category, framing how students learn mathematics with technology, includes frameworks that can characterize the ways students utilize technology tools in math courses, the processes by which they learn math concepts with technological aids, and the cognitive aspects involved in their use of technology for learning mathematics. The second category, framing the design and evaluation of mathematics technology tools and tasks, focuses on frameworks that can guide teachers in designing/curating, selecting, or evaluating technology-based tools and activities for teaching mathematics. The third category, framing how teachers use technology to teach mathematics, encompasses frameworks that characterize and evaluate how teachers utilize technology during mathematics instruction. The last category, framing how teachers learn to use technology to teach mathematics, includes frameworks that describe the specialized knowledge teachers need for integrating technology into mathematics teaching, as well as how they acquire that knowledge. While not necessarily intended for direct use by teachers, these frameworks can guide the design of teacher learning experiences like courses or professional development. They can also inform research aimed at identifying or developing teachers' knowledge for teaching math with technology.

However, instead of looking at technology integration frameworks in general, this study focuses only on the geometry topic. Thus, this synthesis aims to answer the following questions:

1. What technology integration frameworks were used in selected studies focused on geometry education during the years 2016-2022?
2. How were these technology integration frameworks used or framed in the context of geometry classes/education?

Table 1: Inclusion and Exclusion Criteria

No.	Criterion	Inclusion	Exclusion
1	Type	Full peer-reviewed journals	Non-peer-reviewed journals, book chapters, books, preliminary reports, short papers, brief reports
2	Topics	Studies focused geometry and computer technology (CT) integration	Studies focused on non-geometry topics, geometry without technology, technology integration in non-geometry topics
3	Type of CT	Exploratory environments	CT other than exploratory environments (e.g., tutors or tutorials, communication media, tools)
4	Populations	Research on technology integration in geometry classes at all levels, including stakeholders (e.g., in-service teachers)	Non-human subject research on geometry and technology even if in an educational context (e.g., meta-analyses, bibliographic studies, syntheses)
5	Language	Studies written in English	Studies in other languages
6	Framework	Technology integration framework	No framework, or framework that is not specific to technology integration

3. Methodology

Research synthesis, according to Chalmers et al. [14] and Gurevitch et al. [15], refers to the process of gathering findings from multiple related studies on a topic, assessing those results, and then integrating them to gain an overall understanding of the problem and identify sources of variation in the outcomes. In this synthesis study, I examine peer-reviewed journal articles on the use of computer technology in geometry teaching and learning that were published between 2016 and 2022. The data for this study were reselected based on the search from an earlier study [16]. In that study, we conducted an article search using academic databases, particularly Google Scholar and ERIC, with these keywords, "technology," "dynamic geometry software," "geometry education," "geometry

learning," and "geometry teaching" all together. To retrieve articles published in a specific year, the search was customized by setting the year range. The initial pool of search results was then screened by using the first five inclusion and exclusion criteria in Table 1. This screening process involved reviewing article titles, abstracts, mathematical topics covered, language, as well as the integration of CT and how it was used to see whether or not CT integrated fit into the category of exploratory environments. As a result, out of the 338 articles from the initial search, only 64 articles (18.95%) met the criteria in our initial study [16]. In this study, however, I added one criterion of inclusion: the availability of technology integration frameworks (criterion number 6 in Table 1). The details of the criteria for articles that were included or excluded are set out in Table 1. As a result, out of the 338 articles from the initial search, only 14 articles (4.1%) met our criteria.

Data analysis

After conducting the screening and applying exclusion and inclusion criteria, we carefully analyzed the frameworks from the selected articles and then categorized them under key themes based on McCulloch et al.'s framework categorizations: a) how students use and learn with technology, (b) the design and evaluation of technology tools and tasks, (c) how teachers use technology, and (d) how teachers learn to use technology.

4. Results

RQ1: What technology integration frameworks were used in selected studies focused on geometry education during the years 2016-2022?

Not all published articles related to technology integration in geometry education use a technology integration framework. Indeed, while each peer-reviewed published article has a theoretical framework(s), only a small percentage of them, 4.1%, were about technology integration. Out of the ten technology integration frameworks, the majority, five of them, are Technological Pedagogical Content Knowledge (TPACK or TPCK). The rest of the frameworks are unique to each study. Consequently, since TPACK or TPCK frameworks relate to teachers' specialized knowledge to teach with technology, in terms of research subjects, the majority of the subjects in the selected articles are in-service teachers (nine of 14 articles) and pre-service teachers (three articles). Only two articles have subjects focused on students with grade ranges from 5th to 8th (e.g., grade 6th in [17]; 7th and 8th graders [18]), and one article, [19], focuses on textbook analysis.

The majority of computer technology that utilized technology integration frameworks is GeoGebra. This CP was used in nine out of the fourteen selected studies, as a standalone CT [20, 21, 22, 23, 24, 25, 26, 27] or with other CT such as Geometer's Sketchpad (GSP) [28]. The second most dominant CT used is Geometer's Sketchpad (GSP), where one of them was used either a standalone CT [29] or in combination with other CT like GeoGebra [28]. On the other end, whereas only one study, [18], integrated AR that that provides a technology integration framework. There is a study that does not specify the kind of CT since they investigated DGS-related (e.g., using GeoGebra, GSP, Cabri, etc.) tasks in mathematics textbooks [19].

In terms of framework categorizations, based on McCulloch et al. [1], only one article, [18], framed how students learn mathematics with technology along with another category. Two articles [19, 17] were framed on the design and evaluation of mathematics technology tools and tasks, two articles [29, 30] framed how teachers use technology to teach mathematics, and most of the articles, nine of them [22, 25, 28, 26, 27, 24, 23, 21, 20], were framed on how teachers learn to use technology to teach mathematics, with the majority of these articles, five of them, utilizing Technological Pedagogical and Content Knowledge (TPACK or TPCK) to frame their study.

RQ2: How were these technology integration frameworks used or framed in the context of geometry classes/education?

This section describes the framework categorizations according to McCulloch et al. [1] and presents which frameworks fall into each category based on our analysis. We give brief descriptions of the categories and frameworks in each, together with a short summary of every framework and how it had been framed based on selected articles in this study.

Category 1: Framing how students learn mathematics with technology.

According to McCulloch et al. [1], these are the frameworks which characterize the way students learn mathematics with technology and can be used for characterizing the ways in which students use technology tools in courses of math, the processes by which they learn math concepts with technological aids, and cognitive aspects involved in their use of technology to learn mathematics. According to our classification, in this category there is only one framework that can be included: Framework of the Lines and Angles module of ScholAR in Sarkar et al.'s [18] article.

Framework of the Lines and Angles module of ScholAR

The Framework of the Lines and Angles module of ScholAR, developed by Sarkar et al. [18], focuses on framing how students learn geometry concepts, specifically lines and angles, through augmented reality (AR) technology. The primary goal of this framework is to frame how students can learn and apply geometric concepts in real-world contexts using AR technology. It aims to facilitate the development of geometric thinking aligned with van Hiele's theory levels. This framework frames the progression of students' geometric thinking through three AR activities, aligned with van Hiele's levels. The use of AR in these activities was designed to help learners achieve the first three levels of van Hiele's theory of geometry thinking - visualization, analysis, and informal deduction/abstraction. The integration of contextual problem-solving with AR interactions frames how students can apply their knowledge of definitions, properties, and formulas related to types of angles and polygons in an immersive learning environment. In this context of study, Sarkar et al. [18] included three learning activities with AR (see Figure 4.1 for details).

Activity	Activity 1: Types of Angle	Activity 2: Pairs of Angles	Activity 3: Interior and Exterior Angles of a triangle
Learning Objective	To be able to identify and distinguish between acute, obtuse and right angles in their surrounding objects.	To be able to classify the different pair of angles: complementary, supplementary, adjacent and linear pairs of angles in the examples of real-life objects.	To be able to locate a polygon in their surroundings and practice calculating its sum of interior angles
Learning Theory	Experiential Learning, Embodied learning		
Van Hiele Level	Level 0: Visualization	Level 1: Analysis	Level 2: Informal Deduction/ Abstraction
Problem Solving Method	Repetition: Multiple Possible Solutions	Contextual Problem	Breaking into smaller problems
Target Actions	Recall → Visualize → Identify → Mark		
Technology Support	AR on tablets using ARCore		
Interactions Defined	One Finger Tap – Draw on the screen like a pencil Two Fingers Tap – Get the House on the detected plane, zoom in and out Three Fingers Tap – Erase the content drawn on the screen		

Fig. 3 The Framework of Lines and Angles module of *ScholAR*

Figure 4.1 The Framework of Lines and Angles module of ScholAR ([18], p. 8)

Category 2: Framing the Design and Evaluation of Mathematics Technology Tools and Tasks
These are frameworks, according to McCulloch et al. [1], that frame the design and evaluation of math technology tools and tasks. More specifically, they look into frameworks that are in a position to help teachers in the designing/curating, selection, or assessment of technology-based tools and activities for teaching mathematics. Our analysis results that there are only two frameworks identified to fall into this category: Dynamic Geometry Task Analysis in Ulusoy & Turuş [19], and a model for Planning-Practicing-Evaluation (PPE) in Dikkartın Övez & Kıyıcı [17].

Dynamic Geometry Task Analysis
The foregoing framework of Dynamic Geometry Task Analysis was developed by Trocki [31] for the assessment of quality in tasks developed to be used with DGS in mathematics education. In Ulusoy & Turuş [19], it was used to analyze DGS-related tasks in middle school and secondary school mathematics textbooks in Turkey. The goal is to provide a structured way to investigate how far DGS tasks combine mathematical depth with technological affordances in a way that will help students learn geometry.

It focuses on two dimensions along which dynamic geometry tasks can be analyzed: 1) mathematical depth, drawing on cognitive demand, and 2) types of technological action (see figure 4.2). The mathematical depth aspect is based on Smith and Stein's [31] cognitive framework and other research that classifies the tasks from code 0, mathematical fidelity to code 5, discovering relationships and generalizing. Lower codes involve simple manipulations from 1 to 2. Codes 3–5 refer to conjecture making, testing hypotheses, and explaining ideas in mathematics. Whereas the technology actions describe the ways in which the features of DGS are used within a task. Codes A-C represent traditional geometric actions that were transposed into DGS, code D is dragging objects, and codes E-F somehow relate to exploring invariant relationships and extreme cases. In this framework, high-quality tasks are considered to be those combining a variety of technological actions enriched with important depths of mathematics.

Allowance for Mathematical Depth	
Hierarchical Levels	**Descriptions**
N/A	Prompt requires a technology task with no focus on mathematics.
0	Sketch does not have mathematical fidelity required to respond to prompt.
1	Prompt requires student to recall a mathematics fact, rule, formula, or definition.
2	Prompt requires student to report information from the construction. The student is not expected to provide an explanation.
3	Prompt requires student to consider the mathematical concepts, processes, or relationships in the current sketch.
4	Prompt requires student to explain the mathematical concepts, processes, or relationships in the current sketch.
5	Prompt requires student to go beyond the current construction and generalize mathematical concepts, processes, or relationships.

Types of Technological Action	
Affordances	**Descriptions**
N/A	Prompt requires no drawing, construction, measurement, or manipulation of current sketch.
A	Prompt requires drawing within current sketch.
B	Prompt requires measurement within current sketch.
C	Prompt requires construction within current sketch.
D	Prompt requires dragging or use of other dynamic aspects of the sketch.
E	Prompt requires the creation or consideration of multiple examples from which one can generalize.
F	Prompt requires a manipulation of the sketch that allows for recognition of emergent invariant relationship(s) or pattern(s) among or within geometrical object(s).
G	Prompt requires manipulation of the sketch that may surprise one exploring the relationships represented or cause one to refine thinking based on themes within the surprise. (Adapted from Sinclair [2003, p. 312])

Figur 4.2 Dynamic Geometry Task Analysis Framework [31, p. 703]

Planning-Practicing-Evaluation (PPE) model

The PPE is a model of integration of information and communication technology in mathematics education developed by Yildiz [33]. The planning, practicing, and evaluating processes implemented to assist pre-service mathematics teachers to effectively teach mathematics and integrate ICT practices in teaching are encompassed within the PPE model. In the geometry classes especially in Dikkartın Övez & Kıyıcı [17] study, this framework guided some technology-enhanced lesson designs, implementations, and assessments focused on specific geometric problems, in this case, heights problems using GeoGebra. It has three stages, with *Planning* being the strongest stage. In which at this stage, regulation concerning the content regulation, together with selecting the teaching methodology and choosing relevant ICT tools. At the second stage of *Practicing*, the teaching material is integrated into the learning process, this stage split into three phases: before, during, and after the lesson. *Evaluation*, the last step, does not consider being a phase on its own, but an ongoing process throughout the practice. It evaluates the activities, corrects the deficiencies, and makes them better for future iterations.

Category 3: Framing How Teachers use Technology to Teach Mathematics

In this category, McCulloch et al. [1] say that the included frameworks frame how teachers use technology to teach mathematics and reach to include frameworks that characterize and evaluate what teachers do with technology during mathematics instruction. The analysis result showed that two frameworks were included within this category: Technology acceptance model, TAM, in Huang et al. [29] article, and Technology Acceptance Scale for Teachers: T-TAS level of technology use in Kiyici & DiKkartin Övez [30] article.

Technology acceptance model (TAM)

One of the most widely known frameworks describing and predicting technology integration is the Technology Acceptance Model. It focused on mainly two factors: the perceived usefulness and the perceived ease of use, which influence attitude and intention regarding the use of technology. The model establishes causal relationships between these variables to explain how technology interaction with user behavior are determined. In their research, Huang et al. [29] used TAM to examine the acceptance and intentions of secondary mathematics teachers in the adoption process of Geometer's Sketchpad software for instructional use. Their variables were based on TAM: perceived usefulness, perceived ease of use, attitude towards the use of the software, and intention to use. The study aimed to elicit insights into the beliefs, attitudes, and intentions that underpin teachers' decisions to integrate GSP technology into their classroom practice. This framework is under category of *framing how teachers use technology to teach mathematics*, since it provides some type of organized scheme for understanding influences of technology adoption in mathematics education.

Technology Acceptance Scale for Teachers: T-TAS level of technology use

Technology Acceptance Scale for Teachers: T-TAS level is the teachers' acceptance level of the incorporation of technology in education. In their study, Kiyici & DiKkartin Övez [30] adopted the framework of T-TAS to assess the overall technology acceptance levels among mathematics teachers in the setting of distance education embedded during COVID-19. This framework was based on the article from Ursavaş [34] explaining the progression of the educators' competence and assimilation of technology in their teaching. In the first level, *Introduction*, the teachers are described as beginner but willing users of technology. In this stage, there is less or no use of technology in their teaching. This is a start of their path to technology integration. The second level is *Adoption*, where teachers begin using technology as an aid to traditional method. However, the integration of technology at

this point normally stays rather quite separate from the core learning-teaching process and not very much as constituent instruction. It is when educators start moving towards *Adapting* that they begin incorporating the use of technology meaningfully into learning activities. They start to notice the value of technology in enhancing their productivity as well as that of their students; therefore, they begin utilizing the technological tools available more intentionally. In *Exploring* level, the teachers feel comfortable to use the technology. They more total to incorporate the technology into the curriculum of the school; thus, this level depicts a much fuller and creative use of technological tools. At *Advancing* level, teachers creatively think of ways in which to use technology to transform their routine activities. Teaches who have advanced this level explore new ways that the technology might be used, extending practice beyond familiar uses, and seeking innovative applications of technological tools in education.

Category 4: Framing How Teachers Learn to Use Technology to Teach Mathematics

Frameworks that are included in the last category were framed on how teachers learn to use technology to teach mathematics, including frameworks that describe the specialized knowledge teachers need for integrating technology into mathematics teaching, as well as how they acquire that knowledge [1]. While not necessarily intended for direct use by teachers, these frameworks can guide the design of teacher learning experiences like courses or professional development. They can also inform research aimed at identifying or developing teachers' knowledge for teaching math with technology [1]. Five frameworks were included within this category: Technological Pedagogical and Content Knowledge (TPACK or TPCK) in Yildiz & Baltaci [22], Yildiz & Gokcek [25], Özçakir [28], Segal et al. [26], and Benning [27] articles; the levels of teachers' utilizations of technology in Kirikçilar et al. [24] article; GeoGebra-based Professional Development (PD) in Kul [23] article; Instrumental genesis in Alqahtani & Powell [21] article; and Diffusion of innovation model (DIM) in Mwingirwa & Marguerite [20] article.

Technological Pedagogical and Content Knowledge (TPACK or TPCK)

The TPACK framework builds upon studies by Mishra and Koehler [35], Niess [36], and Niess et al. [37], based on the extension of Shulman's [38] work in PCK. It has added an extra dimension called TPACK, referring to Technological, Pedagogical, and Content Knowledge, which gives definition to the kinds of specialist knowledge needed for teaching with technology. This framework combines three essential knowledge areas: Technological Knowledge (TK), Pedagogical Knowledge (PK), and Content Knowledge (CK), which suggests these intersect in complex ways. All this complexity is reflected in the combined model components: Technological Content Knowledge (TCK) and Technological Pedagogical Knowledge (TPK). TPACK represents an intersection of kinds of knowledge independent of the subject area and became a valuable tool in teacher education for understanding technology integration [39].

Five articles we included in this study reveal both commonalities and distinctions in their utilization and framing of TPACK to geometry education. Yildiz & Baltaci [22] and Yildiz & Gokcek [25] both focused on in-service teachers and used TPACK to guide professional development, but the former emphasized fractal geometry while the latter adapted TPACK specifically for high school geometry. In contrast, Özçakir [28] and Segal et al. [26] centered on pre-service teachers, with Özçakir [28] examining how existing TPACK levels influenced technology use, while Segal et al. [26] explored how a specific task developed TPACK components. Benning [27] stood out by not explicitly framing the study around TPACK, instead using it as an underlying principle for professional development. All studies incorporated dynamic geometry software, particularly GeoGebra, but their approaches varied: Yildiz & Baltaci [22] used collaborative lesson studies,

Özçakir [28] focused on software selection and problem-solving, Segal et al. [26] employed an inquiry-based task, and both Benning [27] and Yildiz and Gokcek [25] utilized professional development programs.

Levels of Teachers' Utilizations of Technology

This framework, adapted from Hughes [44] and Demir [42], is used for classifying teachers according to their level of technology use. It was employed in a study [24] alongside the TPACK framework to categorize in-service middle school math teachers by their level of technology usage (see Figure 4.3), specifically their use of GeoGebra when teaching geometry. The teachers were categorized through semi-structured interviews, and the researchers selected one teacher from each level for their study.

Level 0	These are situations where technology is not used during the teaching process. It is just like using computers for playing games.
Level 1	This technology level refers to using one type of file and reflecting it on the board during the teaching process.
Level 2	In this level, the technology for carrying out the teaching process quickly and efficiently. For instance, instead of drawing the parabola on a board, describing it using technological applications can be given as an example for this level.
Level 3	In this level, dynamic software, which guide learners in their learning of a subject, are used for studying a subject in-depth. For example, according to this level, one can show the slope of a curve that is drawn at a given point to a function is equal to the derivative of this function at that point.

Figure 4.3 The levels of teachers' utilizations of technology ([24], 2018, p. 104)

GeoGebra-based Professional Development (PD)

The GeoGebra-based Professional Development (PD) was designed by Kul [23] and it consists of three progressive development stages in the teachers' proficiency regarding the use of GeoGebra in teaching geometry. In this study, the initial stage will be set up to familiarize participants with GeoGebra through basic activities. Where the interface and general features of the GeoGebra are presented to the teacher, upon which more complex applications will be developed. The real heart of the program, represented by the second phase, consists of worksheets enriched with pre-designed tasks. These tasks are directly related to mathematics topics that can be found in textbooks and even on the GeoGebra website. This phase allows both individual exploration and small group work; it gets very rich debates between teachers about mathematical concepts and their possible representations with GeoGebra. The final stage of the framework is characterized by independent work. The teachers are given some investigation-type mathematical tasks for work at home, supposed to be used to gain practical experiences and evaluate their knowledge about GeoGebra.

Instrumental genesis

Instrumental genesis examines how the learners, i.e., in-service teachers in Alqahtani & Powell [21] study, interact with technological tools in goal-oriented, instrument-mediated actions. Instrumental genesis encompasses two dialectical processes: instrumentalization, where learners enrich artifact properties, and instrumentation, focusing on the learner's development and adaptation of utilization schemes. In Alqahtani & Powell [21] study, the framework was used to analyze how teachers developed both usage schemes (basic operation of the tool) and instrument-mediated action schemes (using the tool for geometric exploration and problem-solving). It was also used to explain how interactions with the GeoGebra shaped the teachers' geometric understanding. Lastly, the framework helped track the progression of how teachers used the Geogebra tools, from basic dragging to more sophisticated uses for problem-solving.

Diffusion of innovation model (DIM)

The Diffusion of Innovation Model (DIM), developed by Everett Rogers [43], is a theoretical framework that frames the process of how new ideas or technologies spread and are adopted within a social system over time. In the context of Mwingirwa and Marguerite [20] study, it specifically frames how mathematics teachers adopt and integrate new educational technology (GeoGebra) into their geometry instructions. Specifically, they used DIM to frame "teachers' perspective towards training and eventual use of GeoGebra as a tool to enhance learning of mathematics" [20, p. 286]. They applied the framework to interpret "teachers GeoGebra uptake tendencies" [20, p. 287], framing the process of how Kenyan mathematics teachers adopted this new technology for teaching geometry. This framework served as a conceptual lens through which the researchers examined teachers' GeoGebra uptake tendencies. DIM framework categorizes the teachers or adopters into different groups based on their tendencies to adopt the innovation of GeoGebra - innovators, early adopters, early majority, late majority, or laggards. Hence, using this DIM framework, Mwingirwa and Marguerite [20] were able to frame and interpret the varying rates at which the Kenyan mathematics teachers adopted and implemented GeoGebra in their classrooms after receiving training on it.

5. Discussion

The most striking finding is the predominance of frameworks focused on teachers rather than students. Of the 14 studies analyzed, 11 centered on how in-service (N = 9) and prospective (N = 2) teachers use or learn to use technology in geometry instruction. However, this finding was quite opposite to what we have found in our previous study [16], before we did a screening for the availability of technology integration framework, the majority of the subjects of the study was the students in the grades of 6th to 8th, and followed by pre-service teachers, and the subject of in-service teachers was the second least subjects that the studies in integrations of technology in geometry focused on.

The widespread use of the TPACK framework (N = 5) further emphasizes this trend of technology integration framework shows more teacher-centric approach. This finding aligns with McCulloch et al.'s [1] study, where they found that more frameworks are teacher-centric, with a total of seven frameworks focused on how teachers use or learn to use technology. This makes sense in their study since it focused on examining frameworks used by mathematics teacher educators to prepare secondary mathematics teachers to teach with technology, compared to this more general study that is not intentionally focused on prospective teacher preparation. However, within each category, our findings differ from McCulloch et al.'s [1] study. While they found only two frameworks focused on how teachers learn to use technology, we found five out of ten frameworks in geometry topics focused on this aspect.

McCulloch et al. [1] found five out of seventeen frameworks from their survey focused on how students use and learn with technology, which was still fewer than the teacher-centric frameworks. In our study, on the other hands, only one framework, the Lines and Angles module of ScholAR [18], explicitly framed how students learn geometry concepts through technology. This happened since the other framework from studies that focused their subjects on K-12 students, i.e., Planning-Practicing-Evaluation (PPE) model [17] fell under *Category 2: Framing the Design and Evaluation of Mathematics Technology Tools and Task*.

Such a trend may indicate that the frameworks on integration technology in geometry education were over-focused on how teachers use technology, rather than on how students learn mathematics with technology which shows a significant literature gap in both studies. Further studies are needed for the development and application of frameworks about how students interact and learn from technology-enhanced geometry instruction.

Moreover, out of the ten technology integration frameworks found in this study and seventeen in McCulloch et al.'s [1] study, only two frameworks appear in both studies, i.e., Dynamic Geometry Task Analysis and TPACK. Although McCulloch et al. [1] did not focus on a specific mathematical topic like this study focusing on geometry, this finding shows the gap between integration technology frameworks that are taught for prospective teachers in the university [1] and for the integration technology frameworks that are used for research purpose.

Regarding mathematics-specific contexts, only a few frameworks, such as Dynamic Geometry Task Analysis and the Framework of the Lines and Angles module of ScholAR, are tailored to specific mathematics content. Some frameworks in this study are more computer technology-specific, such as the Framework of the Lines and Angles module of ScholAR and GeoGebra-based PD. Notably, the Framework of the Lines and Angles module of ScholAR is the only framework that is both mathematics-specific and computer technology-specific. On the other hands, most of the technology integration frameworks in this study have a broader perspective or originated in general education context and are not mathematics-specific or computer technology specific, such as the TPACK framework. However, McCulloch et al. [1] argued that the lack of mathematical context or technology tool specificity in frameworks makes framework less useful for curriculum development, teaching instruction or adoption, and research activities related to teaching mathematics with technology. To make them more useful, these general education context frameworks can be adapted to more mathematics or technology-specific contexts [1].

6. Limitations

This synthesis study is part of a previous study [16] investigating trends in the integration of computer technology in geometry classes. The article search was limited to peer-reviewed articles published between 2016 and 2022, and therefore likely did not capture all relevant technology integration frameworks that have been used or developed for other mathematical topics, published in different years, or in non-peer-reviewed articles. Moreover, as the search was conducted only in Google Scholar and ERIC journal databases, there is a high possibility that additional technology integration frameworks exist in other databases.

In addition, similar to McCulloch et al. [1] findings, we noticed that our study also did not find any technology integration frameworks that specifically address equity issues related to technology accessibility or that provide guidance on when and what technology to integrate into the classroom. Moreover, we did not find any single technology integration framework that framed how to elicit and use students' thinking when they are learning with or without technology, nor did we find one that specifically addresses classroom mathematics discourse.

Therefore, a future study on investigating research-based technology integration frameworks in a broader scope need consider more databases, in addition to non-peer-reviewed articles and tapping into a much larger time frame. This would likely discover more relevant frameworks of technology integration and provide a better understanding of technology integration in mathematics education, especially in geometry topics. In addition, there is also a need for research into developing and studying frameworks that focus on student learning processes with technology in geometry, as at present the trend is mostly teacher-centered. In addition, developing more mathematics-specific and technology-tool-specific frameworks, especially for geometry education, is needed as they would afford more targeted guidance for curriculum development, teaching instructions, and research activities. This would bring more balance to our understanding of both the teaching and learning aspects of technology integration in mathematics education, especially in geometry topics. Furthermore, future studies should also fill this gap of the frameworks, dealing with issues of equity that are bound to pop up in mathematics education related to accessibility to technology. In the same

breath, framework development focused on eliciting and leveraging students' thinking in teaching mathematics with or without technology as well as those attending to mathematics discourse in technology-enhanced environments is long overdue.

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Automated reasoning tools for dealing with elementary but intriguing geometric loci

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Abstract

A common response to new educational technology is to suggest banning it, arguing that it could replace the development of certain skills and knowledge with the capabilities of the new tool. To counter this argument, it is essential to provide examples demonstrating how the wise use of new instruments can enhance the teaching and learning of mathematical competencies.

In our present contribution, we address the situation described above through an example in Geometry which incorporates the following elements:

a) the automated reasoning tools of GeoGebra Discovery, an experimental version of the mathematical software GeoGebra;

b) the development of mathematical reasoning and proof competencies through elementary geometry problems, such as loci computation; and

c) a concrete geometric construction as triggering event: given a triangle ABC, find the locus of points P such that $\angle ABP$ and $\angle ACP$ are congruent.

This construction can be quickly done using GeoGebra Discovery, but what does "finding" mean here? Is it just creating a visual image or finding an equation with coefficients based on the positions of A, B, and C? Our goal is to understand the geometric locus both symbolically and geometrically. As we explore with the help of algebra and geometry software, we'll discover various connections to geometric concepts that will deepen our understanding of elementary geometry.

In summary, our goal is to describe the challenges that arise in this elementary, yet highly inspiring and intriguing context, as an example of the methodological protocols and clear advantages associated with new technologies in mathematics education.

1 Introduction

1.1 On a geometric construction related to an Olympiad problem

In the field of Mathematics Education, the introduction of new technologies in the classroom has spurred a debate on the advantages and pitfalls that tools like Dynamic Geometry Software (DGS) or Computer Algebra Systems (CAS) are bringing in the teaching of Mathematics. The authors of this paper have been involved in exploring new automatic reasoning tools (see for example [9], [13]), mainly implemented in an experimental version of the software GeoGebra (GG), GeoGebra Discovery (GGD), mantained by Kovács Zoltán ([8]).They also want to convey the idea that these new technologies can help students to gain understanding and stimulate their curiosity when confronting geometric challenges.

With this intention in mind, the first author stepped on a Geometry problem posed in the 60th Spanish Mathematical Olympiad (OME):

Problem 1 *Let ABC be an scalene triangle and P a point in its interior such that $\angle PBA \cong \angle PCA$. Lines PB and PC intersect the interior and exterior angle bisectors through A at points Q and R respectively. Let S be the point such that $CS \parallel AQ$ and $BS \parallel AR$. Show that Q, R, S are collinear.*

The solution to this problem can be found in the web site of the LX OME ([11], [12]), but from the statement of this problem a related question arises concerning a geometric locus:

Problem 2 *Let ABC be a triangle. Find the geometric locus L' of all points P such that $\angle PBA \cong \angle PCA$.*

Computing geometric loci using algebraic symbolic tools has been an active area of interest for the research team of the first author ([2, 3, 4, 5, 6]). The software GGD, through its command `LocusEquation( , )`, allows the automatic computation of loci in a wide variety of geometric constructions, and Problem 2 was an inviting proposal for checking again the power of the automatic reasoning tools implemented in GGD.

Let us remark, very roughly speaking (see more details in the mentioned references), that locus computation through Dynamic Geometry Systems can be approached in quite different ways: just plotting some positions of the tracing point that builds the locus; or describing the locus through a numerical equation obtained from a large collection of such positions and with some probabilistic assumptions; or finding a more precise -yet approximate- equation of the locus by performing some symbolic elimination algorithms depending on the numerical coordinates of the points involved in the figure; or considering symbolic coordinates for the points in the figure and then performing elimination, yielding in this way an exact locus equation.

In this context, as purely symbolic computations are sometimes not performing well, GeoGebra Discovery `LocusEquation( , )` algorithm chooses to output, by elimination, an equation with coefficients depending on the *numerical* coordinates of the points in the construction, so it is not possible to use in a straightforward way the automated, symbolic, reasoning tools of GeoGebra Discovery to check the purely symbolic validity of the output. In the next sections we will develop the specific approaches we have developed to deal with this, apparently, simple, but intriguing locus. Let us construct an arbitrary triangle ABC in GGD, and let us add an arbitrary point P. We can then consider the angles $\delta = \angle PBA$, $\epsilon = \angle PCA$ (in what follows,

as is usual in elementary geometry, we will consider non-oriented angles, i.e. all angles will be non-negative and less than π). The command `LocusEquation( , )` can admit as parameters a Boolean expression f relating geometric elements of a construction, and a moving point P involved in f, producing as output the locus of points P making true the given Boolean expression (see [10] for more on this command and other automatic reasoning tools in GGD). In our case, solving Problem 2 within the GGD environment amounts to type the simple command

`LocusEquation(`$\delta == \epsilon$`, `P`)`

The output after introducing this command is shown in Figure 1a. We will call L this automatically obtained locus (to distinguish it from our target locus L'). Notice that dealing with angles through symbolic computation requires handling some subtle issues: one, the transcendental character of the notion of angle (with respect to the coordinates of defining points); two, the need to deal with signs—and thus with real algebraic geometry—if approaching angles through trigonometric functions such as sine, cosine, tangent, etc.

After trying different configurations for the triangle ABC and inspecting the graphic and algebraic expression (with numerical, approximate coefficients) of this locus we come up with several observations:

Observation 1 (O1): The locus L seems to be the union of a circumference L_1 and a hyperbola L_2.

Observation 2 (O2): The circumference L_1 seems to be the circumcircle of ABC.

Observation 3 (O3): The hyperbola L_2 seems to contain the vertices A, B, C.

As the reader can appreciate, the use of the software GGD has allowed us to quickly establish a sequence of conjectures almost from scratch. The appearence of the circumcircle is unsurprising, because of the elementary properties of inscribed angles in a circle. In any case, these initial observations stir up our interest in the problem and give rise to a variety of questions that can lead to further exploration:

Question 1 (Q1): Does L really solve Problem 2? That is, $L \cong L'$?

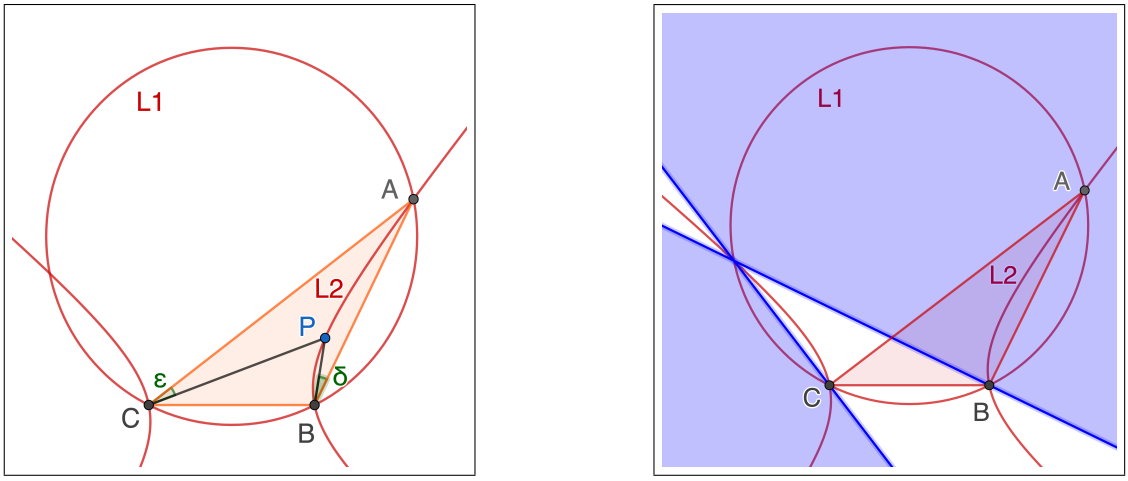
Question 2 (Q2): If our previous observations O1-O3 are correct, the circumference L_1 and the hyperbola L_2 meet in general at four points, being A, B, C three of them. What can we say about the fourth point of intersection?

Question 3 (Q3): How can we characterize/construct the elements of the hyperbola L_2 (center, radius, axes, asymptotes, vertices and foci)?

Question 4 (Q4): Can we say anything else about this locus?

2 Proving observations O1-O3

As a first consideration, the fact that vertices A, B and C belong to the locus arises some concern, because when point P coincides with any of these points the equality $\angle PBA = \angle PCA$ becomes degenerate. Moreover, some experimentation moving the point P along L reveals that



(a) (b)

Figure 1: (a) Initial construction and locus L; (b) Fixing up locus L

on some arcs the equality $\angle PBA = \angle PCA$ seems to hold, while on others we rather have $\angle PBA + \angle PCA = \pi$. The reason behind this behaviour comes from the fact that GGD handles the angle equality $\delta = \epsilon$ in symbolic computation terms, an involved approach, as already remarked. To see this, let us choose appropriate coordinates to simplify calculations. Since angles are preserved by homothecies and isometries, we can assume $B = (1, 0)$, $C = (-1, 0)$. Let us set $A = (a_1, a_2)$, $P = (x, y)$. Now, the equality $\delta = \epsilon$, by using the scalar product formula, becomes:

$$\delta = \epsilon \quad \Leftrightarrow \quad \cos\delta = \cos\epsilon \quad \Leftrightarrow \quad \frac{\vec{BA} \cdot \vec{BP}}{\|\vec{BA}\| \cdot \|\vec{BP}\|} = \frac{\vec{CA} \cdot \vec{CP}}{\|\vec{CA}\| \cdot \|\vec{CP}\|}$$

By squaring both sides of the last equality and removing denominators we finally get the polynomial identity

$$\begin{aligned} f :=& (\vec{BA} \cdot \vec{BP})^2 (\|\vec{CA}\| \cdot \|\vec{CP}\|)^2 - (\vec{CA} \cdot \vec{CP})^2 (\|\vec{BA}\| \cdot \|\vec{BP}\|)^2 \\ =& -(a_2 y + (a_1+1)(x+1))^2 (a_2^2 + (a_1-1)^2)(y^2 + (x-1)^2) \\ &+ (a_2 y + (a_1-1)(x-1))^2 (a_2^2 + (a_1+1)^2)(y^2 + (x+1)^2) = 0 \end{aligned}$$

Notice that the squaring process allows the possibility $\cos\delta = -\cos\epsilon$, and this means we can also have $\delta + \epsilon = \pi$ (this will bring us to Question Q1 later). The left side can be factorized (the GG CAS can perform this easily) to give us

$$f = -4(a_1^2 y - x^2 a_2 - y^2 a_2 + y a_2^2 - y + a_2)(a_1^2 xy - a_1 x^2 a_2 + a_1 y^2 a_2 + a_1 a_2 - xy a_2^2 - xy).$$

Let us consider the equations

$$l_1(x, y) := -a_2 x^2 - a_2 y^2 + (a_1^2 + a_2^2 - 1)y + a_2 = 0, \tag{1}$$
$$l_2(x, y) := -a_1 a_2 x^2 + a_1 a_2 y^2 + (a_1^2 - a_2^2 - 1)xy + a_1 a_2 = 0. \tag{2}$$

Under the assumption $a_1 a_2 \neq 0$ (which amounts to say that triangle ABC is non-degenerate and A is not contained in the Y-axis), these equations correspond to a circumference and a

hyperbola, so that we have just verified O1, with L_1 and L_2 corresponding respectively to $l_1 = 0$ and $l_2 = 0$. When A lies on the Y-axis the hyperbola L_2 degenerates into the union of the coordinate axes, and from now on we will be assuming this is not the case, which is easy to handle. It is also straightforward to check now that $l_1 = 0$ on A, B and C, so that $l_1 = 0$ is the circumcircle of triangle ABC, as well as to check that the hyperbola $l_2 = 0$ contains the vertices of the triangle. Therefore, observations O1-O3 are correct.

3 Answering questions Q1-Q4

3.1 Answering Question 1

In the previous section we already mentioned that, strictly speaking, the locus L' we are looking for does not coincide with L, because of two reasons:

i) The vertices B, C are degenerate points in the sense that equality $\delta = \epsilon$ loses its meaning;

ii) for certain points $P \in L$ we might have $\delta + \epsilon = \pi$ instead of $\delta = \epsilon$.

In order to clarify this last assertion we can proceed as follows: To preserve the equality $\cos\delta = \cos\epsilon$ we can add to the above locus equation $f = 0$ an extra condition (carrying our computations into the realm of computational real algebraic geometry, of greater complexity) which avoids the problem of loosing control on the signs of these cosines after squaring, that is

$$(\vec{BA} \cdot \vec{BP})(\vec{CA} \cdot \vec{CP}) \geq 0,$$

so that

$$(a_2 y + (a_1 + 1)(x + 1))(a_2 y + (a_1 - 1)(x - 1)) \geq 0.$$

The left side is a product of two linear expressions in x, y and after some reflection we conclude (see Figure 1b):

Proposition 3 *The solution locus for Problem 2 is formed by two arcs: The circular arc BAC containing vertex A, together with a hyperbolic arc BAC (with two branches, since it crosses the line at infinity) containing A. We must exclude the vertices B, C from this locus.*

3.2 Answering Question 2

In general, the circle L_1 and the hyperbola L_2 meet at four points, and we already checked that three of them correspond to the vertices of $\triangle ABC$. What can we say about the fourth point? After some experimentation in the graphic view of GGD, we establish a conjecture to work with: *The fourth intersection point of $L_1 \cap L_2$ is the symmetric D of point A with respect to the circumcenter of $\triangle ABC$.* It is a simple exercise to find the coordinates of point D:

$$D = \left(-a_1, \frac{a_1^2 - 1}{a_2}\right).$$

And we can also check through the GG CAS that D belongs to the hyperbola L_2 (and obviously also to L_1).

Since we already have four points of the hyperbola L_2, and having in mind that five points determine a conic, it is natural to look for a fifth point in L_2. If we think of the triangle centers, the orthocenter E of $\triangle ABC$ comes up as a suitable candidate, since $\angle EBA, \angle ECA \in \left\{\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha\right\}$, where $\alpha = \angle BAC$. Computing the coordinates of E gives us

$$E = \left(a_1, \frac{1 - a_1^2}{a_2}\right),$$

the symmetric point of D with respect the midpoint O of BC (!). So we came up with these two facts:

Proposition 4 *The hyperbola L_2 contains the vertices of the triangle, its orthocenter and the symmetric point D of A with respect to the circumcenter.*

Proposition 5 *In $\triangle ABC$, the symmetric point of a vertex with respect to the circumcenter and the orthocenter are symmetric with respect to the midpoint of its opposite side.*

Under a suitable construction in GGD, we can also verify the truth of this last statement by using the GGD commands `Prove(AreCollinear(D,O,E))` and `Prove(DO==OE)` (or the more informative command `ProveDetails()`). We also invite the reader to try, in the latest version of GGD, the command `ShowProof()` as in `ShowProof(AreCollinear(D,O,E))`, which delivers a step-by-step proof of the geometric statement "The points D, O, E are collinear".

3.3 Answering Question 3

The powerful tools included in the GeoGebra software allow us to explore easily the hyperbola L_2 and represent its main elements: center, foci, vertices, asymptotes and axes (see Figure 2a and [14, First Construction]). In particular, this facilitates the task of detecting invariant properties. So, after playing a while by dragging vertex A, we guess that

i) The center of the hyperbola is the midpoint of BC.

ii) The asymptotes are parallel to the angle bisectors at A.

The first assertion readily follows from the expression of the center of a conic in terms of the coefficients of its equation (see for example [1]). For the second assertion, we start by obtaining the equations of the angle bisectors at A. Let $Q = (x, y)$ be a point on one of the bisectors at A. Then

$$\cos(\angle BAQ) = \pm\cos(\angle CAQ) \quad \Leftrightarrow \quad \left(\frac{\vec{AB} \cdot \vec{AQ}}{\|\vec{AB}\| \cdot \|\vec{AQ}\|}\right)^2 = \left(\frac{\vec{AC} \cdot \vec{AQ}}{\|\vec{AC}\| \cdot \|\vec{AQ}\|}\right)^2$$

and we obtain the polynomial equation

$$\begin{aligned} g(x, y) := &- 4a_2(-a_1a_2x^2 + a_1a_2y^2 + (a_1^2 - a_2^2 - 1)xy \\ &+ (a_2^3 + a_1^2a_2 + a_2)x - (a_1^3 + a_1a_2^2y - a_1)y - a_1a_2) = 0. \end{aligned} \tag{3}$$

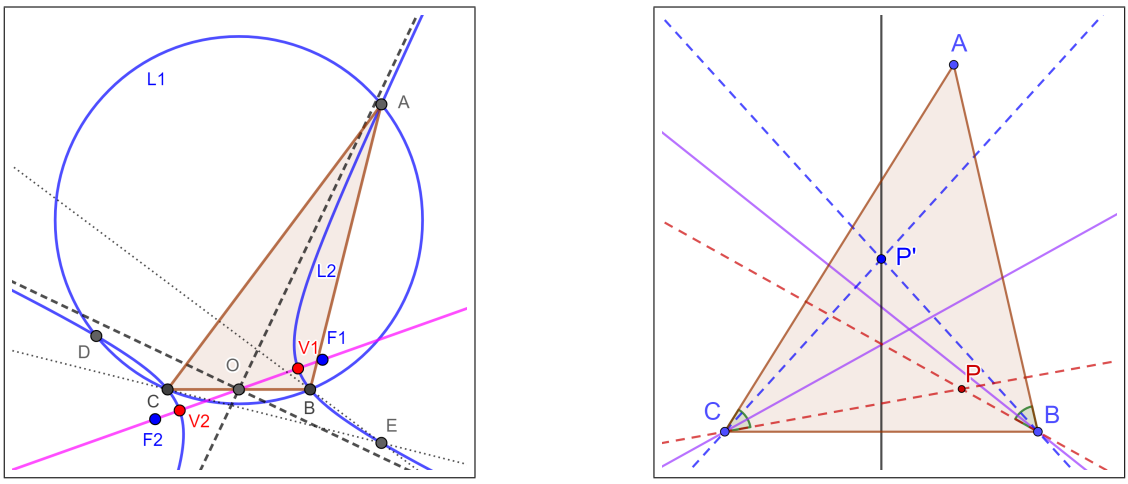
Equating to zero the second degree part $g_1(x, y) := -a_1a_2x^2 + a_1a_2y^2 + (a_1^2 - a_2^2 - 1)xy$ of the left hand side gives us the parallel lines to the angle bisectors through the origin. But g_1 does

coincide with the second degree part in (2), which gives the asymptotes r_1, r_2 of L_2. From all this we infer that L_2 is an equilateral hyperbola, and the axes of L_2 are the bisectors of the right angles formed by r_1 and r_2.

In regard to the vertices and foci of L_2, no simple expressions for its coordinates were found, and the GGD CAS could not provide direct explicit output for them in our trials. Since the center of L_2 is at the origin of coordinates and the points $(x, y) \in L_2$ such that the their normal line contains the origin are only the vertices of L_2, these are determined by the real solutions of the system

$$\begin{cases} l_2(x,y) = 0 \\ \dfrac{\partial l_2}{\partial y} \Big/ \dfrac{\partial l_2}{\partial x} = y/x \end{cases} \implies \begin{cases} -a_1a_2x^2 + a_1a_2y^2 + (a_1^2 - a_2^2 - 1)xy + a_1a_2 = 0 \\ \dfrac{2a_1a_2y + (a_1^2 - a_2^2 - 1)x}{-2a_1a_2x + (a_1^2 - a_2^2 - 1)y} = \dfrac{y}{x} \end{cases}$$

Solving this system leads to cumbersome expressions that we will not reproduce here.



(a) (b)

Figure 2: (a) The hyperbola L_2 and its elements; (b) Isogonal conjugation

3.4 Answering Question 4

By using GGD we have been able to determine many properties of the locus L arising from Problem 2, but perhaps we can try to delve deeper and gain a better understanding of its nature. And here the human intuition still plays a role, sometimes based in prior knowledge, sometimes based in... luck? Consider the lines BP and CP, where P is such that $\angle PBA \cong \angle PCA = \delta$, and construct the symmetric lines BP' and CP' with respect to the internal angle bisectors at $\angle ABC$ and $\angle BCA$ respectively, which intersect at P'. It is straightforward that $\angle P'BC \cong \angle P'CB = \delta$, so that $\triangle P'BC$ is isosceles with $P'B \cong P'C$, and therefore P' lies in the perpendicular bisector of BC. In other words, the transformation $P \mapsto P'$ takes points in the hyperbola to points in the perpendicular bisector of BC, and viceversa. But this transformation has a name in plane geometry: *isogonal conjugation (with respect to a triangle)*, see for example [7]. In this context, a new statement concerning the hyperbola L_2 arises which makes clearer its appearance in our initial approach, and allows for an easy way to geometrically construct

points of L_2 from points in the perpendicular bisector of BC (see [14, Second Construction], where by dragging point P' you can get points in L_2):

Proposition 6 *The hyperbola L_2 is the isogonal conjugate of the perpendicular bisector of the side BC.*

It is worthwhile to mention here that for well trained Geometry students (of the kind that receive intensive training for participating in mathematical contests such as the International Mathematical Olympiad) this connection to isogonal conjugacy could have been perceived in an initial exploration of Problem 2, and from this initial realization many of the properties deduced above become consequences of the isogonal conjugation properties. But, in general, isogonal transformations do not form part of the standard mathematics curriculum in secundary education, and our initial problem can become a starting point that leads to beautiful Geometry new topics for the interested students. The isogonal approach allows this informal interpretation of L: Let us consider the locus $\hat{L}$ of points whose distance to vertices B, C are equal. Usually, we infere that $\hat{L}$ is the perpendicular bisector l of BC. But if we think in projective terms and allow points at infinity, we could say that $\hat{L}$ also contains the line at infinity l_∞, since a point in infinity has the same distance from B and C (which is ∞!). And the isogonal conjugate of $\hat{L} = l_\infty \cup l$ is precisely $L = L_1 \cup L_2$ because from well-known properties of this transformation the isogonal conjugate of the line at infinity is the circumcircle of $\triangle ABC$!

4 Tricks and treats: Two constructions

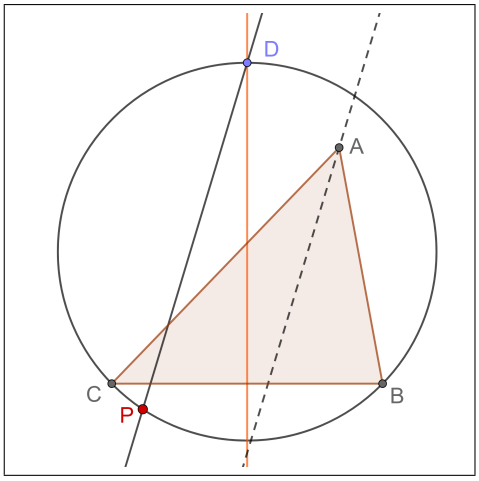
As a final treat, we propose to the readers two constructions which came up along this work and allow for further exploration and reflection.

First construction: One of the simplest constructions we devised to construct points of L_2 is represented in Figure 3a (see also [14, Third Construction]). Place a point D on the perpendicular bisector m of BC. Construct the circumference BCD and the internal angle bisector at A. Draw a parallel to this angle bisector through D, which intersects circumference BCD at other point P. Show that the locus of points P as D moves along m is the hyperbola L_2.

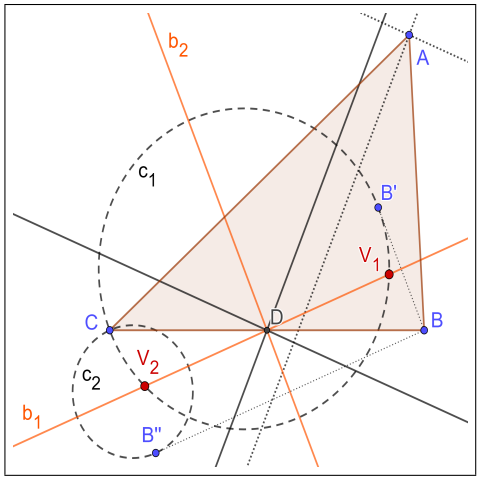
Second construction: Notice that we used the powerful commands of GG to represent notable points such as the foci and vertices of L_2, but we did not actually gave a geometric construction of them. Here we propose a construction of the vertices of L_2 (since this hyperbola is equilateral, getting the foci from the vertices is straightforward) and we invite the readers to check its validity. Construct parallels through O to the angle bisectors at A. These parallels divide the plane in four right angles. Trace the bisectors of these angles b_1, b_2 and find the symmetric points B' and B'' of B with respect to these lines. Trace the circles c_1, c_2 with diameters $B'C$ and $B''C$. Now we claim (see Figure 3b and [14, Fourth Construction]):

- One of the bisectors b_1, b_2 intersects both circles c_1, c_2.
- One of the circles c_1, c_2 meets both bisectors b_1, b_2.
- The intersection points of the circle meeting both bisectors with the bisector meeting both circles determine the vertices of L_2.

(Let us remark, again, that in the above we assume A does not lie in the Y-axis).



(a)



(b)

Figure 3: (a) Construction of points of L_2; (b) Construction of vertices of L_2

5 Conclusion

In summary, we think that we have shown how an apparently trivial geometric question: to describe the locus of points P such that $\angle PBA \cong \angle PCA$, for a triangle ABC, can give rise to a collection of—each time more involved—observations, then conjectures, and finally, mathematical statements.

What is notorious here is to remark that the use of a technological tool such as GeoGebra Discovery, that provides simultaneously plotting devices, dynamic geometry dragging possibilities, automatic answers for loci equations, computer algebra manipulation of the obtained equations..., fosters our imagination, and demands insistently to launch a joint cooperation with our mind to successfully deal with such problems.

This is, we think, the main message of our contribution, the great possibilities provided by the technological tools, if we organize to use them "to do better things, instead of for just doing (old) things better" [1].

6 Acknowledgements

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On a Study of Improvements of the Quadratic Curves Addition Method: Consideration of Theorem 8

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Abstract

In a quest to find the centers of circles inscribed or circumscribed by multiple circles, the author has developed a practical and applicable solution-the quadratic curves addition method. This method, which involves the addition of ellipses, hyperbolas, and parabolas to a drawing made with a ruler and compass, is not just a theoretical concept. The author has constructed the quadratic curves addition method with eight theorems, with Theorem 8 being the focus of this study. Its relation to other theorems is discussed, and we will demonstrate its practicality by using it to draw mathematically correct figures for Wasan and Sangaku, using Cinderella and KETCindy.

1 Introduction

The author believes that the quadratic curves addition method finds the center of a circle inscribed or inscribed in multiple circles by adding ellipses, hyperbolas, and parabolas to a drawing made with a ruler and compass.

The author presented a paper entitled "Figure Drawing using KETCindy and its Application to Mathematics Education" at the 20th ATCM International Conference held at Leshan Vocational and Technical College, Leshan Normal University, China ([1]).

The presentation described a drawing system using Cinderella and KETCindy developed by Prof. Setsuo Takato's research group. This system is capable of drawing figures in both GUI and CUI. For example, the author presented the problem of finding the center of a circle using parabolas, ellipses, and hyperbolas on the subject of Wasan and Sangaku figures. However, the author had not yet devised the eight theorems using the quadratic curves addition method.

The author's in-depth study of quadratic curves allowed him to construct the quadratic curves addition method using eight theorems. As a result, the author established a method for finding the center of a circle inscribed or circumscribed by multiple circles ([2], [3]).

In Section 2, the author outlines the eight theorems that comprise the quadratic curves addition method, quoting from his previously published papers ([3],[4]).

In Section 3, the author discusses the results of deeper considerations of Theorem 8.

Section 4 presents conclusions and future work.

2 About the Quadratic Curves Addition Method

For example, the center of a circle tangent to a definite circle, and a definite line is on a parabola, so we specify the foci and directrix that define the parabola. The center of a circle tangent to two constant circles should also detect the foci that make up an ellipse or hyperbola and specify the sum or difference of the distances from the two foci.

In this presentation, the quadratic curve used to determine the center of a circle is represented as an equation on the coordinate plane based on the definition. It is shown that the coordinates of the circle's center can be determined as the solution of the simultaneous equations and the circle's radius. Note that CAS(Computer et al.) was actively used to solve the simultaneous equations.

The quadratic curve is not the equation $f(x, y) = 0$, but a relational expression based on defining a quadratic curve in **elementary geometry**.

The dynamic geometry software Cinderella and KETCindy were used to draw the figures. All figures in this paper were created using Cinderella, KETCindy, and LaTeX.

2.1 The three fundamental theorems

2.1.1 Theorem 2: Use of parabola

Theorem 1 Figure 1 shows a straight line ℓ and a fixed point not on F. F that does not lie on ℓ. The center of circle P lies on a parabola (dashed line) with ℓ as the semilinear line and F as the focus. In other words,

$$\mathrm{PF} = \mathrm{PH}$$

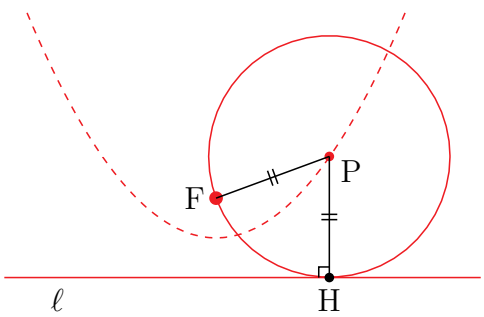


Figure 1: Circle P is tangent to line ℓ through point F.

Example: Problem with parabola

Problem 1 As shown in Figure 2, there is a square ABCD and a quadrant B. Find the center P of the circle.

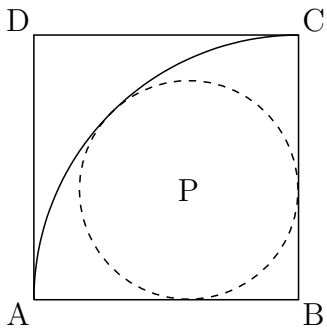


Figure 2: Circle P is tangent to the quadrant, sides AB and BC.

Solution1 : As shown in Figure 3, let F be the intersection of the diagonal BD and the quadrant; P lies on a parabola with F as the focal point and AB as the quasi-line. P also lies on the diagonal BD.

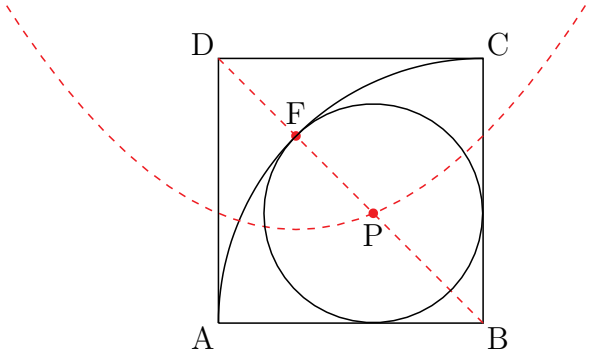


Figure 3: Point of intersection of the diagonal with a parabola.

Solution2 : As shown in Figure 4, let F be the intersection of the diagonal BD, and the quadrant P lies on a parabola with F as the focus and AB as the quasilinear. It also lies on a parabola with F as the focal point and BC as the semilinear line.

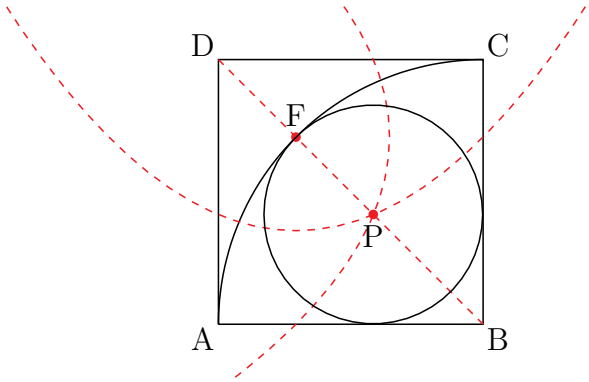


Figure 4: Point of intersection of two parabolas.

Solution3 : As shown in Figure 5, let F be the intersection of the diagonal BD, and the quadrant P lies on a parabola with F as the focus and AB as the quasilinear. It also lies on a parabola with F as the focal point and BC as the semilinear line.

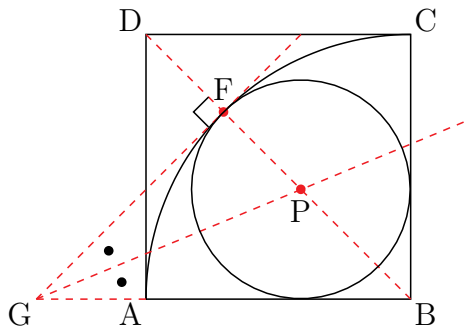


Figure 5: Bisector of the angle between the tangent to the quadrant and the line AB.

The third solution is not directly related to the quadratic curves addition method but is included for the reader's reference.

2.1.2 Theorem 2: Use of ellipse

Theorem 2 Figure 6 shows a fixed circle F and a fixed point F′ inside the circle. The center of circle P passing through the point F′ and tangent to the circle F lies on the ellipse (dashed line) with two points F and F′ as foci. The sum of the distances PF + PF′ is the radius r of the circle F. In other words, the radius r of the circle F is the sum of

$$\mathrm{PF} + \mathrm{PF}' = (\mathrm{FQ} - \mathrm{PQ}) + \mathrm{PF}' = r$$

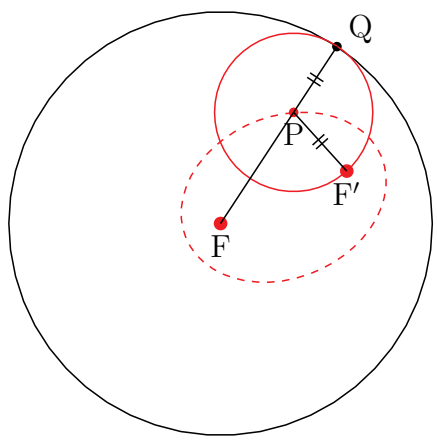


Figure 6: Circle P is inscribed in circle F through point F′.

Example: Problem with ellipse

Example 2 As shown in Figure 7, two quadrants, A and B, exist in the interior of square ABCD. Find the center P of the circle.

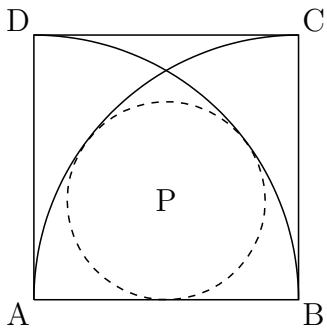


Figure 7: Circle P is tangent to two quadrants and sides AB.

Solution 1 : As shown in Figure 8, the circle P passes through the midpoint M of the side AB. Therefore, the center P of the circle lies on the perpendicular bisector of the side AB and on the ellipse, whose foci are the two points A and M.

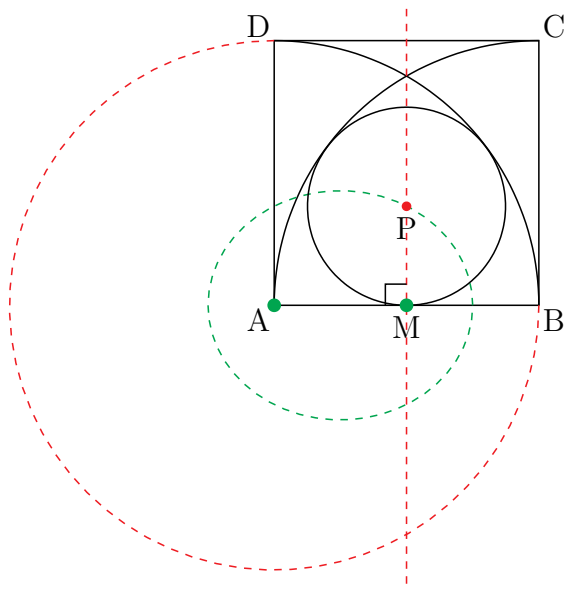


Figure 8: The center of circle P lies on the perpendicular bisector of side AB and the ellipse.

Solution 2 : As shown in Figure 9, the circle P passes through the midpoint M of the side AB. Thus, the center P of the circle lies on the ellipse whose foci are the two points A and M and on the ellipse whose foci are the two points B and M.

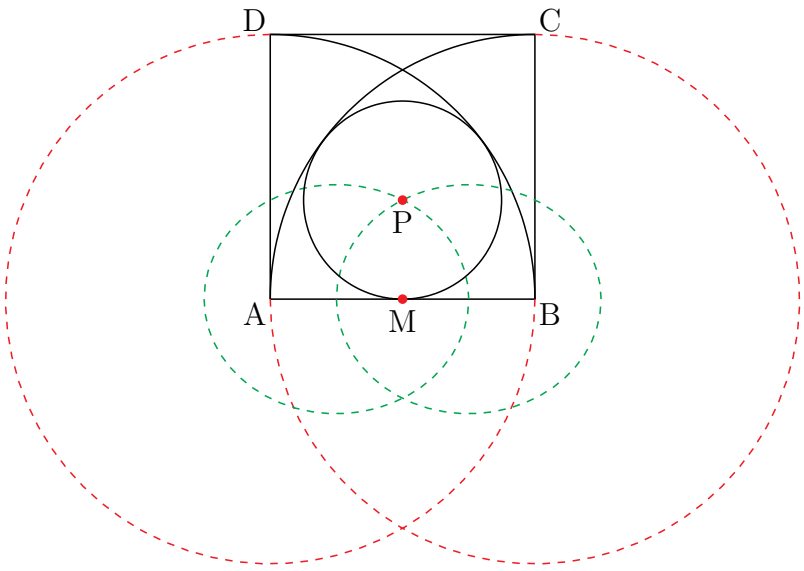


Figure 9: The center of circle P is the intersection of two ellipses.

2.1.3 Theorem 3: Use of hyperbolic curves

Theorem 3 As shown in the following Figure 10(a) and (b), there is a fixed circle F and an external fixed point F′. The center of circle P passing through F′ and tangent to the circle F lies on the hyperbola (dashed line) with two points F and F′ as foci. The difference $|\mathrm{PF} - \mathrm{PF}'|$ is the radius r of the circle F in both (a) and (b). In other words

(a): $|\mathrm{PF} - \mathrm{PF}'| = \mathrm{PF} - \mathrm{PF}' = (\mathrm{PQ} + \mathrm{FQ}) - \mathrm{PF}' = (\mathrm{PQ} + r) - \mathrm{PF}' = r$

(b): $|\mathrm{PF} - \mathrm{PF}'| = \mathrm{PF}' - \mathrm{PF} = \mathrm{PF}' - (\mathrm{PQ} - \mathrm{FQ}) = \mathrm{PF}' - (\mathrm{PQ} - r) = r$

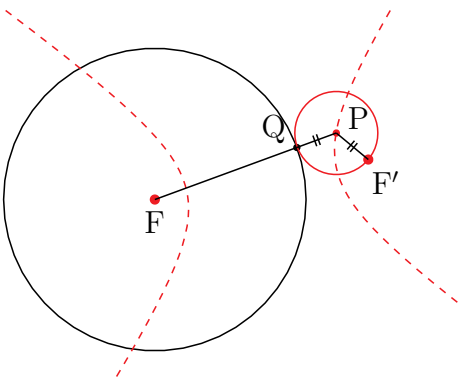


Figure 10(a): Circle P passes through the point F′ and is circumscribed by circle F.

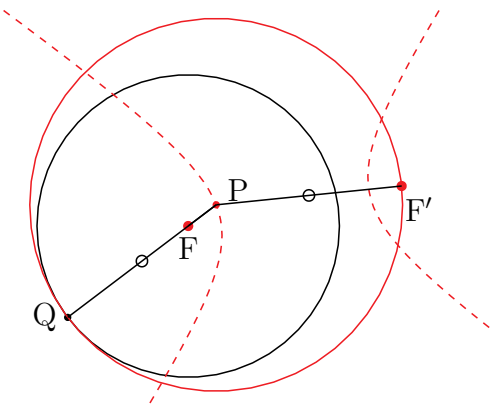


Figure 10(b): Circle P passes through the point F′ and is inscribed in circle F.

Example: Problem with hyperbola

Problem 3 Figure 11 shows two circles, A and B, of equal radii in a circle O. Find the centers P and Q of the circles.

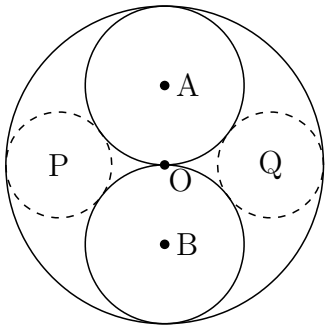
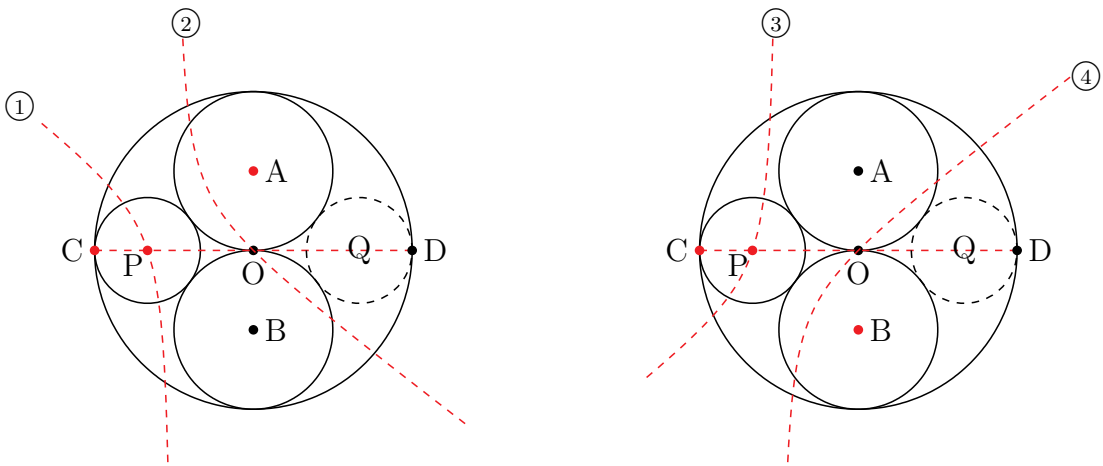


Figure 11: Circles P and Q are inscribed in circle O and circumscribed by circles A and B.

Solution 1 : As shown in Figure 12(a), the circle P passes through the endpoint C of the diameter CD. Thus, the center of circle P is the intersection of the hyperbola whose foci are the diameter CD and the two points A and C(hyperbola curve ①). It can also be obtained as the intersection of the hyperbola with diameter CD and two focal points, B and C(hyperbola curve ③), as in Figure 12 (b).



(a): Hyperbola with foci A and C. (b): Hyperbola with foci B and C.

Figure 12: The center of circle P lies on the perpendicular bisector of side AB and the ellipse.

Solution 2 : The center of circle P is the intersection of the two hyperbolas(hyperbola curve ① and hyperbola curve ②) in Figure 12(a) and (b). The center of circle Q can be obtained in the same way.

Consideration : In Figure 12(a), the hyperbola curve ② passes through the center of circle O, as in (b) of Theorem 3 (Figure 10(b)). In Figure 12(b), the same is true for hyperbola curve ④.

2.2 The five applied theorems

2.2.1 Theorem 4: Application of parabola

Theorem 4 As shown in the following Figure 13, there is a fixed circle F and its tangent line ℓ. The center P of the circle tangent to both the circle F and the line ℓ on the same side as the circle F lies on a parabola (dashed line) with the point F as the focal point and the line ℓ' as the quasi-line, in other words.

$$\mathrm{PF} = \mathrm{PQ} + \mathrm{QF} = \mathrm{PH} + \mathrm{FT} = \mathrm{PH} + \mathrm{TL} = \mathrm{PH} + \mathrm{HK} = \mathrm{PK}$$

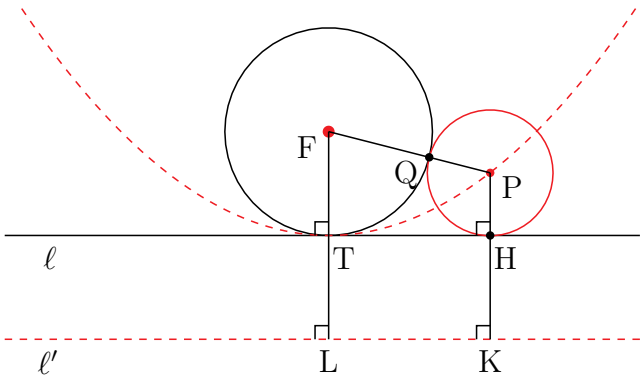


Figure 13: Circle P is tangent to circle F and straight line ℓ.

2.2.2 Theorem 5: Application of Ellipses

Theorem 5 As shown in the following Figures 14 (a) and (b), there is a constant circle F and a constant circle F′ inside F. The center of circle P tangent to both of these two circles lies on the ellipse (dashed line) whose foci are the two points F and F′. The sum of the distances from the two points F and F′ to P, and PF + PF′ is as follows (a), (b).

(a): $\mathrm{PF} + \mathrm{PF}' = (\mathrm{FQ} - \mathrm{PQ}) + (\mathrm{PR} + \mathrm{F'R}) = (r - \mathrm{PQ}) + (\mathrm{PR} + r') = r + r'$

(b): $\mathrm{PF} + \mathrm{PF}' = (\mathrm{FQ} - \mathrm{PQ}) + (\mathrm{PR} - \mathrm{F'R}) = (r - \mathrm{PQ}) + (\mathrm{PR} - r') = r - r'$

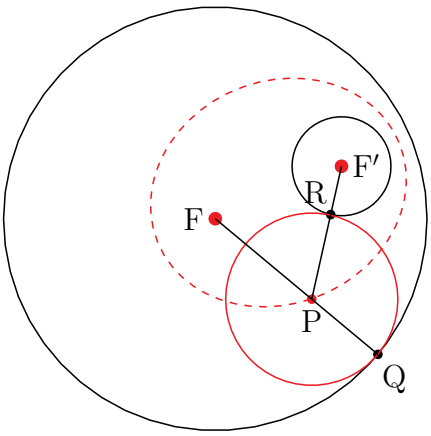


Figure 14(a): Circle P is inscribed in circle F and circumscribed in circle F′.

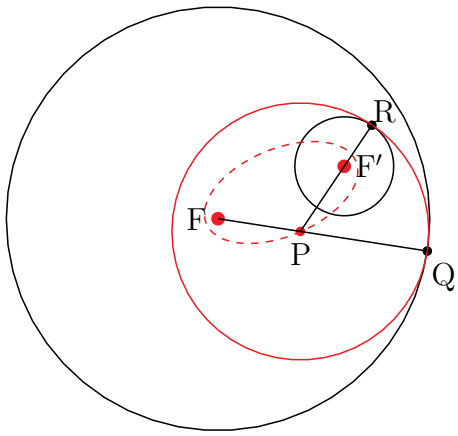


Figure 14(b): Circle P is inscribed in circle F, F′.

2.2.3 Theorem 6: Application of Hyperbola 1

Theorem 6 As shown in the following Figure 15 (a) and (b), there are two circles F, F′ that are external to each other, and the center of circle P that touches both circles lies on a hyperbola (dashed line) with two points F, F′ as foci. The difference $|\mathrm{PF} - \mathrm{PF}'|$ between the two points F, F′ and P is $r - r'$ in both (a) and (b).

(a): $\mathrm{PF} - \mathrm{PF}' = (\mathrm{PQ} + \mathrm{FQ}) - (\mathrm{PR} + \mathrm{F'R}) = (\mathrm{PQ} + r) - (\mathrm{PR} + r') = r - r'$

(b): $\mathrm{PF}' - \mathrm{PF} = (\mathrm{PR} - \mathrm{F'R}) - (\mathrm{PQ} - \mathrm{FQ}) = (\mathrm{PR} - r') - (\mathrm{PQ} - r) = r - r'$

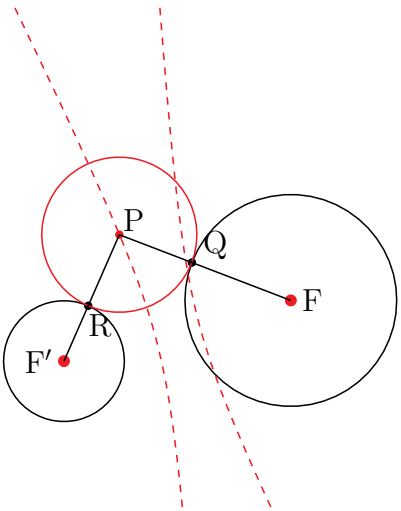


Figure 15(a): Circle P is circumscribed by F, F′.

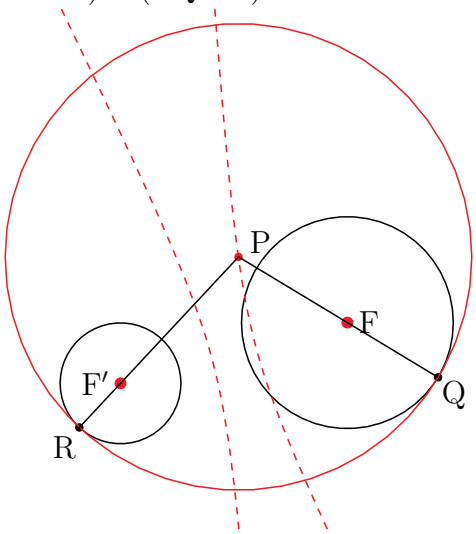


Figure 15(b):Circle P is inscribed in F, F′.

2.2.4 Theorem 7: Applications of Hyperbola 2

Theorem 7 As shown in the following Figure 16 (a) and (b), there are two circles F, F′ that are external to each other. The center P of the circle inscribed in one circle and circumscribed in the other circle lies on a hyperbola (dashed line) with two points F, F′ as foci. The difference $|\mathrm{PF} - \mathrm{PF}'|$ between the two points F, F′ is $r + r'$ in both (a) and (b).

(a): $|\mathrm{PF} - \mathrm{PF}'| = \mathrm{PF} - \mathrm{PF}' = (\mathrm{PQ} + \mathrm{FQ}) - (\mathrm{PR} - \mathrm{F'R}) = (\mathrm{PQ} + r) - (\mathrm{PR} - r') = r + r'$

(b): $|\mathrm{PF} - \mathrm{PF}'| = \mathrm{PF}' - \mathrm{PF} = (\mathrm{PR} + \mathrm{F'R}) - (\mathrm{PQ} - \mathrm{FQ}) = (\mathrm{PR} + r') - (\mathrm{PQ} - r) = r + r'$

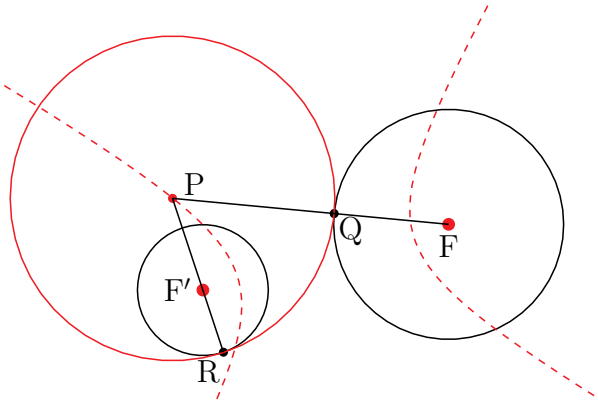


Figure 16(a): Circle P is circumscribed by F and inscribed by F′.

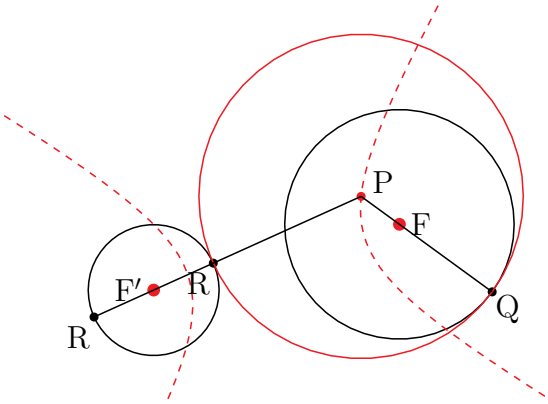


Figure 16(b): Circle P is inscribed in F and circumscribed by F′.

2.2.5 Theorem 8: Applications of Hyperbola 3

Theorem 8 As shown in the following Figure 17, when two circles F and F′ intersect at two different points, the center of circle P tangent to the standard part of the two circles F and F′ is on the hyperbola (dashed line) with two points F and F′ as foci. However, the difference |PF − PF′| that is between the two points F, F′ and P is $r - r'$.

$$\begin{aligned}|\mathrm{PF} - \mathrm{PF}'| = \mathrm{PF} - \mathrm{PF}' &= (\mathrm{FQ} - \mathrm{PQ}) - (\mathrm{F}'\mathrm{Q}' - \mathrm{PQ}') \\ &= (r - \mathrm{PQ}) - (r' - \mathrm{PQ}') = r - r'\end{aligned}$$

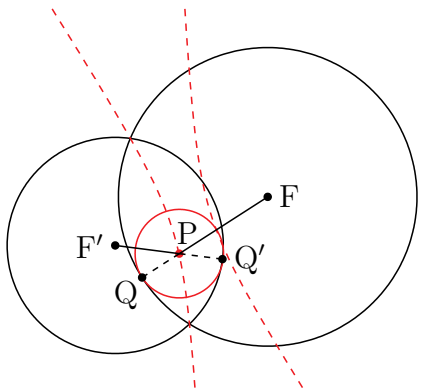


Figure 17: Circle P is inscribed in both circle F and circle F′.

3 On the consideration of Theorem 8

Problem 1: "As shown in Figure 18, there are 3, 4, and 5 right triangles. When drawing semicircles with sides AB and BC as diameters, find the maximum area of the circle inscribed in the common part of the semicircles."

Answer: $\frac{1}{4}\pi$

This is a problem created by a first-year junior high school student who imitated Sangaku.

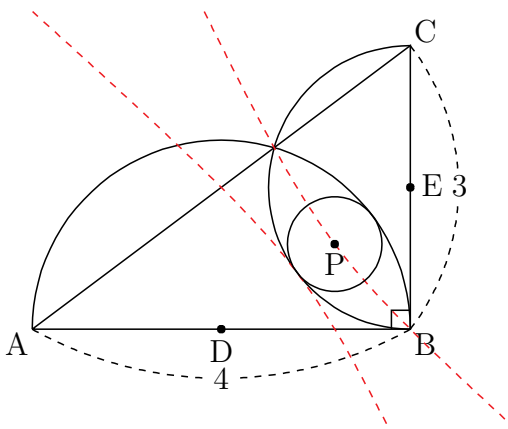


Figure 18: Circle P is inscribed in the common part of the semicircles.

The author became interested in the relationship between Theorem 5, Theorem 6, Theorem

7, and Theorem 8 after encountering the problem created by this student.

The author would like to investigate Theorem 8 in more detail[1].

3.1 Considerations

Consideration 1 As shown in Figure 19 (a), if the center of circle P lies on the extension of the hyperbola in Theorem 8, then we have the case of Theorem 6 (a). (See Figure 15 (a) and (b).)

That is when two circles circumscribe circle P.

Consideration 2 As shown in Figure 19 (b), if the circle's center is on the other hyperbola, then Theorem 6(b) is the case. (see Figure 15 (a) and (b)).

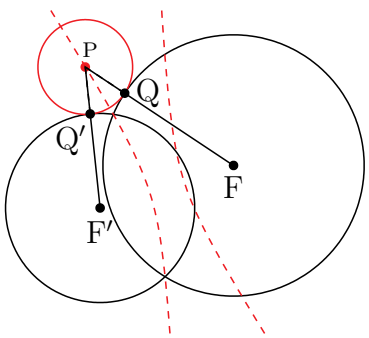


Figure 19 (a): Circle P is circumscribed by both circle F and circle F′.

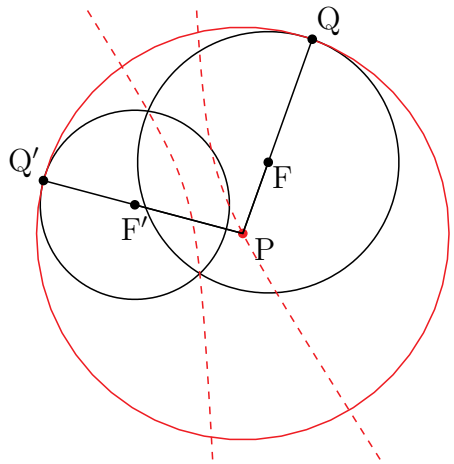


Figure 19 (b): Circle P is inscribed in both circle F and circle F′.

Conclusion 1: From the above, we see that Theorem 8 is a particular case of Theorem 6.

Consideration 3 As shown in Figure 20, it is inscribed in the circle F and is circumscribed by the circle F′.

The center P of the circle is the case of Theorem 5 (a).

The case of circle P that is circumscribed by circle F and inscribed in circle F′ is also the case of Theorem 5 (a).

[1]Other questions include: of the two semicircle intersections (one of which is self-evidently point B), is it true that the other one also appears to be on the oblique side AC? Furthermore, does this hold for right triangles with side lengths other than 3, 4, 5, and so on.

Conclusion 2: As shown in the following Figure 20, a circle with two points F and F′ as foci and a point P on the ellipse whose sum of the distances from F and F′ is the sum of the radii of the circles F and F′ can be made to be inscribed in one of the two circles and circumscribed in the other. However, the intersection of the two circles is excluded.

This can be considered the case of Theorem 5 (Figure 14 (a)).

$$\mathrm{PF} + \mathrm{PF}' = (\mathrm{PQ} + \mathrm{QF}) + (\mathrm{F}'\mathrm{Q}' - \mathrm{PQ}') = r + r'$$

However, in this case, the ellipse whose sum of the distances from F and F′ is the difference of the radii of the two circles F and F′ as in Theorem 5 (Figure. 14 (b)) cannot be drawn.

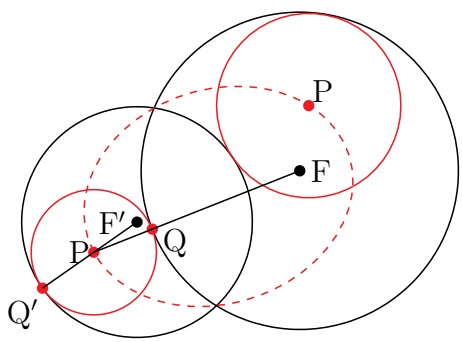


Figure 20: Circle P is inscribed in one of the two circles and circumscribed in the other.

4 Conclusion and future work

The quadratic curves addition method is a plotting method realized by considering where to take the focal point and the semilinear line, which are characteristics of a quadratic curve, for a plot by ruler and compass. Since there are two hyperbolas, we have divided them into three theorems. Therefore, Theorems 6, 7, and 8 of the hyperbola exist while interrelated. In addition, some cases can be explained either by hyperbola or ellipse. We plan to create a collection of cases that can be drawn using quadratic curves from Sangaku problems. In doing so, our future task is to draw the diagrams accurately and provide them to students as teaching materials. The diagrams used in this study were created using LaTeX, Cinderella, and KETCindy. We want to deepen our practice in this respect as well.

Using the proposed quadratic curve for drawing is expected to contribute to teaching mathematics classes in upper secondary schools as inquiry-based learning.

Although the quadratic curves addition method is classified into eight theorems, consistency, and teaching materials that align with the high school curriculum are needed. As teaching materials, we believe quotations from Sangaku's problems would be helpful.

Acknowledgements

The author would like to thank the reviewers for their time and valuable comments.

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Mathematical Modeling in a 3D Printing Project

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Abstract: *This paper presents an experience in mathematical modeling conducted with master's degree students in the Innovation in Education program, focusing on a project involving 3D modeling and printing. The project aimed to integrate hands-on mathematical activities into teacher training. The positive feedback from students and the mathematical competencies developed during the project support the inclusion of such activities in the teacher education curricula. This study highlights the potential of 3D modeling and printing projects to enhance mathematical understanding and pedagogical skills among future educators.*

1. Introduction

Mathematical modeling is a process that allows to connect mathematics with the world around us, applying and inventing mathematics to solve problems [6]. It is configured as a mathematical competence developed through the creation of mathematical models explaining real-life phenomena [14].

The inclusion of digital tools produces new opportunities for the learning and teaching of mathematical modeling, as they allow multiple and interactive representations of abstract concepts, providing new ways of visualizing, understanding, evaluating, and interpreting real-world situations [10]. Research on the educational potential of digital technologies in mathematical modeling is attracting recent interest, providing evidence of their usefulness in the modeling process, and revealing that the benefits outweigh the difficulties in successfully using the technology in modeling [5].

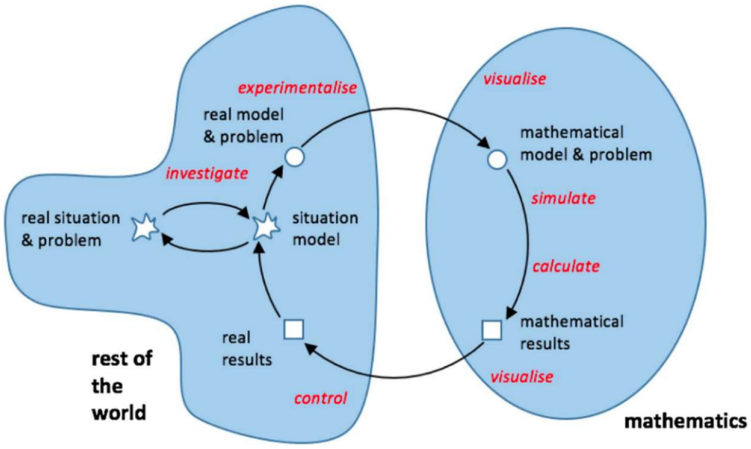

Figure 1. Modeling cycle of Greefrath [9], based on Blum and Leiß [4].

The interrelation between the mathematical and the real world can be described from the model of Blum and Leiß [4] by means of the so-called modeling cycle (see Figure 1). The model consists of 7 steps: starts with a real situation or problem that, to be solved, requires an understanding and simplification of the situation to establish a *real model* of the problem. This model is translated into the mathematical world through *mathematization*, and the mathematical work (calculations, reasoning, equation solving, etc.) produce *mathematical results,* which can be interpreted and validated as *real results* in the real world. This process is cyclic, since the validation of these results will determine whether the desired result has been reached, either concluding with the presentation of these results, or with the reformulation of the model when another round must begin [2,3].

The employment of digital tools during a modeling process influences each part of the cycle [9]. Some examples are shown decorating in red the modeling cycle in Figure 1, including the steps *investigate*, *experimentalize*, *simulate*, *calculate*, *visualize* and *control,* as well as the steps of *construct*, *draw* and *measure* from Figure 2 [10, p. 236] in the context of a Dynamic Geometry Software.

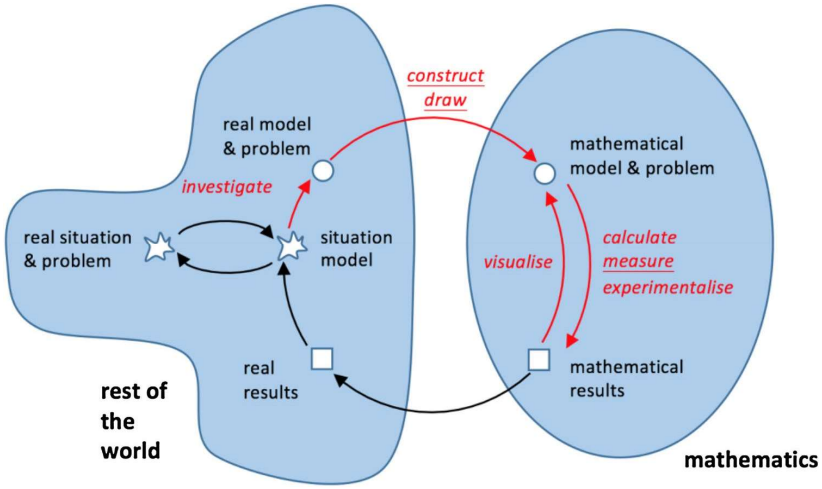

Figure 2. Modeling cycle of Greefrath [10], in the context of a Dynamic Geometry Software.

An example of activity that develops mathematical modeling skills are the so-called 3D modeling and printing activities [12]. These activities propose to students a real problem that they have to solve through the creation of prototypes or three-dimensional models with 3D design tools that can be replicated and validated thanks to the use of 3D printing. This type of projects is considered as tools with great educational potential that improve both participation and motivation in the learning process, especially in the context of STEAM education [1,12].

From a mathematical point of view, these are activities that encourage students' visualization and spatial reasoning using computer-aided design CAD software [7,13]. They allow students to work collaboratively, combine theory with practice, and provide teachers with a tool to design manipulative materials [8]. In addition, the process of validating the models with real objects allows students to self-assess their own results [1].

Finally, we would like to emphasize the need to promote teacher training programs that include STEAM activities and projects that improve knowledge about this approach and provide the necessary tools to implement it in the classroom [15]. In particular, and related to the present work, several authors argue that both modeling and 3D printing activities represent enriching activities for future mathematics teachers [11].

The primary objective of this study was to investigate the development of mathematical modeling and visualization skills in graduate students through a project-based learning approach using the 3D modeling software Tinkercad and printing technology.

2. Methodology

2.1 Participants

This study involved a group of 13 graduate students enrolled in the subject *The use of space in Mathematics Education* of the Master of Innovation in Specific Didactics at Universidad Autónoma de Madrid (UAM) during 5 weeks in the academic year 2023/24. Among the participants, 11 of them were graduated in Early Childhood and/or Primary Education and the other 2 had no background in Education. The group consisted of 9 females and 4 males, with ages ranging from 22 to 35 years. None of the participants had previous experience with 3D printing or with the software Tinkercad.

2.2 Materials

Two resources were used to carry out the project: the three-dimensional modeling program Tinkercad and 3D printers. Tinkercad (https://www.tinkercad.com/), an online 3D CAD design tool, was used by students to create and simulate their 3D models. This software is specifically tailored for educational purposes, offering a user-friendly interface to quickly learn and apply the 3D principles of design. Its accessibility via web browsers eliminates the need for high-performance computing equipment, making it ideal for educational settings. The software supports direct printing capabilities, allowing designs to be easily exported and prepared for 3D printing. Two 3D printers (a Creality Ender-5 and a Prusa i3 Mk3/Mk3s) were made available in the university's technology lab, providing students the opportunity to bring their digital models into tangible forms. These printers were capable of printing designs with PLA plastics, which are commonly used in educational environments due to their cost-effectiveness and ease of use.

Students compiled dossiers that included detailed accounts of their design processes, mathematical calculations, and iterations of the 3D models. These dossiers served as a concrete record of the application of mathematical concepts and the evolution of their project work, from initial design to final validation. The project dossiers also included reflective journals; in them, students talked about some of the difficulties, challenges and lessons learned.

2.3 Procedure

The students were asked to carry out a modeling project of their choice, where they had to jointly choose a real space/object to replicate in 3D. All the selected spaces/objects must have a clear relationship among them, as well as to keep the proportion. The project was called *Foods of the World*, where each student had to recreate a typical dish from their region or country in order to reflect and share the different customs and cultures represented in the group. Throughout the course, 4 of the sessions were used to resolve doubts, to correct frequent modeling errors and to reorganize the size and scale of the figures so that the overall project would have coherence. All the objects were shown in a final exhibition.

The main objective was for the students to learn how to use the Tinkercad tool and to perform modeling cycles (Figure 2) throughout the design process: choosing the object and obtaining its real measurements, designing the 3D model, 3D printing, validating the model, re-designing the 3D model, re-printing the 3D model, re-validation, etc., based on the Greefrath model [10]. The process was reflected in a dossier that included the steps followed, and the mathematical elements contained in their 3D designs.

Throughout the sessions the students had a constant experimentation and knowledge of both the Tinkercad software and the 3D printing process (3D design, lamination and printing phase). Most of the time the students worked autonomously on their models and when they finished a model it was printed and proceeded to validation. A modeling cycle is considered to have been completed when the student obtains the 3D printed object and can therefore carry out its validation with the real starting object and with the rest of the objects in the project.

2.4 Data Collection and Analysis

Participants documented the various phases of the modeling process in their project dossiers. These dossiers included screenshots from the Tinkercad software, capturing the different stages, refinements and re-designs of their 3D models. Additionally, participants included photos of the real-world objects they aimed to model, alongside images of the 3D printed versions of their designs. Dossier projects detailed the types of geometric figures intended for modeling and provided insight into the cognitive and affective experiences of the participants. This comprehensive visual documentation was accompanied by reflective journals, where participants recorded their thoughts and experiences throughout the design process. These reflective journals not only offered students an opportunity to articulate their learning experiences but also provided researchers with qualitative data.

The qualitative analysis was focused on identifying the different phases of the modeling process undertaken by the participants. This involved a detailed examination of the modeling and redesign cycles generated by the students. The analysis drew on data from the project dossiers and observational notes taken during lab sessions. These notes provided context and additional insights into the participants' interactions, problem-solving strategies, and use of technology.

3. Results

The project "*Foods of the World*" was done during year 2023/24, Figure 3 shows some of the dishes chosen by the students, such as a wine set from Valladolid (Spain), "*humita"* from Chile and a "*espeto"* (skewer) of sardines from Málaga (Spain).

Figure 3. Final productions of the 3D modeling project.

In addition to choosing a typical dish from their region, the students made different elements to complement and decorate their final presentation. For example, the student who chose wine products made a bottle of wine, a wine glass, a corkscrew and a bunch of grapes.

3.1 An example of a modeling cycle: the bottle of wine

The modeling process began with the selection of the real object and its measurement on which they would base their model. They made some very restrictive initial simplifications since they were not very familiar with the software either. As the modeling cycles were carried out, more complex details were added in order to obtain objects that were closer to the initial object.

Figure 4. Modeling process of the bottle.

For example, Figure 4 shows the process developed to make the wine bottle. At first, very simple figures were chosen, only cylinders and cones. Cylinders were used to make the body and the two parts of the neck of the bottle. A truncated cone was used to resemble the curved top of the body. At this stage, the inclusion of letters as a label and other details were not included.

The 3D printing of the object allows the validation of the model obtained. At this point, its resemblance to the real object is verified and changes are proposed to be made (by the teacher or by the students themselves) either to make it more similar to the real object or to correct errors. For instance, Figure 4 shows in the case of the bottle the first model made using a truncated cone. In the next iteration of the cycle, it was replaced by a semi-sphere which generated a smoother edge and a closer resemblance to the real object. Finally, after some feedback and one more modeling cycle, the letters were added as a label to give the final model. The label was made by intersecting the text with a cylinder wider than the body of the bottle to obtain the desired curvature.

Figure 5 below illustrates a simplified scheme of the modeling cycle used by the student during this process following the Greefrath modeling cycle (Figure 2).

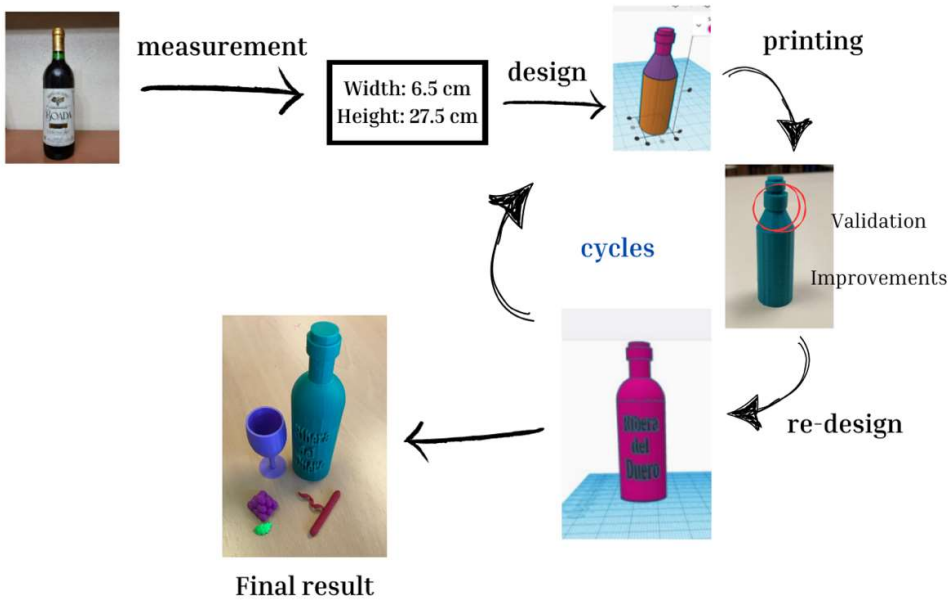

Figure 5. Modeling cycle of the wine bottle design.

The student started by taking measurements of a real object. Using the 3D modeling tool Tinkercad, he made a preliminary design that allowed him to re-enter the modeling cycle with an improved perspective. This design was subsequently materialised through 3D printing, which facilitated the evaluation and validation of the results initially obtained. Based on this evaluation, a new modeling cycle was conducted, which allowed the wine bottle model to be further refined and detailed. The final results obtained are in line with the cycle proposed by Greefrath, where 3D printing not only serves as a bridge between the mathematical results and their real-world application, but also as a means of continuous iteration within the mathematical modeling process.

3.2 Modeling cycles of the whole class

During the realization of the 3D modeling and printing project, student participation was high and, in general, students were involved. Table 1 shows the topic each student worked on, with the number of objects they designed, and the average number of modeling cycles performed. It can be observed that all the students made at least 2 modeling cycles, indeed the average of the whole class was 2.4.

Table 1. Summary of the objects and cycles realized by the students.

Project	Nos. Objects	Cycles (average)
"*Humita*" from Chile	6	2.5
"*Almogrote*" from Canary Islands	3	2.7
Calamari "*bocadillo*" from Madrid	3	3
"*Empanada*" and "*emboque*" from Chile	3	2
Chinese desserts	4	2.25
Madrilean "*cocido*"	6	2
Wine products from Valladolid	5	2
Chocolate and churros from Madrid	4	2.5
Mexican taco	4	2.25
"*Pintxo*" from the Basque Country	2	2
Canary Islands drinks	2	3
Chinese fruit brochette	6	3
"*Espeto*" (skewer) of sardines from Málaga	6	2.2

3.3 Some special cases of the modeling cycle

Although this group of students was facing for the first time the use of Tinkercad and a 3D design process, it is worth highlighting the mathematical communication developed by the students and the spatial reasoning skills they put into operation. For example, the student who made the "*humita*" from Chile used a varied selection of three-dimensional objects and constructive geometry operations (alignment, union, difference, intersection) to make her final model: the cord surrounding the "humita" was made by a rectangular prism, which she intersected with a smaller one to obtain a frame. To curve the corners, she firstly used a shape called "tube" from Tinkercad, but since it was hollow in the center, the expected result was not obtained. Finally, she decided to use a torus, which allowed her to cut the desired shape (see Figure 6).

Figure 6. Rope thread bending process.

In order to obtain the reliefs that resembled the "*humita*" leaf, she tried to use joined planes, but the result obtained was not the desired because the cuts could hardly be seen when printed. So, finally, a predefined model of a wood plank in Tinkercad had its own reliefs that allowed the intersection of both pieces to obtain the desired result (see Figure 7).

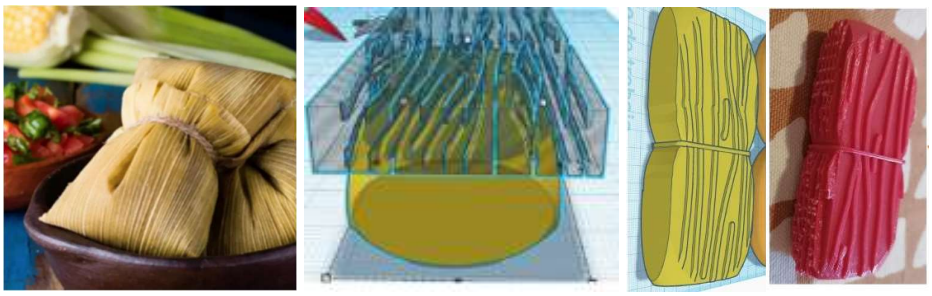

Figure 7. Image model of the real humita and relief process performed to give texture to the leaf.

It should be noted that some of the students were not satisfied with the 3D models as they came out of the printer, and they wanted to give it a more realistic touch by painting the pieces. In the case of the student who made the "*espeto"* of sardines, after painting it she obtained a result much closer to the reality (Figure 8).

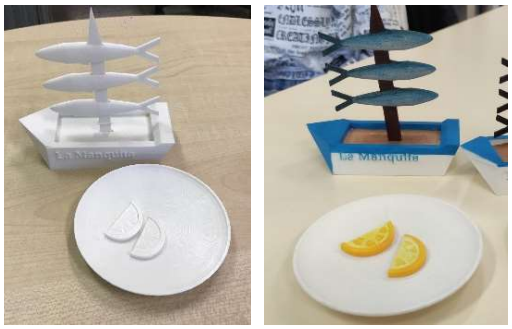

Figure 8. Final result of the "*espeto"* of sardines.

It should be noted, that, in a different activity, they took advantage of the knowledge they acquired with 3D modeling and printing. In order to give Primary School Students coming to the University

some gifts, they decided to create pirate medals with Tinkercad. After printing them, they also decided to paint them to have a more professional touch.

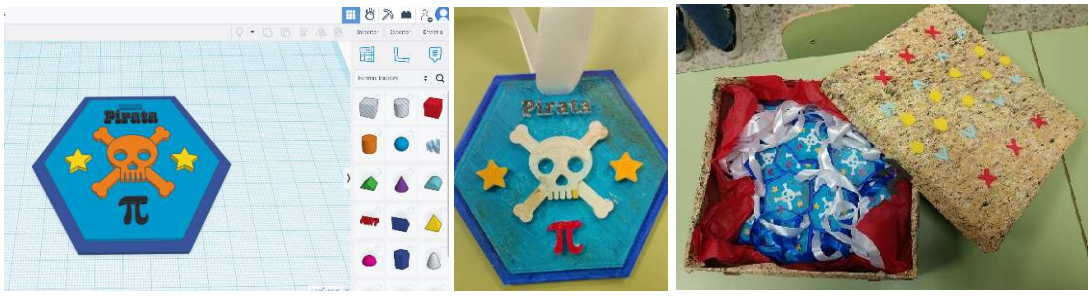

Figure 9. The pirate medals.

4. Conclusions

The 3D modeling and printing activity generated a high level of engagement and interest among the master's students. They were able to enhance their modeling skills through iterative cycles aimed at refining their outputs. As mentioned in [6] and [14], mathematical modeling linked abstract mathematical concepts directly to real-world applications, enhancing both understanding and practical engagement. Despite their initial limited expertise in both mathematical concepts and technological tools, the students showed significant growth and produced an outstanding mathematical work. This aligns with findings by Greefrath et al. [10] and Blum and Leiß [4], who emphasize the enhancement of cognitive processes through digital tools in the modeling cycle, making geometrical abstract concepts more tangible and comprehensible. The good results obtained from the experience support the relevance and suitability of including this type of mathematical modeling activities in teacher training.

There is evidence of the fundamental role played by visualization processes and skills during the creation of 3D models, reflecting the educational potential of these technologies to enhance participation, motivation, and the learning process, particularly within STEAM education frameworks [1,12]. Moreover, the use of CAD software, as noted in [7] and [13], supports the development of crucial spatial reasoning and visualization skills, further validating the integration of such digital tools in mathematical education. The inclusion of more guided activities and reflection on what kind of modeling projects can foster deeper mathematical work emerge as lines to be considered in future research.

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E-Learning material creation system that utilizes existing teaching materials

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Abstract

This paper describes a new framework for creating E-Learning materials. The new framework is intended for use by school teachers and can transform an existing teaching material into an E-Learning system utilizing HTML5 technology. The E-Learning system created is a web application consisting of a single HTML file, and can be used not only on PCs, but also on tablets and smartphones.

The E-Learning system created with this framework features a two-layered screen structure: existing teaching materials are placed on the lower layer of the two-layered structure, and text boxes and buttons are added on the upper layer, allowing existing teaching materials to be converted to E-Learning systems. In addition, by incorporating Algebrite ([4]), a JavaScript library for symbolic computation, it is possible to perform symbolic computation of mathematical expressions and automatically grade students' answers.

Since this paper is an extension of reference [3], we first briefly describe the contents of the reference. Then, after explaining the newly added functions, the flow of actually constructing an E-Learning system using this system will be explained step by step.

1 Introduction

With the spread of the Internet and the development of computer technology, E-Learning is playing an increasing role in education. However, E-Learning in schools is not yet widespread

enough. There are many reasons for this, one of which may be that E-Learning systems are difficult for teachers to use.

Current E-Learning systems are sold as stand-alone packages and are not designed for teachers to use in their own classes. As a result, E-Learning is often used as a stand-alone drill material, separate from the classroom, and is not fully effective. To solve this problem, references [1] and [2] proposed a framework for building E-Learning systems that can be used by teachers in elementary, middle, and high schools in conjunction with their classes. The system created with this framework consists only of HTML files and runs on a browser. Therefore, the system can be used not only on PCs but also on other browser-enabled devices such as tablets and smartphones. Furthermore, since no knowledge of HTML syntax is required to create an E-Learning system using the system, even ordinary teachers without special knowledge can create E-Learning systems.

Reference [3] extends the system to allow for the use of existing teaching materials. Here, existing teaching materials are assumed to be drill printouts. More concretely, the system in [3] makes it possible to convert existing teaching materials into an E-Learning system. In this paper, we make improvements to the system in [3] to enhance its functions. By listening to the voices of actual teachers in the field, we are enhancing the functionality that is actually needed in the field of education. In the latter half of this paper, examples of actual teaching materials are presented.

The structure of this paper is as follows. First, in Section 2, a brief description of [3], on which this paper is based, is given, followed by a description of the enhanced functions in this paper in Section 3. Next, in Section 4, we build a simple e-learning system using this framework. Finally, in Section 5, we conclude and discuss future prospects.

2 Overview of the our previous work

Reference [3] presents a framework for building an E-Learning system using existing teaching materials. The E-Learning system created with this framework is a web application consisting of a single HTML file, which can be used on a PC, tablet, or smartphone. The features of the framework are describe below;

2.1 Features

Dual-layer screen structure: The most distinctive feature of this framework is that the screen has a dual-layer structure shown in left figure of Fig. 1. The bottom screen displays existing teaching material, and the top screen displays additional buttons and text boxes. In this way, buttons and text boxes can be added to the screen without directly modifying the existing educational materials displayed on the bottom.

Use of JavaScript libraries: By using the symbolic computation library Algebrite ([4]), the system performs symbolic computation of mathematical expressions and provides an automatic grading function. The system also utilizes libraries such as Pagedown ([5]), which supports Markdown-style text input, and MathJax ([6]) for displaying mathematical expressions.

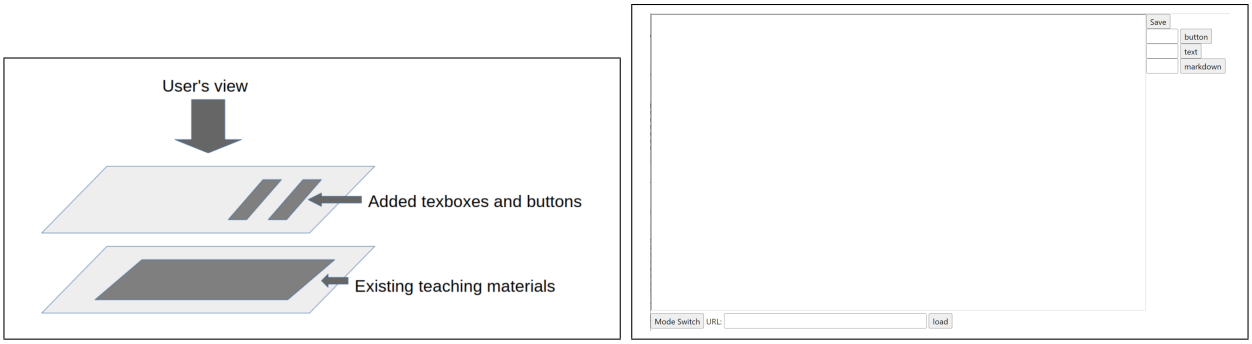

Figure 1: Dual-layer structure (left), screen at start (right)

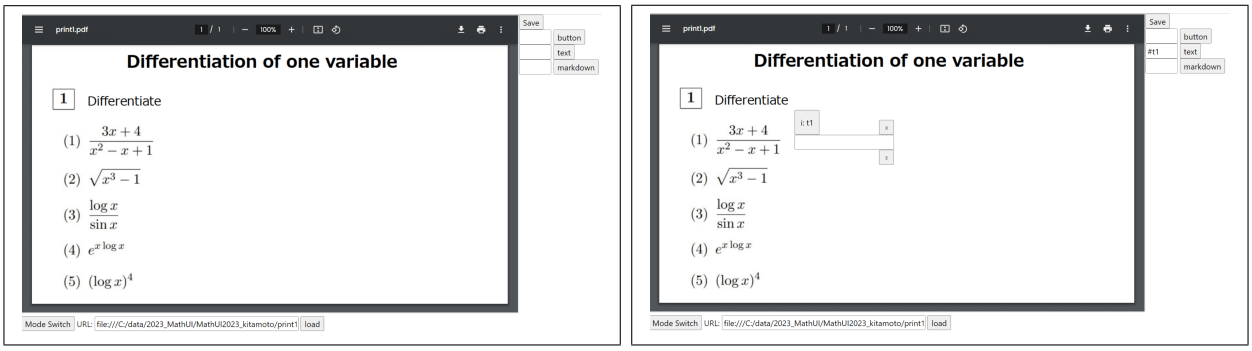

Figure 2: Existing teaching material is loaded (left), text box is inserted (right)

2.2 How to use the framework

When the system is activated, the blank window shown in right figure of Fig. 1 appears on the screen. The functions of button in the right figure of Fig. 1 are as follows;

Save: Downloads the on-screen system as a single HTML file (you click the button when creation of the E-Learning system is complete).

text: Displays a text box that allows you to adjust its position and size on the screen.

button: Displays a button that allows you to adjust its position and size on the screen. You can also assign JavaScript code to the button.

Markdown: Displays an editor for entering text in Markdown format. You can adjust its position and size on the screen.

Mode Switch: Switch between edit mode (used by teachers) and user mode (used by students). In the user mode, buttons for moving and resizing are hidden.

load: Displays existing educational material on the screen; if you enter the name of a PDF file and click this button , then its content will be displayed on the lower layer of the screen.

Below. we will show the process of actually creating an E-Learning system using this framework step by step.

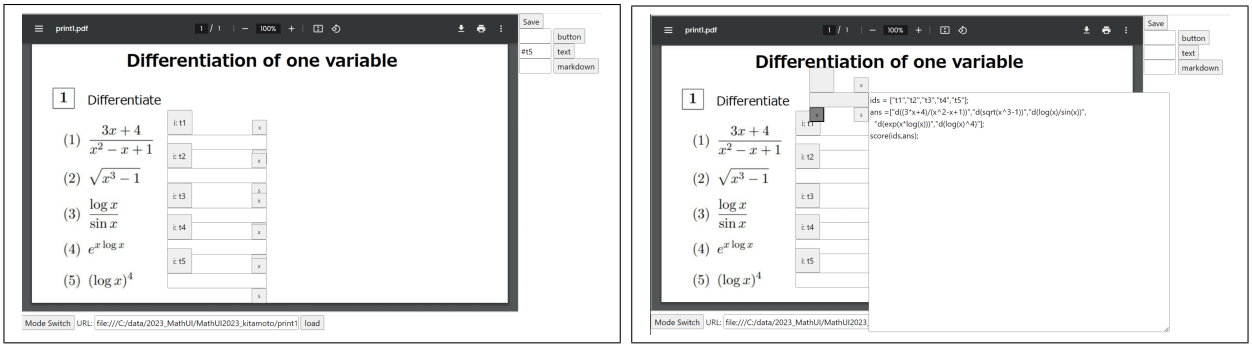

Figure 3: text boxes are inserted (left), button to execute JavaScript code is inserted (right)

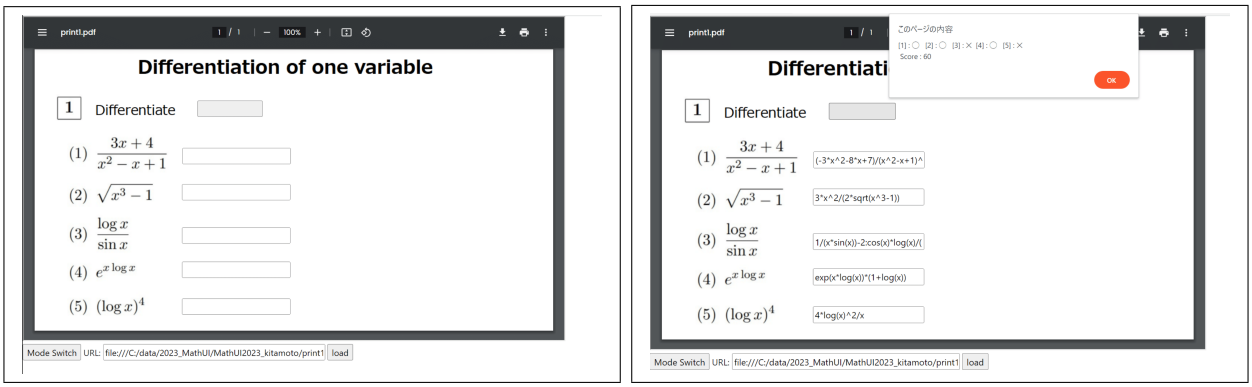

Figure 4: Screen for student use (left), screen displaying the results of scoring (right)

Step 1: First, we scan existing course materials to create a PDF file. Next, we enter the PDF file name in the text box at the bottom of the screen and click the "load" button. Then the PDF file is displayed as shown in left figure of Fig. 2.

Step 2: Next, we create a text box for entering the answers to the questions. Specify a name for the text box and click on the "text" button to display the named text box. The text box can be moved by dragging the button at the upper left corner. We place this text box to the right of the question (see right figure of Fig. 2). Follow the same procedure to place answer text boxes for the other questions (see left figure of Fig. 3).

Step3: Next, we create a button for grading and assign a JavaScript program to the button to implement the automatic grading function (see right figure of Fig. 3). This JavaScript program is executed by clicking on the button in gray in left figure of Fig. 4.

Step 4: Next, we click the "Mode Switch" button to hide unnecessary parts and display the screen for student use (see left figure of Fig. 4).

Step 5: Finally, we click the "Save" button to download the created E-Learning system as an HTML file. Once the downloaded HTML file is opened, it is ready for students to use.

You now have a simple E-Learning system. Students enter their answers in the text box on

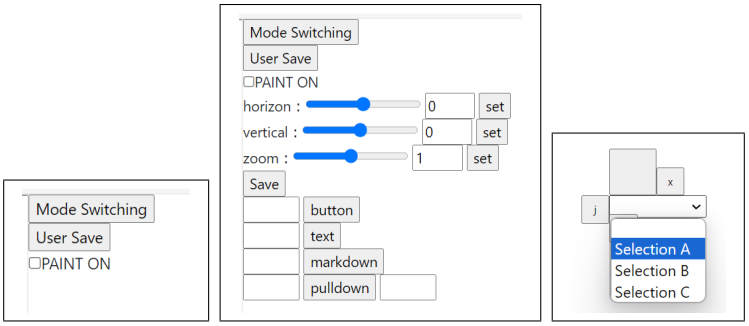

Figure 5: Menu in user mode (left), menu in edit mode (middle), pull-down menu sample (right)

the right figure of each question. When students click the button in gray in the left figure of Fig. 4, the system will automatically grade and display the results (see right figure of Fig. 4).

3 Enhanced Features

This section focuses on features extended from the system in [3]. The left figure of Fig. 5 shows the menu in user mode, and the right figure of Fig. 5 shows the menu in edit mode. Compared to the system in [3], the enhancements of the current system are as follows.

Function to adjust the position of PDF displayed on the lower layer of the screen: When using the teaching materials on tablets, PCs, Google Chromebooks, etc., the position of the existing teaching materials displayed on the lower layer sometimes shifted depending on the device. For this reason, we added a function to adjust the position of the teaching materials displayed on the lower layer and a function to zoom in and out. These adjustments can be made by moving the slide bar next to horizontal, vertical, and zoom in middle figure of Fig. 5

Function to add pull-down menus: The actual E-Learning materials used include many questions for which the answers are not numerical, and these questions cannot be scored using our previous scoring system. In order to deal with these problems, pull-down menu is added on the menu (see middle figure of Fig. 5). For a sample of the pull-down menu created by the system, see right figure of Fig. 5.

Function for drawing on the screen In response to the comments from teachers that it would be useful to be able to handwrite calculation processes and pictures when using teaching materials, we have made it possible to handwrite on the screen. This function is realized by a JavaScript program using the canvas functionality. When the check box "PAINT ON" in the left and right figures in Fig. 5 is marked, the Paint menu as shown in right figure of Fig. 5 will appear, and the user can handwrite on the screen using the menu.

Saving with the drawing retained: We have added the ability to save a file while maintaining the handwriting on the screen above and the student's answers. To use this function, click on the "User Save" button on the left and right figure of Fig. 5. Using this function,

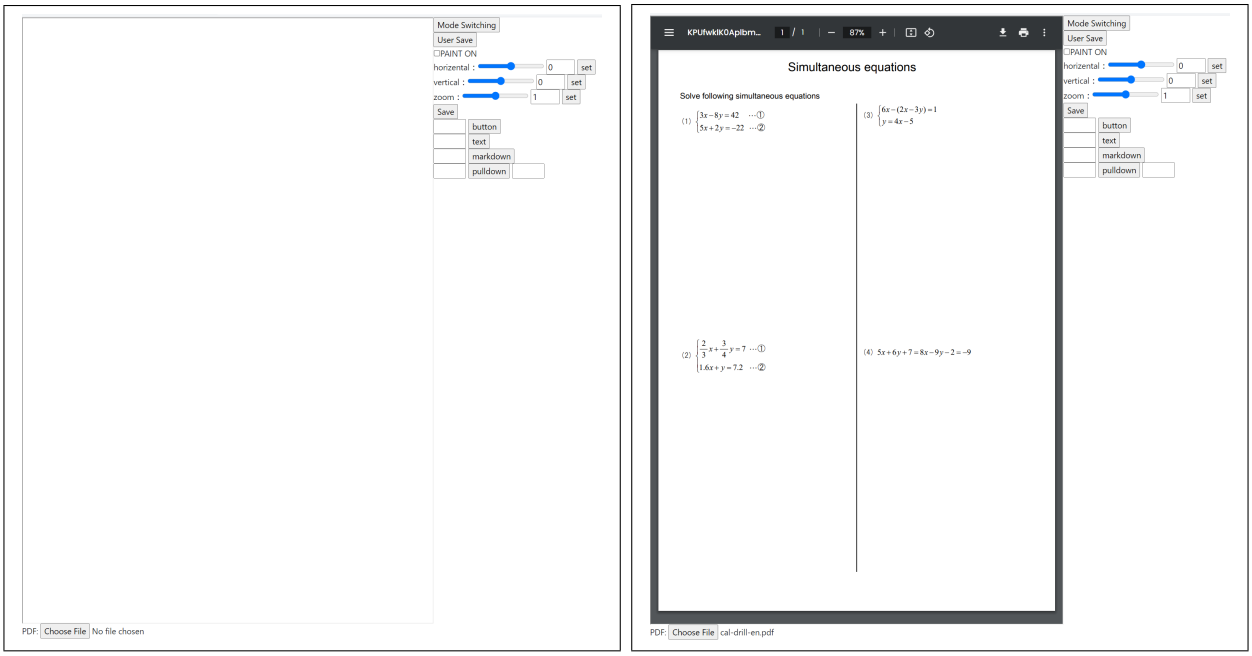

Figure 6: Menu in user mode (left), menu in edit mode (right)

students can save their answers as they are in the file and send them to the teacher by e-mail or other means.

4 Illustrative example

4.1 How to create E-Learning materials

This section describes the step-by-step process of creating a simple E-Learning material using the system described in this paper.

Step1: First, we scan a drill printout into a PDF file.

Step2: After launching the system, we will see a screen in left figure of Fig. 6. We click on the "Choose File" button at the bottom and select the PDF file created in Step1 above, and the contents of the file will appear on the screen (see right figure of Fig. 6).

Step3: We add $x =$ and $y =$ where students enter the solution. We click the "markdown" button in right figure of Fig. 6 to add a markdown editor to the screen. Clicking the button marked "c" of the editor opens a text box as shown in left figure of Fig. 7, where we type `$x=\\y=$` to display the mathematical expression as shown in left figure of Fig. 7. We click the button marked "c" again to close the text box, and move the markdown editor to the position shown in right figure of Fig. 7 by dragging the button in the upper left corner. We perform this operation for the other problems.

Step4: The next step is to add text boxes where the answers should be entered to the screen. We click on the "text" button in right figure of Fig. 7 to add a text box to the screen

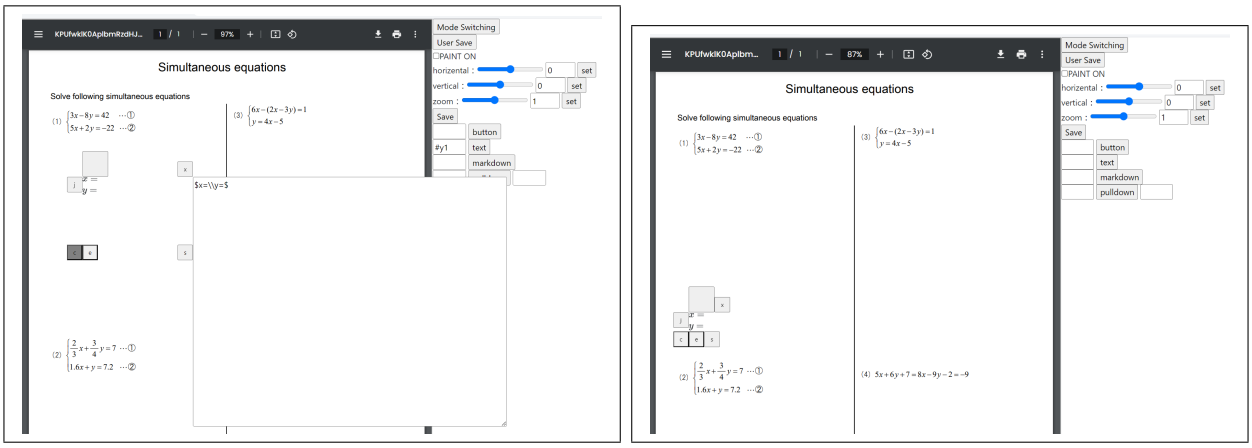

Figure 7: Markdown editor with text area open (left), deployed markdown editor (right)

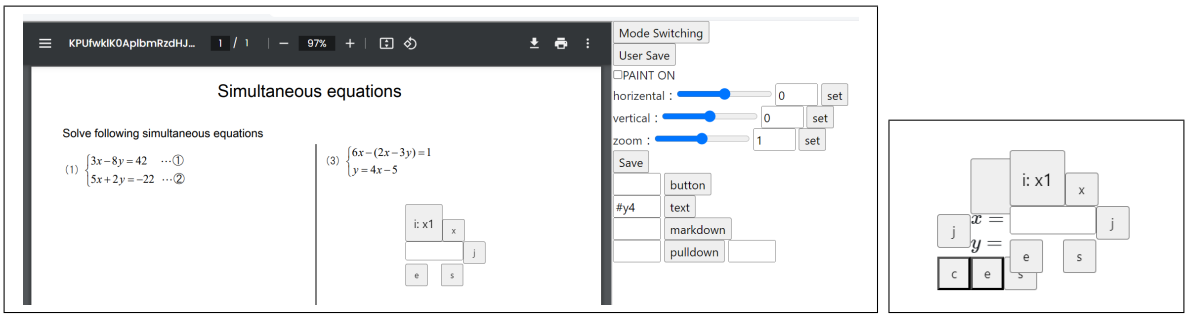

Figure 8: Text box on the screen (left), deployed text box (right)

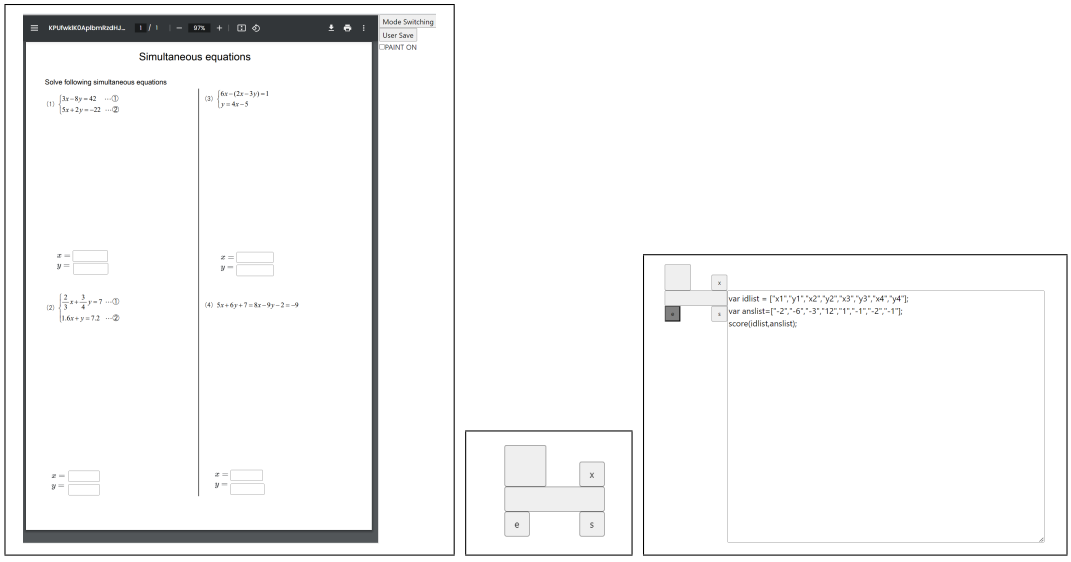

Figure 9: The screen in user mode (left), button created (middle), button with text area open (right)

(see left figure of Fig. 8). This text box can also be moved by dragging the button in the upper left button, and can be resized by the button marked "s" in left figure of Fig. 8. We placed the text box next to "$x =$" as is shown right figure of Fig. 8. Then, we placed another text box next to "$y =$" in the same way. Text boxes where answers for other problems should be entered are placed similarly. This completes the placement of the text boxes for the answers. Now, if we switch to the user mode, the screen will look like left figure of Fig. 9.

Step5: Next, we add a button for scoring. We click on the button labeled "button" in middle figure of Fig. 5, and the button in middle figure of Fig. 9 will appear on the screen. Clicking on the button marked with "e" opens a text area as shown in right figure of Fig, 9. We can write JavaScript code there and the JavaScript code entered will be executed when the button in the center in middle figure of Fig, 9 is clicked. The JavaScript program to perform the scoring is as follows;

```
var idlist = ["x1", "y1", "x2", "y2", "x3", "y3", "x4", "y4"];
var anslist=["-2","-6","-3", "12", "1","-1","-2","-1"];
score(idlist,anslist);
```

In the above program, the variable `idlist` holds the name of the text box into which the student's answers should be entered, and the variable `anslist` holds correct answers to the problems. Function `score()` is a system-defined function that uses Algebrite, a JavaScript Computer Algebra system. The function checks students' answers and displays the results on the screen. We place this button for scoring in the upper right corner of the screen (unlabeled gray button in left figure of Fig. 10).

Step6: Finally, we click the "Save" button in middle figure of Fig. 5, which downloads the E-Learning system as a single HTML file.

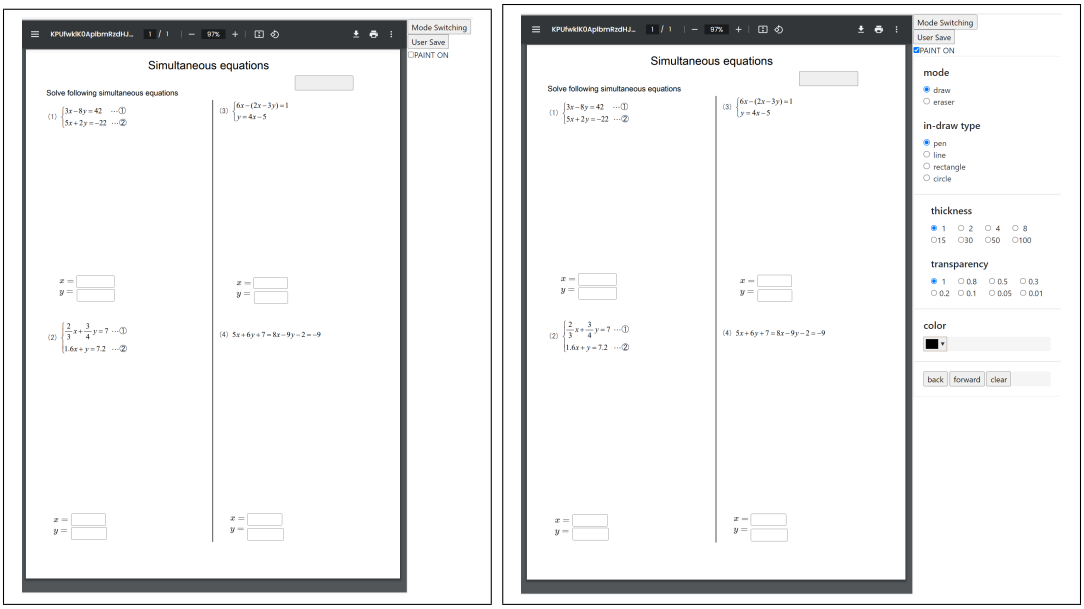

Figure 10: Screen at start up (left), screen when "PAINT ON" is checked (right)

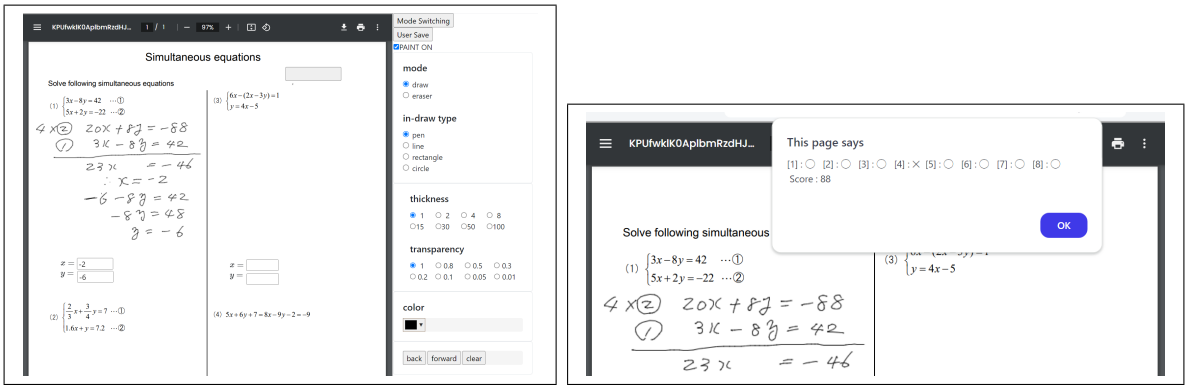

Figure 11: Screen with the calculation process handwritten (left), screen when scored (right)

4.2 How to use E-Learning materials created

When the student opens the E-Learning material we have created above, the screen in left figure of Fig. 10 is shown. When students check "PAINT ON" in the menu on the right figure, handwriting function is enabled as shown in right figure of Fig. 10. Students may use the space below the problem statement to handwrite calculation process as in left figure of Fig. 11. After entering the answers into the text boxes, students click score button (unlabeled gray button on top). Then, the answers are automatically scored and the results are displayed as shown in right figure of Fig. 11.

When students finish their work, they can click "User Save" button in the menu on the right figure of the screen, and their answers and handwriting work will be saved in a HTML file (the handwriting is preserved as an image file in png format, which is converted to a string and embedded in the HTML file).

The student can send the HTML file to the teacher via e-mail, and when the teacher opens it on his/her computer, the student's handwriting calculation process and answers are displayed as the student wrote them as shown in left figure of Fig. 11. In this way, the teacher can check the calculation process as well as the student's answers.

5 Conclusion

Reference [3] proposes a system for creating E-Learning materials that utilize existing drill printout. Using the system, an existing drill printout can be transferred to an E-Learning systems. The system can be operated interactively on a browser, and does not require programming nor special IT knowledge, making it possible for ordinary teachers to use the system.

In this paper, we extend the system by adding new functions of adjusting the screen position, pull-down menus, and handwriting on the screen. We have created a sample E-Learning material using the system and showed a demonstration to handwrite the calculation process into the E-Learning material.

Currently, we are working for two future targets. The first target is to increase the number of E-Learning materials created by our system, and the second target is to further enhance the system's functionality and expand its range of applications by utilizing JavaScript libraries and other tools.

Acknowledgment

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Towards a conceptual framework for integrating mathematical digital competencies in digital project-based learning of mathematics

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Abstract: *In this paper, we will consider the theoretical connections between Project-Based Learning (PBL) and Geraniou's Mathematics Digital Competency (MDC). PBL is generally defined by its constructivist theoretical grounding, where students are expected to answer a big question by forming groups and answering complex questions using knowledge and skills from multiple subject matter. Studies have shown that PBL contains five distinct features: collaborative learning, disciplinary subject learning, iterative learning, authentic learning, and student engagement. An evolution in PBL is Electronic or Digital Problem Based Learning (E-PBL), which grew out of the COVID-19 pandemic and the need to adapt. The researchers seek to enhance E-PBL and its features through the MDC by mapping their main features to each other in an effort to enhance and improve student innovation, creativity, and empowerment. Findings show that E-PBL and MDC intersect on the basis of motivation, instrumentation, and competency. The paper suggests that integrating MDC into E-PBL would be a robust framework that encourages collaboration, research, and innovation towards strengthening techno-mathematical literacy and fluency among students.*

1. Project-Based Learning

Project-based learning (PBL) is a collaborative, student-centered, constructivist pedagogy where students work in groups to construct knowledge and gain mastery of content in different areas of learning [1]. In PBL, students work on a project for an extended period of time and are expected to solve or answer a complex question using their knowledge and skills. As a culmination, students are expected to publicly deliver a product or presentation for an actual audience.

In 21st century learning, there is a great demand for continued innovation, critical thinking, analysis, and creativity. To help address these needs, schools and studies have resorted to PBL in part due to its impact on increasing creativity, problem-solving, critical thinking, communication, collaboration, etc. in students [2]. The digital revolution has entrenched itself in education, and the existence of information and communication technology is an opportunity that educators can capitalize on. The current era of educational development has been rooted in technological development; hence, students and teachers alike will need to be prepared [3] (as cited in [2]). Studies such as those by Rohman et al. [4] (as cited in [2]) point out that the PBL model—especially when integrated in Science, Technology, Engineering, Art, and Mathematics (STEAM)—can develop the ability of students to investigate and make connections in knowledge by using technology to design products with accurate calculations and artistic value. This makes the PBL model an ideal pedagogical model that shapes students to be ready for the real-world and the complexities that come with it. Innovation and collaborative professionalism [5] are all noted to be skills of the VUCA (Volatile, Uncertain, Complex, Ambiguous) world.

As teacher-centered methods begin to fade out with the rise of student-centered learning, innovative methods have borrowed from John Dewey's principle of learning by doing [6] (as cited in [5]). These thoughts are echoed in the findings of Knoblauch [7], who found that the freedom

and flexibility of PBL, as well as its ability to develop skills like collaboration and organization were highlighted as the drivers of learning. There is higher intrinsic motivation because students own the project, and because they know their teachers are there to provide close mentoring.

From a teacher development perspective, Knoblauch [7] noted that features of PBL are able to foster key aspects for future teachers: cross-cultural competency, communication, and collaboration. Other literature [8] also state the need for teacher leaders to galvanize fellow teachers to action in order to ensure the end goal of professional development and the creation of meaningful learning experiences for both students and teachers alike.

1.1 Features of the PBL Approach

PBL [9] is informed by Seven Essential Project Design Elements: a challenging problem or question, sustained inquiry, authenticity, student voice and choice, reflection, critique and revision, and public product. These are accompanied by Seven Project Based Teaching Practices: designing and planning, aligning to standards, building the culture, managing activities, scaffolding student learning, assessing student learning, engaging, and coaching.

A framework proposed by Almullah [10] cites potential relationships between five main aspects of the PBL Approach: collaborative learning, disciplinary subject learning, iterative learning, authentic learning, and student engagement. Despite numerous takes on the nature of PBL, many proponents have agreed on some basic features: it is cooperative and research-based [11], an effective pedagogy to develop 21st-century skills, and important in improving problem-solving, innovation, and creativity [12]. It was found that PBL and its features engage student learning due to its *constructivist* nature of repetitive cycles, meaning-making, and reflection. PBL is a learner-centered method founded on three constructivist principles: context-specific, active student involvement in learning, and achieving goals through social interaction, knowledge sharing, and understanding. Further, it is inquiry-based, where students need to learn concepts to solve the real-world and open-ended problems presented [13].

Problem-based learning and project-based learning are often used interchangeably, but differ in the fact that problem-based learning focuses on process, while project-based learning focuses on results [14]. Regardless, both problem-based and project-based learning are collaborative in nature and include the important elements of experiential learning, reflection, and active participation. To prevent confusion in the paper, the researchers have decided to use the terms interchangeably, with the rationale that there exist no significant difference between both pedagogies, the supposed focus on results and process notwithstanding. Projects must have five essential characteristics: centrality, driving questions, constructive investigation, autonomy, and realism. These must be accompanied with learning aspects on student participation, reflection, reorganization, and presentation. [14]

1.2 PBL and E-PBL in Mathematics

In the face of the COVID-19 pandemic, educators and education researchers needed a jumping off point and direction to guide the practice of mathematics education. The pivot towards the different spaces of learning was acknowledged, as well as a radical shift to virtual spaces [15]. Initial solutions included online learning to prevent the spread of the virus. Classes were facilitated via online conferencing platforms such as Zoom, Google Meet, etc. Notable issues with online learning platforms include the distance, the inability to interact, and the discomfort experienced by teachers and students with the use of computers and online platforms in learning [16].

A learning model that helped address student boredom and build interesting learning activities is Electronic Project/Problem-Based Learning (E-PBL; also referred to as Digital PBL). It aims to

provide students with opportunities to learn by experiencing real-world conditions. Applying inquiry learning within a digital environment allows students to explore solutions within the digital sphere to help address their project or problem [17]. E-PBL requires students to approach research and learning from a rigorous academic approach: formulating questions, finding sources, gathering valid and reliable information, analyzing and synthesizing findings, and applying information. These do not stray heavily from the basic elements of PBL: analysis of authentic problems and communication among peers. Steps introduced by Ronteltap and Eurelings [18] suggest that students can first meet in groups to establish a shared starting point, do individual studies, then activate knowledge from individual research to discuss potential solutions. Asynchronous communication is important to ensure students get timely feedback and discuss further topics.

In E-PBL, teachers are expected to act as facilitators and subject experts, not necessarily the main and only source of knowledge. Communication between teachers and students is important when dealing with open-ended problems that require the application of concepts. Learning materials and technology provided by the teacher must be relevant and must motivate students to take interest in the subject matter. Interaction with students is doubly important, as it can increase student understanding and motivation [19]. The design and construction of learning tasks affect the productivity of interactions as well as the initiation of social interaction [18]. Experience gained by students through E-PBL shapes work environments and careers of students due to the interaction, while also helping students learn how and when to use certain technology and information [17].

Studies by Lazić et al. [20] in teaching measuring and measurements in third grade elementary schools in Serbia showed significant differences in academic achievement between the group learning through PBL and the group learning through traditional teaching. The findings showed that PBL is a valid and feasible alternative to other teaching concepts, and showed potential for methodological empowerment of teachers in practice. Longitudinal studies in 2016 by Holmes and Hwang [21] cited studies finding that PBL encouraged students to become challenge seekers [22] (as cited by [21]), to deepen their conceptual understanding, and to be motivated to tackle mathematics problems [23] (as cited by [21]). Though the study [21] did not yield any significant difference between PBL and traditional methods, it was found that students showed signs of thinking and active construction of mathematics knowledge. This was corroborated by findings that students in the PBL group were increasing the use of high cognitive skills such as organization, rehearsal, elaboration, and self-rehearsal.

2. Mathematical Digital Competency (MDC)

In their work on defining levels of techno-mathematical literacy and techno-mathematical fluency for students, Geraniou et al. [24] proposed the Mathematical Digital Competency (MDC) and viewed this as integral for students in the 21st century. Their extended study on MDC for Teaching (MDCT) [25] pointed out that teachers need to keep up with teaching and guiding students in their journey to developing MDC. Their initial findings on MDC were guided by the Mathematical Problem Solving with Technology (MPST) Framework by Jacinto & Carreira [26], which outlined the importance of techno-mathematical fluency and literacy by combining mathematical and technological knowledge. Both the MDCT and MPST frameworks are similar in that they cover what teachers need to know about how content and pedagogy inform each other, how technology informs content and pedagogy, as well as awareness of students' a priori epistemology and student responses to certain content.

To this, Geraniou et al. [24] outlined the following three-point MDC:

- **MDC 1:** Engage in techno-mathematical discourse.

- **MDC 2:** Being aware of which digital tools to apply within different mathematical situations and context and being aware of the different tools' capabilities and limitations.
- **MDC 3:** Being able to use digital technology reflectively in problem solving and when doing mathematics.

A study in Peru showed that integration of technology and a virtual setup favored the learning of digital and learning competences similar to the competences outlined by Geraniou et al. [24]. The digital setup allowed objects to be represented, encouraged students to be creative, and afforded opportunities to work with dynamic objects [27]. The researchers found that integrating learning strategies that adhere to the mixing of digital and mathematical competences allows cognitive development through the stimulation of creativity and sustained attention [28] (as cited by [27]). These findings corroborate the fact that technology is becoming fundamental to learning, and technology allows mathematics students to develop their mathematical competences. Research in Guangxi Normal University experimented on how PBL with technology can improve mathematical capability and creative thinking among future mathematics teachers. The work done shows promise in the use of dynamic mathematics software in improving student-teachers' communication skills, problem-solving ability, creative thinking, and self-confidence [29]. These findings highlight the potential of the MDC framework [24], where students can showcase a "reflected use of digital technology for learning".

3. Research Questions

In sum, the researchers wish to find theoretical intersections between the E-PBL and the MDC, and explore how enhancing E-PBL with the MDC will have an effect on the creative thinking and problem-solving abilities of students. We hope to contribute to existing academic literature on the application of the MDC framework in facilitating student learning.

In this regard, we ask the following research questions:

- Which key characteristics of E-PBL and MDC relate with each other?
- How can the integration of MDC with E-PBL be modeled?
- What possible effects on learning could be deduced from the proposed model?

The researchers aim to answer these questions by establishing a conceptual framework that matches the key characteristics of E-PBL with MDC and their possible corresponding effects.

4. Methods

The present paper aims to find common ground between E-PBL and the MDC as represented by a conceptual framework. To do this, we first qualitatively analyzed existing literature pertinent to PBL and E-PBL (as discussed in the preceding sections) and identified their salient features that would serve as foundational constructs of the proposed conceptual framework. Afterwards, we analyzed the characteristics of each MDC and matched them with the relevant foundational constructs, thereby modeling how E-PBL and the MDC could possibly be related with each other. Lastly, we referred back to literature related to E-PBL and the MDC and identified how interweaving these two could result in possible effects on learning mathematics.

5. Results and Discussion

Through an analysis of literature relevant to PBL and E-PBL, the researchers identified four salient features of the E-PBL that would serve as foundational constructs of the proposed conceptual framework: (1) authentic and context-specific problems, (2) social interaction and knowledge sharing, (3) active involvement and self-inquiry, and (4) the use of technology [1, 2, 9-10, 13, 18].

To model the possible connections between E-PBL and MDC, we attempted to match each MDC with the four aforementioned constructs. The researchers identified MDC 1 to be connected to the foundations of constructing authentic and context-specific problems, social interaction, and knowledge sharing. The construction of problems and the understanding of student interaction is consistent with the pedagogical and meta-pedagogical facet of MDC 1, where students are expected to "orchestrate student engagement in a techno-mathematical discourse" through the "assessment competency" [24]. By managing this, students are accorded opportunities to be innovative and creative in the proposed project output and context, as well as anticipating interactions among themselves and with technology [17].

As for MDC 2, the researchers found it to be possibly connected with the foundations of active involvement, self-inquiry, and selection of technology. By reflecting on the selection of technology and how students would fully utilize a technological resource in an E-PBL scenario, teachers manage to carry out a "process of bringing resources into a classroom and orchestrating student explorations of [the resource] affordance and limitations" [24]. By connecting MDC 2 with these foundations of E-PBL, students are provided with the platform to be innovative and creative in finding technology that allows inquiry and active learning towards addressing the project or problem assigned to them [17, 21-23].

With respect to MDC 3, the researchers found it could be connected to the foundation of social interaction and knowledge sharing. In their role as subject expert and facilitator, teachers act as experts that share knowledge and interact with students in findings solutions to the E-PBL. Conversations between the teacher, as a subject expert and facilitator, and their students may guide the latter in the process of inquiry and solution-finding [24]. Research has shown that this interaction empowers students to openly share their knowledge and ability to use technology in solving problems [27, 28], therefore being more reflective of their work — a key characteristic of MDC 3.

Creativity and innovation are defined as the ability to find ways to solve problems, create new technology, or create new branches of knowledge [2]. These traits are noted as important factors in education in order to adapt to the dynamic and uncertain 21st century world [2]. Over time, research has shown that creativity and innovation are not innate, and can be developed through projects that allow students to design solutions to real-world problems [2]. Knoblauch [7] notes that creativity and empowerment are manifested in student appreciation of the independence that E-PBL allows for the integration of individual interests with perspectives from different learning disciplines. Likewise, innovation stems from the "broadening of horizons", which allows students to adapt to diverse perspectives by creating flexible solutions that integrate technology and different learning disciplines [7]. Thus, PBL and E-PBL environments enhance student ability to apply knowledge in real-world scenarios, which empowers and motivates students to be creative in approaching tasks and solving problems [7, 21].

In line with this, empowerment is an important output to consider. This is operationally defined as students owning their voice in the process and taking control of their learning [2, 7, 10, 14]. Students need to be confident and empowered in their skills, especially in settings that involve

collaboration and interaction within groups [2]. The active participation of students in the learning process encourages empowerment by giving students autonomy to take control of their learning and progress [20]. Given the freedom to pursue creative and innovative thinking, students are empowered to reconcile different disciplines into one solution. Ronteltap and Eureling's findings show that increased social interaction empowered students to express learning needs more clearly as well as their learnings from different resources [18]. By accepting responsibility in the group setup of PBL and E-PBL environments, students are motivated and empowered in facing their individual strengths and deficiencies [18].

After analyzing the aforementioned literature on PBL, E-PBL, and the MDC, the researchers identified the possible outputs on learning mathematics in intertwining E-PBL and MDC. These are as follows: student innovation, creativity, and empowerment [2, 7, 8, 10, 14, 18-23].

5.1 Integration of E-PBL and the MDC

By laying out the important features of the E-PBL and MDC, we propose a conceptual framework, as shown in Figure 5.1, that interrelates the two as a student learning framework. The research-identified learning effects of creativity, innovation, and empowerment (henceforth referred to as outputs) are drawn with bidirectional arrows to signify the cyclic nature of the framework. The researchers envision the implementation of the framework demonstrating the outputs affecting the inputs and vice versa. Simply put, as these outputs grow within students, they are able to enrich their learning experience and cognitive development in MDC-enhanced E-PBL situations [10,13]. This symbiotic relationship will allow for empowered students to collaborate and apply technology to find solutions to their mathematical problems and projects.

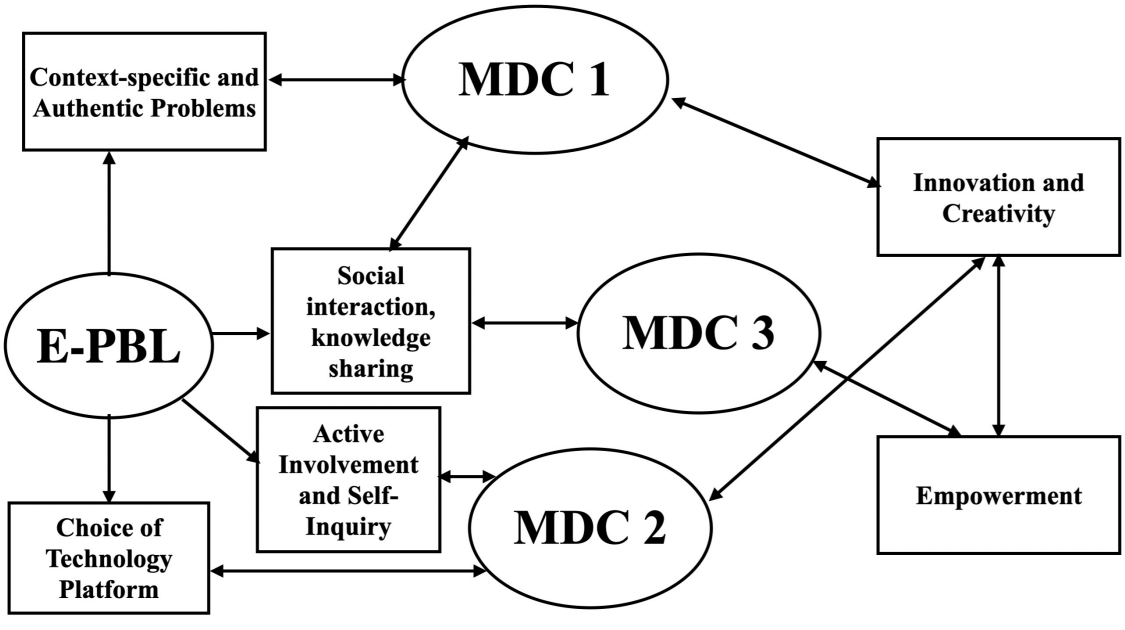

Figure 5.1 Proposed Conceptual Framework integrating MDC and E-PBL

5.2 A lesson and teaching-learning experience designed using the proposed framework

To illustrate the conceptual framework, the researchers propose a lesson and teaching-learning experience in this section. An ideal mathematics topic that would manifest this would be a project on the laws of sines and cosines and circle mensuration, which are topics studied in Grade 10 in the Philippines [30]. These topics tend to be taught using traditional methods, with students often being lectured to, without understanding the dynamism and applications of the topics.

The researchers envision the E-PBL taking place once the students have a familiarity with the lessons in question. An authentic and contextual driving question for the students would need to be set in the Filipino context, specifically: "How would you create a model for affordable housing given a specific land area?" This open-ended question covers the first connection between context-specific and authentic problems with MDC 1 leading towards innovation and creativity.

In order to cover the second and third connections between social interaction and active involvement to MDCs 2 and 3, submissions will be done in milestones that cover different facets of the driving question: research literature, methodology, prototype, and a public presentation. Each milestone covers students needing to interact amongst themselves and communities to gather data, while also leading to questions that revolve around maximizing space using different shapes and geometries. These resulting questions enact the potential for students to share knowledge and engage in self-inquiry that will guide them towards further inquiry, to collaborate amongst themselves, and to consult with the teacher as facilitator and subject expert in order to find ways forward in their potential solution. Milestones will have mini-assessments to check for conceptual and procedural understanding, while specific periods will be utilized for milestone planning.

To cover the construct of technology platform, students will be introduced to different educational technology applications that can be utilized to create their prototype. These may include GeoGebra, Desmos, Scratch, or any 3D shape graphing technology to do calculations efficiently and visualize their ideas as needed. As part of MDC 2, students choose the most applicable technology to create their prototypes and present them for the public to see.

6. Conclusion

In summary, the paper analyzed available literature to propose a conceptual framework that integrates the E-PBL and MDC frameworks. By identifying and matching foundational constructs from E-PBL and MDC to appropriate outputs, the researchers constructed a framework that seeks to strengthen student innovation, creativity, and empowerment in mathematics through the appropriate use of educational technology in mathematics.

In response to the first research question, the E-PBL and MDC are frameworks that can complement each other to develop a student learning framework. The PBL characteristics in the realms of context-specific problems, social interaction, active inquiry, and choice of technology [9-12] line up with MDC 1, 2, and 3 [24] on the characteristics of engaging techno-mathematical discourse, being aware of the appropriate digital tools, and being able to use digital technology in the solving and reflection on mathematics.

As for the second question, the researchers propose a framework with a cyclic nature that allows for constant feedback and assessment [5-7, 17-19, 21-23] through active guidance and communication between students and the teacher-as-facilitator. The integration of MDC and E-PBL matches the respective foundational constructs with outputs that put a premium on extending content knowledge and strengthening techno-mathematical literacy and fluency to create situations for future success among both teachers and students [25]. Further, the model framework

encourages teachers to produce authentic and engaging projects and problems for students to fully utilize technology to drive innovation, creativity, and empowerment in their mathematics learning.

With respect to the last research question, the researchers hope this proposed conceptual framework for an integration of MDC and E-PBL begins a conversation on the improvement and revamping of student learning frameworks. As suggested in previous studies [5-8, 20-21], there is potential in the use of PBL to develop innovation, creativity, and empowerment among students in order to take control of learning in the field of mathematics through techno-mathematical literacy and fluency [27]. In light of mathematics education research gearing towards mathematics education in physical and virtual contexts, the researchers aim to contribute to the academic literature on online assessment and addressing the mathematics education goals of deploying evidence, reasoning, argumentation, mathematical thinking, and creative thinking [15]. Instead of repetitive lectures telling students how to do math, the hope is that this framework allows teachers to provide opportunities for students to put theory into action using technology and authentic situations situated within the current and future landscape of technology.

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Research on the Use of Dynamic Geometry via Blended Learning

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***Abstract**: Teaching geometry through a blend of distance learning, IT support, and in-person consultations has proven to be an effective approach. The use of appropriate software allows for personalized learning experiences, replacing traditional knowledge with the development of valuable habits and strategies. Over two years, we conducted statistical evaluations of students' knowledge and attitudes after utilizing a blended learning method that actively incorporated dynamic geometry software.*

1 Introduction and motivation

Blended learning is an educational approach that combines traditional classroom instruction with online learning. This method has been widely acknowledged for over ten years as an effective approach that overcomes many limitations of conventional teaching methods. It is becoming increasingly popular, especially at the university level [8], [14].

Blended learning is a teaching approach that integrates a variety of activities to enrich the overall learning experience. It involves a combination of traditional face-to-face classroom instruction, interactive live e-learning sessions, and self-paced learning modules [4], [8].

In educational transformation at the institutional and individual teacher levels, a four-phase process is commonly observed: rejection, resistance, exploration, and loyalty. The resistance phase is particularly noteworthy as it marks the point at which teachers grapple with the realization that the methods and learning materials, they have painstakingly developed over the years, must now undergo updating or replacement [7]. Effectively implementing blended learning with the support of digital tools helps save teachers' and students' time. Well-configured classroom management tools replace administrative and routine work, thereby eliminating manual test creation and correction. Additionally, students appreciate the personalized approach and the flexibility to determine their own pace of learning and concentration on individual problems [1]. This is more beneficial than traditional teaching methods since it allows for adjustments based on students' different levels of abstract and logical thinking skills. An important benefit is the increased opportunity for students to explore and visualize mathematical concepts and ideas [14], [16].

1.1 The peril of excessive reliance on digital technology

The use of digital technologies in education has several benefits, but some teaching challenges also need to be considered. Digital tools can easily distract students and divert their attention from the learning material. According to Spitzer [19], there is no evidence that modern information technology used in schools improves teaching. The learning process requires mental effort and deep engagement in learning. Spitzer argues that computer-based learning replaces the depth of mental work with digital superficiality.

Cognitive distortion, also referred to as metacognitive slippage, can arise when students mistakenly believe they have a complete understanding of the material as a result of hints and simplified explanations [6]. This misconception stems from an illusion of comprehension. When individuals find information meaningful, they may assume they grasp it fully. However, in truth, their understanding may only be superficial. Moreover, excessive use of technology can present health risks and constrain opportunities for interpersonal interaction and the development of social skills [6], [17].

Recognizing the perils of digital misuse, cognitive decline, and digital memory loss, we are cautious about swiftly changing our educational tools. Nevertheless, regardless of expert advice and empirical research, individual learning is rapidly evolving in its distinct manner. How do we show children the benefits of mastering arithmetic operations when even the most affordable mobile phones can perform them? Similarly, how do we underline the significance of understanding concepts like calculating the extrema of functions to students, when a simple "extrema" command in an online app can do the job? Digital technologies fundamentally and irreversibly impact our activities, shaping our attitudes, competencies, and relationship to education. If we resist this trend by prohibiting calculators, mobile phones, and tablets, we only emphasize the ease of substituting the skills and knowledge required. Equally unwise are lessons filled with mandatory copying in a paper exercise book or prohibiting the use of the internet and artificial intelligence (AI).

Convincing students that access to information is not a substitute for real knowledge can only be achieved through their own experiences. It is one thing to be able to access information, another to actively seek it out, another to become familiar with it, and quite another to understand its meaning. The internet offers solutions to all questions, but the teacher must instruct students to ask questions, critically evaluate the answers, and seek connections.

2 The Role of Computer Algebra Systems in Modern Education

With the help of information technologies, we can largely change the approach of teachers and students, we can bring school mathematics closer to truly applied mathematics. When solving real problems, no expert calculates on paper or draws graphs on a square paper. Computer Algebra Systems (CAS) have been an essential part of applied mathematics for more than half a century. They must become an integral part of primary and secondary school mathematics as well.

The more digital technologies are incorporated into the classroom, the more important the role of the educator becomes. The role of the teacher is to find a balance between technology application, creative thinking, and rote memorization. Excessive use of and reliance on digital technology is harmful, and care must also be taken with independent thinking, deep understanding, and long-term memory [14]. The role of the teacher is changing, as is the proportion of time spent copying from the blackboard and correcting papers. It is up to each teacher to decide how to use the time with their students.

Above all, the teacher should be a partner in the journey of discovery; frontal teaching should be reduced to a minimum. Our pupils belong to the so-called alpha generation, which is characterized by multitasking, Nintendo logic in trial and error, and a preference for typing over handwriting [1]. Let us not fight their way of perceiving the world but let us use it. Problem-based learning supported by CAS has the potential to inspire students to experiment. The more concrete experiences pupils gain, the more firmly the associated abstract model is anchored.

2.1 GeoGebra: Transforming Visual Learning in Mathematics

The most widely used computer algebra system (CAS) in the education system is GeoGebra [5], software for dynamic geometry. With over a decade of experience, we have found that GeoGebra plays a crucial role in enhancing the teaching and learning of intricate mathematical concepts. Interactive manipulation and visualization foster a deeper understanding, enhance student engagement, and promote inquiry-based learning. These features help bridge the gap between abstract mathematical concepts and their concrete applications, making math accessible and enjoyable for students [16].

GeoGebra helps visualize complex mathematical concepts that might be abstract when taught through lectures alone. Students can create visual representations of differential geometry concepts like Frenet frame, curvature, or transition curve.

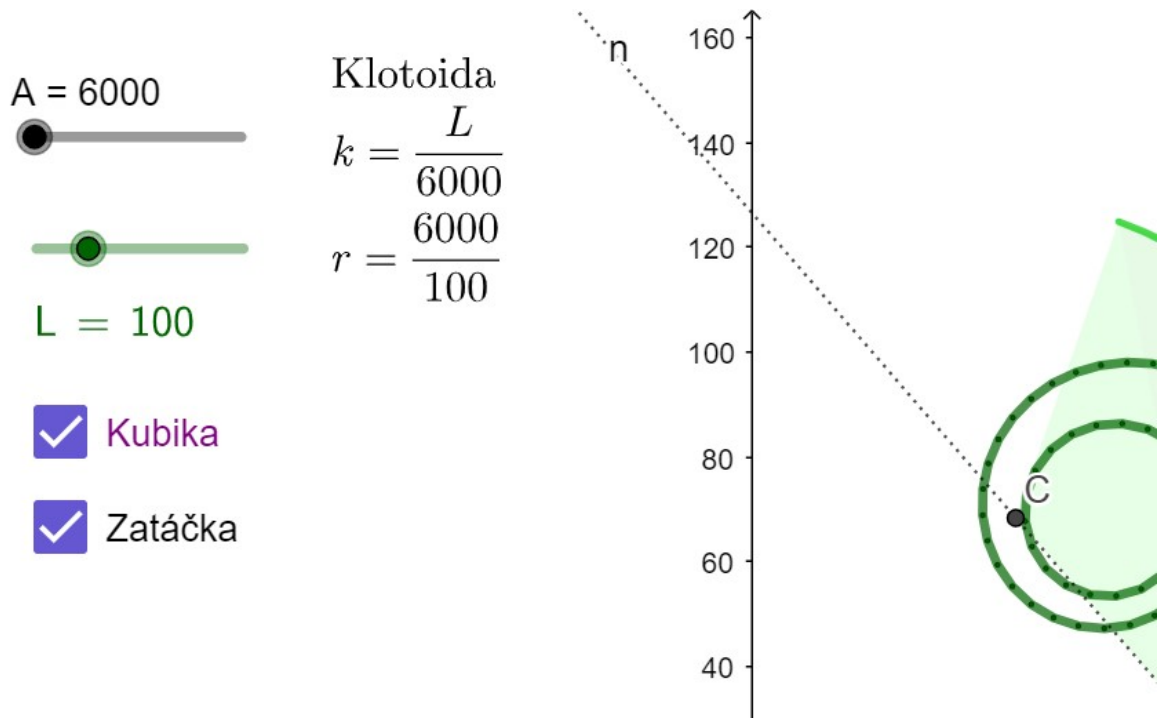

Figure 1. Screenshot from GeoGebra applet Euler spiral (clothoid). Slider *A* for coefficient *k = L/A* influences the shape of the Euler spiral. Slider *L* determines the length of the transition curve and resulting turn radius *r = A/L*. Cubic approximation, often used in train transport, is plotted in purple.

For instance, when studying an osculating circle, students can graph the curve with a moving point on the curve to see how the curvature radius changes at different positions. Then it is easy to demonstrate the construction of the Euler spiral and its use as a transition curve, see Figure 1. In the figure, the Euler spiral is given by a linear relationship between curvature *k* and arc length *L*, measured from the origin *O*. Once the desired radius *r = 1/k* of the curve is reached, the road follows the osculating circle of the Euler spiral. If the vehicle travels at a constant speed and the

driver turns the steering wheel uniformly, the trajectory follows an Euler spiral. The constant A determines how sharply the driver turns the steering wheel as a function of speed.

3 Study of the Effectiveness and Weaknesses of Blended Learning

Based on the results of nine independent studies across various school systems, it has been found that blended learning has a positive impact on the development of student's knowledge, skills, and attitudes [2], [3], [7], [9], [10], [12], [14], [18], [20]. These studies specifically focused on mathematics instruction. Teaching quality was evaluated based on instructional time, clarity of instruction, content structure, and stimulus. Additionally, motivation was assessed through student engagement, opportunities for individualization and choice, promotion of collaboration, and creating a positive culture around learning from mistakes. The evaluation of cognitive activation is based solely on practicing formative tests with tasks of different difficulty levels. Grades and other grading tools are typically given more importance, while informal knowledge is often overlooked. It's challenging to measure the quality of understanding in real-world contexts and applications; none of the reviewed studies focused on this aspect.

For distance feedback, the potential of using prepared problem sets for homework and classroom testing is still somewhat underestimated. Nevertheless, experience with various forms of formative assessment can be beneficial for students and can provide teachers with inspiring ideas and valuable feedback. By using suitable learning platforms, continually monitoring students' educational progress becomes much easier and more effective than with traditional exams and written tests.

3.1 Advantages of blended learning over the classical approach

Many studies have shown that blended learning provides several benefits to students, such as providing theoretical learning with time flexibility, increased self-assessment, responsibility, engagement, motivation and interest in learning [9], [14], creative and critical thinking, collaborative ability, group work, and improved learning outcomes [12].

In addition to the above, the positive effects of blended learning on students' attitudes have also been elucidated in most studies. Students actively engage in learning [18] and are satisfied with the course [15]. Participation in blended learning courses stimulates mathematics teachers' readiness for self-directed learning and awareness of technology integration [7].

In addition to the benefits of developing students' knowledge, skills, and attitudes in mathematics, three studies confirmed the time and cost benefits of blended learning in the classroom [3], [13],[20]. There are also benefits in terms of reasonable learning costs for students due to the free support of many online learning platforms and low internet fees [10].

3.2 Problems Associated with Blended Learning

Issues related to technological factors include insufficient computer equipment, unreliable internet connections, and the absence of control systems to prevent fraudulent behavior. Another significant challenge is the selection of appropriate tools by the learning organizer [10], as well as the language used in online platforms for teaching and learning. It is essential to conduct studies on the development or use of tools that support blended learning and to clarify how these tools function, along with their advantages and disadvantages. The selection of appropriate tools for different learning objectives and environments will be possible while overcoming technical language or

technical limitations. Mathematics teachers need to actively learn different tools and improve their ability to use these tools effectively [13]. Teachers should ideally possess a firm grasp of English technical terms. This will enable them to effectively equip their students to engage with foreign materials and navigate online learning platforms.

There are significant challenges in ensuring the quality of course content, preparing activity plans, and designing appropriate questions for the effective application of blended learning [7]. Teachers need to ensure that learning activities are appropriate to the learners' level to help them connect existing and new knowledge. Particularly in the preparation and pilot phases, increased demands on the time of the teachers involved must be anticipated. Math teachers had trouble assessing students' comprehensive grasp of the material and the informal aspects of their knowledge. Statistical analysis of formative test scores is limited in its usefulness; conducting oral tests is crucial for identifying and addressing any incomplete mental models early on.

4 Quantitative Research on Blended Learning at CTU

At the Czech Technical University's Faculty of Transportation, we have been using the GeoGebra software to support our Geometry course for over a decade.
The essential platform for all courses in our faculty is the Learning Management System (LMS) Moodle. Thus, the application of a teaching approach combining full-time and distance learning using online materials was directly offered.

Moodle allows random generation of tasks from the database according to set parameters, analysis of the difficulty of tasks based on submitted solutions, and basic descriptive statistics for didactic research. If you already have a database of tasks, creating any test, whether random or deterministic, with a prescribed procedure or graded clues, is a matter of minutes. Appropriate selection and setup of test questions and tests by providing feedback and preparing students to take tests can contribute to an overall improvement in understanding of the taught matter, as shown in [4]. For teachers, creating the database is the most time-consuming task. To ensure truly random tests that assess knowledge rather than memorization, at least two hundred problems are required. We recommend using parametric task assignment, where the system automatically generates different values for marked parameters. Unfortunately, there is not yet a shared repository where teachers can download assignment files to import into their courses. If Moodle worked openly with a reputation system for grading assignments by verified users, it would be a full replacement for paid learning portals.

The topics of the Geometry course at the CTU are not available on Czech teaching portals and, as far as we know, not even in foreign applications. Therefore, we have created our hypertext study guides using GeoGebra applets and automatic test generation tools in LMS Moodle, see Figure 1. We have published study materials for students on the GeoGebra server [5] in the GeoGebra Book "Geometry FD ČVUT" [21]. The advantages are interactivity, public accessibility, and easy referencing of the study materials with the online help of LMS Moodle. Most tasks are designed in a way that cannot be solved with a single query to the AI chatbots. We try to incorporate non-traditional practical problems that focus on transportation problems, using mathematical frameworks and geometric principles. These hands-on problems are designed to challenge students to rethink standard methodologies, thereby encouraging critical thinking and creative solutions to the logistical and navigational problems facing modern transportation scenarios. We have integrated GeoGebra activities directly into the Moodle learning materials, which supports consistent learning in different formats and quick feedback for students. The problem in Figure 2 can be easily solved

by using GeoGebra by command “OsculatingCircle”. If the student solves the problem correctly, more difficult application problems (graded series of problems) follow.

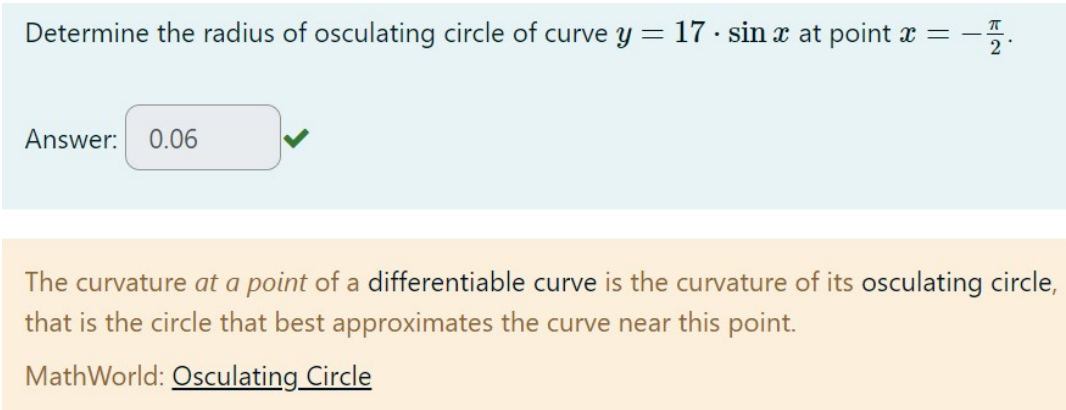

Figure 2. Student preview of the assignment in Moodle. Along with the evaluation, we can add a note on the recommended command, a short explanation, or a link to the answer

The teaching materials developed are designed in such a way as to enable students to have the opportunity to create concepts, procedures, and principles in mathematics through a variety of learning activities, including activities that are individual, group, or in pairs, to class activities. They learn to recognize which mathematical models to apply to solve the problem and are encouraged to work independently and think critically.

During the semester, students can take up to six thematic formative tests dedicated to specific practical problem scenarios. Attendance is not compulsory, and face-to-face meetings are devoted to consultation of problematic assignments. After students are made aware of the availability of all materials and the possibility of distance feedback, attendance stabilizes at approximately 30%. Most students thus voluntarily choose the blended learning method. Students demonstrate their knowledge in the final exams, which are open-book standardized examinations.

4.1 Survey Methodology with Moodle

We use the educational design research [11] approach to set up a more effective structure of graduated tests for formative online assessments. Online tests and in-person exercises provide ongoing feedback to students and instructors. Moodle records most of the students' learning activities, whether they are completing assignments, reading materials, or solving tests. We used this information for subsequent statistical analysis.

Survey objectives:

- Compare the results of traditional face-to-face and blended learning by measuring correlations between attendance, course activity, and final exam results.
- Examine the impact of various activities and instructional methods on student engagement and academic performance.

Sources of data collection:

- Moodle attendance report: Correlation of attendance with assessment results. The normality and variance homogeneity tests were first carried out before deciding on the statistical test.
- Moodle “All logs” tool: Frequency and timing of student interactions with course materials (Geogebra book, lecture handouts, quizzes).
- Moodle quiz statistic report: Results of formative and summative tests, a tool for analyzing the difficulty of individual problems in the "Question Bank".
- Official CTU student survey: Student feedback on which activities they find most beneficial, how they rate the difficulty of the course, and the fairness of the final classification.

In our survey, we found surprising connections between teaching methods, student attitudes, and their performance on the final test. We recognize that evaluating teaching effectiveness goes beyond just exam scores and student feedback. Nevertheless, this input will be essential for enhancing the structure of the Geometry course in the future. Consistent with the findings of other studies, we observed positive effects on student attitudes and significant improvements in student engagement. In surveys, students appreciate the possibility of individualization, time flexibility, and fairness of the graded final test.

4.2 Factors affecting the result of the test

The basis of knowledge is error, which we gradually refine.
In the end, we know nothing, but we know it right.
Jára Cimrman, fictional genius, the greatest ever Czech

Students' success on the exam is heavily influenced by their knowledge from high school, particularly in math and digital literacy. Motivation, analytical thinking, and time availability also play significant roles. Our analysis focused on factors that can be measured through Moodle results and that we can address for future learning approaches. The survey was conducted over two years with a total of 374 participating students.

We compared whether attendance at lectures and tutorials influences exam results. The correlation coefficient for the dependence of percentage attendance on exam scores is 0.32, and even Spearman's test of independence p-value of $p = 0.58$ does not allow us to reject the hypothesis of independence of the variables under study. Slightly better values are shown by the dependence of the student's achievement in the exam on the scores of the continuous formative tests in Moodle, but even here Spearman's test did not show dependence ($p = 0.24$). The independence of the variables is also shown by the score plot in Figure 2. Points on the x-axis represent students who dropped out during the semester and were not included in the survey.

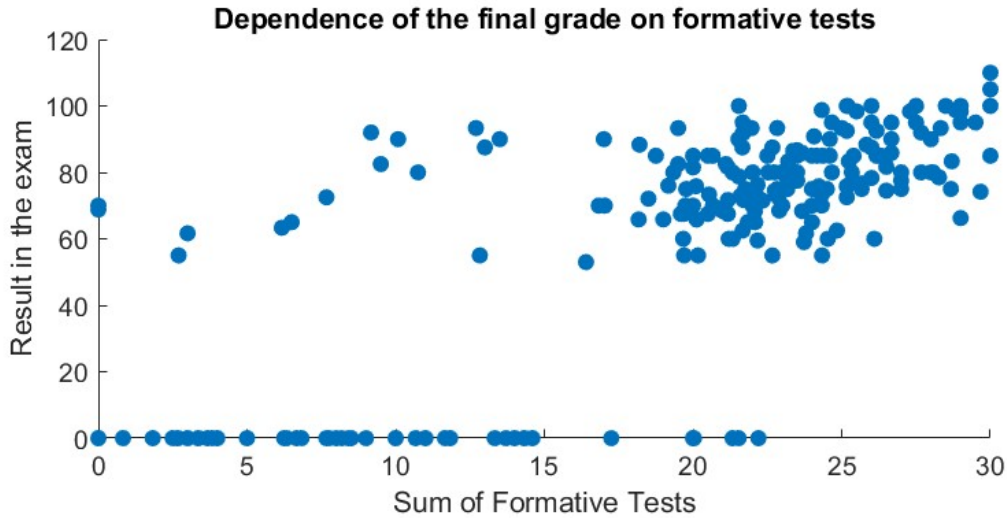

Figure 3. Scatter plot of the correlation between summative results and formative tests

We were surprised by the result because both values quantify the success of solving a selection of examples from the same database. We explain this by the mandatory minimum score threshold for formative tests and the low motivation of students to try to achieve the best possible result. Students can prepare for the exam in other ways than solving formative tests.

A similarly inconclusive result was obtained when examining the effect of time spent on learning supports in Moodle. With a low correlation coefficient of 0.34, we rejected the possible linear dependence of the final grade on the number of accesses in Moodle, which measures the intensity of learning activity throughout the semester.

The exercises are led by three teachers with different teaching methods. Half of the courses were conducted using frontal teaching, and the other half by the flipped classroom method and problem-oriented activities. A one-factor ANOVA confirmed the null hypothesis that teaching style did not affect final grades ($p = 0.65$), which is consistent with the previous conclusion that attendance at the exercise does not predict exam performance.

In the second year of the survey, we compared traditional face-to-face teaching with blended learning. The study groups must have comparable course completion rules and the same number of hours of face-to-face instruction. The study groups voluntarily chose between classical teaching with compulsory attendance and blended method with compulsory solving of six formative tests in Moodle. Three classes chose mandatory attendance, while the other ten classes preferred compulsory take-home tests. The first three study classes served as the control group. Again, different types of learning did not affect the final grades. The only statistically significant difference was found when comparing two consecutive years of teaching. The probability that the estimated means of the results are equal came out to 0.03; thus, the unpaired test rejected the null hypothesis of equality at the 5% significance level. Surprisingly, students in the second year of testing performed better. The “Question bank” was expanded to include non-trivial application examples for precisely those problem types that showed the greatest difficulty in the first year of testing. This improvement can be attributed to the release of a complete database of 530 examples and the inclusion of helpful hints linking to relevant study materials.

5 Conclusion and future work

Our two years of research indicate that with adequate learning materials and feedback tools, distant learning is just as effective as traditional face-to-face teaching. Of the observed factors (teaching style, class attendance), none significantly affected the quality of the skills acquired by the students. This result is undoubtedly influenced by the use of GeoGebra, which allows students to experiment and test hypotheses independently. Thus, the student learns actively, with firmly anchored mental models with a long-term memory trace. When working with dynamic geometry software, the role of the teacher changes. Instruction and continuous frontal explanation should be replaced by the role of mentor and guide through a constructivist approach.

Blended learning, with a quality set of interactive learning aids and ample self-evaluation opportunities, is more appealing to students. It helps develop self-study skills and yields results comparable to traditional face-to-face teaching. According to student surveys, the benefits include time savings and more objective assessment. By leveraging technology, blended learning can be an effective alternative method for teaching and learning mathematics, thereby motivating students.

The teaching process is quite complex, which makes it difficult to draw definitive conclusions from most educational research. It is important to continue analyzing the results over an extended period and consider additional contributing factors. Reflecting on student and colleague feedback over time will help to evaluate teaching methods objectively and develop alternative strategies. In a future survey, we would like to expand the measurement of student feedback by measuring the level of agreement with statements related to engagement, the perceived value of the course, and the link between participation and student success with Likert scale questions.

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An Initial Analysis on the Reliability of Using Chatbots in Solving Word Problems in Probability

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Abstract: *This study explores the reliability of using chatbots, specifically ChatGPT-4o and Copilot, in solving word problems in probability. The integration of artificial intelligence in education has transformed instructional methods, offering tools for personalized learning and instant feedback, which are critical for developing problem-solving skills. Chatbots leveraging natural language processing (NLP), simulate human conversations to assist in educational settings. This research focuses on evaluating the accuracy of chatbots in solving probability problems. A comparative analysis approach was employed, wherein probability word problems involving conditional probability were posed to both chatbots. The solutions provided by the chatbots were then compared to the correct solution using a specific evaluation criterion. The results highlight the potential and limitations of chatbots in educational applications. While chatbots demonstrated the ability to engage in meaningful conversations and provide correct solutions in some instances, issues such as misunderstanding, complex queries, and the necessity for continual algorithm updates were also observed. This study's findings contribute to improving chatbot design and functionality, aiming to enhance educational outcomes and problem-solving skills. By identifying metrics for accuracy, this research provides insights into the potential of AI-powered chatbots as reliable educational tool.*

1. Introduction

In the age of digital transformation, the integration of artificial intelligence (AI) into various educational processes has become increasingly prevalent. It has significantly transformed educational practices, providing new tools and methods for instruction. AI has been touted to have the potential to offer personalized learning experiences, adapt to individual student needs, and provide instant feedback, which is crucial in improving problem solving skills [1]. One of the applications of AI is the use of chatbots, defined as a computer program created to mimic conversations with human users, particularly online, by employing natural language processing (NLP) to enable interactive and automated communication [2]. Educational chatbots can serve as effective tutors, because they provide explanations, answer questions, and help in the problem-solving process, along with the possibility of engaging in meaningful conversations [3].

While chatbots offer several advantages, such as availability and personalized feedback, they also have limitations, such as the chatbot's inability to understand complex queries or the need for continuous updates to the underlying algorithms [4]. These factors can affect the reliability and overall user experience. In addition, research specifically addressing chatbots in the context of probability is limited. Thus, this study aims to explore the reliability of using chatbots in solving word problems in probability --- assessing their accuracy as an educational tool. The study also seeks to understand the potential of chatbots to contribute to the pool of knowledge in enhancing learning outcomes in studying probability. The significance of this paper is in the fact that students may use these AI tools to supplement their mathematics learning; thus, we have to check the reliability of such tools in providing accurate and sound mathematical solutions and answers. ChatGPT-4o and Copilot are the chatbots used in this study; a referee of the present paper utilized Gemini, to which the results are briefly reported in this paper as well.

2. Problem Statement

The application of chatbots in educational settings has shown promise in enhancing student engagement and providing personalized assistance. Several studies have been reviewed which highlighted the benefits of using chatbots in education, such as providing instant feedback and facilitating personalized learning experiences [3]. The reliability of chatbots in accurately solving probability word problems remains uncertain, since there is a lack of comprehensive research specifically focusing on the accuracy and reliability of chatbots in solving word problems in probability. This gap in the literature necessitates an investigation to determine whether chatbots can be relied upon to provide accurate solutions and to identify the limitations they may face in understanding and processing complex probabilistic queries [5]. Thus, this paper aims to describe the quality and accuracy of output provided by chatbots in solving word problems in probability.

3. Theoretical and Conceptual Underpinnings

The theoretical framework for this study is based on reliability theory. The evaluation criteria utilized by Kwon et al. [6] based on Frieder et al. [7] align closely with the principles of reliability theory, which focuses on the consistency and accuracy of a system's performance over time. Reliability theory emphasizes the importance of a system's ability to perform its intended function without failure, and this is mirrored in the scoring system [6, 7]. Their criteria range from a score of 0, indicating a complete failure to process the prompt, to a score of 5, indicating a completely correct answer. This evaluation method allows for an assessment of the chatbot's performance, identifying areas where the chatbot reliably provides accurate solutions and areas where it fails. By identifying errors and assessing accuracy, the criteria support the continuous improvement and validation of the chatbot's reliability, ensuring it can be a dependable educational tool. This connection highlights how reliability theory can guide the development and refinement of AI-driven educational technologies.

4. Methodology

This study employed a comparative analysis approach to evaluate the reliability of chatbots in solving word problems in probability. A comparative analysis approach involves systematically evaluating and contrasting different subjects or systems to highlight their similarities and differences, facilitating a deeper understanding of each [8]. The research process involves posting probability word problems into the AI chatbot, in which we sampled five problems. The word problems selected in the study involve conditional probability as based in the reference material [9], because high conceptual knowledge is needed in order to set up the solution.

The accuracy of the chatbot solutions were analyzed by comparing them to the correct solutions. Each solution were evaluated for accuracy using the evaluation criteria as shown in Table 4.1 [6, 7].

Table 4.1 Evaluation Criteria

Scores	Description
0	Chatbot fails to process the prompt at all.
1	Chatbot fails to answer in the context of the prompt
2	Chatbot answers in the context of the prompt, but the answer is completely wrong.
3	Chatbot answers in the context of the prompt, but the answer is only partially correct.
4	Chatbot answers in the context of the prompt, and the answer is almost correct.
5	Chatbot answers in the context of the prompt, and the answer is completely correct.

5. Results and Discussion

In this section, we present the responses provided by ChatGPT, Copilot, and Gemini (as provided by the referee) to the five probability word problems sampled in this study. We likewise provide a brief discussion on the chatbot's output and the score assigned to it.

Probability problem 1: A biased coin, twice as likely to come up heads as tails, is tossed once. If it shows heads, a chip is drawn from Urn 1, which contains 3 white chips and 4 red chip; if it shows tails, a chip is drawn from Urn 2, which contains 6 white chips and 3 red chips. Given that a white chip was drawn, what is the probability that the coin come up tails.

Correct solution. We define the following events:
A1: the biased coin comes up tails (and a chip will be drawn from Urn 2)
A2: the biased coin comes up heads (and a chip will be drawn from Urn 1)
B: a white chip is drawn

From the given information, we know the following probabilities: P(A1) = 1/3; P(A2) = 2/3; P(B|A1) = 6/9; P(B|A2) = 3/7. We are asked to find P(B|A1). Using Bayes' Theorem, P(B|A1) = (2/3 * 1/3) /((3/7 * 2/3) + (2/3 * 1/3)) = 7/16. Thus, the probability is 7/16.

Solution and answer generated by ChatGPT
To solve this problem, we'll use Bayes' theorem. Let's define the events
H: the coin comes up heads T: the coin comes up tails W: a white chip is drawn

We need to find P(T|W). From the problem, we know: The coin is twice as likely to come up heads as tails. Thus P(H) = 2/3 and P(T) = 1/3. If heads, a chip is drawn from Urn 1, which has 3 white chips and 4 red chips. So, P(W|H) = 3/7. If tails, a chip is drawn from Urn 2, which has 6 white chips and 3 red chips. So, P(W|T) =6/9.

We need to use Bayes' theorem which is given by P(T|W) = $\frac{P(W|T) * P(T)}{P(W)}$. First, we need to find P(W), the total probability of drawing a white chip. This can be found using the law of total probability. P(W) = P(W|H) * P(H) + P(W|T) * P(T)
Substituting the known values: P(W) = (3/7 * 2/3) + (2/3 * 1/3)) = 32/63
Now we can find P(T|W): P(T|W) = $\frac{P(W|T) * P(T)}{P(W)} = \frac{\frac{2}{3} * \frac{1}{3}}{\frac{32}{63}}$. Therefore, the probability is $\frac{7}{16}$.

Solution and answer generated by Copilot
Let's calculate the probability step by step: 1. The probability of getting a white chip from Urn 1 is 3/7. 2. The probability of getting a white chip from Urn 2 is 6/9. 3. The overall probability of drawing a white chip is the weighted average of the 2 urns. Where is the probability of getting heads (which is 1/2). 4. Since we all know that a white chip was drawn, we can set up the following equation: P(white) = P(white from Urn 1) * $\frac{1}{2}$ + P(white from Urn 2) * $\frac{1}{2}$. 5. Solving for P(tails), Therefore the probability is 3/7

Solution and answer generated by Gemini
Understanding the Problem: We're asked to find the probability that the coin landed on tails given that a white chip was drawn. This is a conditional probability problem.

Defining the Events;
H: Coin lands on heads T: Coin lands on tails W: A white chip is drawn

Setting Up the Probabilities
P(H) = 2/3 (since heads is twice as likely as tails) P(T) = 1/3
P(W|H) = 3/7 (probability of white chip given heads)
P(W|T) = 6/9 = 2/3 (probability of white chip given tails)
Applying Bayes' Theorem. We want to find P(T|W), the probability of tails given a white chip. Bayes' Theorem states: P(T|W) = P(W|T) * P(T) / P(W). We know P(W|T) and P(T), but we need to find P(W), the probability of drawing a white chip.

Finding P(W), Using the Law of Total Probability: P(W) = P(W|H) * P(H) + P(W|T) * P(T) = (3/7 * 2/3) + (2/3 * 1/3) = 32/63, Calculating P(T|W), Now we can plug in the values into Bayes' Theorem: P(T|W) = (2/3 * 1/3) / (32/63) = 7/16. So, the probability is 7/16.

For this problem, ChatGPT provided a complete and step-by-step solution and a detailed explanation of the process. Moreover, it arrived at the correct answer, thus scoring 5 points. Meanwhile, Copilot answer is far from the correct answer but it somehow showed some solution which are partly correct, thus scoring 3 points. Gemini got the correct answer and solution; however, it made an arithmetic slip in the first attempt, thus scoring 4 points.

Probability problem 2: Urn 1 contains 2 red chips, 4 white and 2 black. Urn 2 contains 3 red, 1 white and 1 black. A chip is draw at random from Urn 1 and transferred to Urn 2. What is the probability that the chip from Urn 2 is red?

Correct solution. We define the following events:
A1: Red chip is transferred A2: Red chip not transferred B: red chip drawn from Urn 2
We thus know that P(A1) * P(B|A1) + P(A2) * P(B|A2) → $= \frac{2}{8} * \frac{4}{6} + \frac{6}{8} * \frac{3}{6} = \frac{13}{24}$

Solution and answer generated by ChatGPT
<u>Step 1:</u> Determine the probability of drawing each type of chip from Urn1, Urn 1 contains 2 red chips, 4 white chips, 2 black chips. The total number of chips in Urn 1 is 2 +4 + 2 = 8. Thus, the probabilities of drawing each type of chip from Urn 1 are:
P(Red from Urn 1) = 2/8 P(White from Urn 1) = 4/8 P(Black from Urn 1) = 2/8

<u>Step 2:</u> Determine the composition of Urn 2 after transferring each type of chip.
Urn 2 initially contains 3 red chips, 1 white chip, 1 black chip. When a chip is transferred from Urn 1 to Urn 2, the total number of chips in Urn 2 becomes 6.
<u>Case 1:</u> A red chip is transferred to Urn 2 → Urn 2 will have 4 red chips, 1 white chip, 1 black chip. The probability of drawing a red chip from Urn 2 in this case is P(Red from Urn 2| Red transferred) = 4/6
<u>Case 2:</u> A white chip is transferred to Urn 2. Urn 2 will have 3 red chips, 2 white chips, 1 black chip. The probability of drawing a red chip from Urn 2 in this case is P(Red from Urn 2|White transferred) = 3/6
<u>Case 3</u>: A black chip is transferred to Urn 2. Urn 2 will have 3 red chips, 1 white chip, 2 black chips The probability of drawing a red chip from Urn 2 in this case is: P(Red from Urn 2| Black transferred) = 3/6

Step 3: Use the law of total probability. The overall probability of drawing a red chip from Urn 2 is the weighted sum of the probabilities of drawing a red chip in each case, weighted by the probabilities of transferring each type of chip from Urn 1. P(Red from Urn 2) = $\frac{2}{8}\ x\ \frac{4}{6} + \frac{4}{8}$ x $\frac{3}{6} + \frac{2}{8}$ x $\frac{3}{6}$ = $\frac{13}{24}$. Therefore, the probability that the red chip drawn from Urn 2 is red is $\frac{13}{24}$

Solution and answer generated by Copilot
Let's calculate the probability that the chip from Urn 2 is red after the transfer. We'll denote the events as follows: R1: Chip drawn from Urn 1 is red. R2: Chip drawn from Urn 2 is red.

We want to find (P(R2 | R1)), which represents the probability that the chip from Urn 2 is red given that the chip from Urn 1 is red. First, let's find the combined probability of drawing a red chip initially: From Urn 1, the probability of drawing red is 2/8 since there are 2 red chips out of 8 total chips. From Urn 2, the probability of drawing red is 3/5, since there are 3 red chips out of 5 total chips. The combined probability of drawing red initially is (P(R_1) 2/8 * 3/5 = 3/20
Next, let's find the probability of drawing a red chip from Urn 2 after the transfer: If we draw a red chip initially, the chip transferred to Urn 2 will be red with certainty, since Urn 2 only contains red chips. Therefore, (P(R2 | R1) = 1). Finally, we can calculate the overall probability: [P(R_2 | R_1) = \ P(R_2) * P(R_1 | R_2)}{P(R_1)}] Since (P(R_2) = 1) and (P(R_1 | R_2) = 1), we have: [P(R_2 | R_1) = 1 * = 20/3 . Therefore, the probability is approximately 6.67%.

Solution and answer generated by Gemini
Step 1: Determine the possible outcomes for the first draw (from Urn 1)
Red chip: Probability = 2/8 = 1/4 White chip: Probability = 4/8 = ½
Black chip: Probability = 2/8 = 1/4

Step 2: Calculate the probability of drawing a red chip from Urn 2 for each possible outcome of the first draw. If a red chip is drawn from Urn 1: Urn 2 now contains 4 red, 1 white, and 1 black chip.
Probability of drawing a red chip from Urn 2 = 4/6 = 2/3
-If a white chip is drawn from Urn 1: Urn 2 now contains 3 red, 2 white, and 1 black chip.
Probability of drawing a red chip from Urn 2 = 3/6 = 1/2
-If a black chip is drawn from Urn 1: Urn 2 now contains 3 red, 1 white, and 2 black chips.
Probability of drawing a red chip from Urn 2 = 3/6 = 1/2

Step 3: Calculate the overall probability of drawing a red chip from Urn 2.
We use the Law of Total Probability: P(Red from Urn 2) = P(Red from Urn 1) * P(Red from Urn 2 | Red from Urn 1) + P(White from Urn 1) * P(Red from Urn 2 | White from Urn 1) + P(Black from Urn 1) * P(Red from Urn 2 | Black from Urn 1)
P(Red from Urn 2) = (1/4 * 2/3) + (1/2 * 1/2) + (1/4 * 1/2) = 13/24,

For this problem, both ChatGPT and Gemini broke the problem into step-by-step procedures and arrived at the possible cases which led to the correct answer, thus scoring 5 points. Meanwhile, Copilot's answer is very far from the correct answer, with the solution requiring to be compiled using LaTeX for the symbols to appear properly. Thus, it scores only 2 points.

Probability problem 3: Two fair dice are tossed. What is the probability that the sum of the rolls is 6, but neither roll is twice the other?

Correct solution
The possible outcomes are (1,5), (5,1), (3,3) (4,2), (2,4). We remove (4,2) and (2,4) because of the restriction. Thus, the probability is 3/36 = 1/12

Solution and answer generated by ChatGPT
To solve this problem, we first determine the possible outcomes when two dice are rolled and their sum equals 6. Then, we exclude the cases where one roll is twice the other.
Step 1: Find all possible outcomes where the sum is 6. The possible outcomes for rolling two dice that sum to 6 are: (1, 5) (2,4), (3,3), (4,2), (5,1)
Step 2: Exclude outcomes where one roll is twice the other. From the list above, we need to exclude any pairs where one number is twice the other. These pairs are: (2, 4), (4,2). So, the remaining valid pairs are: (1, 5), (3,3), (5,1)
Step 3: Calculate the probability: The total number of possible outcomes when two dice are rolled is 6×6=36. The number of favorable outcomes (those that sum to 6 but do not have one roll twice the other) is 3.

P = $\frac{\textit{Number of favorable outcomes}}{\textit{Total number of possible outcomes}} = \frac{3}{36} = \frac{1}{12}$, Therefore, the probability that the sum of the rolls is 6, but neither roll is twice the other is $\frac{1}{12}$.

Solution and answer generated by Copilot
Possible Ways to Get a Sum of 6: To get a sum of 6, we need the following combinations:
(1, 5) or (5, 1) (2, 4) or (4, 2) (3, 3)
Total Number of Possible Outcomes: Each die has 6 faces, so there is a total of 6 × 6 = 36 possible outcomes when rolling two fair dice. Excluding Doubles (Neither Roll Is Twice the Other): We need to exclude the case where both rolls are the same (doubles). There are 6 ways to roll doubles (e.g., (1, 1), (2, 2), …, (6, 6)).
Desired Outcomes: We have 5 favorable outcomes (sum of 6 without doubles). Probability: The probability of getting a sum of 6 without rolling doubles is 5/36, Therefore, the probability is $\frac{5}{36}$

Solution and answer generated by Gemini
Gemini got Problem 3 wrong first time! But telling Gemini that its favorable outcomes were wrong, it fixed it.
Corrected Solution
Total possible outcomes: 6 * 6 = 36 Favorable outcomes: (1, 5), (3, 3), (5, 1) = 3 outcomes
Probability: P(sum of 6, neither roll is double the other) = 3/36 = 1/12

For this problem, ChatGPT explained the context of the problem first and provided the step-by-step process of inclusion and exclusion criteria which led to the correct answer, thus scoring 5 points. Meanwhile, Copilot's answer and solution is somehow close to the correct answer but it failed to process detailed information in the problem, specifically "but neither roll is twice the other", thus scoring only 3 points. Gemini initially misinterpreted the problem, so one more prompt was needed to get the correct answer, thus scoring 4 points.

Probability problem 4: Consider a standard deck of cards. What is the probability of holding an ace given that the card drawn is a spade?

Correct solution. We define the following events: A: I have an ace. B: I drew a spade.
Thus, P(A|B) = $\frac{P(A \cap B)}{P(B)}$ = 1/52 ÷ ¼ = 1/13

Solution and answer generated by ChatGPT
Step 1: Identify the relevant sets
Total number of cards in a standard deck: 52, Number of spades in a standard deck: 13 (one for each rank from Ace to King), Number of aces in a standard deck: 4 (one for each suit: spades, hearts, diamonds, clubs). Number of aces that are spades: 1 (Ace of Spades)
Step 2: Define the events
AAA: the event that the card drawn is an ace SSS: the event that the card drawn is a spade
We need to find P(A|S), the probability that the card is an ace given that it is a spade.
Step 3: Use the definition of conditional probability. P(A|S) = $\frac{P(A \cap S)}{P(S)}$, P(S): the probability of drawing a spade. Since there are 13 spades in a deck of 52 card. P(S) = 13/52
P(A ∩ S): the probability of drawing the Ace of Spades. There is only 1 Ace of Spades in a deck of 52 cards. P(A ∩ S) = 1/52
Step 4: Calculate the conditional probability. P(A|S) = $\frac{P(A \cap S)}{P(S)}$ = 1/52 ÷ ¼ = 1/13.Therefore, the probability of holding an ace given that the card is a spade is 1/13

Solution and answer generated by Copilot
First, let's find the individual probabilities: The probability of drawing an ace card from the deck is 4/52, since there are 4 aces in a standard deck of 52 cards. The probability of drawing a spade card is P(S)= 13/52, since there are 13 spades in the deck. Now, we want to find the probability of drawing an ace given that the card drawn is a spade. Let's denote this as P(A|S). Using conditional probability, we have: [P(A|S) = {P(A \cap S/P(S)}]
The numerator represents the probability of drawing both an ace and a spade. There are two ways this can happen: Draw the ace of spades first, then any other ace: 1·3 /51. Draw any other ace first, then the ace of spades: 3/52
Combining these two cases, we get: [P(A \cap S) = 2 * 3/52 *{1/51]
The denominator is simply the probability of drawing a spade: P(S) = \13/52} = 1/4. Now, let's calculate the conditional probability: P(A|S) = 3/52 *1/51 1/4 = 1/17

Solution and answer generated by Gemini
Understanding the Problem: We're looking for the probability of drawing an ace, given that we already know the card is a spade. This is a conditional probability problem.

Solution: Total number of spades: 13 Number of ace of spades: 1
Therefore, the probability of drawing an ace, given that it's a spade, is: P(Ace | Spade) = Number of ace of spades / Total number of spades. P(Ace | Spade) = 1 / 13. So, the probability is 1/13 or approximately 0.0769. This means there's a 7.69% chance of drawing an ace, given that you know the card is a spade.

For this problem, both ChatGPT and Gemini provided step-by-step procedures. In each step, there is a complete and easy to understand explanation of the context. In addition, each arrived at the correct answer. Thus, ChatGPT and Gemini scores 5 points. Meanwhile, Copilot arrives at the wrong answer; however, there are parts of the solution that explained the context of the problem well, such as "4 aces in a standard deck of cards", thus scoring 3 points.

Probability problem 5: A fair coin is tossed, if a head turns up, a fair die is tossed; if a tail turns up, two fair dice are tossed. What is the probability that the face (or the sum of the faces) showing on the die is equal to 6?

Correct solution. We define the following events:
A1: Coin toss comes as heads A2: Coin toss comes as tails B: The sum of rolls is 6
Thus, P(B) = P(B|A1) * P(A1) + P(B|A2) * P(A2)
= ½ * 1/6 + ½ * 5/36 = 11/72

Solution and answer generated by ChatGPT
We need to consider the two scenarios based on the coin toss outcome: 1. If the coin shows heads, a single fair die is tossed. 2. If the coin shows tails, two fair dice are tossed. We will calculate the probability for each scenario separately and then combine them using the law of total probability.
Scenario 1: Coin shows heads. If a single die is tossed, the probability that the face showing is 6 is: P(Single die shows 6) = 1/6. The probability that the coin shows heads is: P(Heads)= 1/2
Scenario 2: Coin shows tails. If two dice are tossed, we need to find the probability that the sum of the faces is 6. The possible pairs that sum to 6 are: (1, 5), (2,4), (3,3), (4,2), (5,1). There is a total of 6×6=36 possible outcomes when two dice are tossed. The number of favorable outcomes (pairs that sum to 6) is 5. Thus, the probability that the sum of the faces is 6 when two dice are tossed is: P(Sum of two dice is 6)= 5/36. The probability that the coin shows tails is: P(Tails) = ½.

Combining the probabilities. Using the law of total probability, the overall probability that the face (or the sum of the faces) showing on the die (or dice) is equal to 6 is: P(6)= P(Heads) * P(Single die shows 6) + P(Tails) * P(Sum of the two dice is 6). Substituting the known probabilities: P(6) = ½ * 1/6 + ½ * 5/36 = 11/72, Therefore, the probability is 11/72

Solution and answer generated by Copilot
1. Coin Toss: With probability ½, you toss a head and then roll one die. The probability that the die comes up 6 is 1/6 Therefore, the overall probability is ½ * 1/6 = 1/12
2. Tail Toss: With probability ½. you toss a tail and then roll two dice.
3. Sum of Faces Equal to 6: To get a sum of 6 with two dice, we need the following combinations: (1, 5) or (5, 1) (2, 4) or (4, 2) (3, 3) 4. Total Number of Possible Outcomes with Two Dice: Each die has 6 faces, so there is a total of 6 × 6 = 36 possible outcomes when rolling two fair dice.
4. Probability: The probability of getting a sum of 6 with two dice is: [P(\text{sum of 6}) = \ {favorable outcomes}}{\text{total outcomes}} = 5/36
5. Overall Probability: The overall probability of getting a total of 6 is the weighted average of the coin toss and tail toss probabilities: [P(\text{total of 6}) = ½ * 1/6 + ½ * 5/36 = 11/72 Therefore, the probability is 11/72

Solution and answer generated by Gemini
Understanding the Problems: We have a two-step experiment:
Toss a fair coin. If heads, toss one die; if tails, toss two dice.
We want to find the probability that the result of the die or dice is 6 (either a single die shows 6, or the sum of two dice is 6).
Solution Step 1: Define Events
H: Coin lands on heads T: Coin lands on tails D6: The result of the die or dice is 6
Step 2: Calculate Probabilities,
P(H) = 1/2 P(T) = 1/2 P(D6|H) = 1/6 (probability of getting a 6 on one die)
P(D6|T) = 5/36 (probability of getting a sum of 6 on two dice)
Step 3: Apply the Law of Total Probability. P(D6) = P(H) * P(D6|H) + P(T) * P(D6|T)
= (1/2 * 1/6) + (1/2 * 5/36) = 11/72, Therefore, the probability that the face (or the sum of the faces) showing on the die is equal to 6 is 11/72.

For this problem, both ChatGPT and Gemini provided a complete explanation and likewise, a step-by-step procedure in arriving at the possible scenarios. This led to the correct answer, thus scoring 5 points. Meanwhile, Copilot arrived at the correct answer and explained the step-by-step procedure well; however, there are some symbols or text which could affect the reader's view and interpretation of the solution. Thus, we give a score of 4 points to Copilot for this problem.

The evaluation of chatbots in solving word problems in probability, based on the criteria provided [6, 7], reveals insights into their reliability. The analysis compared the performance of ChatGPT and Copilot across five questions, with scores ranging from 0 to 5 based on their ability to process and accurately respond to prompts. A summary of these scores is presented in Table 5.1.

Table 5.1 Results of the Evaluation Criteria

	Evaluation Criteria Scores per Question					
Chatbots	**Question 1**	**Question 2**	**Question 3**	**Question 4**	**Question 5**	**Total Score**
ChatGPT	5	5	5	5	5	25
Copilot	4	2	3	3	4	15
Gemini[1]	4	5	4	5	5	23

ChatGPT consistently scored the highest marks, achieving a perfect score of 25. This can be attributed to its consistency in delivering detailed explanation and step-by-step procedure which may support student learning. In contrast, Copilot scored a total of 15, reflecting its partial correctness and occasional inaccuracies in the problems. Gemini scores a total of 23. The high scores of ChatGPT demonstrate its potential as a reliable educational tool, capable of enhancing student learning outcomes in probability. However, the gap in scores between the two chatbots also suggests that there is room for improvement, particularly for other chatbots like Copilot.

6. Conclusion

The reliability of chatbots in solving word problems in probability is a crucial factor that can significantly impact their effectiveness as educational tools for several reasons, such as their

[1] A referee noted that the performance of Gemini was similar to that of ChatGPT, and Wolfram Alpha is good at doing the sums but not at understanding when fed a whole question.

accuracy, efficiency, and opportunity to enhance the learning experience. The findings of this study indicate that chatbots have the potential to provide consistent, accurate, and timely support. The future of chatbots in education, particularly in solving probability problems, looks promising. However, continued advancements in AI and natural language processing (NLP) are necessary to further improve the reliability and capabilities of these chatbots. Future research should focus on developing more sophisticated algorithms, enhancing user interaction, and conducting large-scale studies to validate and refine the reliability of chatbots in educational settings. This study offers valuable insights into the current state of chatbot reliability, setting the stage for future developments in AI-based educational tools.

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Online learning incorporating a mathematics online test system and a dynamic geometry system

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Abstract

This research reports on the creation of rich mathematical content for secondary education in Japan using the open-source e-Learning platform "Moodle". The study focuses on utilizing the "page" and "quiz" functions to incorporate complex mathematical expressions and dynamic mathematical graphs as teaching materials. Additionally, the "KeTCindyJS" is used for generating dynamic graphs, and the "STACK" system is employed for automatic grading to evaluate learners' mathematical knowledge. The primary goal is to develop quizzes within Moodle that integrate advanced mathematical graphs.

1 Introduction

This paper discusses research activities aimed at developing comprehensive mathematical content for e-Learning services using the open-source platform "Moodle".

The focus is on mathematics education for secondary and higher education institutions in Japan. At our affiliated higher education institution, we use learning management systems like "Microsoft Teams" and "Moodle". Moodle, in particular, is used to provide a platform for studying mathematical concepts that include complex and advanced two-dimensional mathematical expressions (e.g., x^2+1, $\sin(x^2+1)$, e^{x^2+1}, etc.) , and mathematical graphs (e.g., plane curves like $y = x^2 + 1$, etc. anytime and anywhere.

For example, the "Page" feature under "Resources" in Moodle allows users to access electronic materials with two-dimensional mathematical expressions and images (static images)

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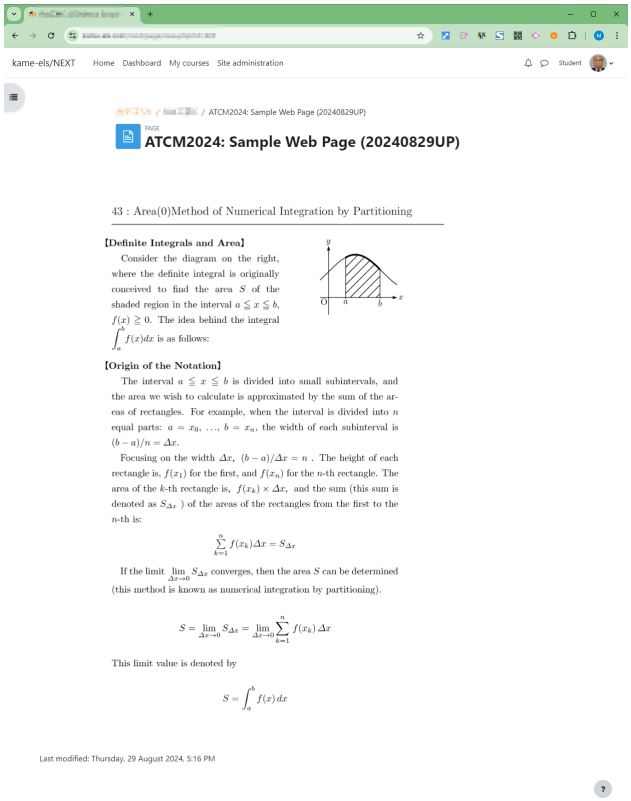

Figure 1: Sample textbook by the Page function of Moodle

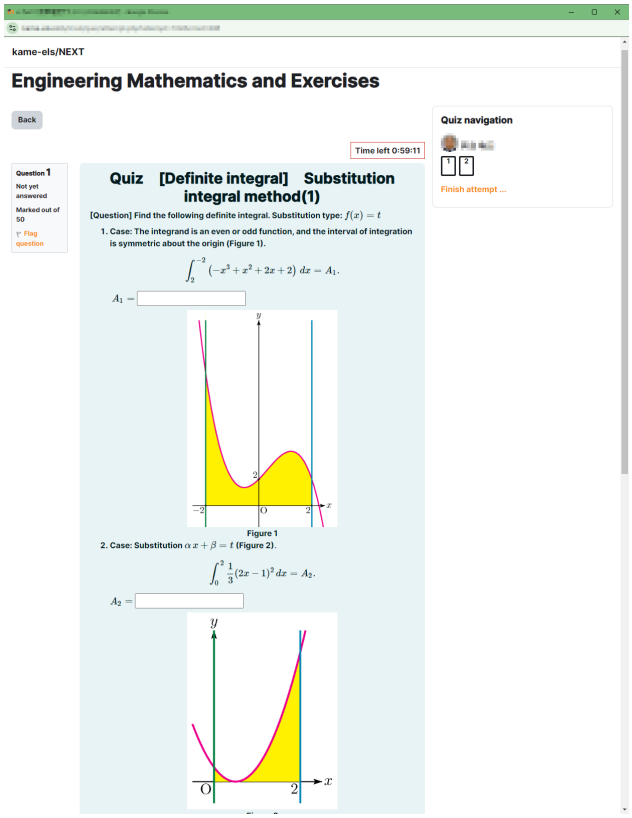

Figure 2: Sample quiz by the Quiz function of Moodle

online or print them in a physical format. Figure 1 shows a section of a webpage that explains the definition of a definite integral using basic mathematical concepts, simple two-dimensional expressions, and a straightforward reference diagram. Additionally, the "Quiz" feature under "Activities" in Moodle offers an e-Learning service with automatic grading, multiple attempts, and the option to review grading results, both in and outside the classroom. Figure 2 displays a quiz provided on a webpage that asks about definite integrals of basic functions, including a simple color-coded reference diagram.

Secondly, mathematical graph materials are utilized according to the following steps to help students visually understand mathematical concepts:

- Since around 2006, we have been using the Graph Generation System "KETpic", which adopts a simple yet not difficult method for paper-based graph drawing
- Since around 2014, we have been using the Graph Generation System "KeTCindy", based on the dynamic geometry software Cinderella, to provide this mathematical material in e-Learning services
- Since around 2018, we have been using the Script-type Graph Generation System "KeTCindyJS" to add dynamic properties to existing generated graphs

Ultimately, we aim to use advanced two-dimensional mathematical expressions and dynamic mathematical graphs as teaching materials on the web pages of the e-Learning service.

Next, the “Quiz” function is used to evaluate learners’ mathematical knowledge. We particularly regard “visualization”, “random question generation”, and “automatic grading” as core characteristics in mathematics education within this e-Learning service. To achieve these features, we utilize the ”multiple choice questions”, “essay questions”, and the “mathematics online assessment system STACK” in Moodle’s quiz feature

The immediate goal is to create quizzes with combined features of dynamic mathematical graphs in Moodle.

2 About KeTCindyJS

KeTCindyJS is a library of the interactive graph generation software “Cinderella”, written in Java programming language, and it allows the creation of dynamic mathematical graphs (TeX-based) that can be easily embedded in web pages. For installation and usage instructions regarding KeTCindy, please refer to the website [2] and follow the supplementary explanations carefully.

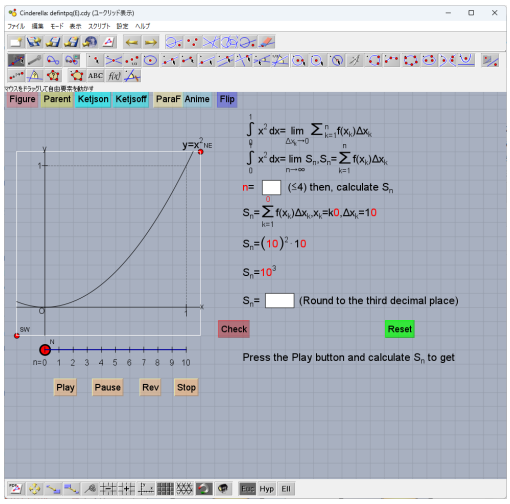

Figure 3: An initial screen of the content created in Cinderella by partitioning for definite integrals

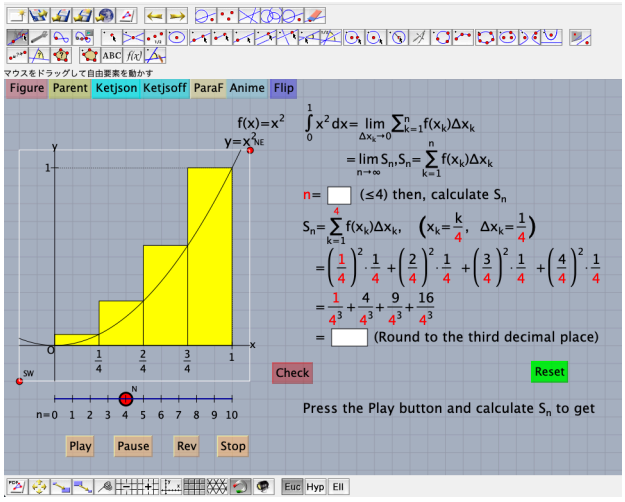

Figure 4: An intermediate screen during the operation of the created content in Cinderella by partitioning for definite integrals

In practice, we can create images in the KeTCindyJS format (hereafter referred to as “sample.cdy”) by installing the following recommended and necessary software in addition to Cinderella and KetCindy on their Windows 11-based computer. Skip the installation if similar software is already in use.

- Typesetting System “TeX Live” [5]: Creates paper-based documents containing advanced and complex mathematical expressions.
- Statistical Programming Language “R” [6]: An engine that performs complementary internal processing.

- Mathematical Processing System “Maxima” [7]: Handles advanced mathematical processing.
- PDF Viewer “SumatraPDF” [8]: Provides a compatible environment for viewing PDF documents with KeTCindy.

At this point, we begin by loading a temporary format in Cinderella’s editing screen to create a KeTCindy graph related to the method of numerical integration by partitioning for definite integrals. As a result, Figures 3 shows the initial screen of the content created in Cinderella, and Figures 4 shows an intermediate screen during the operation of the created content in Cinderella. This visually demonstrates that a graph with dynamic properties has been generated in the created Cinderella content.

The KeTCindy graph (Figures 5 and 6) related to the method of numerical integration by partitioning, which we will discuss next, uses a modified version suitable for the ”Quiz” feature in Moodle, where the creation process in the Cinderella software remains the same.

Next, using the “Ketjson” feature (button) in the KeTCindy software (Figure 4), the “sample.cdy” image file is converted to an HTML format web file (hereafter referred to as “samplejson.html”). This HTML file “samplejson.html” is then adjusted to be accessible from the internet (hereafter referred to as `https://www.anydomain/samplejson.html`). Figure 5 shows the HTML file “samplejson.html” created.

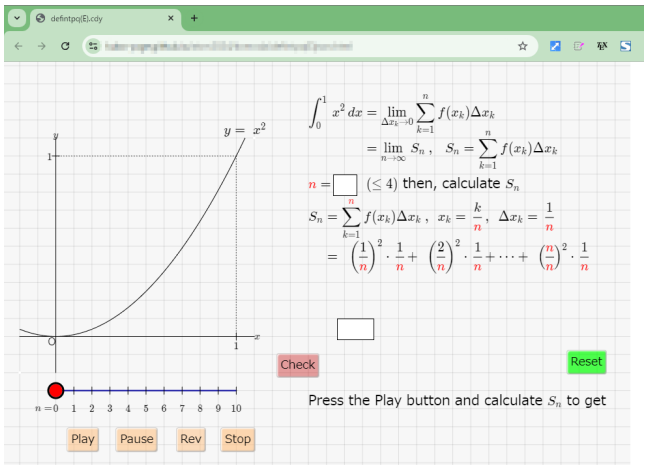

Figure 5: The HTML file “samplejson.html” included the KeTCindy graph

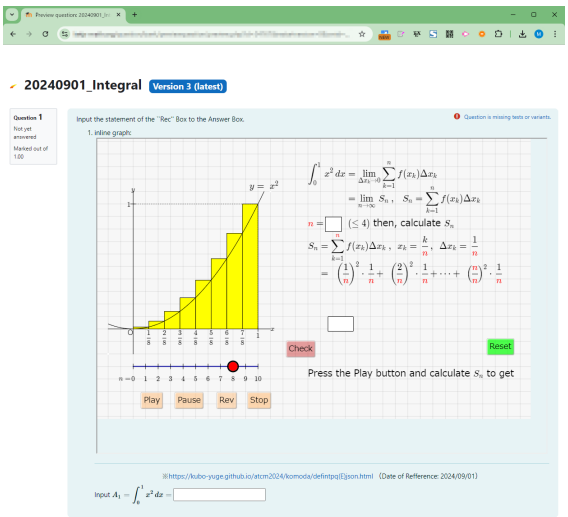

Figure 6: The Moodle’s quiz included the KeTCindy graph

3 Using Moodle Environment

Create a quiz content (here, named “question”) associated with the mathematical graph of the dynamic image file “sample.cdy” and configure it in Moodle’s quiz. Then, use the “inline frame (iframe)” tag to embed this question content text. Specifically, specify the link

information `https://www.anydomain/samplejson.html` in the src attribute of the iframe tab (pay particular attention to the adjustments of the width and height attributes).

Figure 6 shows the screenshot of content quiz with the dynamic mathematical graph "sample.cdy" created through these steps.

4 Future Research

Akaike has developed the following interesting quiz [9], the quiz (Figure 7) created up to this point is a quiz content that incorporates related KetCindy graphs and asks learners to answer numerical values or mathematical expressions, and it is an e-Learning service that automatically grades the answers after the quiz is taken.

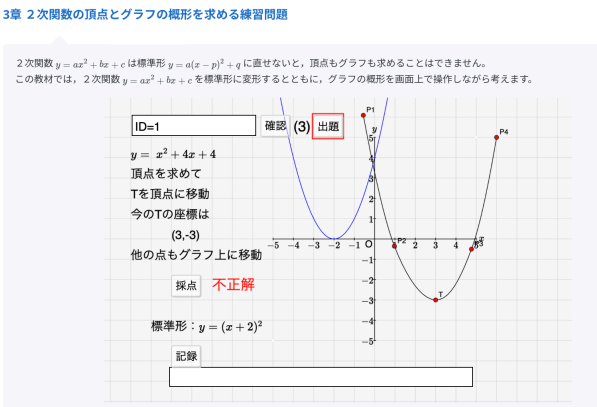

Figure 7: A quiz using the KeTCindy graph introduced here (Japanese notation in parts)

This problem (Figure 7) with the KeTCindy graph directly features a web page originally created in Japanese. The overview of this problem can be translated into English as follows: For a randomly generated quadratic function, the task is to find its standard form, and then draw its graph using five movable points within the diagram. Based on the positions of the five moved points, a beige curve is automatically drawn, and the mathematical expression related to the drawn graph is calculated in the background.

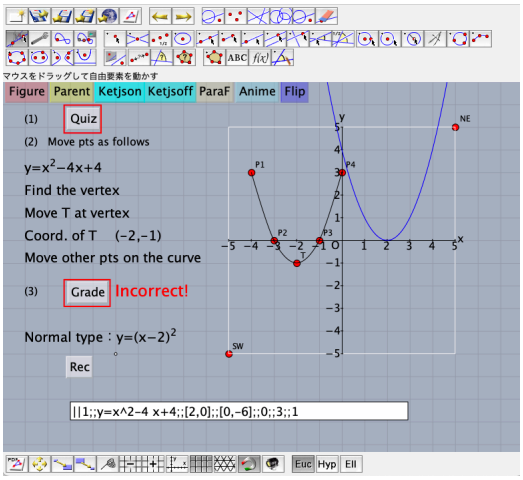

Figure 8: A KeTCindy graph in the Cinderella system

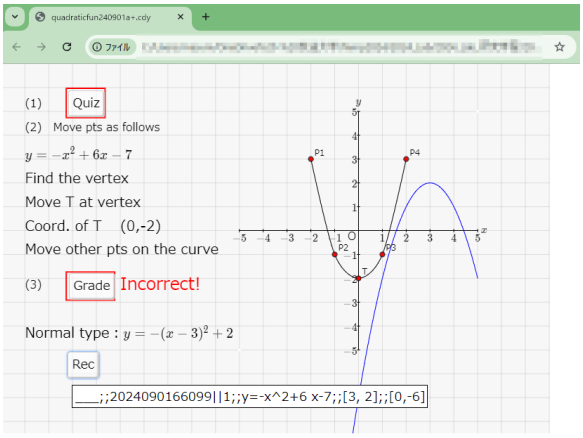

Figure 9: A standalone KeTCindy graph displayed in a web browser

On the other hand, in the literature, there are also quizzes (single questions) that use KetCindy graphs and can be automatically marked. In this quiz, the learner moves the points on the KeTCindy graph canvas freely, and the system automatically draws the curve by calculating from the positions of the points and following the generated curve. Furthermore, it is possible to generate specific mathematical information from the geometric situation of the curve, and to perform mathematical processing on that specific mathematical information to determine whether the curve drawn is correct or not.

One of the advantages of this KeTCindy graph-based problem is that, unlike the content (quizzes) we have developed so far, which primarily rely on algebraic processing for correct/incorrect judgments, this KeTCindy graph-based problem allows for determining correctness based on geometric properties obtained from freehand movable points, which are then processed mathematically.

However, because this KeTCindy graph-based problem is presented as a standalone question, it is considered insufficient for a comprehensive learning management service.

To address this limitation, we decided to develop the following in our e-Learning service. First, the existing KeTCindy graph-based problem is transferred to the Cinderella system (Figure 8). This KeTCindy graph in the Cinderella system (Figure 8) that enables equivalent functions such as problem presentation, freehand graph generation, correctness evaluation, and mathematical processing of geometric properties.

Second, the transferred content is edited and output as an HTML file (Figure 9).

Third, the HTML file is published on the network. Finally, the published HTML file is integrated into Moodle's quiz feature using the "iframe" command (Figure 10).

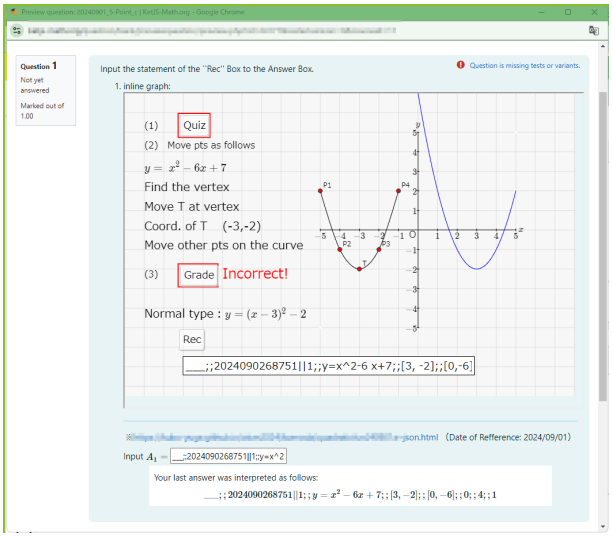

Figure 10: A Moodle quiz that evaluates the correctness of a KeTCindy graph with geometric properties

Through these steps, we created a Moodle quiz that allows for correct/incorrect judgments based on geometric properties.

5 Acknowledgements

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Development of a Specification Management System for Multidimensional PISA-like Mathematics Items

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Abstract: *This paper stems from the primary author's ongoing dissertation, which encountered a technological-logistical issue during the implementation of an item-writing study. The central problem discussed involves discrepancies between the table of specifications (TOS) for multidimensional PISA-like mathematics items created by teachers and the item specifications indicated in their individual item submission forms. These inconsistencies posed challenges in consolidating item specifications, impacting data collection and analysis (e.g., inaccurate alignment ratings due to mismatches between the TOS and item specifications). To resolve this, the authors developed an item specification management system in MS Excel, which automates processes like verifying incorrect entries and identifying unfulfilled constraints. Testing the system with hypothetical item specifications showed its ability to perform data validation, identify input errors, and detect entry conflicts; hence making it a viable and efficient alternative to the workflow used in the original study. These findings are significant for future studies replicating the primary author's dissertation, as the system mitigates potential issues in the manual input of item specifications, particularly for multidimensional PISA-like items with multiple dimensions such as item format, content category, context category, and process category. Such a specification management system can be very useful on a larger scale, as independent item-writing projects in educational institutions can utilize this without having to endure the use of several TOS's simultaneously which can inevitably cause confusion among the item-writers which makes the item specifications prone to manual errors; and without having to rely on commercially available item-banking software.*

1. Introduction

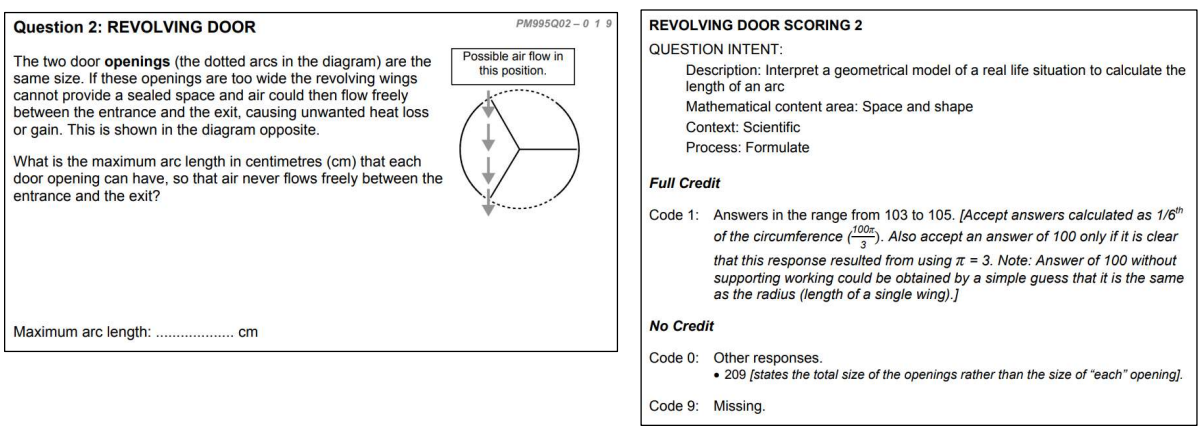
This paper is an extension of a larger study which is the primary author's ongoing dissertation project, where mathematics teachers from the Philippines teaching high school students were tasked to write PISA-like mathematics items. The goal of doing so was to measure students' mathematical literacy as defined by the PISA 2022 mathematics assessment framework, which meant that the item specifications (and distribution of the content domain categories) must comply with those stipulated in the assessment framework. As a researcher, the primary author encountered technical challenges in consolidating the items given that there were nine participants who were tasked to construct eight items each; and that the teachers have to declare the specifications for each item before writing the actual item. Hence, the aim of this paper is to capitalize on the technological-logistical challenges in the consolidation of the items to develop a specification management system in MS Excel that will alleviate identical problems that might be encountered in similar future research projects.

1.1 PISA Mathematics Assessment Framework

As defined by OECD [1], mathematical literacy is the ability of 15-year-old students to apply mathematics in real-world scenarios, specifically "to reason and mathematically and solve problems in a variety of 21st century contexts" (p. 6), by using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. Released PISA mathematics items (see [2])

indicate that each is categorized according to dimensions such as item format (multiple choice or constructed response) and content domain categories (i.e., content categories, context categories, and process categories). The test-takers of PISA math items are expected to demonstrate skills in the form of reasoning and problem-solving, which is further classified as formulating, employing and interpreting. Hence, there are four process categories in the framework: Reasoning (RSN), Problem-Solving: Formulate (PSF), Problem-Solving: Employ (PSE), and Problem-Solving: Interpret (PSI). The categories classified as content categories describe the mathematical construct measured by each item, namely: Quantity (QNT), Change & Relationships (C&R), Space & Shape (S&S), and Uncertainty & Data (U&D). As for the types of contexts presented in the items, the PISA mathematics assessment framework provides a list enumerating (but not in an exhaustive manner) likely scenarios in which the contextual real-life problems take place. The four context categories are as follows: Personal (PER), Occupational (OCC), Societal (SOC), and Scientific (SCI) which differ in terms of the scope or reach of the problem's level of impact or significance. Each of the four content categories share the same percentage allotment (25%), and this applies to the context categories and process categories as well. It is also worth noting that there are almost the same number of multiple choice items and constructed response items in the PISA mathematics assessment framework. (For a thorough description of the content, context, and process categories, please see [1].)

As shown in [2], an example of an item from the PISA 2012 released mathematics items indicates the item format, content category, context category, and process category – which shall henceforth be collectively referred to as *item specifications*, that the item is classified under. Figure 1.1.1 below demonstrates the item specifications for an item about a revolving door.



Question 2: REVOLVING DOOR *PM995Q02 – 0 1 9*

The two door **openings** (the dotted arcs in the diagram) are the same size. If these openings are too wide the revolving wings cannot provide a sealed space and air could then flow freely between the entrance and the exit, causing unwanted heat loss or gain. This is shown in the diagram opposite.

What is the maximum arc length in centimetres (cm) that each door opening can have, so that air never flows freely between the entrance and the exit?

Maximum arc length: ..... cm

REVOLVING DOOR SCORING 2

QUESTION INTENT:

Description: Interpret a geometrical model of a real life situation to calculate the length of an arc

Mathematical content area: Space and shape

Context: Scientific

Process: Formulate

Full Credit

Code 1: Answers in the range from 103 to 105. *[Accept answers calculated as 1/6th of the circumference ($\frac{100\pi}{3}$). Also accept an answer of 100 only if it is clear that this response resulted from using π = 3. Note: Answer of 100 without supporting working could be obtained by a simple guess that it is the same as the radius (length of a single wing).]*

No Credit

Code 0: Other responses.
- 209 *[states the total size of the openings rather than the size of "each" opening].*

Code 9: Missing.

Figure 1.1.1 "*Revolving Door*" – a sample item from the PISA 2012 released mathematics items [2]

Following the legend indicated in Figure 1.1.2 (as implemented in the primary author's ongoing dissertation project), the item specifications for item shown in Figure 1.1 would be CR, S&S, SCI, PSF – with each specification representing a certain aspect or dimension of PISA-like mathematics items.

LEGEND:

Item Specification	Code	Meaning
Item Format	MC	Multiple Choice
	CR	Constructed Response
Content Category	QNT	Quantity
	C&R	Change & Relationships
	S&S	Space & Shape
	U&D	Uncertainty & Data

Item Specification	Code	Meaning
Context Category	PER	Personal
	OCC	Occupational
	SOC	Societal
	SCI	Scientific
Process Category	RSN	Reasoning
	PSF	Formulate
	PSE	Employ
	PSI	Interpret

Figure 1.1.2 The legend of item specifications used to classify the PISA-like mathematics items

1.2 Rationale for this Paper

The data-gathering procedure for the primary author's ongoing dissertation project involved the use of three pertinent forms: (1) the Table of Specifications or TOS Submission Form (TSF), (2) the Item Submission Form (ISF), and (3) the Expert Evaluation Form (EEF), and all of these are MS Word templates which were distributed, accomplished and collected online via submission in Google Drive and through email. The participants hence were to fill out the TSF first, before drafting their items in the ISF, with each of the items labelled from Q1 to Q8. However, the participants were given the liberty to revise their TSF should there be corresponding changes in item specifications as reflected in their ISFs. Each ISF consists of three tables that show the item specifications for all eight items constructed by one participant. The first table is TOS #1, which displays the items' content categories vis-à-vis process categories, including the item formats. Meanwhile, the second table is TOS #2, which shows the plotting of the items' content categories against their context categories. A final table would be necessary to consolidate all the eight items' item specifications – hence the Summary of Item Specifications (SIS). The accomplished ISFs were forwarded by the primary author to expert evaluators who were experienced in the administration of International Large-Scale Assessments (ILSAs) in the Philippines. These experts were requested to provide both quantitative and qualitative feedback on the items submitted by the participants as shown in their ISFs. The quantitative feedback came in the form of Likert scale ratings from 0 to 5, where 0 denotes no alignment while 5 denotes a perfect alignment with the item specifications. The participants were provided a filled out TSF (a hypothetical TSF) for their reference, as shown in Figure 1.2.1 below.

Table of Specifications (TOS #1):

Content Category \ Process Category	Item Format	Reasoning (25%)	Formulate (25%)	Employ (25%)	Interpret (25%)	Subtotal	Total No. of Items
Quantity (25%)	MC				Q5	1	2
	CR	Q1				1	
Change & Relationships (25%)	MC	Q4				1	2
	CR			Q7		1	
Space & Shape (25%)	MC			Q2		1	2
	CR		Q3			1	
Uncertainty & Data (25%)	MC				Q6	1	2
	CR		Q8			1	
Total No. of Items		2	2	2	2		8

Table of Specifications (TOS #2):

Content Category \ Context Category	Personal (25%)	Occupational (25%)	Societal (25%)	Scientific (25%)	Total
Quantity (25%)	Q1		Q5		2
Change & Relationships (25%)	Q4	Q7			2
Space & Shape (25%)		Q2		Q3	2
Uncertainty & Data (25%)			Q6	Q8	2
Total	2	2	2	2	8

Summary of Item Specifications (SIS)

Item Placement	Item Format	Content Category	Context Category	Process Category
Q1	CR	QNT	PER	RSN
Q2	MC	S&S	OCC	PSE
Q3	CR	S&S	SCI	PSF
Q4	MC	C&R	PER	RSN
Q5	MC	QNT	SOC	PSI
Q6	MC	U&D	SOC	PSI
Q7	CR	C&R	OCC	PSE
Q8	CR	U&D	SCI	PSF

Figure 1.2.1 A hypothetical accomplished TSF used as reference by the participants

Given that the MS Word template for the TSF is static and that each participant must input all entries in the tables manually, what is highly crucial in the process of filling out the TSF is the accuracy of the match among the categories indicated. In other words, any conflicting entries will have corresponding effects on the rating of the participants' items, given that the expert evaluators' main duties involved rating each item based on their alignment with the item specifications in the SIS. The following figure, Figure 1.2.2, displays a portion of the ISF of one participant.

Item Placement:	Item Format:	Content Category:	Context Category:	Process Category:
Q1	MC	C&R	PSE	PER

Question:

Ellie's cellular plan let's her choose the amount of data is included per month. The table shows the plan's monthly cost (y) for a given amount of data (x).

Data (GB) (x)	2	5	7
Cost (Php) (y)	500	800	1100

What is the estimate cost of plan if there is 15 GB included?

Response Options:

A.) The estimate cost of plan which includes 15GB is Php 1,700.	B.) The estimate cost of plan which includes 15GB is Php 2,300.	C.) The estimate cost of plan which includes 15GB is Php 3,200.	D.) The estimate cost of plan which includes 15GB is Php 2,975.

Reason for using each response option:

* (3, 500), (5, 800)

$m = \frac{800-500}{5-3}$

$\boldsymbol{m = \frac{200}{2}}$

$m = 100$

* SIF (3, 500)

$y = mx + b$

$500 = 100(3) + b$

$b = 220$

* If 15 GB:

$y = 100(15) + 200$

$y = 1700$

* (5, 800) , (7, 110)

$m = \frac{1100-800}{7-5} = 150$

* SIF (5, 800)

$y = mx + b$

$800 = 150(5) + b$

$\boldsymbol{b = 725}$

* If 15 GB:

$y = 150(15) + 725$

$y = 2975$

* (3, 500) , (5, 800)

$m = \frac{800-500}{5-3} = 150$

* SIF (3, 500)

$y = mx + b$

$500 = 150(3) + b$

$\boldsymbol{b = 950}$

* If 15 GB:

$y = 150(15) + 950$

$y = 3200$

Correct Answer:

* (3, 500) , (5, 800)

$m = \frac{800-500}{5-3} = 150$

* SIF (3, 500)

$y = mx + b$

$500 = 150(3) + b$

$b = 50$

* If 15 GB:

$y = 150(15) + 50$

$\boldsymbol{y = 2300}$

Figure 1.2.2 A sample TSF made by one of the participants, showing item Q1

It can already be noted that in the item specifications, the participant had incorrectly placed PSE as the context category (instead of process category), and PER as the process category (instead of context category). This minor mishap occurred in the succeeding items Q2, Q3, and Q4 as well, prompting the assigned expert evaluator to give her items Q1 to Q4 a rating of zero for the alignment as shown in Figure 1.2.3.

EVALUATION FORM FOR EXPERT EVALUATORS

PART A:
Please rate each test item of the participant based on its alignment with PISA's Content Category, Process Category and Context Category. Please use the 6-point Likert scale shown below.

NOTE: Comments and feedback must be based on the Summary of Item Specifications. Thank you!

5 – Perfectly Aligned	2 – Somewhat Aligned
4 – Moderately Aligned	1 – Poorly aligned
3 – Fairly Aligned	0 – Not aligned at all

Item	Rating based on the alignment with **CONTENT Categories**	Rating based on the alignment with **CONTEXT Categories**	Rating based on the alignment with **PROCESS Categories**
Q1	5	0	0
Q2	1	0	0
Q3	5	0	0
Q4	5	0	0
Q5	5	5	5
Q6	5	4	2
Q7	5	5	5
Q8	1	5	5

Figure 1.2.3 The accomplished EEF for the participant who made errors in the TSF

Such ratings would adversely affect the measure of item quality, given that item alignment is part of it; and this would consequently muddle the results and findings of the original study. The errors exhibited by the participant were not an isolated case, as there were several other instances of conflicting entries among the tables in TSF that went unnoticed by the participants or the expert evaluators. The collective effort of the authors mirrors the primary author's aim to help improve data collection procedures for any similar studies that might arise in the future. Such kinds of errors in the filling out of the TSF is deemed preventable, hence the authors are convinced that automatizing the process helps ensure correct alignment among the entries in the TSF. This reason became a driving force to develop a dynamic template using MS Excel that will check for conflicting entries. This template – a specification management system, is envisioned to detect errors in the TSF and prompt the item-writer about them in a very timely manner. Hence, the assumptions behind the development of the specification management system in this paper include the following: (1) Each teacher has to write eight items each; (2) the distribution of items (and categories) follows the PISA math assessment framework; (3) each item can be classified into any of the categories with minimal overlap; (4) no two items share the same exact set of item specifications.

1.3 Conceptual Framework and Related Literature

Items with various aspects or dimensions such as those described in the prior sections have been termed by Turner and Carlson [3] as *multidimensional items*. As multidimensional items measure multiple constructs (or skills) concurrently, one aspect of assessment that may be overlooked in the process of item development is the individual validity of each item. This was termed by Rovinelli and Hambleton [4] as *item validity*, which is subsumed under content validity. The item-writing task in the original study in which the present study is based on, exhibits the four principles of content validity identified by Bridge et al. [5]: (1) establishment of purpose for intended use (writing PISA-like mathematics items), (2) ensuring representativeness across the domain of interest (the use of TOS displaying the content domain categories), (3) ensuring clarity and quality of test items; and (4) an external review of items by field experts (items and TOS were submitted to PISA experts in the Philippines for qualitative feedback). Similarly, these steps (as well as those discussed in sections 1.1 and 1.2) reflect the initial components of the test development process proposed by Lane et al. [6], namely: overall plan, domain definition, content specifications, item development, and test design and assembly. Fives & DiNonato-Barnes [7] argue that the TOS functions as a test

blueprint by helping align objectives, instruction, and assessment – which is a fairly common occurrence in constructing traditional summative tests. Hence, the present study subscribes to *item-objective congruence* [4, 8] (as cited in [3]) as its conceptual framework. Item-objective congruence refers to the level of match between an item and the learning objective it is built upon, as reflected in the TOS [9] (as cited in [3]).

The production of the evidence of item validity is usually conducted by an independent expert panel, and as mentioned in the earlier sections, there may be difficulties in monitoring the item validity when there are many items, each with several dimensions to keep track of. This is especially applicable in the case of writing PISA-like mathematics as shown in the work of Garcia, Gaylo, & Vistro-Yu [10]. A plethora of technology-aided item-banking tools have been discussed in the works of Weiss [11] and Muckle [12]; and such facilitate and streamline processes such as item development, test assembly, and test delivery. Weiss [11] discusses that technological advances in the recent decades (e.g., the emergence of the use of personal computers and the internet) have played a role and greatly influenced the way that modern item-banking is conducted. The caveats of the manual production of tests may have driven the need for automatizing certain aspects of the process. Weiss describes such production as "labor intensive, tedious, and prone to errors" (p. 198), while commending the modern ways of item-baking due to their increased accuracy and efficiency. In a similar vein, Muckle's [12] work discusses in great detail the recent developments in item-banking, as well as the extensive features of web-based item-banking, such as remote authoring, administrative features, and metadata and queries. Despite the undeniable power of and the wide range of applications posed by the use of such sophisticated item-banking tools, they are highly unapplicable in school contexts due to the specificity of the needs of teachers and administrators, as well as lower level of stakes associated in the use of such assessments. Weiss [11] and Muckle [12] both present commercially available item-banking software with too elaborate workflow processes including delegation and assembly; and with too advanced computations of test item statistics for an average class teacher or school administrator to comprehend. As the functionalities of these tools are aimed at facilitating the primary tasks of commercial task developers or for-profit assessment organizations, the advanced features of item-banking tools described by Muckle render them irrelevant in school contexts. This is considering that (1) teachers collaborate and communicate mostly through verbal means, without the assistance of automated functions, considering that the size of the department is not as large as that of a testing organization; (2) learning the functionalities of such software will entail devoting time and effort on the part of teachers, which is something they can't afford to do given the amount of teaching and non-teaching duties that they are already preoccupied to begin with.

It is thus imminent to aim to develop a similar software – one which has simplified capabilities and is well-suited to the practical needs of test creators unaffiliated from any commercial testing organization. An attempt of doing this is seen in the work of Manaligod [13] as he developed a prototype software for item analysis and item-banking system applicable to the local context in the Philippines. For an accuracy check, he compared the descriptive statistics and item analysis results of his software ItemMan, with those from the Statistical Package for Social Sciences or SPSS software and found that both yielded identical results. However, ItemMan mostly produces item statistics and data post-facto. Attention to the item-level data is equally crucial prior to and during the drafting of the items. Ensuring the alignment of the items with content-validity standards is seen in a very recent study by Ismail & Zubairi [14], where they evaluated reading test items in Sri Lankan locale. They did this by inviting subject matter experts who judged whether the reading items reflected the skills or standards outlined in the test blueprint, and consequently computing for the index of item-objective congruence for all items. However, one major critique in their procedure is that the validation of items entirely rests on the hands of the subject matter experts, leaving minimal

opportunity for the item writer to double-check the validity of his or her items against the TOS he or she has initially formulated.

2. Methodology

2.1 Software Requirements Analysis

The scope of the specification management system for PISA items is based on the existing TSF that the participants have used. The TSF is primarily a bookkeeping instrument for managing multidimensional items and does not directly assess item validity. Instead, as mentioned in the rationale for the paper, the TSF form assists in presenting item specifications to participants and expert evaluators, who can then assess item validity. Furthermore, the specification management system does not need to concern itself with item security, test assembly, test delivery, or item statistics.

The authors combined the tables present in the MS Word TSF into a single MS Excel sheet, and a data flow was established by identifying redundant information present in the existing MS Word forms. This is done by first listing down all the dimensions and categories within each dimension tracked by each table. The MS Word TSF consists of three tables, whose functions in terms of their contribution to the test blueprint are described below.

1. Table of Specifications (TOS #1): maps each item to an Item Format, Content Category, and Process Category
2. Table of Specifications (TOS #2): maps each item to a Content Category and Context Category
3. Summary of Item Specifications (SIS): lists the Item Format, Content Category, Process Category, and Context Category for each of the items.

Since TOS #1 and TOS #2 can be fully derived from the information in the SIS, to minimize data conflicts, the specification management system could have participants fill up the SIS with all relevant information, from which TOS #1 and TOS #2 can be populated. Since participants do not provide the same dimension more than once, no data conflicts can occur. This data flow is illustrated in Figure 2.1.1.

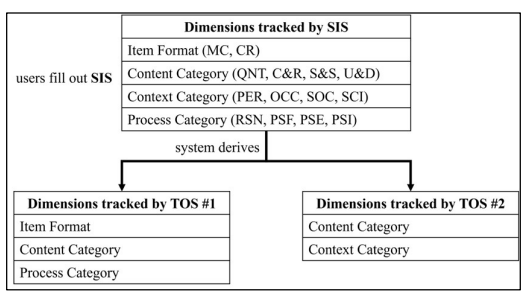


Figure 2.1.1 A diagram showing a possible data flow with no potential for data conflicts

However, this is reversed from the current implied workflow in the MS Word template, where participants start by filling out the TOS #1 and TOS #2 tables, and then summarizing their inputs in the SIS. This is because the TSF tracks the counts of items satisfying each category in each dimension, and participants are tasked with providing appropriate items to satisfy the desired counts for each

category. As seen in Figure 1.2.1, it is the TOS #1 and TOS #2 tables which explicitly express these counts, and hence participants may find it more intuitive to fill out TOS #1 and TOS #2 first. This data flow, presented in Figure 2.1.2, immediately shows that participants are tasked with providing the same dimension more than once.

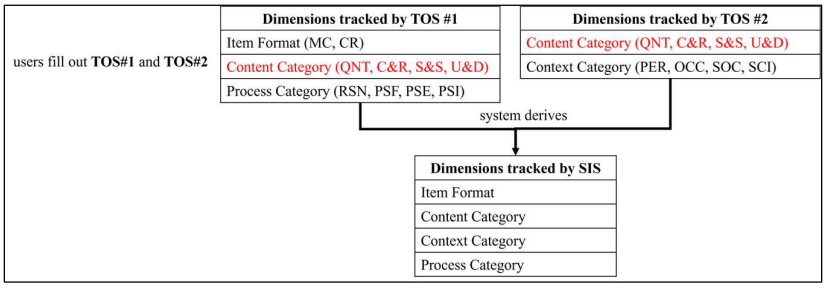


Figure 2.1.2 A diagram showing the data flow used in the MS Word template where redundant inputs are highlighted in red

The authors remark that deciding on how to structure the data flow for each dimension is a design decision which emerges naturally when managing multidimensional items. Users find it easier to work with tables which present less dimensions, but when there are multiple dimensions that must be kept in sync with each other, a specification management system must implement rules to avoid conflicting information. The two data flows present the following design decision.

1. Sacrifice ease-of-use in favor of data integrity by adopting the data flow in Figure 2.1.1.
2. Maintain ease-of-use by adopting the data flow in Figure 2.1.2, but alert users to conflicting information.

In line with the rationale for this paper, the authors opted to employ the data flow in Figure 2.1.2 in designing the MS Excel spreadsheet, as it mirrors the familiar data flow already in use in the MS Word template for the TSF. The steps in developing the system then proceed as follows:

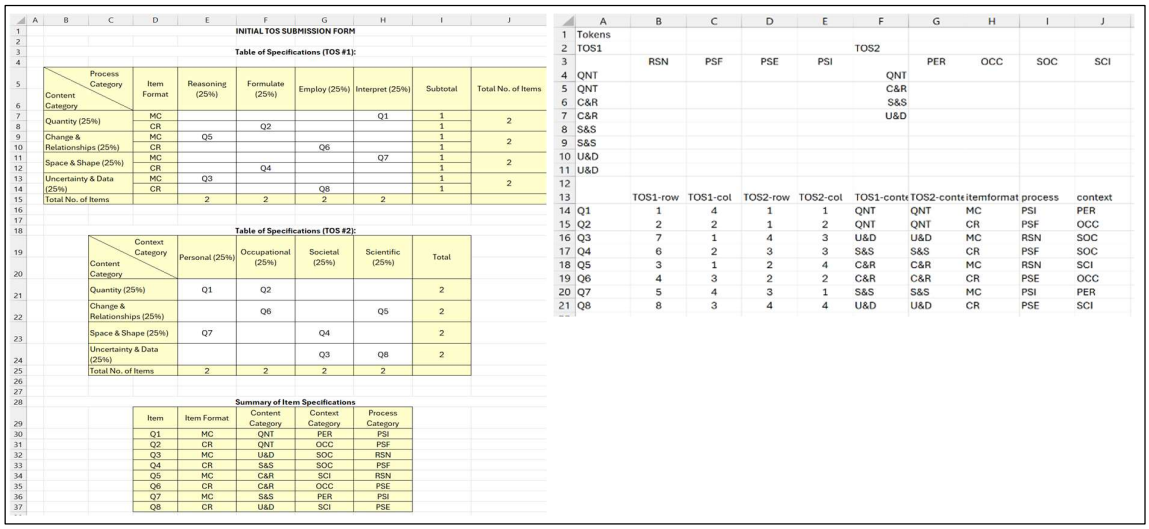
1. Identify all possible sources of errors or data conflicts.
2. Implement an input interface allowing users to fill up TOS #1 and TOS #2.
3. Implement a data validation layer that catches all possible errors.
4. Construct the SIS in the case that the data validation succeeds.

2.2 Software Implementation and Validation

The MS Excel spreadsheet has two sheets, shown in Figure 2.2.1. The first sheet (data entry) mirrors the MS Word template for the TSF, allowing participants to fill up TOS #1 and TOS #2. As this also includes information regarding correct item counts for each category, conditional formatting and formulas are employed to highlight cells in red if the desired item counts are not satisfied. The second sheet (data validation) uses MS Excel formulas to perform the following:

- Collect all information from the data entry sheet into a single table reporting, for each item, the provided categories for each dimension. Since the Content Category dimension is entered twice by the user, the reported Content Category from TOS #1 and TOS #2 are tracked separately.

- Identify and report data conflicts.
- Consolidate the reported categories into the form to be reported in the SIS.



INITIAL TOS SUBMISSION FORM

Table of Specifications (TOS #1):

Content Category \ Process Category	Item Format	Reasoning (25%)	Formulate (25%)	Employ (25%)	Interpret (25%)	Subtotal	Total No. of Items
Quantity (25%)	MC				Q1	1	2
	CR		Q2			1	
Change & Relationships (25%)	MC	Q5				1	2
	CR			Q6		1	
Space & Shape (25%)	MC				Q7	1	2
	CR		Q4			1	
Uncertainty & Data (25%)	MC	Q3				1	2
	CR			Q8		1	
Total No. of Items		2	2	2	2		

Table of Specifications (TOS #2):

Content Category \ Context Category	Personal (25%)	Occupational (25%)	Societal (25%)	Scientific (25%)	Total
Quantity (25%)	Q1	Q2			2
Change & Relationships (25%)		Q6		Q5	2
Space & Shape (25%)	Q7		Q4		2
Uncertainty & Data (25%)			Q3	Q8	2
Total No. of Items	2	2	2	2	

Summary of Item Specifications

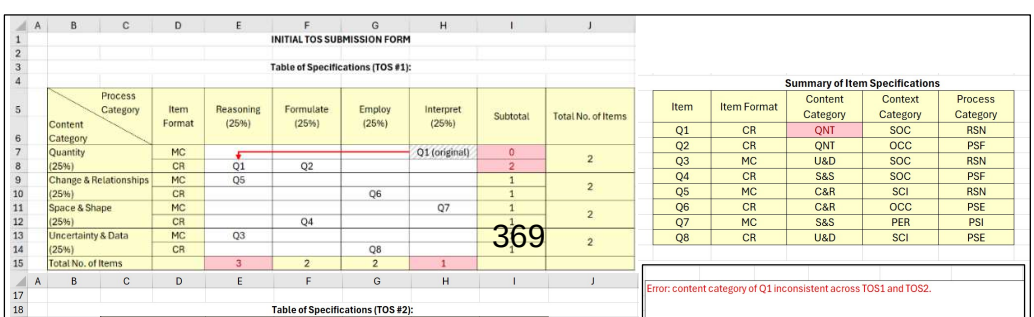
Item	Item Format	Content Category	Context Category	Process Category
Q1	MC	QNT	PER	PSI
Q2	CR	QNT	OCC	PSF
Q3	MC	U&D	SOC	RSN
Q4	CR	S&S	SOC	PSF
Q5	MC	C&R	SCI	RSN
Q6	CR	C&R	OCC	PSE
Q7	MC	S&S	PER	PSI
Q8	CR	U&D	SCI	PSE

Tokens									
TOS1					TOS2				
	RSN	PSF	PSE	PSI		PER	OCC	SOC	SCI
QNT					QNT				
QNT					C&R				
C&R					S&S				
C&R					U&D				
S&S									
S&S									
U&D									
U&D									
	TOS1-row	TOS1-col	TOS2-row	TOS2-col	TOS1-conte	TOS2-conte	itemformat	process	context
Q1	1	4	1	1	QNT	QNT	MC	PSI	PER
Q2	2	2	1	2	QNT	QNT	CR	PSF	OCC
Q3	7	1	4	3	U&D	U&D	MC	RSN	SOC
Q4	6	2	3	3	S&S	S&S	CR	PSF	SOC
Q5	3	1	2	4	C&R	C&R	MC	RSN	SCI
Q6	4	3	2	2	C&R	C&R	CR	PSE	OCC
Q7	5	4	3	1	S&S	S&S	MC	PSI	PER
Q8	8	3	4	4	U&D	U&D	CR	PSE	SCI

Figure 2.2.1 The data entry sheet (left) and data validation sheet (right) of the MS Excel specification management system

2.3 Testing Entry Conflict Detection

A simple demonstration of entry conflict detection can be done by moving one of the entries (e.g., Item Q1) to another cell (from cell H7 to cell E8 in TOS #1, and from cell E21 to cell G22 in TOS #2). It can be seen in Table 2.2.2 below that TOS #1, TOS #2 and SIS simultaneously flag concerned cells which show that the item placement did not comply with the constraints related to the distribution of content, context, and process categories. In the case of SIS, it simply showed a signal that there was an error related to the content category of Q1, where the error description field shows the message “Error: content category of Q1 inconsistent across TOS #1 and TOS #2” as a warning to the item writer regarding how to make the necessary amendment to fulfill the constraints.



INITIAL TOS SUBMISSION FORM

Table of Specifications (TOS #1):

Content Category \ Process Category	Item Format	Reasoning (25%)	Formulate (25%)	Employ (25%)	Interpret (25%)	Subtotal	Total No. of Items
Quantity (25%)	MC				Q1 (original)	0	2
	CR	Q1	Q2			2	
Change & Relationships (25%)	MC	Q5				1	2
	CR			Q6		1	
Space & Shape (25%)	MC				Q7	1	2
	CR		Q4			1	
Uncertainty & Data (25%)	MC	Q3				1	2
	CR			Q8		1	
Total No. of Items		3	2	2	1		

Summary of Item Specifications

Item	Item Format	Content Category	Context Category	Process Category
Q1	CR	QNT	SOC	RSN
Q2	CR	QNT	OCC	PSF
Q3	MC	U&D	SOC	RSN
Q4	CR	S&S	SOC	PSF
Q5	MC	C&R	SCI	RSN
Q6	CR	C&R	OCC	PSE
Q7	MC	S&S	PER	PSI
Q8	CR	U&D	SCI	PSE

Error: content category of Q1 inconsistent across TOS1 and TOS2.

Table of Specifications (TOS #2):

Figure 2.2.2 The display of the specification management system when Item Q1 is moved to another cell in both TOS #1 and TOS #2

2.4 Limitations

In the implemented MS Excel specification management system, the authors note that this approach can scale poorly with the number of items, number of dimensions, and desired counts for each category. It can be noted that the number of items assigned to each participant was based on the design of the logistical considerations of the primary author's ongoing dissertation project. Eight items would be the ideal number per participant as it would allow the participants to write two items corresponding to each of the four content categories, and concurrently two items corresponding to each of the four context categories, as well as process categories. Hence, increasing any of these parameters will affect the layout of the data entry tables, sacrificing ease of use. This would also increase the amount of information handled in the data validation layer. However, the dimensions and categories for the PISA items are fixed and known in advance. Hence, the developed system only scales with respect to the number of items assigned to each participant.

3. Conclusion and Recommendations

To summarize, this paper presents the development of an application which is a specification management system for test blueprints involving multidimensional PISA-like mathematics items. The initial step was the identification of the problem from a technical-logistical point of view, which emerged from the primary author's implementation of his ongoing dissertation project as the participants of the study encountered minor troubles in properly indexing their items, which resulted to significant implications in terms of the external evaluation of the items for alignment or item-objective congruence. Ensuring such alignment is important in test development because it provides evidence of item validity, which ultimately contributes to content validity. The workflow was then discussed in detail, particularly the need to make the process of filling out the TSF efficiently and less prone to errors brought about by manual input in the cells.

The implemented MS Excel specification management system in this paper caters to the dimensions and item quantities needed for larger projects involving PISA item creation. The number of dimensions and categories in this use case is quite small compared to other use or broader cases,

such as the categorization of items for standardized school-level internal assessments. While MS Excel can serve as a quick and accessible method to create a prototype management system, a more proper tool would be to use relational databases, as they are particularly suited to managing data structures and views similar to those in Figure 2.1.1 and Figure 2.1.2. One obvious recommendation that the authors of this paper have to offer for future similar studies is to create the same specification management system that caters to a more flexible set of item specifications – for instance, a nonuniform percentage distribution among the item dimensions. This poses a significant challenge for the developer as changes in percentage distributions will likely produce decimal values for the number of items in each dimension. Additionally, researchers who might be interested in modifying or extending the capabilities of the system presented in this paper might bank on expanding the number of dimensions (e.g., any set of multidimensional items with more than four dimensions), as well as an added feature of streamlining item authorship assignments.

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Constructing Multi-layered Acrylic Tiling Models Using Laser Engraving and Cutting

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Abstract: *This work discusses the creation of tiling or tessellation models that support the teaching of mathematics. The geometric models are first constructed using a computer algebra system and then realized as tangible manipulatives using laser engraving and cutting.*

1. Introduction

The use of 3D models for teaching mathematics has become more widespread with the emergence of digital fabrication technologies such as 3D printers and laser-cutters. There have been studies on the use of models through 3D printing in teaching and learning mathematics (see [4, 7, 12], for example). In addition, patterns and designs on various materials created by laser engraving or cutting technology offer new methods in teaching and learning in schools. For instance, teachers produce intricate geometric shapes and models that help students visualize mathematical concepts like fractions [17] and tessellations [6]. Some benefits of this technology include, among others, encouraging creativity and innovation, enhancing hands-on-learning, promoting collaboration and teamwork, and bridging the gap between STEM and the arts [20]. Due to the potential of fabrication technologies in mathematics education, symposia and training workshops on the use and impact of these technologies for teachers have become more widespread [3, 9].

In this paper, we present our work on the use of geometric models in the teaching of fundamental linear algebraic concepts that were designed using a computer algebra system (CAS) and realized through laser engraving and cutting technology. In the literature, the use of technological tools in teaching linear algebra involves computer algebra or digital geometry software [2, 16, 18, 22]. This paper contributes to other existing strategies and approaches in teaching linear algebra and other mathematics courses such as geometry, crystallography, and group theory.

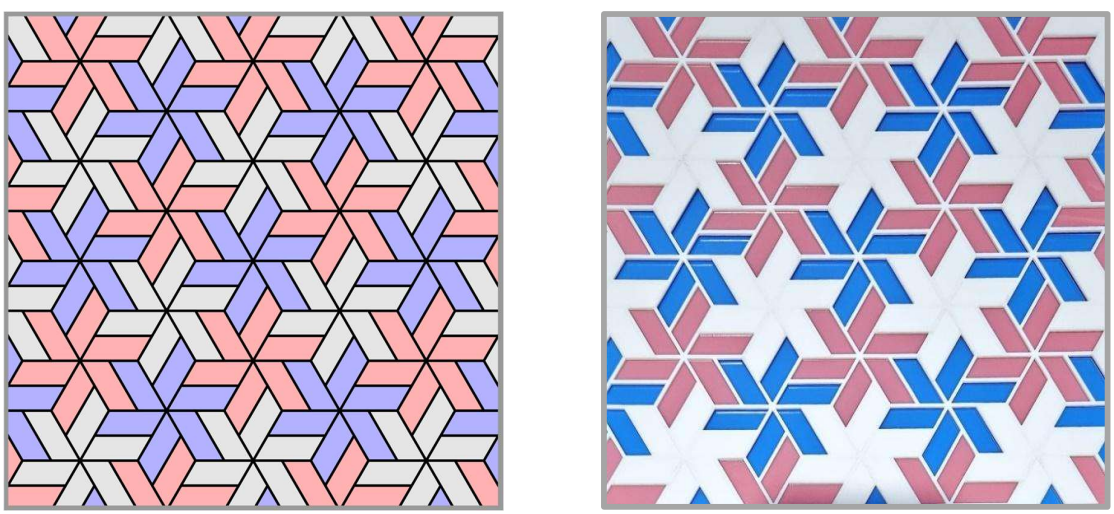
2. Laser Engraving and Cutting Technologies

Laser engraving or cutting is the process of using a laser to vaporize material. *Laser engraving* carves a specific design or pattern onto a material. *Laser cutting*, on the other hand, creates clean cuts into the material and separates it into two or more parts based on a file containing a design or a pattern. High powered laser beams are employed to either carve a design or cut through a given material such as cardboard, acrylic, or even wood with high precision and efficiency. However, before starting the process, a file must be prepared and uploaded first to the laser platform's software. The machine then reads the file to determine the laser's positioning. Supported file types include vector image files where areas to be engraved or cut are indicated by different colors. For laser engraving, the machine is usually set up with lower power and higher speed. By increasing the power and reducing the speed, the laser will stay at a particular point for a certain amount of time, enough for it to pierce through the material and cut it with precision. The time it takes to create a single laser project varies depending on several factors like the intricacy of the design, the speed and power of the laser, and the material being used.

Digital fabrication technologies are being made more and more available in schools and universities through *makerspaces* [1, 8, 15]. These are dedicated areas that allow students and educators to explore and innovate. Makerspaces provide an environment where abstract concepts can be turned into physical tangible realizations. By integrating digital fabrication into the classroom, educators can bridge the gap between theoretical knowledge and practical applications, preparing students for a future where technology and creativity are intertwined.

3. Process of Constructing Multi-layered Acrylic Tiling Models

Given a *planar tiling* or *tessellation* $\mathcal{T}$, which is a collection of subsets of the infinite plane $\mathbb{R}^2$ called *tiles* whose interiors do not intersect and whose union is $\mathbb{R}^2$, our goal is to discuss the entire process of creating a multi-layered acrylic model representing a *patch* or a smaller portion of $\mathcal{T}$.



(a) (b)

Figure 3.1 (a) A square patch of the tiling $\mathcal{F}$ of the plane $\mathbb{R}^2$ by congruent parallelogram tiles, arranged as petals of six-fold flowers. (b) A multi-layered acrylic tiling model representing the patch in (a).

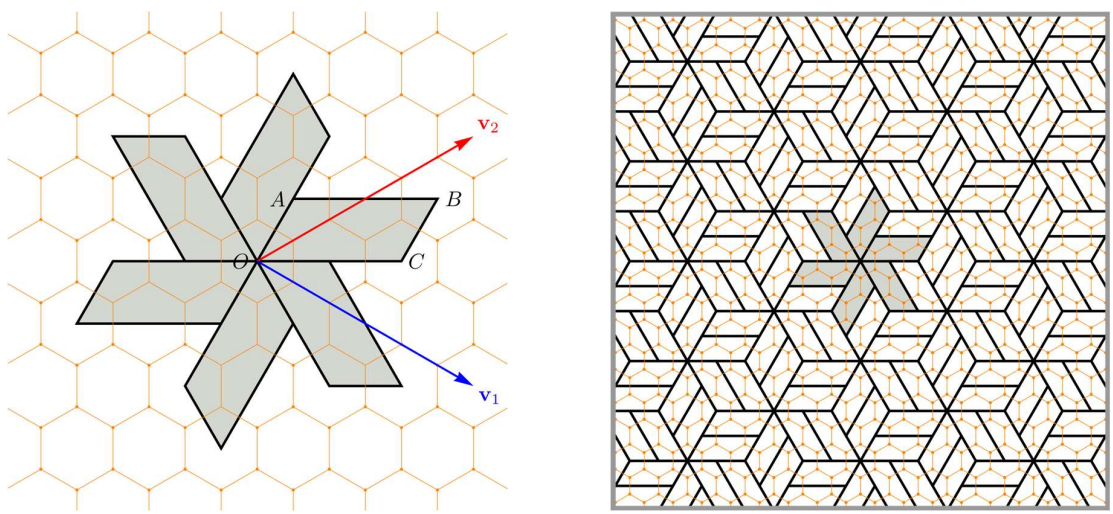
A computer-generated image depicting a patch of the planar tiling $\mathcal{F}$ whose tiles are congruent parallelograms arranged around six-fold rotation points, like flower petals, is shown in Figure 3.1(a). For ease of visualization, we introduced colors to the interiors or *faces* of the tiles. Beside it in Figure 3.1(b) is a photo of the 25cm x 25cm constructed acrylic model. This model is made up of three layers of colored acrylic sheets stacked and glued together. Each layer has either been laser engraved or cut at pre-planned areas to emphasize either the faces of the tiles or its boundaries or *edges*.

We explain the full process of creating an acrylic model of a tiling $\mathcal{T}$ from generating a vector image file of a patch of $\mathcal{T}$ to sending this file to a laser platform and performing the actual engraving and cutting of acrylic sheets. We subdivided the process into three subparts below. To illustrate the steps in each subpart, we use the tiling $\mathcal{F}$.

3.1. Creating a Tiling Vector Image File

The first of this three-step process is to create a computer-generated image of a patch of the tiling. For this, we use Wolfram Mathematica [21], a proprietary software that is capable of generating and manipulating user-created images via a series of programmatic instructions. For a free and open-source software alternative, we recommend using SageMath [13] or its web-based cloud version, CoCalc [14].

Since $\mathcal{F}$ is an example of a *periodic* tiling, we can follow the procedure discussed in [11] to generate the image in Figure 3.1(a) wherein we first draw each tiling motif and replicate them through translations in two linearly independent directions. In the case of the tiling $\mathcal{F}$, the sole motif we choose is the parallelogram $\square OABC$ shown in Figure 3.2(a). The vertices of $\square OABC$ are conveniently located at the centers of four tiles in a hexagonal tiling of the plane. The other five parallelograms in the figure are simply rotations of $\square OABC$ about the origin O by multiples of 60°. These rotations of $\square OABC$ yield the six *petal tiles* with common center point O. Translating these petals using a finite number of integer linear combinations of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and introducing some cutoff parameters to show only a square patch with black a frame, yield Figure 3.2(b).



(a) (b)

Figure 3.2 (a) The parallelogram tile motif $\square OABC$, its six rotations about the origin O, and the translation vectors $\mathbf{v}_1, \mathbf{v}_2$, superimposed on a hexagonal tiling of the plane. (b) A patch of the tiling $\mathcal{F}$ with the six petal tiles in (a), shaded.

Producing Figure 3.1(a) from Figure 3.2(b) is then a matter of selecting any six petal tiles with a common center point from the latter; coloring them using the same color; translating these colored petals using the vectors $\mathbf{w}_1 = 2\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{w}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2$; and then repeating this process two more times with a different color choice per repetition.

The Mathematica code that produces the tiling patch in Figure 3.1(a) and saves it as an svg vector image, the file type that the user interface of a laser platform usually requires, is presented in Code 3.1 below.

```
(*Translation Vectors*)
o = {0, 0};
v1 = N[{3*Sqrt[3], -3}];
v2 = N[{3*Sqrt[3], 3}];
w1 = 2v1 - v2;
w2 = -v1 + 2v2;

(*Tiling Motif*)
OABCD = N[{o, {-0.5*Sqrt[3], 1.5}, {0.5*Sqrt[3], 4.5}, {Sqrt[3], 3}}];

(*Colored Tiles*)
M = 15;
parsVerts[t_] := Flatten[Table[RotationTransform[2k Pi/6][v] + t + n1*w1 + n2*w2,
   {n1, -M, M}, {n2, -M, M}, {k, 0, 5}, {v, OABCD}], 2];
blueTiles = {EdgeForm[{Thickness[0.005], Black}], FaceForm[Lighter[Blue,0.7]], Polygon[parsVerts[o]]};
whiteTiles = {EdgeForm[{Thickness[0.005], Black}],FaceForm[Darker[White,0.1]], Polygon[parsVerts[v1]]};
redTiles = {EdgeForm[{Thickness[0.005], Black}], FaceForm[Lighter[Red,0.7]], Polygon[parsVerts[v2]]};
boundingEdge = {EdgeForm[{Thickness[0.02], Lighter[Black,0.6]}], FaceForm[None],
   Polygon[{{-M, -M}, {M, -M}, {M, M}, {-M, M}}]};

(*Tiling Vector File*)
patch = Graphics[{blueTiles, whiteTiles, redTiles, boundingEdge}, PlotRange -> {{-M,M}, {-M,M}},
   Axes -> False]
Export["patch.svg", patch]
```

Code 3.1 Mathematica code to generate the tiling patch in Figure 3.1(a) and save it as an svg file.

3.2. Post-processing of the Vector Image File of a Tiling Patch

After generating a vector image file of a tiling patch, the next step is to process this file to produce new vector images where an image corresponds to the layout of an acrylic sheet with clearly marked areas for either engraving or cutting.

For the three-layered acrylic tiling model in Figure 3.1(b), we produced the three separate vector images in Figures 3.3(a – c). The bottommost layer corresponds to (a); the middle layer corresponds to (b); and the topmost layer corresponds to (c). In each of these images, we marked with dashed lines the portions that will be engraved and colored with white the areas that will be cut away. When these three layers are stacked together according to the aforementioned order, the cut areas on a layer will allow one to see certain portions of each layer below it. Note that the thin strips of the uncut acrylic sheet depicted in Figure 3.3(c) will serve as edges of the tiles on the layers below it. These strips will also serve as connectors of the petal tiles on the topmost layer.

We remark that the images in Figures 3.3(a – c) serve as colored blueprints of the layers in the tiling model. They also allow one to plan and visualize the tiling model after the layers have been stacked together. To make these files readable by the software of a laser platform, however, one must follow the platform's own rules in marking the areas to be engraved or cut. This usually means coloring areas to be engraved with one color and coloring those to be cut away with a different color. In Figures 3.3(d – e), we modified the images in Figures 3.3(a – c) by marking areas to be engraved using solid black and those to be cut using solid red.

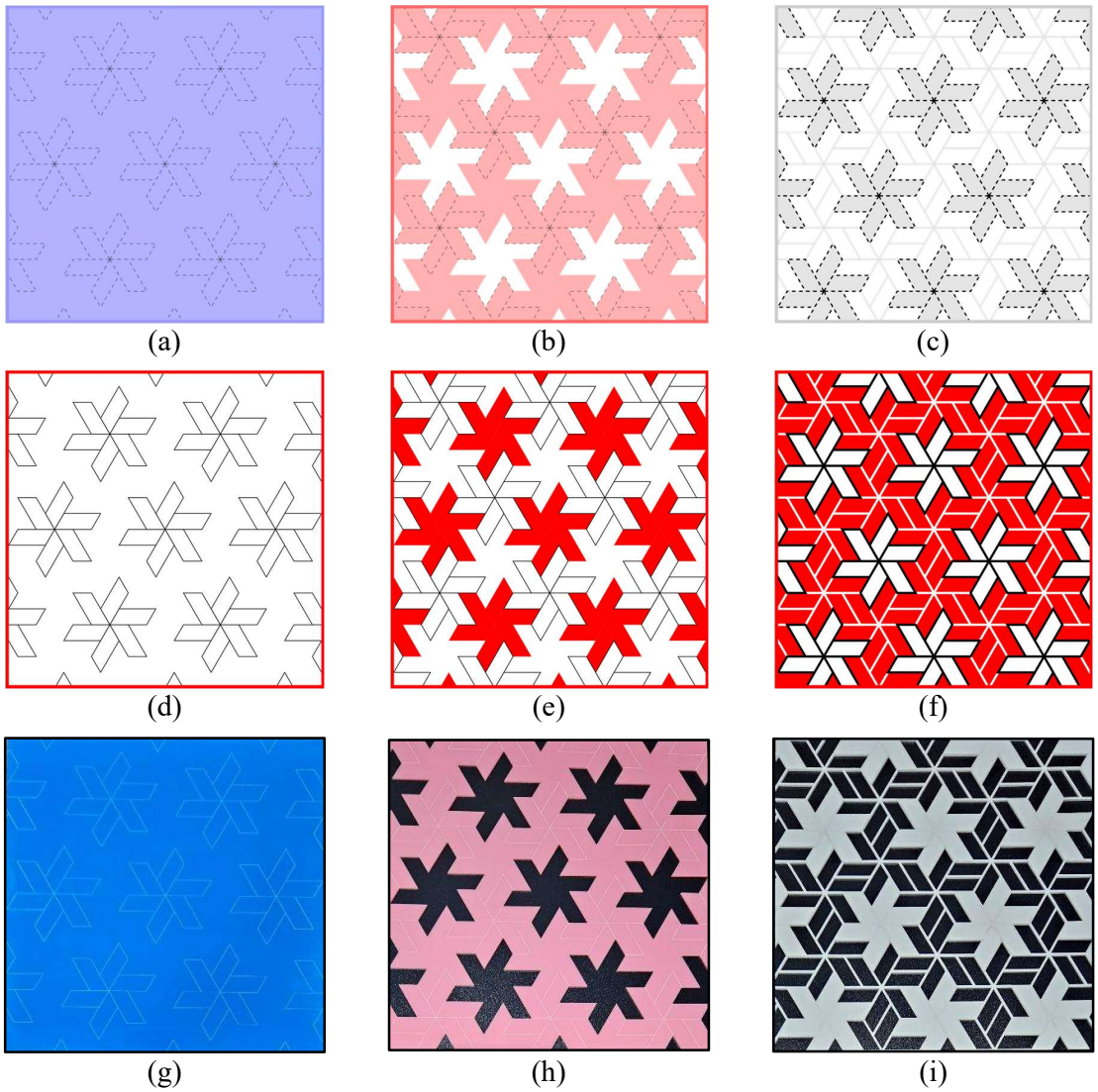


Figure 3.3 Vector images generated from Figure 3.1(a) illustrating (a – c) the layouts of three acrylic layers and their (d – f) laser platform-ready versions. (g – i) The resulting colored acrylic sheets after laser engraving and cutting.

There are at least two ways to produce the vector images in Figure 3.3(a – f). One can either modify Code 3.1 to generate these images programmatically (see Code 3.2, for example) or import Figure 3.1(a) to a vector graphics editor to manually edit or directly manipulate it. In Inkscape [10], a free and open-source vector graphics editor, after importing the file and ungrouping the tile objects, we right-click a tile and choose "Select Same > Fill and Stroke" on the context menu (see Figure 3.4) to select all tile objects of the same fill and stroke color, and then recolor or restyle these selected objects. We also need to resize the new vector images according to the dimensions we want the final multi-layered output to have prior to sending these files to the laser platform.

```
(*Recolored Tiles*)
blueTiles = {EdgeForm[{Thickness[0.007], White}],
   FaceForm[Red], Polygon[parsVerts[o]]};
whiteTiles = {EdgeForm[{Thickness[0.007],Black}],
   FaceForm[White], Polygon[parsVerts[v1]]};
redTiles = {EdgeForm[{Thickness[0.007], White}],
   FaceForm[Red], Polygon[parsVerts[v2]]};
boundingEdge = {EdgeForm[{Thickness[0.02], Red}],
  FaceForm[None], Polygon[{{-M, -M}, {M, -M},
  {M, M}, {-M, M}}]};

(*Tiling Vector File*)
patch = Graphics[{blueTiles, redTiles, whiteTiles,
   boundingEdge}, PlotRange -> {{-M, M}, {-M, M}},
   Axes -> False]
Export["patch.svg", patch]
```

Code 3.2 Modified version of the last six lines of Code 3.1 to generate the vector image in Figure 3.3(f).

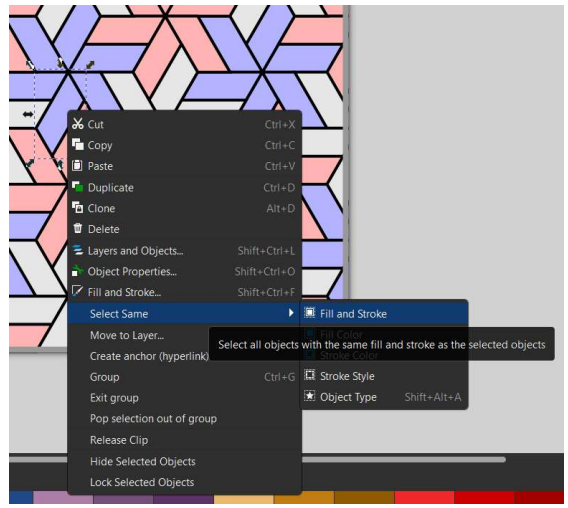


Figure 3.4 A useful command in Inkscape to turn the vector image in Figure 3.1(a) into one of vector images in Figure 3.3(a – f).

3.3. Laser Platform Settings for Engraving and Cutting

The final step is to import each vector image in Figures 3.3(d – f) to the user interface of the laser platform to commence the process of engraving and cutting. For the tiling models that appear in this paper, we used the Universal Laser Systems (ULS) VLS 6.75 laser platform [19] (see Figure 3.5), which is housed at The Eugenio Lopez Jr. Makerspace, our university's makerspace [5].

For the bottommost, middle and topmost layers, we used colored acrylic sheets with thicknesses 3mm and 2mm, respectively. In the laser platform, we set the power to 60.0% and the speed to 100% for engraving, and the power to 90.0% and the speed to 8.0% for cutting. Applying these settings resulted in the three acrylic sheets in Figures 3.3(g – i). After the laser engraving and cutting process, which took a total of 1 hour and 40 minutes, we stacked these acrylic sheets and glued them together using super glue to get the multi-layered acrylic tiling model in Figure 3.1(b). Photos of other tiling models produced by the authors' research group are found in Figures 4.1(a – b).

(a) (b) (c)

Figure 3.5 Views of The ULS VLS 6.75 laser platform at The Eugenio Lopez Jr. Makerspace.

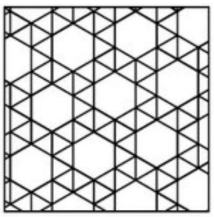
4. An Example Student Activity

We now present a sample student group activity that applies the process discussed in the previous section to construct a multi-layered acrylic model of a given planar tiling. This activity is suitable for students who already have enough experience in the use of a CAS (or any geometry software with the same capabilities) and who can translate geometric objects and processes in terms of vectors and vector operations. The instructor must have already demonstrated, and the students must have already learned, the full construction process of a tiling model including the use of an accessible laser platform. This activity expects the students to work together as a group and to receive minimal assistance from their instructor.

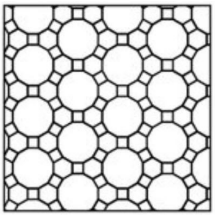
Student Group Activity. This assessment task is worth 60 points and will be graded according to the following assignment of points:

- Code Accuracy and Efficiency – 25 points
- Code Output – 10 points
- Code Readability – 5 points
- Tiling Model Output Requirements – 10 points
- Tiling Model Output Quality – 5 points

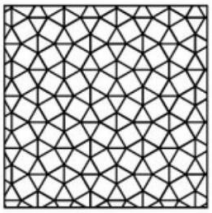
Your group's main task is to create a multi-layered acrylic tiling model of your assigned periodic tiling (see below for the assignment) that satisfies the specifications or requirements stated below.



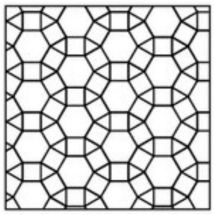
Group 1



Group 2



Group 3



Group 4

Code Specifications. Upload a CAS file with your code and its output images that follow the specifications below.

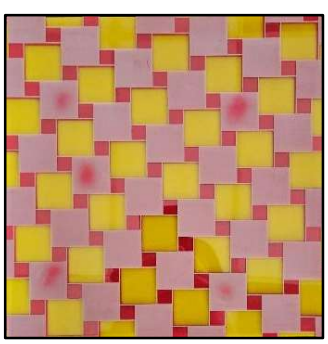
1. Your group must produce <u>two versions</u> of the tiling: black-and-white and colored.
2. Each version must be truncated so that only a square patch with a black frame appears. For the frame dimensions, you may freely choose the bounding values. Just be sure that each version must show at least as many tiles as the corresponding figure above shows.
3. You may assign the size of the tiles on your own. Make sure that all the tiles are regular polygons (congruent sides and equal angle measures).
4. For the colored version, the tiling must use at least <u>three colors</u>. You may choose any combination of colors. Make sure that the coloring is NOT random and MUST adhere to a rule that is based on two linearly independent translations of your choice.
5. All computations pertaining to the vertices of the tiling and translation vectors must be accomplished using a CAS (and not manually). The CAS file must contain the <u>vertices</u> of all the relevant tiles in the motif(s) you have chosen and all relevant <u>translation vectors</u>.

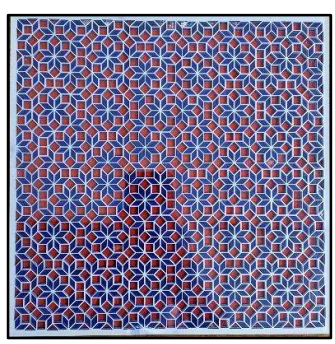
6. The main codes for generating the black-and-white version of the tiling must be found under the comment "Black and White Tiling".
7. The main codes for generating the colored version of the tiling must be found under the comment "Colored Tiling".
8. Include brief comments to describe important portions of your codes.
9. Points, vectors, and vector operations must be used prominently in your code. You will be graded based on how you have used these concepts.

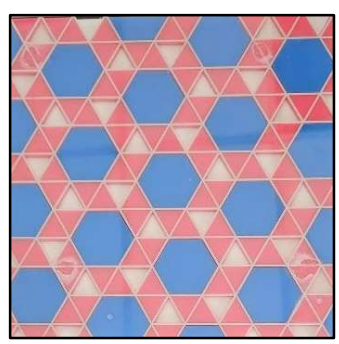
Tiling Model Requirements. Submit the actual tiling model that follows the requirements below.

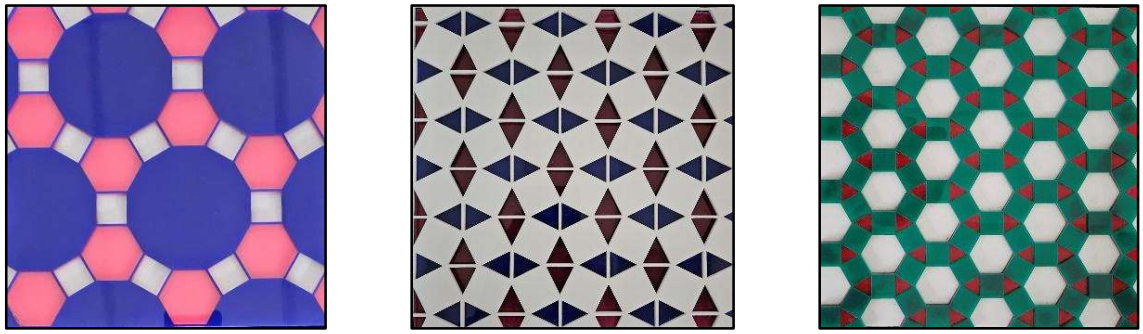
1. For the tiling model, you need to have colored acrylic sheets laser engraved or cut. Glue these sheets together to create a single multi-layered acrylic tiling model.
2. Engrave the surnames of your group members at the bottom right of the back of the bottommost layer. Any font style of size 12 will do.
3. The number of colors for the tiling model must match the number of colors you used in the CAS file above. Depending on availability, you may use a different set of colors for the acrylic sheets. Choose a color combination that you believe is visually appealing and pleasing to the eyes. Opaque white and black are considered as colors.
4. The physical output must be a multi-layered square of dimensions 20cm x 20cm.
5. Before you use the laser platform, be sure to consult with your instructor your plans for the tiling model.
6. For materials and laser platform service fees, each group member may spend only a maximum of 3.50 USD.

A slightly modified version of the above student activity was given by the first author to his class in *Linear Algebra for Games Programming* during the first semester of SY 2023 – 2024. The course is an introduction to linear algebra with emphasis on applications for computer graphics and games programming. It is taken by university students who are majoring in digital games design and development. The students in this class were well-versed in using different programming languages and have been using Mathematica since the start of the semester. This activity was given toward the end of the semester and served to reinforce a student's understanding and mastery of important linear algebra concepts such as vectors and vector operations, span and linear independence, and linear transformations and their matrix representations through computer-generated images and tangible manipulatives. Students learned, for instance, that applying two linearly dependent translations to a tiling motif will not generate the entire tiling. Photos of the tiling models submitted by the student groups in this class are found in Figures 4.1(c – f).


(a)


(b)


(c)



(d) (e) (f)

Figure 4.1 Multi-layered acrylic tiling models from the outputs of (a – b) the authors' research group and (e – f) students in a linear algebra class.

On average, each group of five students spent a total of 8.50 USD to construct a tiling model.

5. Conclusion

Digital fabrication technologies such as laser engraving and cutting have a huge potential as an innovative tool in teaching and learning mathematics. Activities designed using such a technology can foster a student's creativity, collaboration skills, and higher-order critical thinking skills. A next step in this work is to explore more ways on how these technologies can be used in other mathematics classes and then study its impact on student learning and increased engagement.

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Exploring the Influence of Educational Robotics in Geometry to Students' Computational Thinking Skills

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Abstract: *Computational thinking (CT) skills are generally adopted in various subjects, including mathematics. Geometry, however, remains a subject in mathematics that students barely understand. This research was conducted to examine how educational robotics devices facilitate the achievement of students' CT skills while learning in a geometry class setting. The study employed a qualitative approach involving participants from grades 4 to 6 at nature-based schools. The participants were nominated purposively for a set of predetermined criteria. The instruments used to determine the impact of robotic devices in geometry teaching and the achievement of CT aspects during learning were observations and interviews. Data from the observations and interviews were analysed using descriptive and content analysis. Based on the study's results, it can be stated that Computational Thinking (CT) skills in geometry learning using robotic devices can be well facilitated in problem-solving, creativity, innovation, efficiency of problem-solving with algorithms, and further learning of geometry.*

1. Introduction

Computational thinking (CT) is a vital skill in the 21st century, particularly in problem-solving. It can be seen as a cognitive process that bridges the gap between humans and technology. CT is instrumental in managing cognitive activity and understanding and solving problems in various contexts, not just in computer science but in all disciplines. This research aims to explore the role of CT in mathematics learning, particularly in geometry, and how it can be facilitated through the use of robotics devices [1].

CT learning is now recommended to be applied in the early years of school, such as elementary school, because it can be used in the future. [1]. CT is believed to improve students' ability to solve problems. By understanding these abilities, students can formulate, solve, and express solutions through computer science-based information processing required by most other scientific fields [2]. In addition, other factors cause low CT skills, such as the learning model carried out by teachers not giving freedom for students to develop CT skills, lack of teacher understanding in innovating learning, and mathematics learning that requires students to memorise formulas [3]. Although CT is developing in education, it still has gaps or constraints, such as limited active student engagement and low student engagement, which urgently require further empirical investigation [1].

Researchers agree unanimously that CT skills can be applied to subjects such as mathematics [3]. Geometry, however, remains a subject in mathematics that students barely understand [4]. The majority of students find it difficult to distinguish whether a square is a plain rectangle or a kite, nor can they state that all three shapes belong to a parallelogram [4]. Even though geometry is taught in schools to help improve logical thinking and correct generalisations [5]. This is also despite that geometry has an important role because, in everyday life, we often find things or problems related to geometry [5]. To understand arithmetic, algebra, calculus and others better, students must master geometry concepts in depth because the idea of geometry plays a role here.

Educational robotics is a tool to improve students' ability to acquire knowledge and skills through programming activities or robot development. Learning delivery using robot facilities can increase student motivation, learning, and interaction. The use of robots in mathematics materials

makes students understand conceptually so that they can improve student learning outcomes in national standard assessments [6]. Using robots in education makes students learn concretely, arouses curiosity about the subject, and helps them learn better through more fun activities [7]. Although the use of robots in learning can provide many benefits, the use of robots in education is still not too crowded. Robots are often used in extracurricular activities in assessment and practice [6]. A study stated that using educational robots in mathematics education is necessary to explore the integration of robotics and mathematics education [8]. On the other hand, some factors can affect the use of robots in education, namely the lack of skills of some people in the field of specialisation and skills in the development and research of technology to increase the attractiveness of students and teachers so that the application of technology can be carried out in learning [9].

Based on this background, the purpose of the research to be carried out is to find out how the developed robotics device can facilitate *Computational Thinking* skills in geometry learning. The use of educational robotics as a learning medium and the influence of educational robotics on the development of CT skills in geometry learning will be further analysed. Robotics device innovations with the help of Lego Mindstorm EV3 will be developed, then tested robotics devices and the use of robotics devices in elementary school students' learning. Thus, it is hoped that we can learn more about the influence of educational robotics.

2. Literature Review

CT in School Education

CT is not constrained to using computers continuously and as a problem-solving tool. Wing [10] argued that CT is the process of finding an effective solution to a problem with the help of a computing device. Several major complex issues are associated with CT, including algorithmic thinking, data collection, and analysis. With these more complex challenges, the demand for systematic and logical thinking has become higher to be better prepared to solve problems requiring critical and innovative thinking. CT can continue to be developed in education, especially in STEM. STEM education is a teaching and learning strategy integrated with science, technology, engineering, and mathematics that teaches how these sciences are interconnected and interact. In education, two elements in CT, namely explaining concepts in theory and practice, can help thinking patterns. [11]. Theoretical understanding can provide a solid foundation for students to think systematically and logically. However, the practical aspect is no less critical. With practical learning activities, students can apply learning concepts directly in the real world. Eventually, both student's understanding of the theory and their problem-solving practical skills improved. not only does the understanding of theory improve, but so do the practical skills needed to solve real problems, work in teams, and innovate. Students are prepared to face the challenges of the ever-evolving modern world.

The Role of Robotics in the Delivery of Geometry Materials

In recent years, the popularity of robotics has increased and shown a positive trend. Several literatures say that using robotics as a means of student learning can improve students' interest in the subject [12]. Some researchers also agree that CT skills can be applied to subjects such as mathematics [3]. One of the parts of mathematics that is very weakly absorbed by students in school is geometry [4]. The use of robotics as a facilitator in learning can make students' understanding of complex and abstract concepts more explicit and more specific. With the help of robotics technology in education, students will get a direct learning experience, not only deepening understanding but also increasing student involvement in learning so that student interest can grow. The introduction of robotics in education can align with the development of CT, which is starting to be applied in the

current independent curriculum. The application of robotics in learning can be done in two ways: as a learning object and as a means of learning subjects. [13]. Robotics will be very important if used in learning activities, especially when the subject needs a direct example. Using robotics as a learning tool will make the concept of complex and abstract learning materials easier for students to understand. Robotics allows students to see directly in natural form so that students can connect between theoretical concepts and actual practice more easily.

The Influence of Robotics on CT Skills Development

CT skills utilise computing devices as problem-solving tools, including robotics. CT has begun to be applied to the independent learning curriculum to encourage students to think critically and systematically in various challenges. Applying CT skills in learning is often associated with robotics as an additional tool. Using robotics as a learning medium can improve students' thinking skills. Robotics is used to learn about mechanisms and technologies and can help students understand basic concepts in science and mathematics through hands-on learning. Using robotics in education offers opportunities for students to explore, create, and apply knowledge to solve real-world problems to practice CT skills [14]. For example, when designing and programming a robot, students must break down a significant problem into smaller steps, identify patterns, and develop algorithms, all of which are essential components of computational thinking. CT, which implies the ability to solve problems algorithmically and logically, is one of the factors in digital learning [15]. Through robotics media, students can see practical learning directly and apply these concepts in learning. For example, students can program robots to perform tasks requiring a deep understanding of programming algorithms and logic. In this way, students can break down complex problems into simple steps that robots can implement. This is similar to the algorithmic problem-solving process in CT.

3. Methodology

Research Procedure

This research was conducted using a qualitative method. This research was conducted using a qualitative method to explore students' expressions and perceptions during the learning process [16]. The study adopts two commonly used qualitative data collection methods: participant observation and interviews [17]. Participant observation allows the researcher to be directly involved in their lives, observing their behaviour in a natural setting. At the same time, interviews provide more profound insights from the participants regarding their experiences with geometry learning through robotics. These two methods complement each other well, as they allow data to be gathered from multiple perspectives, both through direct observation and through students' personal understanding conveyed in interviews [17].

Sampling Technique

Participants are selected based on their suitability to the research characteristics [18]. Purposive sampling is used because it can select subjects with characteristics or experiences relevant to the research objectives [19]. This method ensures that the selected participants provide valuable, specific insights aligning with the study's goals.

The data from this study are related to students' behaviour, activities, expressions, and perceptions during the use of educational robotics in geometry learning. Qualitative research data sources are generally 5-6 people, and it can be less if sufficient data is obtained [20]. The data in this

study will use sources from 6 students who attend Sekolah Alam Solo Raya in the young class (grades 4-6) because 6 students are enough to meet this study.

Nature schools are schools built for educational development efforts carried out in nature with learning from the surrounding environment directly. This research was conducted with project-based learning that is relevant to real life. In high school grades, elementary school children (4, 5, and 6) have begun to think critically so that positive things and an understanding of what they learn can be accepted and then will be developed according to the child's abilities [21]. Therefore, this study took a sample of students who attended nature schools at the elementary school level at the youth level (grades 4-6).

Participants	Class
1	4th grade
2	4th grade
3	4th grade
4	4th grade
5	5th grade
6	6th grade

Table 3.1. Participant Data

Data Collection and Validity Testing Techniques

There are two techniques that will be used, namely observation and interview. The observation method applied is through observation of students' expressions during geometry learning using Lego Mindstorm EV3. In this process, interesting or unique actions taken by participants are recorded so that a recording can be produced that reflects this. The interview method was conducted by providing in-depth questions about the influence of the use of educational robotics in geometry learning. The purpose of the interview is to assess the perception of students during learning.

The strategy to check the validity of participant statements or the validity of data in the study will use triangulation techniques. In this study, three triangulation techniques were used, namely triangulation of data collection methods, triangulation of data sources, and triangulation of data. With this approach, it is hoped that various perspectives will be obtained to complement the information from each participant, so that it can deepen the analysis and interpretation of research results.

Data Analysis Techniques

In this study, two data analysis techniques are used, namely qualitative descriptive analysis techniques and content analysis techniques. Descriptive analysis will be used for the observation data. Meanwhile, content analysis will be used for the results of the interview transcript. This technique is used in order to provide an overview and summary of the data that has been produced. This is done, to find new facts and provide an overview or description of each research data source. This analysis is expected to help simplify the data and make it easier to understand.

4. Findings

During learning using Lego Mindstorm EV3, students are observed to analyze the application of aspects of CT, namely abstraction, algorithms, generalizations, decomposition, and modularity. A teaching module (https://drive.google.com/file/d/1aO4GrkW7pUW2i3KPuWZ0BseR4FiolvBC/view) was created as a guide for learning activities throughout the session. The activity began with the initial stages of learning, such as self-introduction and the introduction of material on plane geometric shapes, their components, and how to calculate the area and perimeter of these shapes. The students started using Lego Mindstorm EV3 to assemble the initial stage of the robot according to the provided manual. After assembling the robot, the students continued by programming it using Classroom EV3

software. The components of the program were introduced in a simple way, and then the students were asked to try programming themselves. They were also encouraged to explore the existing program to create different sizes of plane geometric shapes. Once finished, the session ended with a reflection activity to review the material covered during the lesson.

They then began assembling the robot, where the aspect of abstraction was evident when all participants were able to sort out the necessary Lego parts. The decomposition aspect was also identified because students divided the Lego components into parts before assembly. The algorithm aspect arises when students follow the robot's assembly instructions well, while generalizations are seen in some students who can combine the already collected Lego parts. However, the modularity aspect was not identified at this stage. Next, the focus of learning shifted to assembling architectural robots according to the guidebook.

At this point, the aspects of abstraction, generalization, algorithms, and decomposition are still identified, but modularity does not appear. Some students still collect Lego parts and follow building instructions, while the modularity aspect is not applied. In the final stages, when students program robots using Classroom EV3 software, the modularity aspect becomes dominant, with students trying different sizes of shapes to draw flat figures, except for one student who doesn't. All of these activities show the application of various aspects of CT, with a focus on changes along with the development of learning and activities carried out.

Observation Results					
Participants	Abstraction	Generalisation	Algorithm	Modularity	Decomposition
1	✓	✓	✓	✓	✓
2	✓	✓	✓	✓	✓
3	✓	✓	✓	✓	✓
4	✓	✓	-	✓	✓
5	✓	✓	✓	-	✓
6	✓	-	✓	✓	✓

Table 4.1 Observation Results

The analysis of the interview content was carried out by analysing the answers from the participants, then mapped to answer *research questions* in the form of the influence of the developed robotics device that can facilitate CT capabilities in geometry learning. Based on the analysis results, *the research questions* were mapped into several themes, namely further learning geometry, efficiency of solving problems with algorithms, problem-solving processes, creativity, and innovation in the form of robots. For more details, the mapping results can be seen in Figure 4.1.

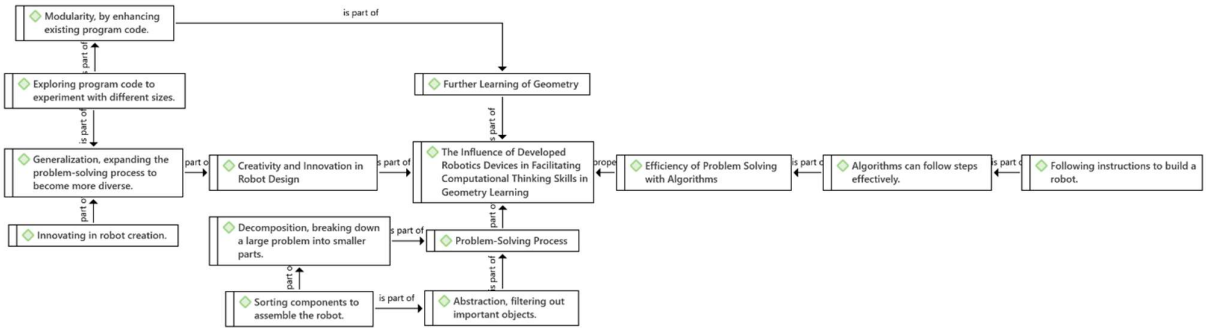

Figure 4.1. Data Mapping (Extracted from Atlas. ti)

Based on the mapping results that can be seen in Figure 4.1, several themes can be produced that implement the five CT aspect values. Each theme can represent one or more aspects, more details can be seen in Figure 4.2.

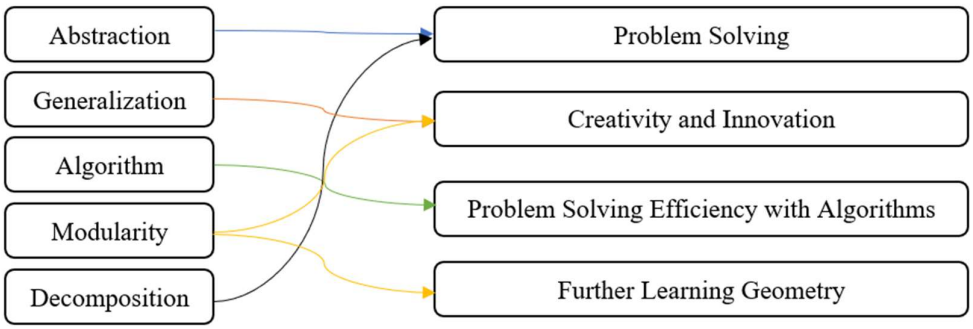

Figure 4.2. Generated theme

5. Discussion

a) Problem Solving

Based on the results of the observational analysis, the six participants identified skills to sort complex information into simpler ones. Students try to solve problems by making complex things simpler to make them easier. This is evidenced by students sorting out the Lego parts needed to make the robot. The observation analysis data was strengthened by the results of interview transcripts from Participant 5.

*"Putting Lego together, building robots**, looking for needed items**, taking turns with others"*

Participant 5 also went through the process of preparing the Lego parts needed to build the robot. Lego parts are grouped based on the parts to be assembled. This will make it easier for students in the process of assembling robots. This result was corroborated by participant 1 in the interview transcript.

"Finding the Lego you need and building it"

In the interview, Participant 1 mentioned the activity of finding the lego needed and arranging it. This activity is a problem-solving process, where participant 1 first describes the problem by looking for the lego needed to make a certain shape, then arranging it. The observation results were corroborated by Participant 2 in the interview results.

*"**Assembling Legos,** geometric flat shapes, flat shapes, and robot components".*

Participant 2 mentioned the Lego assembly process, where in this process participants described the problem before doing the Lego assembly process. So that Lego can be assembled perfectly and function properly.

Other research that uses robotics as a real object in learning can teach students how to find and fix what is wrong without using a step-by-step strategy [22]. This is in line with previous research, where the activity of breaking down or categorizing components is considered a simulation that breaks down complex problems into smaller solutions [23].

b) Creativity and Innovation

Based on the results of the observational data analysis that has been carried out, participants 1, 2, 3, 4, and 5 identified skills to solve problems to be more diverse. Participants explored by assembling the robot they wanted themselves, by adding some additional components, such as swords on the right and left front sides as decorations. The results of the observation were corroborated by the results of interviews from participants 3.

"Assembling different robots"

Participant 3 mentioned other activities that are the implementation of the generalization aspect. Participant 3 mentioned that participants wanted to learn more to make or assemble different robots, meaning that Participant 3 wanted to transfer the robot assembly process into another form of robot.

Previous research mentioned that the generalisation aspect can be used to develop skills in solving problems with one solution to another by identifying familiar patterns [24]. Another study also says that combining learning with robotics interactions can improve children's ideas so they can achieve more complex skills [25]. Another study states that learning with robotics can help students expand existing solutions to a given problem and include more possibilities [26].

c) Efficiency of Solving Problems with Algorithms

Based on the observations made, Participants 1, 2, 3, 5, and 6 can follow the assembly process based on the guidebook in a good and structured manner. So that the robot could be successfully made and function as it should.

A guide to building architectural robots has been prepared so students can follow the steps set out in the guide. In the activity of students assembling robots based on a guidebook, students follow specific steps to carry out the process of arranging architectural robots. The observation results were corroborated by the answers from Participant 5 in the interview.

*"When **to install and how to install it**".*

Participant 5 mentioned an activity related to the algorithm, which is when the participant installs it and learns about how to install it. This answer means that participants know and follow step by step how to install the Lego Mindstorm EV3 robot. The observation results were corroborated by the answers from Participant 3 in the interview.

*"**It's most fun when assembling a robot**, trying different sizes".*

Participant 3 explained that the most fun activity of the learning process was when assembling a robot. In the assembly process, Participant 3 followed step by step based on the manual, so that Participant 3 has solved the problem in order.

This is in line with research that says that students who learn to use robots have higher design skills and algorithm efficiency [27]. Other studies also say that the use of robotics can improve algorithmic thinking in problem-solving and improve CT [28].

d) Geometry Further Learning

Based on the observations that have been made, participants 1, 2, 3, 4, and 6 tried various sizes of geometric flat shapes using the program that has been provided. The results of the observation were corroborated by the results of interviews from Participant 3.

*"The most fun when assembling a robot, **trying different sizes**".*

Participant 3 mentioned in his interview, that participant 3 did activities to try various sizes. The activity of trying different sizes is one of the implementations of the modularity aspect, as participants develop the problem-solving process into various other measures, and participants utilize pre-existing code to be developed to solve other, broader problems.

This is in line with previous research, that teaching programming using robots can improve CT and creativity [29]. By using robotics, the complexity of learning increases as students can visualize and transfer CT skills directly. This results in more sustainable learning outcomes [30].

6. Conclusion

After that, several things were observed, including: CT in learning geometry using robotics, the role of the Lego Mindstorm EV3 module in helping students learn geometry materials, and students' interest in learning. From the results of the analysis and discussion, CT was identified during

learning. Among them are in the problem-solving process activities, namely when students are looking for the materials needed to make robots. Creativity and innovation in the form of robots, which is when students make robots made. Efficiency in solving problems with algorithms, namely when students follow the guidelines for making robots so that they can produce robots that function properly. Further learning is when students try different sizes of flat buildings using existing programs.

Although the results are positive, there are some shortcomings in the study that should be acknowledged for future improvements. Inadequate robot performance is one of the shortcomings in this study. Engineering problems with robots are likely to hinder students' ability to fully engage in the learning process and reduce the effectiveness of teaching. In addition, the collection of interview data still needs to be considered in order to get more accurate and representative answers.

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The Influence of Educational Robotics on Students' Computational Skills: The Case of Physics Learning Material on the Relationships between Work and Energy

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Abstract: *The rapid evolution of technology has revolutionized the educational landscape, underscoring the increasing significance of technology integration in the learning process. The use of robotics in education has emerged as a promising avenue to enrich students' learning experiences. This study delves into the potential of robotics in bolstering Computational Thinking (CT) skills in physics learning, particularly in Work and Energy, in a Non-formal School in Central Java, Indonesia. Employing a qualitative approach with purposive sampling, data were gathered through observations and semi-structured interviews, with triangulation to ensure data validity. The findings reveal the promising role of robots in nurturing CT skills in physics learning. Students showcased CT skills in problem identification through pattern recognition, systematic program development, and problem-solving using deconstruction techniques, albeit with varying levels of proficiency.*

1. Introduction

Computational Thinking (CT) skills are important skills that individuals need in the 21st century [1]. The application of learning that contains aspects of CT can support students in solving complex problems, with logical and structured thinking and can improve critical and creative thinking skills [2]. Despite the importance of these skills, educational institutions often do not prioritize the development of computational thinking skills in students [3]. Educators and practitioners need to strike the right balance between learning content that promotes CT thinking and understanding [4]. Additionally, CT principles can be utilized as guidelines for designing assessments and learning experiences that enhance comprehension [5]. Students learning outcomes and understanding in physics learning in an elementary school can be improved by integrating physics learning with CT [6]. CT can improve students' creative thinking skills to solve problems [7]. Further research is needed to determine the development of CT through the integration of CT in science learning by implementing it in various materials and educational levels [8, 9]. One of the critical topics in physics is the relationship between work and energy, which can be abstract and challenging for students to comprehend without appropriate guidance [10]. This topic offers a valuable opportunity to apply the CT approach through simulation and data analysis. However, it often presents challenges in engaging student interest and ensuring a deep understanding of the material [11, 12].

Based on technological advances that have significantly changed the educational landscape, technology has become increasingly important in the learning process [13]. As a significant component of these technological advances, educational robots provide valuable experiences that empower students to understand physics concepts engagingly and interactively, even without specialized physics lessons [14, 15]. Integrating fun and interactive learning tools like educational robots can effectively develop CT skills [16]. Robots are a teaching technology that enhances students' problem-solving abilities in learning contexts [17]. While the integration of robots in education has demonstrated benefits in developing CT skills, there is still a need for further research

to explore the extent to which educational robots in learning influence the development of students' CT skills [18]. This study examines the impact of using educational robots in facilitating students' CT skills in physics learning.

2. Literature Review

2.1 Robots in Learning

Educational robots have become a significant opportunity in the advancement of learning, playing a crucial role in the development of educational practices [14]. Robots contribute to teachers' teaching planning, trigger children's enthusiasm, stimulate children's knowledge, and stimulate children's agency [19]. Research on robots in education has shown that using robots can significantly enhance student engagement in learning [20]. The use of robots in teaching Newton's laws of motion provides valuable hands-on experiences, which can enhance students' understanding of these concepts even without specialized physics lessons [15].

The effectiveness of educational robots in enhancing student learning outcomes has been recognized, particularly in STEM (Science, Technology, Engineering, and Mathematics) subjects, where it significantly boosts student achievement [21]. Integrating robots into schools' curricula has proven effective in enhancing student engagement, deepening their understanding, and facilitating project-oriented learning [22]. This approach can be adapted for various subjects, using different instructional methods across educational levels. Implementing educational robots proves promising in enhancing children's cognitive abilities by developing CT skills [23]. Studies have shown that students find robots engaging and enjoyable, improve their CT skills, increase their focus, and positively perceive the activity as a form of play. However, further research is necessary to understand how robots influence children's learning and cognitive development. Integrating LEGO EV3 robots into learning Newton's Second Law can significantly enhance students' comprehension of the concept, boost their CT skills, and increase their enthusiasm for learning. This hands-on approach helps students grasp Newton's Second Law's principles more effectively and fosters greater confidence and interest in robots and the learning process [24].

2.2 Development of Computational Thinking Skills through Robots

According to [25], the first person to introduce the term 'computational thinking' (CT) was Seymour Papert in his book Mindstorms: Children, Computers, and Powerful Ideas, published in 1980. CT is an understanding or skill that is the basis for determining how to solve problems using a structured and logical mindset, including the capacity to organize and analyse data, find patterns, articulate problems, and plan and implement solutions by applying algorithms and logical thought processes [26]. The integration of CT learning in schools requires careful attention to the key aspects of CT. These aspects include abstraction, generalization, algorithms, modularity, and decomposition, which have been adapted from various experts in the field [27]. In the study [8] learners utilized concepts such as abstract thinking, algorithmic thinking, decomposition, and evaluation. This study paves the way for future research to explore CT skills using different programs and across various learning materials. CT can improve students' ability in programming and technology development through CT learning with computer simulation methods, application development, modelling, and programming activities [5]. Research findings [17] support the idea that CT is important in education, which is necessary in today's digital era. Utilizing educational robot technology in learning can be an effective approach to developing CT skills [16].

3. Methodology

This research was conducted at a non-formal school, in Central Java, Indonesia with students from

the high-school level, equivalent to middle school by international standards. The participants, aged 13 to 15 years, had not previously learned about the concepts of work and energy. Using a descriptive qualitative approach, this study explored students' perceptions and experiences of using robots in physics learning. Data were collected through observations and semi-structured interviews, involving purposive sampling techniques to select representative participants. The observation instrument uses CT indicators based on [28] which refer to Table 3.1 and the interview instrument is shown in Table 3.1.

Table 3.1 CT Instrument

CT Aspect	Indicators
Abstraction (C1)	Separating unnecessary information Analyzing pattern behaviour in programming contexts between different scripts Identifying abstractions across contexts, i.e. different programming languages
Generalization (C2)	Making connections to abstraction patterns Builds on existing ideas about the problem to extend the problem to a broader case Using variables in the solution
Algorithm (C3)	Stating the steps of an algorithm in detail Analyzes different algorithms for a given problem Generate a suitable algorithm
Modularity (C4)	Uses previous programming code Develop sections of code for use on the same or different problems Produces programming code that is easy to organize
Decomposition (C5)	Deconstructs the overall problem into smaller, solvable segments Divide solvable segments based on specific functions Recombine so that complex problems can be solved

Table 3.2 Interview Instrument

No	Questions
1	What was your first thought when using robotics in learning?
2	What do you think are the functions and objectives of the robotics project you built?
3	Could you describe the process you went through when working on the robotics project?
4	Did the robotics project follow the module provided? If not, what differences did you encounter compared to the module?
5	Were there any difficulties while working on the robotics project? If so, could you explain them?
6	How did you overcome the difficulties you encountered while working on the robotics project?
7	Could you explain how robotics helped you understand the physics lesson on the relationship between work and energy?
8	In general, could you describe how the robotics project has been beneficial to you?

The study involved 3 students over a period of 2 days, with a total of 2 lessons dedicated to the project. The activities included hands-on experiments, problem-solving tasks, and collaborative projects guided by instructional materials designed to integrate robots into the physics curriculum. Data validity was tested using triangulation of sources, data, and methods. In contrast data analysis was conducted through descriptive and qualitative content analysis to identify patterns and themes from the research results.

4. Result and Discussion

On the first day, participants were given a worksheet and introduced to LEGO Mindstorm EV3 to complete the tasks outlined in the worksheet, as shown in Figure 4.1. The participants were tasked with solving physics problems, with the worksheet designed to guide them through the stages of understanding physics concepts while integrating Computational Thinking (CT) into the learning

process of building robots as practical simulation tools.

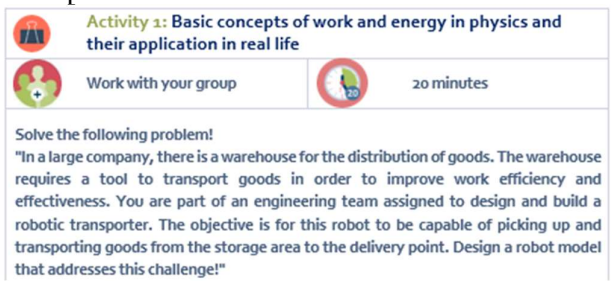


Figure 4.1 Worksheet Participant

On the second day, participants applied CT approaches, such as systematic thinking, strategizing, and modifying robot components based on their function and purpose, to identify and solve the problems on the worksheet. They carefully analyzed each step, from robot assembly to programming. After completing the task, participants were interviewed about their experience. Data mapping from the interviews was conducted by identifying CT codes and their frequency of occurrence, as shown in Figure 4.2.

Figure 4.2 Frequency of Interview Data Codes

The data interpretation of the identified codes was then categorized based on the facts' similarity and relevance to the research questions to build themes. The themes addressing the research problem formulation are presented in Table 4.1.

Table 4.1 Data Interpretation

Code	Data Interpretation	Theme
C1	Utilization of cross-context programming experience	Identify problems through pattern recognition
C2	Visualizing robotics	
C2	Modifying components to create solutions	
C4	Using previous programming code blocks	
C3	Designing systematic programming	Systematic program development
C3	Creating algorithms according to the function and purpose of robotics	
C4	Development of programming code for effectiveness	
C4	Creating programming code that is easy to organize	
C1	Assembling robotics with focus on necessary components	Problem solving with deconstruction
C5	Dividing robotics components based on specific functions	

Physics learning that incorporates robotics can effectively develop CT skills in participants, as

illustrated in Figure 4.2, which depicts participants' behaviors and their engagement with CT skills.

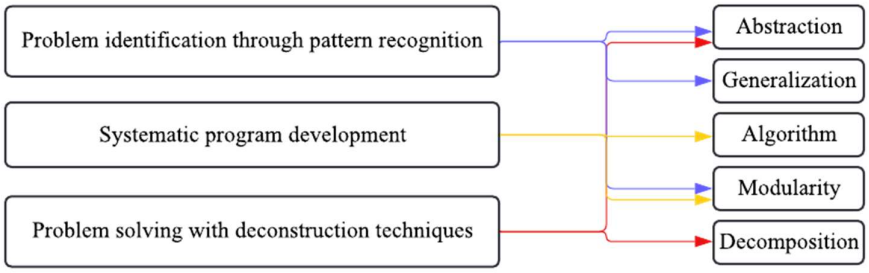


Figure 4.2 CT Skills Capability in Learning

4.1 Problem identification through pattern recognition

Participants utilized their experience related to assembly and programming in a series of robot projects. The observation showed that Participant 1 and Participant 3 were identified as applying the same pattern in programming with different programming languages. As Participant 1 said regarding utilizing his experience in terms of programming,
"...I've seen it on YouTube and I've tried programming too."

Observing tutorials on YouTube and trying to make programs are experiences that Participant 1 did. The same thing was done by Participant 3 related to the experience in programming to develop solutions in problem-solving. As stated in his interview,
"I haven't done the program yet because it's in English. Then I've also been there before, but I don't understand it."

Participant 1 and Participant 3 drew on previous experience in programming to recognize relevant patterns in the new context. Participants who had previous experience with LEGO and programming showed ease in understanding the new concepts introduced. Participants' programming experience helped them analyze in a programming context, albeit a different one, by using previous functions or methods in completing robot programming. This aligns with the opinion of [29] that personal experience in programming may facilitate abstract thinking to solve problems.
Pattern recognition is applied by participants in solving problems. The observation results show that before the assembly stage, participant 1 and Participant 3 visualized the robots that would be realized into their original form in making the robots project.

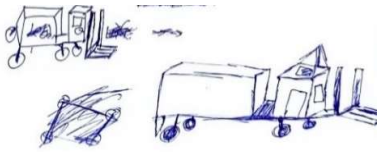


Figure 4.1.1. Robot Visualization by Participant 1

Figure 4.1.2 Robot Visualization by Participant 3

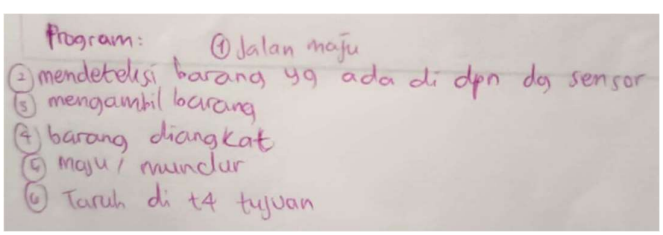
Visualization plays an important role to help students understand how robots will move and function. Participants can also find problems that may occur and find effective solutions. This is

in accordance with [27] that one of the indicators of the generalization aspect is that there is a connection between abstraction and generalization. This is reinforced by [30] that generalization is the ability to apply concepts or patterns that have been learned in different contexts or problems.
Participants in building robots followed the guidelines contained in the module, but there were modifications to the robots built. The observation results showed that participants modified the robot arm that function to pick up items, the items lifted and the sensors used. As expressed by participant 1,
"There are minor changes, in the tools to raise the object."

Participant 1 stated modifications to the robot arm for transporting items. This was also done by Participant 2 in the construction of the robot. As indicated by participant 2,
"It fits but something has been changed, the load is different and the sensor used is also different."
Participant 2 stated modifications to the sensors used with infrared sensors to detect items in place of moving robots. This reflects generalization behavior, where robot components are adapted to specific needs to improve the efficiency of items transport robots, thus the need for modification by linking to abstraction to solve problems [26]. This was also done by participant 1 and participant 3 in using sensor variables in the robot built as seen from the attachment of observation sheet 1. Modification with pattern recognition is applied by participants in the construction of robots. This is an aspect of generalization by recognizing patterns in problems [31], and using variables in robots projects to improve efficiency [27].
The construction of robots is programmed by applying patterns from previous codes. Observations showed that participants used the same programming code blocks for picking up and dropping off items. The use of previous program code can be seen from the results of the code blocks that participants have programmed in the EV3 Classroom application. Making the programming code for lowering an item, participants reused the programming code in raising the item. Participants take advantage of the previously existing program code and modify the direction of the rotating motor. This proves that participants can determine the robots programming code by applying the previous programming code block pattern in the EV3 Classroom application. This is in line with research by [27] that the modularity aspect provides an opportunity for participants to identify programming patterns to reuse previous code blocks to complete the solution.

4.2 Systematic program development through LEGO Mindstorm EV3

Participants systematically designed the programming code in the EV3 Classroom application. The observation results showed that participants were able to create robots program code. Participant 2 expressed robot behavior when writing descriptive algorithms with everyday language in the first step of programming using a sequence of numbers.



Program:
1. Move forward
2. Using sensor to detect objects
3. Pickup object
4. Lifting object
5. Move forward/retreat
6. Put the object at the destination

Figure 4.2.1 Descriptive Algorithm by Participant 2

This shows the use of algorithmic aspects that involve steps in solving problems [32]. Participants succeeded in designing the programming code seen from the programming results compiled by

participants. The algorithm aspect is applied by participants by writing down the stages of robots behavior sequentially so that it can be used in the form of programming code [27]. This is in accordance with the narrative of Participant 3,
"I'm glad to see that it worked..."

Participant 3 showed success in programming the robots built. This was also demonstrated by Participant 1 regarding the ease of completing the programming. As stated by participant 1 in his interview,
"...because I've been there before...the program is easy."

Successful programming is achieved by finding a coherent code suitable for the robots transporting items to attain the goals and functions in answering the problem. The programming code was successfully developed by participants by exploring and experimenting to produce code that suits the purpose and function of the robot [29]. Expansion of the program code was carried out by participants in programming robots. The observation results showed that participants expanded the programming code of the straight-moving robot to pick up items to become an items transport robot by utilizing infrared sensors to detect objects This shows that participants can apply modularity aspects in the process of expanding the robot programming code [27]. Participants were able to create programming codes that were easy to compile, as stated by Participant 2,
"...so I know how to be like this, I have to go into detail..."

The observation results show that Participant 1 created programming code that is easy to compile by developing the code for raising and lowering items using "if" in the retrieval of items. Participant 1 used an "if" control structure in item retrieval. Participant 2 created an easy-to-compile programming code by adding a touch sensor to the programming code. Participant 2 added a touch sensor before starting the robots, so that the robot will start its work if the touch sensor is touched. Participant 1 and Participant 2 can produce code that is easy to compile, while Participant 3 only creates basic programming for the items transport robot. Participant 3 extended the previous code but could not produce a composable programming code. Participant 1 and Participant 2 were able to develop the programming code to be more effective and manageable, resulting in effective and more manageable code [27]. However, participant 3 has not been able to develop effective programming code and can only create basic program code for the items transport robot.

4.3 Problem-solving with deconstruction techniques

The problem deconstruction technique was applied by the participants in assembling the robots. Observations showed that participants assembled the robots by focusing on the necessary components. As stated by participant 1 in his interview,
"...the *LEGO components in the box are already a bit mixed up so it's a bit difficult to find..."*
Participant 1 struggled when all the LEGO components were mixed. LEGO components that have been mixed make it difficult in the early stages of assembly. Participants overcame this by taking the LEGO components needed in making robots.

Figure 4.3.1 Separation of Important Components

Figure 4.3.1 shows participants selecting the necessary LEGO components and sorting them based

on their function in building the robot. Separating robot components by focusing on the essential elements needed by robots is done by participants when designing robots. This shows that abstraction allows students to identify and focus on the substantial ideas in the problem, thus facilitating efficient problem-solving [8]. Observations revealed that participants separated components according to function into smaller units in the robot assembly process. This shows that participants must understand each part of the robot and how each part functions. As Participant 1 stated in his interview,

"*There are step-by-step instructions to build the robot and the components needed.*"

With the help of the robot module, participants can separate components according to function. Thus, Participant 1 divided the components based on the function of which part of the unit to be assembled. Participant 3 also separates between smaller units in the assembly, as stated by participant 3,

"...from building the LEGO of the motorcycle and then combining the right and left wheels..."

Participant 3 divided the wheels into right and left wheels. The unit part is connected to all parts into a single unit as well as providing cables. This is consistent with the research of [8, 32] who emphasized the need to break down problems into smaller components based on function to improve understanding and efficiency while addressing problems.

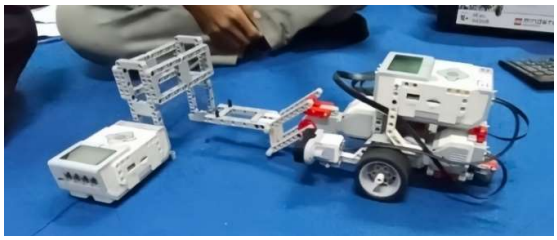


Figure 4.3.2 Finished Freight Transport Robot

Figure 4.3.2 shows the completed transport robot that the participants assembled. Learning using robots influenced the development of CT skills in participants. This is in line with research by [16, 17, 24] that robots can develop CT skills in students. The CT skills observed encompass a broad spectrum of cognitive abilities essential for learning physics through robots. Integrating robots education can enhance students' cognitive abilities by developing their CT skills [23]. Learning with robot technology also helps participants acquire critical thinking and collaboration skills, which are applicable in everyday life [32].

Conclusions

Educational robotics devices effectively facilitate the learning of physics, particularly the concepts of work and energy, through interactive, practical simulations. Participant gain an understanding of how the concept of work is applied when the robot lifts objects, as well as how kinetic and potential energy are involved when the robot moves or lifts objects. Robotics helps students develop CT skills like abstraction, generalization, algorithms, modularity, and decomposition. This is seen in problem identification through pattern recognition, using prior programming experience, visualizing solutions, and reusing code. Systematic program development involves creating effective algorithms and manageable code, while deconstruction is shown in assembling robots by focusing on key components and their functions. Although CT skill levels vary, participants successfully develop these abilities, making robotics a valuable tool for understanding physics and preparing for future scientific and career opportunities.

Implications and Future Work

The findings suggest that robotics can enhance critical thinking and collaboration, prompting educational policymakers to integrate such technologies into the curriculum. Further research is needed with larger sample sizes and varied settings to confirm the impact of robotics on developing computational thinking (CT) skills. Additionally, government funding is crucial for expanding this research, as CT is a key skill for the 21st century [1].

Acknowledgements

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56-73, 2022.

AI, Tensors, and Math Homework

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Abstract: *Many of the current advances in artificial intelligence are based on the mathematics of tensor spaces and optimization. However, most math instructors are not well versed in tensors, as it is often taught as an abstract topic in graduate algebra classes. The purpose of this paper is to help describe tensors and their properties, and how they are used in artificial intelligence. Also, this paper will examine how this informs the behavior of artificial intelligence when it is used to answer mathematics questions.*

1. Introduction

Tensors are often first introduced to students in graduate algebra classes as abstract concepts that apply to vector spaces and modules, or even as a topic in category theory (for example see Algebra by Hungerford [1] or Multilinear Algebra by Merris [2]). But they are often taught only briefly and not used much; so many students never develop a deep understanding of them. However, tensors are important as the primary mathematical tool in transformers, the backbone of deep learning artificial intelligence models like Canva's image generator, Google Translate, and ChatGPT. Studying these examples gives us an opportunity to better understand tensors themselves. Also, as students are more and more likely to turn to generative AI to help them with their homework, understanding what AI is doing can help us predict what kinds of mistakes these tools are likely to make. If we want to incorporate AI into our classrooms, it is very helpful to know enough about the tools to prepare for the kinds of misconceptions that might be developed if students rely on AI for information and problem solving.

2. Tensor Spaces

Tensor spaces stem from an attempt to generalize the idea of linear transformations of vector spaces. We begin by looking at an interesting way to generate matrices from vectors, and then try to generalize the idea.

Anyone familiar with linear algebra is familiar with inner products, the simplest example being the dot product of vectors in $\mathbb{R}^n$. If the vectors are represented as columns, we can get the dot product by using the formula:

$$\vec{v} \cdot \vec{u} = \vec{v}^T \vec{u}$$

from which we get a scalar.

However, there is also an outer product that can be defined by the formula:

$$\vec{v} \otimes \vec{u} = \vec{v} \vec{u}^T$$

from which we get a rank-1 matrix.

For example, we can take the outer product of two vectors as follows:

$$\begin{bmatrix}1\\2\end{bmatrix}\otimes\begin{bmatrix}2\\-1\\3\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}2 & -1 & 3\end{bmatrix}=\begin{bmatrix}2 & -1 & 3\\4 & -2 & 6\end{bmatrix}$$

Which, as a 2×3 matrix, we often use as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ by multiplying the 3-vector on the right of the matrix.

Of course, if we want to describe all linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$, we need more than just the rank-1 matrices, we need all the rank-n matrices, which we get by summing rank-1 matrices. It is not hard to show that every rank-n matrix can be written as the sum of n rank-1 matrices through a process that duplicates row reduction, and that fewer than n such matrices is insufficient.

Given two vectors from $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, we can call the outer product of these vectors the **tensor product** of the vectors, also known as a **pure tensor**, **elementary tensor**, or **decomposable tensor**. Any sum of pure tensors is just called a **tensor**. The collection of all such tensors is called the **tensor product** $\mathbb{R}^n\otimes\mathbb{R}^m$. Here, I'm playing a little fast and loose with the definitions, as tensors and tensor products are supposed to be independent of bases in the same way that linear transformations are. But when talking about linear transformations, once you choose a basis, there is a one-to-one correspondence between matrices and linear transformations in such a way that they basically mean the same thing. In a similar way, pure tensors are basically the same as rank-1 n×m matrices, and the tensor product $\mathbb{R}^n\otimes\mathbb{R}^m$ is essentially the same as the collection of all n×m real valued matrices.

A tensor in $\mathbb{R}^n\otimes\mathbb{R}^m$ can be thought of as a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ by multiplying the m-vector on the right of the matrix, or as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ by multiplying the transpose of the n-vector on the left of the matrix.

While tensors seem to be an overly complicated way to generate matrices, the process lends itself to generalization. We can generate multidimensional arrays using the same kind of process. Let's say we wanted to make a tensor product among three vector spaces, $\mathbb{R}^n\otimes\mathbb{R}^m\otimes\mathbb{R}^k$. We could think of the tensors in this product as three-dimensional arrays of real numbers. These arrays would then be able to serve as several different linear transformations. For example, it could be a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m\otimes\mathbb{R}^k$. It could also be a transformation from $\mathbb{R}^n\otimes\mathbb{R}^k$ to $\mathbb{R}^m$, from $\mathbb{R}^k$ to $\mathbb{R}^n\otimes\mathbb{R}^m$, or other similar transformations.

Let's construct a simple example. Consider a tensor from $\mathbb{R}^2\otimes\mathbb{R}^3\otimes\mathbb{R}^5$ which we will represent as a 2 by 3 grid of vectors from $\mathbb{R}^5$:

$$\tau=\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\\\vec{v}_4 & \vec{v}_5 & \vec{v}_6\end{bmatrix},\vec{v}_i=\begin{bmatrix}v_{i,1}\\v_{i,2}\\v_{i,3}\\v_{i,4}\\v_{i,5}\end{bmatrix}.$$

We could think of this as a transformation from $\mathbb{R}^5$ to $\mathbb{R}^2\otimes\mathbb{R}^3$ in the following way:

$$\tau\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right)=\begin{bmatrix} a_1 v_{1,1} & a_1 v_{2,1} & a_1 v_{3,1} \\ a_1 v_{4,1} & a_1 v_{5,1} & a_1 v_{6,1} \end{bmatrix}+\begin{bmatrix} a_2 v_{1,2} & a_2 v_{2,2} & a_2 v_{3,2} \\ a_2 v_{4,2} & a_2 v_{5,2} & a_2 v_{6,2} \end{bmatrix}+\ldots+\begin{bmatrix} a_5 v_{1,5} & a_5 v_{2,5} & a_5 v_{3,5} \\ a_5 v_{4,5} & a_5 v_{5,5} & a_5 v_{6,5} \end{bmatrix}$$

which mirrors how typical matrix multiplication works, but in more dimensions. Similarly, we could think of this as a transformation from $\mathbb{R}^2 \otimes \mathbb{R}^3$ to $\mathbb{R}^5$ in the following way:

$$\tau(M)=\tau\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right)=a\vec{v}_1+b\vec{v}_2+c\vec{v}_3+d\vec{v}_4+e\vec{v}_5+f\vec{v}_6.$$

Again, we're being a little fast and loose with our definitions. Formally, the matrix in this last example should be written as a sum of pure tensors and then we'd get

$$M=\vec{u}_1 \otimes \vec{w}_1+\vec{u}_2 \otimes \vec{w}_2 ; \tau(M)=\vec{u}_1^T \tau \vec{w}_1+\vec{u}_2^T \tau \vec{w}_2$$

where multiplication looks like scalars multiplied by vectors within the tensor and is performed like matrix multiplication. However, if you chase the elements, it amounts to the same thing as the earlier simpler form. We often find it practical to sidestep the pure tensor notation.

We could do a slightly larger example. Consider a tensor from $\mathbb{R}^2 \otimes \mathbb{R}^3 \otimes \mathbb{R}^5 \otimes \mathbb{R}^6$ which would be a four-dimensional array of numbers. If we wanted to consider this tensor as a linear transformation from three by six matrices to two by five matrices, we could write this tensor as a three by six matrix with entries that are all two by five matrices.

$$\sigma=\begin{bmatrix} M_1 & M_2 & M_3 & M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 & M_{10} & M_{11} & M_{12} \\ M_{13} & M_{14} & M_{15} & M_{16} & M_{17} & M_{18} \end{bmatrix}$$

Now consider a three by six matrix.

$$A=\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{bmatrix}$$

The tensor sigma can act as a transformation that takes matrix A as an input and outputs a two by five matrix as follows:

$$\sigma(A)=a_1 M_1+a_2 M_2+a_3 M_3+\ldots+a_{18} M_{18}.$$

Tensors can be defined in other realms as well, such as modules. For example, we can consider finitely generated abelian groups to be $\mathbb{Z}$ modules. As such, the tensor product of two of these modules would be basically the group of homomorphisms from one module to the other. But we would need some kind of bilinear map to identify which tensors were identified with which

homomorphisms. We can think of this map as being equivalent to choosing a basis in vector spaces, but without the obvious choice of the standard basis.

3. Tensors in Artificial Intelligence

Tensors are used as the primary tool in most of the popular artificial intelligence applications today. They form the primary mathematical component of transformers, the backbone of most deep learning models of AI. For a detailed description of the structure of these things, see the YouTube series on neural networks by 3Blue1Brown [3]. Let's look at a simplified overview of a predictive text AI like ChatGPT, for example. In such examples, there is a dictionary that assigns a high dimensional vector to each word or word part (on the order of thousands of dimensions). When a user provides a prompt for the AI, a matrix is constructed where the columns are the vectors associated with the words or pieces of words from the prompt. Then a series of tensors is used to act on the matrix to produce a vector associated with the space of vocabulary words, where the value of each entry represents the likelihood of that vocabulary word being the most appropriate next word or word part to output, and so is a probability distribution for what word to choose. This distribution is randomly sampled from, and the chosen word or word part is outputted. That word is then appended to the prompt matrix (the oldest word may be dropped if needed) and the process begins again.

Mathematically speaking, this is a very simple process and is not difficult for a computer to perform. To describe what is actually going on, we can look at the applications and tasks where matrices are used and draw an analogy.

One common application of matrices is to study population flow. For example, suppose there are three cities, A, B, and C. During any year, 5% of A's population migrates to B, 10% of A's population migrates to C, and the other 85% stays in A. Similarly, 15% of B's population migrates to A, 5% of B's population migrates to C, and the other 80% stays in B. Also 10% of C's population migrates to A, 15% to B, and 75% stays. So, if the populations of A, B, and C are 1000, 2000, and 3000 respectively at the beginning of the year, at the end of one year the new populations can be calculated by multiplying the matrix as follows:

$$\begin{bmatrix} .85 & .15 & .10 \\ .05 & .80 & .15 \\ .10 & .05 & .75 \end{bmatrix} \begin{bmatrix} 1000 \\ 2000 \\ 3000 \end{bmatrix} = \begin{bmatrix} 1450 \\ 2100 \\ 2450 \end{bmatrix}.$$

The matrix essentially captures the flow of the population in the sense that it acts on the original population to produce the next population distribution. The process can then be iterated to model how the populations change over time.

In the same way, a predictive text AI is a tensor or series of tensors that captures the "flow" of language and acts on a prompt matrix to output a vector of probabilities which can be used to produce an appropriate next word.

This may seem simple, and it is relatively easy for a computer to execute, provided we have the dictionary and the tensors. But coming up with these is very difficult. ChatGPT version 3, for

example, has 175 billion values in these objects that need to be optimized and refined. The process of generating the dictionary and the tensors is usually referred to as **training** the AI.

The way to train an AI is to start with a large set of data which you know follows the patterns you want to capture. For instance, take a large collection of written English text. Now randomly generate a set of dictionary vectors and randomly generate tensors. Take a portion of the data, encode it into a prompt matrix, and act on it with the tensor or tensors. Compare the output to the actual next word in the data. Do this with every portion of your data set. When we use a random dictionary and tensors, the outputs will not likely match the next words in the data set. The object of training the data is to keep changing the dictionary and tensor entries until the outputs more closely resemble the next words in the training data.

Finding more efficient ways to adjust the tensor entries and dictionary vectors can help make this process more efficient (the most popular method is called backpropagation), but there's no easy way to get around the massive amounts of calculations that need to be executed in order to train the AI. It wasn't even possible with computing technology until recent years. The design of modern graphical processing units (GPUs) is well suited to the task, which is why, at the time of writing this paper, the computer graphics company NVIDIA has become the largest company in the world because those interested in AI need NVIDIA's GPUs to refine their tensors.

4. Consequences

A natural question to ask is "Does AI actually have intelligence?" The short answer is no, at least not in the sense that most people think of as intelligence. It's just a very clever application of tensor products that captures the flow of the English language. When trained on enough data, the information contained in that data will naturally be included in the output, even though the training data is no longer used to generate output. However, the AI is basically nothing more than a big multidimensional matrix. It doesn't know anything, or even understand what any of the definitions of the words mean. It's simply an optimized dictionary and tensors. The output is optimized to approximate the data it was trained on, but that data no longer is used in the process. And there's often randomness involved in that process, so it's not common to get identical results from the same prompt.

It is very easy to anthropomorphize the AI, to believe that it is thinking or reasoning things out; but in reality, it is just following the flow of language that it has been trained on. The AI doesn't actually reason, nor does it "know" anything in the way that we usually mean. And this can sometimes lead to some interesting misconceptions.

Sometimes AI outputs results that fly in the face of our understanding, or phrases that sound like statements of fact which clearly aren't true, even though they are the types of words that follow the flow of language. These kinds of outputs that appear to be fabricated facts and other obviously wrong statements are called **hallucinations**. Some of these hallucinations are due to the randomness built into the algorithm. Hallucinations also occur because the training data is gone. There are no facts to draw from anywhere in the algorithm.

Legal briefs, for example, may be a part of the training data, which would include references to case law. When an AI writes something that resembles a legal brief, references to case law within a

brief would be a part of the output, but the actual case law referred to would not be drawn from actual history, but would be entirely composed by the AI, and would not match any existing case law. This actually happened when lawyers used ChatGPT to write a brief in federal court [4].

Similarly, if an AI were prompted to compose an essay that includes a quotation, the AI would not have access to the quotation anywhere in its algorithm unless the exact quote is used often enough in the training data to make outputting the exact quote very likely. In this instance, AI is likely to hallucinate a quotation that might sound appropriate, but is entirely invented by the AI.

5. In Mathematics Class

Many AIs can be used to answer math questions. ChatGPT, for instance, is trained on giant sets of data from many sources which happen to include mathematics textbooks, workbooks, tutoring websites, online discussions, etc. So when a mathematics problem is inputted into ChatGPT, the correct answer is often produced because such answers are commonly available in the training data.

Even proofs, if they are commonly found in textbooks and online sources, are often correctly produced by AI. I assigned students in one of my math classes to ask ChatGPT version 3 to prove that the square root of 2 is irrational, and with some minor variations, each student was provided with the standard proof by contradiction that is included in almost every textbook and online tutorial that is used to teach proofs. It is important to note that this may not always be the case, and ChatGPT can still produce hallucinations with this task.

More obscure mathematics problems can be particularly problematic for AI. Any problems which are not posed in language common to mathematics may also present problems. Furthermore, anything that has many mistaken answers in the training data will of course provide inaccurate results.

It's not too hard to get ChatGPT version 3 to make mistakes. Here are some examples.

As a bonus question on an exam in my class referred to above (on using technology in the classroom) I had my students ask ChatGPT the question "what is the smallest positive integer that is 13 times the sum of its digits?" A little more than half of the time, the correct answer was reached and the method was appropriate and accurate. However, other times there were hallucinations such as one student's result where the method was accurate but when it got to the correct answer, it declared that 117 was not equal to 13×9. So, it kept on searching until 156. Another result was a method which minimized the hundreds place, then the ones place, then finally the tens place which gave an answer of 195.

Another time, I asked ChatGPT "What is the smallest multiple of 12 whose digits do not sum to a multiple of three?" The AI started with an analysis that included the statements that "the sum of the digits of the number is not a multiple of 3" and also "it must be a multiple of 3" which clearly form a contradiction. But rather than conclude that there was no such number, it went on to start testing the multiples of 12 one by one until it finally timed out. It generated output that was correct, but clearly didn't show understanding of that output.

When I asked it to perform a proof by induction, it generally produced a correct proof, but made the mistake "if k is odd, then both k+1 and k+2 are even, again making (k+1)(k+2) even" which didn't

affect the conclusion, but was clearly wrong. This kind of error is not uncommon at all. In another example of verifying an equation with ChatGPT, it correctly calculated the right hand side as 98 but then incorrectly calculated the left hand side as 81.2, but still concluded that they were equal.

All these examples demonstrate the fact that AI is just capturing the flow of language without truly understanding the meaning of the words.

6. Conclusion

The recent wave of AI tools have been designed around the idea of tensor products, so it is helpful to better acquaint ourselves with tensors, what they are, and what they do, if we are to use AI as an effective tool. We can then better understand the limitations of this technology and how it might be appropriately used.

AI is a part of our world now, and students will be turning to AI to assist them with their homework. As math teachers, it may be helpful to incorporate AI into our classrooms, particularly showing students ways it is likely to be helpful and also how to carefully interpret the output and avoid being a victim of its limitations.

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Teachers' Narratives on Integrating GeoGebra for Enhancing Conceptual Understanding in Mathematics

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Abstract: *Integrating information and communication technology (ICT) in mathematics education has been increasingly recognized as a powerful tool for increasing students' interest and performance. It is widely discussed that students' understanding of mathematical ideas is not satisfactory, especially in terms of conceptual understanding. Using narrative inquiry as a research method, we examined the experiences of four secondary mathematics teachers—two men and two women—who successfully integrated GeoGebra while teaching geometry and function, an ICT application, into their regular mathematics classes. These classes typically consist of around 20 students aged 13 to 14 years. Through in-depth interviews and reflective narratives, this study explored the teachers' insights on how GeoGebra transforms the learning environment in school mathematics. The participants reported that GeoGebra significantly advances concept formation by allowing students to visualize and manipulate mathematical ideas and concepts such as geometric shapes and algebraic functions. This interactive engagement clarifies abstract ideas and accelerates students' curiosity and motivation, making learning more enjoyable. Further, participants observed that students are more enthusiastic and willing to engage with challenging mathematical problems when using GeoGebra. The study concluded with recommendations for teachers to use GeoGebra effectively in math lessons. It advocates for professional development programs to equip teachers with the necessary skills to integrate ICT tools like GeoGebra effectively and suggests avenues for further research to explore the long-term impact of such technologies on student engagement and learning outcomes.*

1. Introduction

Teaching mathematics at the school level in Nepal is largely conventional and aligned with one-size-fits-all approaches, such as explaining mathematical concepts by teachers and assigning some tricks and techniques to solve routine mathematical problems [2, 20]. This model of teaching and learning mathematics is reflected even in the result of the Secondary Education Examination (SEE). SEE results are not satisfactory in mathematics. More than 50% of students who appeared in the SEE exam failed in mathematics, with less than 35 out of 100 marks, in SEE 2024. Likewise, the results of the National Assessment of Students' Assessment (NASA) conducted by the Educational Review Office (ERO) were below 50% [6]. It shows that the national level of school mathematics is poor in terms of students' academic achievements in mathematics. It is widely discussed that teachers focus more on procedural knowledge of mathematics, in which students are prepared to solve routine mathematical problems following the predetermined steps [12]. The focus on the procedural aspect of mathematics lacks the students' conceptual knowledge [16].

A procedure or process is a methodical approach by which the correct answers must be accomplished. Our everyday routines often follow a predictable pattern, suggesting that human life might be seen as a routine system. Attempting to understand mathematics exclusively through memorization, following a step-by-step approach to find the correct answer, or relying heavily on pre-defined algorithmic problem-solving methods leads to the development of procedural knowledge [15]. Typically, students create a routine and adhere to it from the start to the end of their day, even when they go to bed. We require a systematic arrangement of all elements, which should be carefully organized. Acquiring knowledge that is acquired through a systematic procedure is referred to as "procedural knowledge." This type of knowledge is obtained through a process, as defined by Hiebert and LeFevre [7]. Procedural knowledge refers to the practice of adhering to a tight and predetermined procedure or process in order to obtain a predetermined answer.

On the one hand, conceptual knowledge in mathematics is formed by building upon prerequisite concepts, ideas, and information. We progressively develop mental representations or schemas of something through interactions with the environment and various individuals. These schemas contain significant capacity to generate other notions. On the other hand, gaining a conceptual understanding of mathematics involves acquiring information about the underlying reasons and principles that explain why mathematical concepts and processes work in specific ways [16]. Such pertains not just to existing knowledge but also to how concepts can be understood. It enhances cognitive flexibility by fostering the development of divergent thinking skills, allowing individuals to consider alternative perspectives while approaching a topic.

Above all, it signifies that a student has many approaches to address the topic of mathematics. Conceptual knowledge refers to understanding definitions, formulas, and methods and the ability to justify them. A conceptual understanding of something (mathematics or other ideas) enables one to generate a logical and sound response. In this regard, the School Sector Development Plan 2016-2023 of Nepal prioritized the integration of ICT in teaching and learning methods [11]. This plan has facilitated the advancement of ICT in educational institutions and other governmental entities. The motives behind all of these policies are to incorporate and seamlessly integrate advanced technology into classroom instructions effectively. It will only be possible when teachers retain a high level of proficiency in the technology tools, equipment, and resources used in actual teaching methods. The Government of Nepal, the Ministry of Education, Science, and Technology, has changed the framework and expanded the use of ICT in education. These include using ICT for communication purposes and as a tool for learning. The utilization of ICT in education can be categorized into four major areas: communication in school administration, facilitation of teaching and learning, capacity building of stakeholders, and the inclusion of a separate discipline in the curriculum to enhance students' life skills. Joshi et al. concluded that ICT enhancement programs are needed for mathematics teachers at secondary schools in Nepal [9].

With the above gap, this article aims to explore the experiences of four mathematics teachers who have incorporated GeoGebra as one of their ICT tools in the classroom. In addition to GeoGebra, numerous ICT tools and applications are available for teaching mathematics. However, GeoGebra is gaining popularity worldwide, including in Nepal, for its ability to facilitate the visualization and manipulation of mathematical concepts. It is a web-based and offline application available in over two hundred languages, allowing users to interact with various mathematical concepts. GeoGebra is also known for its user-friendly interface.

2. Methodology

This study used a narrative inquiry approach to investigate the experiences of four secondary-level mathematics teachers who have used GeoGebra in their mathematical instruction at least five times or more in their mathematics classes. The narrative inquiry is highly appropriate for this research as it enables a thorough investigation of personal and professional experiences, documenting the unique events in which GeoGebra impacts teaching practices and student learning. The participants in this study were purposefully chosen based on their prior use of GeoGebra in their mathematics classrooms. The selection criterion comprised of a) a minimum of three years of teaching experience using GeoGebra at the secondary level, b) proven success in incorporating GeoGebra into their teaching, as indicated by students' verbal comments, c) willingness to engage in comprehensive interviews and share their reflective narratives. Following the three criteria, we carefully chose four participants, specifically two male and two female teachers from two secondary schools in Kathmandu Valley, Nepal. This selection was made to ensure a wide range of opinions

and selection of the research participants. Likewise, narratives were obtained through two methods: comprehensive interviews conducted by the researchers and reflective narratives provided by the participants. During the research process, every participant participated in two consecutive semi-structured interviews, each lasting around 60 to 80 minutes. The interviews were conducted casually, enabling participants to share their experiences and thoughts regarding their encounters with GeoGebra while teaching geometry and functions (to name but a few) based on the themes— (1) students' conceptual knowledge through visualization in GeoGebra, (2) student engagement and ownership and (3) students' learning using GeoGebra. Then, the participants were instructed to compose reflective narratives of their encounters with GeoGebra. These experiences offered further depth and intimate perspectives, enabling participants to express their experiences uniquely. The narratives centered around essential topics such as the acquisition of conceptual knowledge through visualization, student engagement, and the drive to follow further exploration. The narratives produced were subjected to a thematic narrative analysis procedure, which included recognizing, examining, and reporting recurring themes within the data. The scaffolding and More Knowledgeable Others (MKO) ideas of the social constructivist approach [21] have been used for analyzing the narratives and reflective writing of the participants. To ensure the reliability of the results, the study implemented various approaches such as a) Member checking: Participants thoroughly examined the transcripts and produced narratives to confirm the accuracy and authenticity of their reports. b) Triangulation: Data triangulation was done by gathering information from several sources (interviews and reflective narratives) and analyzing these stories to uncover logical patterns. c) Peer debriefing: The study team often held talks to question interpretations and guarantee a thorough analytical process. All participants provided informed permission, indicating their complete understanding of the study's objectives, methods, and entitlements. The participants' names and other identifiable information have been anonymized to ensure confidentiality.

3. Data Analysis and Interpretation

Three key themes emerged after carefully analyzing the participants' narratives and reflective writings. Data presentation and discussion are conducted under each theme. We discovered connections during our review of the theoretical and empirical literature.

3.1 Conceptual knowledge through visualization

A mathematics teacher with more than a decade of experience shared this way;

> *I am always on the lookout for new approaches to help my students understand complex topics such as geometry and function. When I first discovered GeoGebra, I was captivated by its ability to make abstract mathematical concepts more concrete. Over the last three years, I've used GeoGebra in my teachings, particularly when teaching mensuration, focusing on 3D objects and calculating their areas and volumes. In one notable lesson, I used GeoGebra to teach my students about cylinders' volume and surface area. Traditionally, students used to struggle to visualize formulas and grasp why they worked. I used GeoGebra to make a dynamic three-dimensional model of a cylinder. As I adjusted the model in real-time, modifying the height and radius, the students were able to observe how the volume and surface area changed. This visual and participatory approach meaningfully improved their knowledge.*

Next teacher shared about her student;

A student who has previously struggled with geometry showed amazing improvement. Throughout the class, I asked the student to explore GeoGebra on their own devices. He found more engaged as he experimented with the dimensions of various 3-D models. He shouted, "Oh, I get it now!" as he noticed that doubling the radius of a cylinder resulted in a fourfold increase in surface area but an eightfold increase in volume. This moment of exposure had an impact on the entire class, not just on her. I sensed that he developed a deeper level of conceptual understanding.

ICT integration in mathematics pedagogy improves conceptual understanding of mathematics and achieves a higher ability for Bloom's taxonomy, such as creating. However, including ICT in Bloom's taxonomy of learning stages in mathematics education is challenging. For example, suppose a teacher uses GeoGebra to teach mathematical concepts like parabola. In that case, the teacher should know what stages of learning may be completed and what the following stages of learning require. However, visualization techniques help teachers and students understand mathematical concepts and solve problems. There are different ways to develop visualization techniques. One of them is a concept map [24]. While using GeoGebra in teaching factorization, using algebraic tiles in GeoGebra helps students visualize the ideas of factors of algebraic expressions.

Atnafu and Zergaw mentioned that the process of developing students' mathematics visualization involves four steps: (i) creating lessons that incorporate visualization techniques; (ii) instructing students in mathematical concepts using visualization techniques; (iii) providing opportunities for students to practice visualizing mathematical concepts; (iv) assessing students' progress in visualizing mathematical concepts [1]. Next, Lim argued that visualizations are crucial ideas to maximize learning [19]. In one of the reflective notes of teachers, it was found that the students visualized the ideas of vertical height and slant height of a pyramid while showing them through GeoGebra. Furthermore, it was also widely discussed that technology can be used to scaffold mathematical problem-solving [25]. The ideas of Vygotsky, in which scaffolding is one of the effective ways of learning, are well connected to the use of GeoGebra in mathematics learning. Studies such as [4, 10, 18] further elaborated on the use of GeoGebra for visualizing mathematical concepts. For our study, visualization involves the interpretation of meaning from visualization objects or introspective visualizations in relation to “the person’s existing network of beliefs, experiences, and understandings.” [13, p. 26]. While using GeoGebra, secondary teachers shared that both interpretive and introspective visualizations happen for students.

3.2 Student Engagement and Ownership

The student's engagement and ownership were other themes generated based on the teachers’ narratives and reflective notes. One female teacher wrote a reflection this way:

I have found GeoGebra to be a useful tool for promoting student engagement in my geometry course. One day, I assigned my students the responsibility of investigating the surface area and volume of a variety of three-dimensional shapes using GeoGebra. The class was an explosion of activity as students independently found relationships and manipulated shapes. Hem, a student who is known for his reserved behaviors, took the leadership role in his group and proudly shared his findings. In addition to enhancing the mathematical concepts, this hands-on approach additionally created a collaborative learning environment in which students felt empowered and took responsibility for their learning.

While conducting interviews with one male teacher, he responded.

I observed a major shift in my students' learning attitude when I introduced GeoGebra to calculate irregular objects' volume. Students, for example, Bibisha, who typically expected a series of instructions, began to experiment with the GeoGebra independently. Bibisha developed her 3-D models, changing the dimensions to observe their effect on volume. She proudly shared her discoveries with the class, showcasing a sense of ownership and confidence in her learning. The interactive nature of GeoGebra facilitated the development of a more personal and deep understanding of mathematics by allowing students to take charge and explore concepts at their own pace.

Engagement in mathematics is supposed to be an indicator of meaningful mathematics learning. Several researchers in Nepal [22, 23] argued that the lack of engagement in mathematics classrooms is one of the major issues in mathematics education in Nepal. We, as educators, have observed mathematics students for several years, and we have seen that students' engagement in mathematics learning is notably below average. There are several ways to make learning engaging. Ishaq et al. argued that ICT-integrated pedagogy provides an extended collaboration between students and creates ownership in learning [8]. Moreover, learners get several opportunities to explore problems engagingly [3]. Pynos concluded a study and concluded that the use of personal devices contributes to student engagement and that the use of devices by students to help with their learning is largely the same regardless of whether the item in question a personal device or a device is that the school provides [14]. In our study, teachers also mentioned that few students practiced GeoGebra on their personal devices parents at home as the use of GeoGebra grabbed students' attention in the school.

According to Vygotsky, the zone of proximal development (ZPD) is the distance between the actual developmental level of the learner of solving the (mathematical) problem independently and the level of their potential development as determined through problem-solving in mathematics under some guidance or collaboration with more capable peers and teachers [21]. In this study, the interactive exercise supported students' ZPD achievement by making them engaged in learning. In the reflection of a participant teacher, it was written;

Making geometric proofs through logical arguments was difficult for students. When I used GeoGebra and demonstrated the experiment, students started to argue from different perspectives. I felt that it helped to create ownership in learning as they started to search by themselves.

The teachers' narratives focused on how ICT-integrated pedagogy helps create ownership in learning. The use of ICT in teaching mathematics can make the teaching process more effective and improve the students' capabilities in understanding mathematics concepts.

3.3 GeoGebra Motives Students for Learning

This is another theme we developed from participating teachers' narratives and reflective notes. One of the major issues of school education in Nepal is the lack of students' motivation in general and in mathematics learning in specific.

In this context, a participating teacher shared;

The use of GeoGebra in my classroom has immensely affected my students' motivation. Teaching the properties of quadratic functions was one remarkable experience. Students were

enthusiastic about the instant visual representation of the equations they provided. For example, Anil, who usually had difficulties with algebra, was taken in by the capacity to manipulate the coefficients and observe the parabola's transformation in real-time. This interactive experience sparked his curiosity and transformed abstract concepts into tangible ones. He started his own experiments with various equations, driven by a desire to comprehend the fundamental principles. GeoGebra changed a previously considered difficult subject into an engaging and motivating learning experience.

In a similar way, another teacher wrote a short reflective note.

GeoGebra provides a source of inspiration for students and promotes individual inquiry. Following a session on the measurement of the capacity of prisms and cylinders, I observed that several students, including Maya, chose to remain after class to engage in additional experimentation with the program. Maya, who had previously expressed difficulties with mathematics, was now engaged in the creation of her own three-dimensional objects and following the calculation of their volumes. She shared her discoveries with her classmates, inquiring and establishing relations that expanded beyond the confines of the formal curriculum. Her independent investigation demonstrated a notable change in her attitude to learning. GeoGebra inspired her curiosity, motivating her to explore mathematical ideas more deeply.

From the above narratives and reflective notes, it was observed that students feel motivated when they get opportunities to interact with mathematical concepts through GeoGebra software. Rahman et al. conducted a study and found that students were observed to be drawn to the concept of self-access learning and the slider function during the learning process [15]. Moreover, mathematics teachers are encouraged to enhance their teaching and learning methods by utilizing GeoGebra, a dynamic mathematics program. They should build a learning module that integrates this software with written materials. This approach will be particularly beneficial for underachieving students, as it will help to raise their motivation in the classroom.

Mathematics is considered a difficult discipline, and most people do not have positive perceptions of mathematics [20]. One participating teacher wrote:

Students in my schools think that mathematics is made for smart people and can be taught and learned only by a group of smart people. It is too abstract for many people.

As per our experiences, society manifests mathematics as difficult and more abstract. This has demotivated students to learn mathematics. When students don't understand mathematical concepts, it leads to demotivation, and students consider mathematics difficult. On the other side, when teachers use different ICT tools such as GeoGebra, students make sense of mathematical ideas and gradually develop a conception that mathematics is easy and can be learned in a fun way. Thus, ICT-integrated pedagogy helped students to develop positive perceptions of mathematics. Dhakal and Sharma conclude that ICT is helpful in enhancing engagement and interaction [5]. ICT helps students to change their perception of it.

4. Final Remarks

First, the teachers' narratives demonstrated that GeoGebra helps students understand concepts by allowing them to observe and play, in some cases animating with geometric contents and algebraic functions (to name but a few). It helps to bridge the gap between abstract ideas and meaningful understanding. Second, the study found that using GeoGebra makes students interested and responsible for their meaningful learning of mathematical ideas and concepts. The interactive nature of the application motivates secondary-level students to take an active role, not just passively receiving information from the mathematics teachers but also actively constructing their knowledge in their context, pace, and speed. This kind of participation is very important for making the classroom more student-centered. It gets students more involved in the process of learning and makes them more responsible for their learning, as shared by the four secondary-level mathematics teachers. Lastly, one important theme that came up through the teachers' stories was how GeoGebra motivated students. Students experienced fun-filled learning while learning using software because of its dynamic and interactive features. In conclusion, using GeoGebra in secondary math classes in general and other mathematics classes is a comprehensive way to improve students' conceptual understanding, motivation, and involvement for quality learning and engagement. In the context of a highly technology-friendly society, it's becoming more and more important to use technology such as GeoGebra to support students and improve their teaching and learning process. The teachers' positive experiences in this study demonstrate that GeoGebra is useful for reaching these learning objectives by helping students do and perform better in math.

5. Limitations and Future Study

This study has several limitations. Firstly, the small sample size limits the generalizability of the findings. With only four secondary-level teachers participating, the findings may not represent the broader educator population. Additionally, the focus of the study on a specific group of teachers may introduce selection bias, as these individuals might have unique characteristics that do not reflect the diversity of teaching practices and experiences on integrating GeoGebra while teaching conceptual concepts in mathematics. To address these limitations, future research could expand the sample size to include a more diverse group of teachers from various educational settings. This would enhance the generalizability of the findings and provide a more comprehensive understanding of the phenomena under investigation. Moreover, incorporating a longitudinal design could offer insights into how the observed effects evolve over time, providing a deeper understanding of the long-term impacts. Furthermore, future studies could explore additional variables that may influence the outcomes, such as different teaching methods, student demographics, and school environments.

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Preparing Math Teachers for the Future: Analysing the Use of Technology in a Master's Program

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Abstract

Current guidelines for the mathematics curriculum from primary to secondary education in Portugal include Computational Thinking as a cross-curricular theme and advocate for the systematic use of technology. Official mathematics program documents provide methodological suggestions for teachers to develop computational thinking through the use of dynamic geometry environments, such as GeoGebra, internet applets, Scratch, and Python. In this context, it is important to understand how future teachers interpret the use of these technological tools in the first year of a Master's degree in Teaching Mathematics for Basic and Secondary School Teachers in Portugal. This study analyses the work of 12 students during the "Computational Tools for the Teaching of Mathematics" course unit. Supported by the research-based design methodology, the study examines the work carried out over two school years, as well as the adjustments made between the first and second cycles of this intervention, using qualitative data analysis techniques. The results indicate that the integration of content knowledge and technological knowledge among the participants is complex throughout the two intervention cycles. Combining pedagogical content knowledge with technological and content knowledge presents new challenges for the initial training of mathematics teachers. This study offers insights to improve the course unit for future editions and identifies issues to consider for continuous teacher training in Portugal.

1 Introduction

Since the onset of the 21st century, integrating Information and Communication Technologies (ICT) into mathematics education has yielded significant benefits for both educators and students. ICT have been shown to enhance student interest, motivation, and academic performance, while also fostering lifelong learning and positive interpersonal interactions [1]. These

technologies have transformed traditional teaching methodologies, encouraging more active student participation [2]. Furthermore, ICT integration extends beyond the educational sector into technological and industrial domains [3]. Research consistently demonstrates that ICT-based teaching is more effective than traditional approaches, particularly at the secondary education level [4]. However, obstacles such as educators' limited ICT proficiency, inadequate training, and insufficient technical support persist [1]. Addressing these challenges necessitates enhanced training programs and improved technical support systems [1, 3].

ICTs are driving a global transformation in education, reshaping how students learn and aligning education with the demands of the digital workplace [5]. These technologies facilitate active, real-world connected learning and enable personalised instruction through adaptive learning systems [6]. The integration of ICTs has significantly influenced teaching content, methodologies, and administrative processes, redefining the roles of both teachers and students [7]. Successful ICT integration is contingent upon factors such as funding, access to equipment, software availability, and personnel expertise [7].

Recent studies highlight the importance of integrating Computational Thinking (CT) into mathematics education to enhance problem-solving skills and prepare students for the digital era [8, 9, 10]. CT fosters mathematical reasoning, but its implementation faces challenges, primarily due to teachers' limited understanding of CT and its application in mathematics [11, 12]. Effective teacher training and the development of appropriate assessment tools and learning platforms are crucial for maximising the benefits of CT integration [8]. This study seeks to explore how future teachers perceive the use of these technological tools during the first year of a Master's degree in Teaching Mathematics for Basic and Secondary School Teachers in the Department of Mathematics at the University of Coimbra, within the context of a curriculum supportive of technology and CT integration.

2 Framework

This section outlines key sources to understand the integration of CT and technological tools in training in-service and pre-service mathematics teachers, framed by educational theories and models. Driskell et al. identified a significant gap in research on professional development in mathematics education focused on technology, noting that from 1975 to 2015, only 21 out of 1,210 studies addressed this area, despite longstanding advocacy from bodies such as the National Council of Teachers of Mathematics [13]. During this period, lesson study emerged as a promising approach for collaborative planning, observation, and reflection among teacher networks, with increasing interest in its application to professional development in technology-enhanced mathematics teaching [14, 15].

Follmer et al. recently developed a theory-based evaluation framework for lesson study programs, which demonstrates how pragmatic constraints can be turned into opportunities in theory-driven evaluations [16]. Liu et al. found that hands-on experiences with technology in teacher training programs in the United States accelerated classroom adoption and integration [17]. Large-scale initiatives, such as the cluster model in Australia, have successfully built curriculum knowledge and enthusiasm for teaching mathematics, with sustainability supported by professional development content, collective action, and leadership at multiple levels [18].

Integrating various technologies into pre-service mathematics teacher education has been shown to enhance professional knowledge, improve subject knowledge, and foster positive atti-

tudes toward technology use [19, 20]. Incorporating technology-rich methods courses and field experiences can develop Technological Pedagogical Content Knowledge (TPACK), shifting pre-service teachers' perspectives to view technology as a tool for deepening student understanding rather than mere reinforcement [21, 22, 23]. However, preparing pre-service teachers to effectively facilitate student thinking, reasoning, and problem-solving with technology remains challenging and requires further research [20]. Ranellucci et al. [24] found that pre-service teachers' readiness and intentions to use technology are strongly influenced by perceptions of its usefulness, ease of use, supportive conditions, and social norms.

Theodorio's study in South Africa emphasizes the importance of essential support for effectively integrating technology into teacher professional development programmes, including digital and non-digital tools, technical support, and collaborative engagement among educators and IT staff [25]. Furthermore, integrating Education for Sustainable Development (ESD) into mathematics teacher training is highlighted as crucial, with a focus on contextualisation, ecological and mediational competencies, and understanding the relationship between mathematics and the environment [26].

2.1 Models to study acceptance of technology and Innovation

Rogers' Diffusion of Innovations theory [27] provides a framework for understanding the adoption of new technologies in educational settings, encompassing stages such as knowledge, persuasion, decision, implementation, and confirmation. Teacher beliefs and attitudes play a crucial role in this process.

The Technology Acceptance Model (TAM), developed by Davis in 1989, is another widely used framework that explains how users accept and use new technologies [28]. TAM posits that perceived usefulness and ease of use are key factors influencing technology adoption decisions. However, while TAM has high validity and is extensively applied in information systems research, it should be used cautiously in multinational contexts due to cultural variations [28]. Despite its widespread use, TAM has faced critiques and has been modified to address its limitations [29].

In the context of pre-service mathematics teachers, TAM has been employed to study their intentions to use technology, revealing that perceived usefulness, ease of use, and attitude significantly influence this intention [30]. Extended TAM models, incorporating variables such as subjective norm, facilitating conditions, and Technological Pedagogical Content Knowledge (TPACK), have shown a good fit, explaining significant variance in technology use intentions in mathematics education [31]. Nonetheless, actual practice may be hindered by factors such as accessibility and professional development needs [32].

2.2 Instrumental orchestration in pre-service mathematics teachers

Instrumental orchestration in mathematics education refers to the strategic use of technological tools by teachers in their lessons, involving didactical configuration, exploitation mode, and didactical performance [33]. Research on instrumental orchestration has identified five main clusters: managing teaching complexity, designing living resources, teaching with technology, adult learners, and interacting with computers [34]. Recent studies have adapted this framework for online contexts [35] and extended it to instrumental meta-orchestration for teacher education

[36], highlighting the importance of carefully planning and arranging digital artefacts in teaching environments.

In mathematics teacher training, instrumental orchestration has been shown to enhance both content knowledge and pedagogical skills. Studies suggest that pre-service teachers' use of technology is linked to their endorsed sociomathematical norms, with a variety of norms indicating successful technology integration [37]. Additionally, combining instrumental meta-orchestration with the Lesson Study methodology has proven effective in developing pedagogical competencies in pre-service teachers [38].

Nine types of class orchestrations have been identified [39], with further subdivisions in the work-and-walk-by orchestration, including technical-demo, guide-and-explain, link-screen-paper, discuss-the-screen, and technical-support categories [40]. Instrumental orchestrations are particularly effective in ICT-supported environments, especially when using dynamic mathematics software like GeoGebra [41, 42]. These findings underscore the potential of instrumental orchestration frameworks to improve technology integration and pedagogical knowledge in mathematics teacher training programs.

2.3 Technology in teaching maths with a favourable curricula

The integration of technology in mathematics education has garnered significant research interest globally, examining its impact on teacher professional development, student achievement, and classroom practices. Longitudinal studies, such as those on the MathForward program, indicate that technology integration can significantly improve students' mathematics test scores [43]. However, successful implementation in classrooms hinges on factors including teachers' professional identities and their capacity to adopt innovative practices [44]. Pre-service teacher education programs are crucial in preparing educators for effective technology integration, though challenges remain regarding perceived difficulty and inefficiency [45].

In technologically rich mathematics classrooms, student interactions with digital tools can be categorised as Master, Servant, Partner, or Extension-of-self, underscoring technology's potential to empower students both individually and collaboratively [46]. Digital resources can enhance mathematics teaching and learning, yet they may also challenge teachers' practices [39]. In Portugal, technology integration is part of the Essential Learning of Mathematics curriculum for basic education and will be included in secondary education from 2024/2025 [47]. Initial teacher training in technology use presents added challenges, as preconceptions shaped by personal educational experiences and professional aspirations influence the adoption of new pedagogical approaches [48].

3 Methods

The curricular unit "Computational Tools for the Teaching of Mathematics" (CTTM) is a second-semester unit, following two units on specific didactics of mathematics and the History of Mathematics. This study, conducted over the 2022-2023 and 2023-2024 academic years, aims to explore two research questions: how students, as future teachers, perceive the use of technology in mathematics education, and how strategies can be tested to enhance teaching practices.

Across two cycles, students' work in the CTTM was analysed—eight projects in the first cycle (ST1 to ST8) and four in the second cycle (ST9 to ST12). In both cycles, various technologies were integrated into first-semester subjects related to specific didactics in analysis and geometry. These technologies aimed to improve teaching practices by facilitating future teachers' instructional activities, thereby enhancing student learning. Students engaged with a range of tools, including GeoGebra, Scratch, Python Blocks, Graphic Calculators, ASYMPTOTE, and MathCityMaps apps. These tools were applied in tasks designed for both primary and secondary education students, demonstrating their potential to support mathematics teaching and learning.

3.1 First Cycle

In the CTTM, during the second semester of 2022/2023, students were required, for their final assignment, to create a web page that included: i) a brief presentation of their academic background; ii) evidence of having attended a MOC; iii) examples of relationships between mathematics and art from the collection at `https://www.europeana.eu/pt`; iv) a cartoon on a mathematical topic to celebrate International Mathematics Day; v) software they would consider using in their future roles as mathematics teachers; vi) trails created with the MathCityMap application. These items were analysed and discussed in the following section. It is important to note that changes were made to the assignments proposed to students in 2023/2024 based on this analysis. These web pages were individual projects, except for the last item, the creation of a trail in the MathCityMap application, which was a group effort.

3.2 Second Cycle

In CTTM in the second semester of the 2023/2024, students were required to produce a reflective portfolio as their final assignment. This portfolio documented collaborative work conducted in pairs, aiming to enhance learning and contribute to their final grades. Each student reflected on the development of their collaborative work on an assigned theme, with the portfolio comprising both collaborative and individual components, including planning and reflection tools for simulated lessons.

Each student simulated two lessons in the presence of at least one course instructor. The topics were drawn from four available options, such as using Python for visualisation in Real and Complex Analysis or Scratch for teaching geometry and algebra in essential mathematics learning. Pairs collaborated during thematic sessions, receiving support in preparing lesson plans. The simulated lessons, lasting 45 minutes, were presented with the instructor and peers acting as learners, followed by a feedback session, were be filled an evaluation rubric for the student acting as teacher and the supervisor.

The planning models and evaluation rubrics for the lessons were previously discussed among all participants. The analysis of the reflective portfolio was conducted post-course, ensuring the anonymity of the collected data and adhering to ethical research standards.

4 Results

4.1 First cycle

The analysis of results from eight students (ST1 to ST8) reveals diverse approaches to using computational tools in mathematics education. Each student demonstrated engagement with various digital resources, contributing to projects aimed at enhancing teaching practices.

ST1 exhibited proficiency in tools such as GeoGebra, Desmos, Kahoot, and Poly, and actively participated in the Europeana project, creating comics and celebrating Mathematics International Day. ST2 focused on using Poly, mathematics-related comics, images, and completed an online course on active learning. ST3 used Desmos, participated in Europeana, and completed an eTWINNING course, emphasising comic creation.

ST4 integrated GeoGebra, Desmos, Kahoot, and Poly into teaching, contributing to Europeana and developing a MathCityMap resource, while also engaging in social network training. Similarly, ST5 utilised multiple tools, engaged with Europeana, and created a comic. ST6 focused on applying these tools and collaborating in the Europeana project through an eTWINNING course.

ST7 demonstrated advanced engagement by using GeoGebra, Desmos, Kahoot, and Poly, developing a MathCityMap resource, creating a comic, and presenting a lesson plan incorporating Python. ST8 concentrated on developing detailed lesson plans with Python, significantly contributing to the Europeana project, MathCityMap, and an eTWINNING project.

The students' engagement with computational tools and interdisciplinary projects like Europeana and MathCityMap demonstrates their capacity to integrate ICT into mathematics teaching. Their involvement in creating comics, collaborative projects, and continuous professional development through platforms like eTWINNING and NAU highlights a commitment to innovative and effective educational practices.

4.2 Second cycle

The analysis of the results from the four students, ST9 to ST12, are in progress. However considering the aim this study were analised the results getted by the dyads, ST9&ST10 portfolio and ST11&ST12. The portfolios elaborated by students demonstrate a focused integration of computational tools, specifically Python programming (ST9&ST10) to teach secondary mathematics, and Scratch (ST11&ST12) to teach basic mathematics . The students worked both collectively in pairs and individually, with their efforts documented in detailed lesson plans and reflective practices.

In the Portfolio elaborated by dyads ST9&ST10, collaborative part includes justifications for lesson plans on ST9 "Functions Defined by Branche" and "Cubic and Quartic Functions" and ST10 "Quadratic Functions" and "Complex Numbers," emphasising their importance and integration into the curriculum. Individual reflections address simulated lessons, highlighting strengths and weaknesses based on peer and supervisor feedback. The lesson plans progress from basic to complex topics, ensuring alignment with students' prior knowledge. Initial lessons use block programming tools like EduBlocks to introduce Python, while subsequent lessons tackle complex topics, requiring higher programming proficiency (see Figure 1, at left).

The lesson plans effectively integrate technological tools (Python programming), pedagogical strategies (progressive complexity, visual aids), and mathematical content knowledge (func-

tions, polynomial equations, complex numbers). They are designed to enhance CT, algorithmic processes, and logical reasoning. Hypothetical learning trajectories anticipate student difficulties, such as challenges in defining piecewise functions, and suggest strategies like focusing on the function's vertex for better understanding.

Pre-service teachers critically reflect on their lessons, recognising areas for improvement based on feedback. They emphasise the alignment of computational goals with mathematical learning objectives and the importance of clear, step-by-step instructions in programming tasks. The portfolios of dyads ST9 & ST10 demonstrate a thoughtful integration of technology in teaching, reflecting a deep understanding of how programming can elucidate mathematical concepts, fostering CT and problem-solving skills. The reflections underscore a commitment to continuous improvement, vital for effective teaching practice, with Python's use as a teaching tool being particularly significant given its increasing relevance in both educational and professional domains.

The portfolio created by dyads ST11&ST12 includes lesson plans with justifications centred on the organisation of lessons, integrating technological, pedagogical, and mathematical content knowledge, designing hypothetical learning trajectories, and reflections from pre-service teachers. ST11's lesson plans target 8th-grade linear functions, employing Scratch and GeoGebra to enhance understanding through problem-solving and collaboration. ST12's plans focus on the"Pythagorean Theorem" and "Isometries," using Scratch to create scenarios that apply the theorem and develop isometric transformation tasks. Both sets of plans aim to cultivate CT and problem-solving skills. ST11 integrates technology with mathematical concepts, using Scratch for visualising linear functions and GeoGebra for dynamic representations. ST12's approach utilises Scratch to craft interactive problems, emphasising CT and ensuring comprehension of the theorem's application (see Figure 1, at wright).

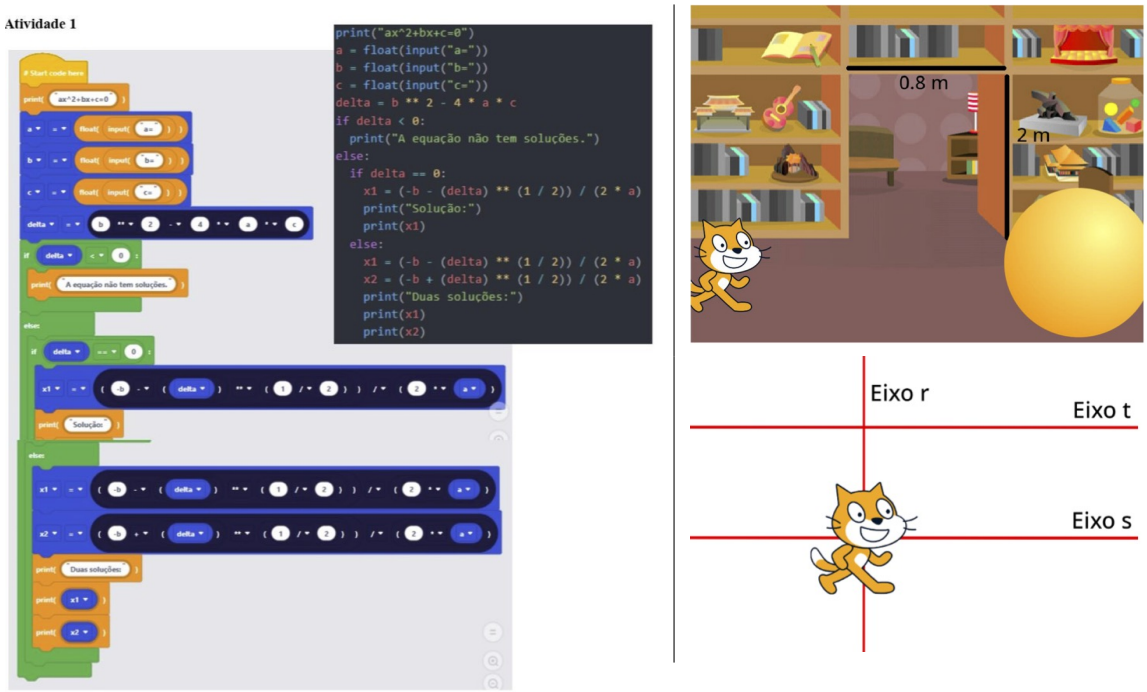

Figure 1: At left, Task 1 of 1st simulated lesson of ST10 - Resolution included in the lesson plan. At wright, scratch scenarios proposed in lessons plans of ST12.

The Hypothetical Learning Trajectories (HLTs) developed by ST11 and ST12 reflect a strategic approach to addressing potential student difficulties in understanding linear functions and the Pythagorean Theorem. ST11 anticipates misconceptions in linear functions, incorporating scaffolded support to mitigate these challenges, while ST12 engages students with the Pythagorean Theorem through storytelling, problem-solving, and Socratic dialogue using Scratch. Both trajectories utilise interactive tools to enhance learning.

The reflections of ST11 and ST12 reveal diverse insights. ST11 highlights the effectiveness of Scratch and GeoGebra in improving student engagement and understanding, while acknowledging areas for improvement based on peer and supervisor feedback. ST12 reflects on the impact of computational tools on students' problem-solving skills and stresses the importance of continuous instructional improvement.

The analysis of the portfolios of dyads ST11&ST12 demonstrates a robust integration of computational tools in mathematics education, emphasising student engagement and understanding. The portfolios showcase distinct approaches: ST9&ST10 focus on programming proficiency and algorithmic thinking, whereas ST11&ST12 prioritise interactive learning and practical applications. Both dyads exhibit reflective practices and a commitment to refining their teaching strategies, underscoring the transformative potential of computational methods in mathematics education.

4.2.1 Computational Thinking in portfolio

The portfolio of dyads ST9&ST10 emphasises CT as fundamental for mathematical problem-solving. Their approach utilises algorithmic processes, focusing on step-by-step instructions and systematic problem-solving with Python and EduBlocks. This methodology fosters logical thinking, particularly in quadratic functions and complex numbers. They emphasise decomposition, breaking down complex concepts to build on existing knowledge. Abstraction is achieved by modelling mathematical problems in Python, enabling generalisation. Automation is promoted through Python for repetitive calculations, enhancing understanding and efficiency. Data representation is facilitated by visual aids and dynamic programming tools, making abstract concepts more tangible.

Conversely, the portfolio of dyads ST11&ST12 integrates CT through interactive problem-solving using Scratch and GeoGebra. These tools enhance logical reasoning and iterative refinement. GeoGebra provides dynamic visualisations, aiding comprehension of mathematical concepts like the Pythagorean theorem and isometries. Decomposition and abstraction guide students to break down complex problems and model real-world scenarios. Pattern recognition is developed by identifying relationships within mathematical problems, while iterative development fosters continuous improvement through solution refinement.

Both portfolios underscore the importance of CT in mathematics education, with ST9&ST10 focusing on algorithmic processes and systematic problem-solving, and ST11&ST12 prioritising interactive problem-solving and visualisation.

5 Discusion

Considering the results of 1st cycle, the integration of ICT in mathematics education, as evidenced by the students' use of GeoGebra, Desmos, and other tools, supports the enhancement

of student interest, motivation, and performance [1, 2]. The students' engagement with projects like Europeana and MathCityMap reflects the transformative potential of ICT in making learning more interactive and connected to real-life scenarios [6].

The emphasis on continuous professional development, highlighted by the students' completion of online courses, resonates with the findings of Driskell et al. [13] regarding the necessity of professional development in the successful integration of educational technologies. The creation of comics and collaborative projects showcases the students' creative and collaborative efforts, supporting the notion that ICT can foster positive interactions and lifelong learning [3, 7] .

The advanced lesson planning by ST7 and ST8, involving Python, aligns with the research suggesting that integrating CT can enhance problem-solving skills and mathematical thinking [8, 9, 10]. This demonstrates the potential of well-designed ICT integration in preparing students for the digital age.

The students' work, in the first cycle, demonstrates a strong engagement with various technological tools and projects, supporting the integration of ICT in mathematics education. Two students reveal more commitment and present lesson planning highlight the potential for enhancing both teaching practices and student outcomes. Further research and continued emphasis on professional development are essential for overcoming the challenges associated with ICT integration [1].

The use of the Technology Acceptance Model (TAM) was not feasible in this study, due to the small number of participants. It should be noted that research conducted with this model indicated that the original framework proposed by Davis in 1987 [49] needed to be expanded with the introduction of Technological Pedagogical Content Knowledge (TPACK) [31]. As the results of the first cycle demonstrate, only two students presented final lesson plans that addressed didactic concerns regarding the use of technology and were sensitive to the cross-cutting theme of CT.

The analysis of these results led to a reformulation of the tasks requested in the CTTM curricular unit, including a mandatory collaborative work, the elaboration of detailed lesson plans, simulations of classes with the entire group, joint analysis of all actions that took place during these classes, and the development of a reflective portfolio in dyads containing both individual and group work. Moreover, this strategy was implemented in all subjects of the second semester of the Master's degree in Teaching Mathematics for Basic and Secondary School Teachers, under the responsibility of the Department of Mathematics.

In fact, the 2nd cycle present substantial differences, the analysis of the portfolios developed by students ST9 to ST12 underscores their integration of technology within the context of established educational theories and practices. This discussion draws upon key frameworks, including Rogers' Diffusion of Innovations theory, the Technological Pedagogical Content Knowledge (TPACK) model, and the concept of instrumental orchestration in technologically enriched mathematics classrooms.

The collaborative efforts of ST9 and ST10 reflect a methodical approach to teaching advanced topics in secondary mathematics, specifically functions and complex numbers. Their lesson plans demonstrate a clear alignment with curriculum standards and a thoughtful justification for the selection of topics. This aligns with Rogers' Diffusion of Innovations theory [27], which highlights the importance of understanding how new educational practices and technologies are adopted within teaching contexts. By incorporating Python programming into their

plans, these students exemplify a commitment to integrating innovative tools into their teaching practices.

The structured progression from simpler functions to more complex concepts illustrates the pedagogical strategies outlined in the TPACK framework [21]. The inclusion of hypothetical learning trajectories indicates a proactive approach to addressing potential student misconceptions, which is essential for effective teaching practice. For instance, ST9 and ST10's focus on the vertex of piecewise functions as a critical learning point aligns with educational models that advocate for anticipatory teaching strategies.

The portfolio of dyads ST9 & ST10 also reflect the principles of instrumental orchestration, particularly the Technical-demo and Guide-and-explain orchestrations. By demonstrating Python techniques and guiding students through complex mathematical concepts, they enhance students' understanding and engagement. This approach is consistent with findings that emphasize the importance of careful planning and arrangement of digital artifacts in teaching environments [37]. Also, The reflective practices displayed in portfolio of dyads ST9 & ST10 further enhance their pedagogical development. Both students engage in critical self-assessment, identifying strengths and areas for improvement. This reflective process is vital, as research indicates that teacher beliefs and attitudes significantly impact the adoption of educational technologies [13]. By emphasizing the alignment of computational goals with mathematical learning objectives, ST9 and ST10 highlight the necessity for clarity and coherence in lesson planning.

The ST11 & ST12 portfolio dyad focus on foundational mathematics topics for 8th-grade students, specifically linear functions and the Pythagorean theorem. The integration of Scratch and GeoGebra into their lesson plans showcases a dynamic approach to teaching, facilitating student engagement through interactive problem-solving. This approach align with the findings of recent studies indicating that engaging with various technologies can enhance pre-service teachers' subject knowledge and attitudes towards technology [19]. The emphasis on collaborative learning strategies in ST11's lesson plans is indicative of a pedagogical philosophy that values active participation. The use of Scratch to visualize linear functions allows students to interact with mathematical concepts in a tangible way, thereby fostering deeper comprehension. This aligns with the educational theories advocating for the integration of technological tools to enhance student learning experiences. Similarly, ST12's focus on the Pythagorean theorem and isometries reflects an innovative pedagogical approach. The incorporation of storytelling in conjunction with Scratch fosters an exploratory learning environment, encouraging critical thinking and problem-solving skills. The application of Socratic dialogue not only promotes student engagement but also aligns with current educational theories that advocate for dialogue-rich classrooms.

The focus on hypothetical learning trajectories in ST11&ST12 dyad portfolio is crucial for identifying and addressing potential student misconceptions. This anticipatory strategy is supported by the literature on teacher professional development, which underscores the importance of preparing educators to facilitate student thinking and reasoning [20]. Their lesson plans also demonstrate the use of various orchestrations such as Discuss-the-screen and Link-screen-board, which facilitate a deeper understanding of mathematical concepts through interactive and visual methods.

Considering the integration of CT, portfolio of ST9&ST10 dyad underscore the significance of CT in mathematics education, albeit through different lenses. The emphasis on algorithmic

processes and systematic problem-solving in ST9 and ST10's work illustrates their understanding of the cognitive processes involved in programming and mathematics. Their approach aligns with the TPACK framework, as they effectively integrate technology to foster students' logical reasoning and problem-solving skills [50]. Conversely, portfolio ST11&ST12 dyad highlight interactive problem-solving and visualization, utilizing tools that encourage students to engage deeply with mathematical principles. Their work exemplifies the key components of CT, including decomposition and pattern recognition, which are essential for mastering complex mathematical concepts. This is consistent with recent findings that suggest integrating technology in teacher education can significantly enhance pre-service teachers' professional knowledge and teaching effectiveness [18].

6 Final Remarks

In comparing the two cycles, the integration of technology and CT in mathematics education revealed both advancements and challenges. The initial cycle demonstrated the effective use of ICT tools such as GeoGebra and Desmos, which enhanced student engagement and performance. Nonetheless, only two students developed lesson plans that adequately addressed the integration of technological and CT, indicating a need for further refinement and professional development. The second cycle showed significant progress, with portfolios reflecting a robust integration of technology through established educational theories, including TPACK and instrumental orchestration. Students adeptly employed Python programming, Scratch, and GeoGebra to teach various mathematics topics, demonstrating comprehensive lesson planning and pedagogical strategies that foster CT. These advancements align with the Portuguese mathematics curricula, underscoring the systematic use of technology and the development of CT as integral components. The second cycle highlighted a more refined approach to technology integration and emphasised the importance of ongoing professional development and reflective practice.

Future research should track students and future teachers through their induction year after completing the CTTM course to analyse their classroom practices. Simulated classes offered valuable insights, yet actual student reactions in mathematics classrooms may differ significantly.

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Applications of GPT and Copilot in Tourism Courses: A Case Study in Chung Hua University

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Abstract: *We utilize artificial intelligence (AI) tools, ChatGPT and Copilot, in one course at the Tourism College, Chung Hua University. These AI tools allow students to achieve their targets through simple instructions and creative prompts. Without working on advanced AI programming languages, students can meet with and even reach success of their performance far beyond their imagination. Utilization of these AI tools in the courses other than science or engineering should be encouraged in such a way that engineering types of learning in various programming languages can be minimized to encourage more students to participate in AI applications. AI-assisted ways to learn mathematics are suggested and should be considered worthy to further study of their effectiveness.*

1. Introduction

In recent years, the development of artificial intelligence (AI) has become popular. With many technologies transitioning from academic research to accessible tools for general usage, we have applied generative AI techniques in education, allowing students to achieve their targets through simple instruction inputs. Upon the new development of these AI techniques, many applications have been generated by using the core of these AI advances to create new services. In modern education communities, how to use the AI tools properly has been discussed extensively worldwide. For the people who are unable to develop AI applications, probably due to the limited training in AI programming, they may find themselves disadvantaged in the job market nowadays. Even though they are not required to have such capability, we believe that anyone from the hospitality industry can develop the capability of AI, no matter if it is from the programming or the direct application, and thus have some definite advantage in the job market. In this report, we describe how Copilot and ChatGPT-3 can be used in a tourism course and students can develop amazing outcomes without prior experience in any advanced Information Science courses. That is one achievement students can benefit from the use of AI tools.

Machine learning (ML) is a field of artificial intelligence where algorithms enable computers to learn from data. ML has two kinds of learning methods, unsupervised and supervised. ML is like teaching a computer to learn. Supervised learning involves providing labeled data so that the computer knows explicitly what is correct and what is incorrect, such as identifying which objects are red and which are green. In contrast, unsupervised learning does not involve labeled data; instead, the computer autonomously discovers patterns within the data. For example, it might group together the words "apple" and "fruit," allowing the computer to identify their characteristics and further uncover relationships and classifications. GTP-1 (Generative Pre-trained Transformer 1) uses semi-supervised learning to train language model with unsupervised pre-training and

supervised fine-tuning [1]; and uses Transformer model architecture to achieve powerful performance on generalization of the shared tasks, such as translation [2]. In other words, GPT-1 attempts to train the model without using supervised learning. To minimize the need for labeled data for fine-tuning in each specific domain and to improve the generalization of the shared tasks, GPT-2 enhances models to achieve these goals. Language models of GPT-2 begin to train models without fine-tuning and supervised learning when trained on a new dataset of millions of web pages. In other words, systems can be developed using data that occur more naturally and can manage a wider range of tasks [3].

GPT-3 is an autoregressive language model with 175 billion parameters, significantly more than GPT-2's 1.5 billion parameters. It scales up language models and improves task diagnostic and few-shot performance. In certain circumstances, the outcomes may be comparable to those achieved through fine-tuning. GPT-3 is utilized without gradient updates or fine-tuning for all tasks to minimize the need for labeled data [4]. By increasing the input size and scaling up the language model—for instance, from being able to input just one sentence to being able to input a hundred sentences—the model naturally gains more context to analyze. This enables it to make better judgments, such as determining how to translate based on the proceeding and the following contexts, leading to improve model performance. GTP-4, a large multimodal model capable of processing both image and text inputs and producing text outputs, is a Transformer-based model pre-trained to predict the next token in a document. It uses the Reinforcement Learning from Human Feedback (RLHF) post-training alignment process results and shows significant improvements over GPT-3.5[5]. We utilize ChatGPT-3 and Copilot, which are applications developed by using GPT-3 and GPT-4. These applications are advancements derived from machine learning algorithms.

Microsoft, in collaboration with OpenAI, published Copilot (Microsoft 365) in 2023. Like ChatGPT, Copilot uses Large Language Models (LLMs) based on GPT-4 and Microsoft Graph, which is an API and SDK to access several data and actions of Microsoft service. It allows users to use commands to generate text content and is capable of directly producing images, music and enables users to quickly create the visual or auditory art [6]. OpenAI published ChatGPT application in November 2022, a chatbot based on the GPT-3.5 model. LLMs can perform a variety of natural language processing (NLP) tasks with just a few examples of the task as inputs. RLHF is added into GPT-3.5, training the language model to follow instructions with human feedback, i.e. fine-tuning with human feedback. The model is called InstructGPT. InstructGPT models are shaped partly by the human feedback collected from contractors, which helps align language models with human intentions [7].

By effectively utilizing AI-enhanced technologies, teachers can unlock new dimensions of teaching and learning, enable a more flexible, interactive, and impactful educational experience for students, and therefore enhance learning outcomes. Celik et al. (2022) suggest that the integration of AI into educational systems could alleviate the pressure faced by teachers and streamline their workload [8]. Chen et al. (2020) explore the beneficial effects of AI on educational processes, highlighting improvements in administrative efficiency, the quality of instruction, and the overall student learning experience [9].

1.1. AI-assisted Courses in Some Universities of Taiwan

So far, there are only few AI courses given in some universities in Taiwan. Quoted from a media news [10], “National Tsing Hua University is the first university in Taiwan to establish a task force dedicated to generative AI. In addition to offering a series of AI general education courses and workshops for all students, the courses of the humanities and social sciences such as history and social cultural analysis, introduced in the new semester, will also integrate generative AI into the teaching. This aims to guide students majoring in Chinese, history, education, law, and management to effectively use AI tools, fostering research and innovation in the new AI era.” In addition, The Center for AI Applications and Development in Humanities and Social Sciences at Tsing Hua University has recorded an online micro-course series titled 'Introduction to AI in Humanities and Social Sciences' to provide foundational AI knowledge for students in these fields. Currently, most AI courses are mostly scheduled in information science-related departments, which still require some programming ability. It is not an easy task for students in the humanities to properly learn AI development. The Tunghai University, a private university in Taichung City, has also provided a Generative AI Resource Network [11]. But they provide information and AI tools in the Education Department only. “However, Tunghai University's mandatory Chinese course for first-year students has undergone reform this year. In addition to teaching students how to ask the right questions to ChatGPT and learn to communicate with AI, the course now incorporates AI text generation content, which has been developed into teaching materials for both teachers and students. It is recommended that the first-year Chinese course should be reformed to integrate traditional writing skills with AI text generation and prompt-based instruction” [12]. Nevertheless, there are still not many courses that are based on the application of AI tools in the current universities in Taiwan. So, more courses developed by using AI tools are still needed for more universities in Taiwan in the future.

1.2. AI-assisted Mathematics Learning

Based on the comment given by GPT, an example of mathematics learning by accessing or cooperating with GPT is outlined in Table 1. Experiments by using these approaches have not been carried out yet, though. However, the fact is that students do not simply reply on the answers given by GPT but can also interactively discuss with GPT. Especially in the subject “Learning Through Teaching” of Table 1, the basic idea can be strengthened by good explanation as if you were an instructor, since GPT can also be used to advise your instructions. It would be interesting to see the real effects by using these approaches. Using GPT in these ways can make math learning more interactive, personalized, and efficient. Due to a lack of opportunities to conduct experiments, the authors strongly recommend young instructors to try working them out, especially in elementary mathematics. The authors believe the experiments can also be conducted in higher education, such as calculus and statistics. Currently AI applications are mostly applied in the courses of programming languages, because GPT can supply code samples in different programming languages. We believe that the success in AI-assisted programming languages courses can provide strong support for other AI-assisted courses, such as mathematics.

Currently, we just apply one “vision inspection” function in Copilot to a mathematics problem (as shown in Figure 1) in the current course. The Copilot is capable of inspecting figures and supplying correct answers.

$$\int_1^3 x^2\,dx$$

Figure 1. A picture to be inspected by Copilot.

Explanation of the process can be given during the course, and it can be used to explain to students the process of problem solving. The use of GPT in mathematics teaching is possible and is strongly recommend for researchers to further explore.

2. Methodology

2.1. Integrating AI Tools into Teaching Practice

This research is based on several assignments in a course titled "Smart Tourism and AI Applications" given at the College of Tourism, Chung-Hua University. The first assignment is a travel proposal for a three-day and two-night trip in Kumano Kodo; the second one is a storyboard for a smart phone; the third one is a video production based on their storyboards; and the last one is a song composing. Among these assignments, students are taught to use Copilot and ChatGPT to draft stories, make photos with proper prompts and "write" their text content of some applications. A midterm was given to make sure that students understood and can apply these functions taught in the course. The outcome is amazingly good. They know what we have gone through during the course. It is encouraging for the instructor to further explore how they manage to use these AI tools. In the following sections, we outline these major assignments by three scenarios and present partial outcomes from the students. Though the outcomes are not uniformly well among all students, there are exceptional good reports that are beyond the instructor's imagination.

2.2. Three Scenarios in the Course

In this report, we outline three scenarios in the assignments, including the purposes of the assignments and the AI-assisted outcomes from students.

2.2.1. Three days and two nights in Kumano Kodo

Based on the answer from ChatGPT, Kumano Kodo is described as follows:

> The Kumano Kodo is a network of ancient pilgrimage routes that wind through the Kii Peninsula in Japan, connecting several sacred sites. These routes have been used for over a thousand years by pilgrims seeking spiritual rejuvenation, and they remain a significant part of Japan's cultural and religious heritage.

It is not easy to know a specific destination thoroughly for a tourist or tour organizer; even though they are specialists, they may not describe the destination so precisely. This is where ChatGPT is useful. We asked students to search for how to describe the destination by accessing these two tools; one is Copilot, and the other is ChatGPT. Since Copilot also cooperates with

ChatGPT, when the data is beyond its accessible range, i.e., after the year 2022, it still can provide the capability of web browsing, which is an original function of Microsoft, making the results given by Copilot usually favorable. In this case, however, ChatGPT provides a better outcome. Actually, a tour proposal is a specialized document that must include several issues, but when the students are not aware of these requirements, AI becomes necessary. In this case, students also asked for assistance from ChatGPT and Copilot. Still, ChatGPT offers a better solution. This allows students to fill in all the necessary materials into the proposal so that a good preliminary proposal can be generated. In addition to the text generation in the proposal, another critical issue is attaching some photos to describe the experience in some specific locations. However, since most photos on the internet are copyrighted, in order to avoid violating copyrights, students were asked to supply photos generated by Copilot or ChatGPT to make the proposal look more professional.

In the end, students do not favor the designated location by the instructor, such as Kumano Kodo, which is initially assigned. Eventually, the choice of location is open to their own preference. In the next section, the travel proposal will be described by their distinctive design photos created with their own prompts. As for the content, since it is written in Chinese, details are not provided here.

2.2.2. A Storyboard for a smartphone

Based on the answer given by ChatGPT, the meaning of a storyboard is given as follows:

> A storyboard is a visual representation of a story or sequence of events, typically used in filmmaking, video production, animation, and other multimedia projects. It consists of a series of drawings or images arranged in sequence, often with accompanying notes or descriptions, which outline the key scenes or moments in a project. Storyboards are used to plan and visualize the flow of a story before the actual production begins, helping to organize ideas, set the pacing, and anticipate challenges.

For students to make a video about a smartphone, they must provide a compelling viewpoint to attract users. Usually, to provide such meaningful information, users must have several years of experience to be able to create such a storyboard. But ChatGPT now can be one resolution to provide a variety of information to make interesting storyboards. Once again, since the sequential events are usually demonstrated in the form of sketches, students are encouraged to use the techniques they learned in photo-making with different prompts to illustrate these sequential events. The outcomes are also impressive. One sample storyboard produced by students is given in Table 2.

2.2.3. A video production based on students' storyboard

To make a video based on students' storyboards, several AI tools are suggested for them to evaluate. The development of these tools has been so progressively fast by the time I introduced only some possibilities to students, many new tools were rapidly available.

Creating AI-generated videos involves a combination of tools that use artificial intelligence for various aspects of video production, such as generating visuals, editing, animating, and even creating voiceovers. Here is a list of popular tools you can use to make AI-generated videos:

- Synthesia:[13]
  - It allows you to create AI-generated videos with avatars that can speak in multiple languages. It is widely used for creating educational content, marketing videos, and training materials.
- Pictory:[14]
  - An AI-powered tool that helps you transform text content into engaging videos. You can input scripts or blogs, and Pictory generates video content with stock footage, music, and voiceovers.
- Lumen5:[15]
  - An AI-driven video creation platform that converts text content into social media videos. It uses AI to match content with relevant visuals and music.
- Runway ML:[16]
  - It offers various AI tools for video editing, including background removal, frame interpolation, and generating new video content. It is a versatile tool for both creative professionals and beginners.

Based on the recommendation, students may make their own choices to achieve favorable outcomes. In the past, it is impossible for students to accomplish such tasks without AI tools, but now ChatGPT and Copilot make these tasks possible. Even though the outcomes may not be directly useful in their future professional career, it is still good training for them to make a video with minimal effort. The use of AI tools is strongly recommended for students.

2.2.4. A song composing

At the end of the course, the instructor encourages students to make a song which could potentially be edited into a video with proper lyrics. That makes the video look more professional. The process of composing a song was very difficult for anyone in the past, but now, by SUNO [17], if you can provide the proper prompt, then a song with unique style and lyrics can be generated. The songs are even exceptionally good to listen to. It is almost impossible to distinguish between a real song and an AI song. It is truly amazing. Since presenting their songs in a paper format is too difficult, the outcomes are omitted here.

3. Main Results

3.1. Presentation of the Experimental Results

In the previous section, we outlined three main scenarios for the course. The following are the students' outcomes. The works presented here are those among the best. Due to the uneven effort put in by the student, those who work harder usually produced better outcomes.

3.1.1. A travel proposal

We only show part of the outcomes, especially the description of specific locations with their AI-designed photos; thus, apparently avoiding copyright issues is the key element of the

assignment. Overall, they may form convincing materials for the customers to imagine where they are going to travel if they will attend the tour. See Figures 2 and 3.

3.1.2. A Storyboard

In this assignment, students will create a storyboard to promote a smartphone. As everyone knows, when considering the functionality of a smartphone, you expect it to be fashionable, waterproof, unbreakable, with autofocus functionality, with high resolution and a great number of pixels, and suitable for all kinds of environments, etc. Of course, it is not easy for a student to think of so many advantages, but now AI can help you in this aspect by setting up these scenarios. Based on this capability, a storyboard can be created, and AI-designed photo functions can assist users in making the drawings. The following are those good outcomes from students. See Figures 4-9, which include the works of three students.

3.1.3. A video making for a top-phone

Based on the students' storyboards, they will try out all kinds of possibilities with the available AI video tools, as mentioned in the previous section. Unexpectedly, they have also developed wonderful works for their assignment. The power of these AI video tools indeed opens their mind in this aspect. Without knowing too much about making commercial videos, they can make a preliminary version of a commercial video, and it is something they can use when they get into the job market in the future. It is a plus for their career. The following is a storyboard content, which includes eight scenes. See Table 2.

Since it is a video format, only some snapshots of the video are attached here. See Figures 10-13. The videos were created by the power of Pictory [14] and Genmo [18] from the works of two students.

4. Conclusion and Recommendation

AI-assisted tools, such as Copilot and ChatGPT, are so powerful that students do not need to have preliminary education for background knowledge; they can still get a basic understanding by accessing AI tools. No matter what the subject is, such as text generation, language translation, photo making, video production, or even song composing, AI can provide satisfactory outcomes. The outlined experiments, including mathematics learning, provide opportunities for teachers to further explore more AI-assisted course developments. This is just the beginning of the AI era; we will have more applications generated or supported by AI capabilities. Educators should be ready for this tremendous impact from the wave of AI. It is not harmful, but we should embrace the challenge. Instead of relying solely on the traditional teaching methodology on the blackboard, educators should seriously consider how to provide better education to students. Nowadays, the various scenarios enabled by AI tools can encourage instructors to apply and prepare good materials for students. It is just the beginning for this new way of education by AI tools.

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Figure 2. Water Puppetry: A traditional art form in Vietnam, where puppeteers manipulate puppets in water to perform folk tales and customs, accompanied by live music and singing from a band and vocalists.	**Figure 3.** Food recommendations in Hanoi: Phở (noodle soup), spring rolls, coffee, grilled fish, fried spring rolls, and grilled bánh mì. The cuisine is diverse and affordable (a meal costs around 50,000 VND, approximately 2 USD).

Figure 4. Narrator: Introducing the top-phone—your everyday hero.	**Figure 5.** Scene 6: At a sports field, an athlete uses the top-phone to time and play music. Narrator: Empowering your healthy lifestyle.

Figure 6. Scene 2: Cut to a traveler on a beach at sunset, using the high-resolution camera of the top-phone to capture stunning scenery and instantly share it on social media.	**Figure 7.** Ending: All the characters gather at a rooftop party in the city, each using the top-phone to enjoy music, photos, and social interactions. The camera pulls back, and the top-phone logo shines brightly in the night sky.

Figure 8.
Scene 3 Auto-focus: She captures a fast-moving taxi, and the Top-phone's auto-focus feature ensures a clear photo.

Figure 9.
Ending: "Top-phone"—your photography partner. Whether it is a sunset, a mountaintop, or a nighttime cityscape, it captures every stunning moment. Waterproof, drop-resistant, and with automatic focus—this is the Top-phone!

Figure 10. Scene 1: "City Skyline" from the first video.

Figure 11. Scene 5: "Night Street" from the first video.

Figure 12. Scene: showing AR capability.	**Figure 13.** Scene: Showing 5 G-level high speed.

Table 1: Instruction to learn mathematics by using GPT.

1. Step-by-Step Problem Solving
• **Practice Problems:** Ask GPT to solve specific math problems step by step. For example, you can input a problem like, "How do you solve 2x+3=72x + 3 = 72x+3=7?" and GPT can walk you through the process. • **Verification:** After attempting a problem yourself, ask GPT to verify your solution or correct any mistakes.
2. Concept Explanations
• **Understanding Fundamentals:** Request explanations of core math concepts, like "What is a derivative?" or "Can you explain the Pythagorean theorem?" GPT can provide clear, concise explanations. • **Analogies and Examples:** Ask GPT to explain concepts with analogies or through multiple examples. For instance, "Can you explain logarithms with a real-world analogy?"
3. Interactive Learning
• **Quiz Yourself:** Use GPT to create quizzes on specific topics, such as algebra or calculus, to test your understanding. • **Challenge Problems:** Request increasingly difficult problems on a particular topic to push your understanding further.
4. Visual Aids
• **Graphing and Visualization:** While GPT can describe graphs and functions, combining it with tools like Desmos or GeoGebra can enhance your learning through visual representation.
5. Learning Through Teaching
• **Teach Back:** Explain a concept to GPT as if you are teaching it. Ask for feedback on your explanation or if any improvements can be made. • **Ask for Teaching Techniques:** You can ask GPT how to teach a concept to someone else, reinforcing your own understanding.
6. Customized Study Plans
• **Personalized Schedules:** Ask GPT to help you create a study plan tailored to your current level and goals, breaking down topics into manageable segments. • **Identify Weak Areas:** Request assistance in identifying your weak points by discussing what topics you struggle with, and then focus on those areas.
7. Discussion of Advanced Topics
• **Deeper Exploration:** If you are interested in advanced math topics like abstract algebra, topology, or advanced calculus, GPT can provide explanations, references, and problem sets to deepen your understanding.

• **Connections Between Topics:** Ask GPT to show how different areas of mathematics connect, like how algebra relates to geometry or how calculus is used in physics.
8. Use of Additional Resources
• **Resource Recommendations:** Ask GPT for recommendations on textbooks, websites, or online courses to complement your learning. • **Reviewing Textbook Problems:** You can also use GPT to discuss problems from textbooks or other resources you are using.
9. Interactive Coding and Math
• **Programming and Math:** For subjects like statistics or calculus, ask GPT how to implement problems using Python or another programming language, which can reinforce your understanding through coding.
10. Clarifying Doubts
• **Quick Doubt Resolution:** Whenever you have a doubt, ask GPT to clarify it immediately, helping you keep the flow of study uninterrupted.

Table 2: A storyboard with Scenes and Narrator for the smart phone "Top-Phone"

A Storyboard Content	
Scene 1: City Skyline: The camera slowly zooms in, showcasing the bustling urban life.	Narrator: In this fast-paced world, you need a phone that can keep up with you.
Scene 2: Close-up of top-phone: The camera switches to a close-up of the top-phone's high-definition display, showing the time and notifications.	Narrator: Introducing the top-phone—your everyday hero.
Scene 3: Office: A professional is using the top-phone to complete work.	Narrator: Equipped with the latest processor to ensure uninterrupted high-performance work.
Scene 4: Café: A young person is using the top-phone for a video call in a café.	Narrator: Connecting with loved ones, no matter the distance.
Scene 5: Night Street: The camera showcases the top-phone's night photography feature, capturing the starry sky.	Narrator: Night mode photography, capturing every shining moment.
Scene 6: Sports Field: An athlete uses the top-phone for timing and playing music.	Narrator: Empowering your healthy lifestyle.
Scene 7: Family Living Room: A family is using the top-phone to watch a movie in the living room.	Narrator: Sharing entertainment time with family, capturing the beautiful moments of life.
Scene 8: Ending Logo: The camera returns to the top-phone, displaying the brand logo.	Narrator: "Top phone," with you every step of the way.

Integrating 3D Modelling into STEAM Education

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Abstract: *3D printers appear at all types of schools more and more frequently. To exploit them effectively, educators must become familiar with their appropriate application in courses. The authors participate in an international ERASMUS+ project that intends to assist future (pre-service) teachers in planning and carrying out their future teaching activities with 3D modelling through a STEAM-based transdisciplinary approach.*
The main aim of our project is to design, develop, and implement appropriate teaching and learning practices using 3D modelling. In order to follow contemporary trends in education, we prefer and promote the learning-to-learn approach based on "learning to do" tasks. They address 3D modelling in order to enhance learners' creativity. To enhance our students' motivation, we invite them to design and develop subjects for their needs.
A MOOC is a part of the project to support both teacher training and school-based STEAM learning. It aims to facilitate instant and worldwide access to STEAM learning. In particular, we pay attention to and respect the specifics of partner countries' curricula and teacher training. The tasks and activities address both in-class and online courses for future teachers. Our paper contains selected examples.
As our long-term aim is to concentrate on transdisciplinary education, we develop not only these activities. Moreover, we create a repository of high-quality teaching and learning resources as well as their evaluation criteria. The dissemination of project results is also premeditated.

1. Introduction

Today's fast technological advancement and constantly changing market needs result in pressing calls for flexible education systems that can provide students with the skills needed to succeed in the uncertain world they face. The progress in Science, Technology, Engineering, Arts, and Mathematics is remarkable and must be reflected in the preparation of people for their jobs. STEAM education is accepted as a correct response to these challenges [1]. Leavy et al. [2] demonstrate the extensive growth of STEAM-based educational approaches and cases. The introduction of relevant content into classrooms must be quick but stepwise, long-lasting, and reflect incoming trends.

One can learn from the current situation in Information Science and its business applications. Cedefop's EU skills forecasting model indicates that about 87% of all EU jobs needed at least a basic digital skills level in 2021 [3]. During the same period, the demand for specific skills grows even faster: GreyViews [4] predicts the growth of the 3D printing industry from 11,5 billion USD in 2020 to 47,5 billion USD in 2028. One can estimate an analog expansion of jobs requiring 3D-related printing knowledge and skills.

Schools and universities have to react to these challenges as soon as possible. The authors have extensive experience introducing innovative content related to programming and information sciences during their careers.

The appearance of programming languages resulted in mini-languages [5]. The introduction of spreadsheets resulted into their incorporation into relevant courses [6].

Often, the initiative comes from departments of Mathematics because various advanced information science courses exploit mathematics as their background [7]. Recently, Artificial Intelligence has served as an example of course innovation [8].

This trend is unstoppable. The current booming onset of 3D printers has resulted in their more and more frequent appearance in all types of schools. To exploit them effectively, educators must become familiar with their appropriate application in classes. The authors are members of an international team collaborating on an ERASMUS+ project which assists future (pre-service) teachers in planning and carrying out teaching activities with 3D modelling through a STEAM-based transdisciplinary approach. Thus, our global aim is to design and develop university teacher training courses that facilitate the 3D modelling skills of pre-service teachers with various technologies and to reach a wider audience and students through such approaches. In addition, they support teacher training in the participating countries and beyond.

Complex solutions require:

- To design, develop, and implement teaching approaches with 3D modelling involving a variety of technologies and STEAM-based transdisciplinary approaches for future teachers;
- To promote the learning-to-learn approach by "learning to do" tasks enhancing creativity and production of 3D models among learners;
- To develop a MOOC to support both teacher training and school-based STEAM learning;
- To facilitate direct access to 3D technology for all interested;
- To support international cooperation in integrating STEAM approaches into their curricula and teacher training.

To achieve optimum results, we have aimed to motivate particular groups of students and be attractive to them and their neighborhoods [9]. In our project, we develop tasks and activities for both in-class and online courses for future teachers related to 3D modelling and STEAM-based transdisciplinary education, exploiting as much from the existing curricula as possible. Then, we evaluate all project activities and resources, train pre-service teachers in each participating country, create a repository of high-quality teaching and learning resources, develop a MOOC for 3D modelling teacher education, and disseminate project results to a wider audience promoting 3D modelling and STEAM education. Finally, we plan to generalize our experiences and formulate a series of proposals for all levels of the educational hierarchy.

In our presentation, we will exemplify the issues and present our solutions.

2. STEAM Education as a Complex Task

In general, the suggested approach to changes in education consists of four stages [10]: First, the constellation of forces leading to the development of 3D-printing-based education is discussed. Second, its state-of-the-art and potential links to educational content are discussed. Third, the appropriate elements of 3D-modelling and printing are incorporated into courses in a way that will depart substantially from patterns characteristic of traditional education. Fourth, the implications for educators, educational researchers, policymakers, and educators are specified.

In the paper, we concentrate on the first two stages: finding optimal areas for introducing 3D modelling and printing in the current curricula and then to enriching them with relevant practical exercises. Simultaneously, exploiting 3D allows us to produce real-life objects and, as such, to enhance the visual side of learning and demonstrate links between theoretical aspects of study content and its application. This means that 3D printers give us a unique opportunity to integrate STEAM approaches and comprehend intricate relationships among individual courses.

At the same time, the course developers are supposed to possess identical qualities, i.e., to make the complexity understandable. That is why our educational strategy begins with the "direct and instant" application of 3D modelling and printing. Below, cross sections of solids serve as an example: the theoretical backgrounds are part of the geometry course; the only novelty is their transfer to the instructions for the printers.

Then, the students are given more freedom. In a stepwise manner, they should proceed to implement their ideas and knowledge to formulate more complex tasks, find their solutions, and come up with their outcomes. From a didactical point of view, this is the most difficult part of the transfer. The team members exploit their previous experience with designing and implementing problems, demonstrating the relationships between Mathematics and its four partners: S, T, E, and A. In our interpretation, the letter E in STEAM stands for Engineering and Economics because students' financial literacy also belongs to our goals.

Therefore, the project is built on the authors' previous experiences with active learning such as:

- Alternative solutions to the same problem [11] by exploring different approaches to solving the same mathematical problem and their contribution to students' understanding and fostering creativity.
- Brute-force solutions [12] demonstrating the application of brute-force methods combined with IT tools to tackle complex mathematical challenges, enhancing students' problem-solving skills.
- Exploitation of spreadsheets for finding large sets of solutions [13] showed how spreadsheets can be effectively used to explore multiple solutions to mathematical problems, thereby integrating computational thinking into the learning process.
- Teaching Mathematics as a managerial task [14], where the management analogy was used to make mathematics more engaging for students, emphasizing the role of organizational skills in education.
- Recognizing non-explicit knowledge in Mathematics [15] focused on how implicit knowledge in mathematics can be recognized and managed to improve educational outcomes.
- Integrating Augmented Reality in Constructional Geometry Lectures [16] explored the use of augmented reality to enhance the teaching of constructional geometry, providing students with a more interactive and immersive learning experience.
- Adapting Geometry Education During the Pandemic [17] by discussing the challenges and solutions for teaching geometry in a remote learning environment, with a focus on maintaining educational quality during the pandemic.
- Creating tools in GeoGebra software [18] and highlighting the development of custom tools within GeoGebra to facilitate the teaching and learning of complex mathematical concepts, supporting the integration of technology into the classroom.
- Utilizing GeoGebra tools for Teaching Monge Projection [19], which focused on the application of GeoGebra in this essential topic in descriptive geometry.
- Developing educational materials for the Monge projection using GeoGebra [20] aims to improve students' understanding through interactive and dynamic materials.

Usually, when discussing mathematical problem solving, Mathematics is primarily connected with one application area, i.e., the relationship "Mathematics: Application Area" can be described as 1:1. The main task of our research project will be to design problems with 1:M or M:N relationships, i.e., to solve problems which require not only application of mathematics but also various interconnections among activities which (formally) belong to different fields of knowledge and are taught in different subjects. An example of such a complex relationship, its formulation, and its solution is described in our paper [21].

3. Trends in teaching Mathematics in Czechia and Slovakia

Mathematics education in the Czech Republic and the Slovak Republic is designed to be comprehensive and interconnected with other areas of study, particularly within the STEAM (Science, Technology, Engineering, Arts, and Mathematics) framework. Mathematics teaching places a significant emphasis on interdisciplinary links, ensuring that students can see the relevance of mathematical concepts in various contexts and subjects. Recent revisions in the educational framework also highlight the importance of digitalization, incorporating modern technologies such as 3D printing to enhance the learning experience and develop students' digital competencies.

3.1. Basic Principles of Mathematics Education

1. *Development of Mathematical Literacy:* Mathematics education aims to develop students' ability to understand and use mathematical concepts and skills in everyday life; it includes solving mathematical problems and the ability to think logically, reason, and work with data.
2. *Progressive Complexity:* Mathematics education is structured to be gradual and adapted to the age and abilities of the students. In primary school, it starts with simple arithmetic operations and gradually progresses to more complex topics such as algebra, geometry, and statistics in secondary school.
3. *Application and Interactivity:* Mathematics is not taught solely theoretically; there is an emphasis on practical applications and interactive teaching methods. They include using modern technologies, group work, and projects that connect mathematics with the real world.
4. *Individual Approach:* Given students' different abilities and needs, an individual approach is important; it includes differentiated instruction and support for students with difficulties and gifted students.

3.2. Curicullum Documents

Several key documents govern mathematics education. The main curriculum document is the Framework Educational Program (RVP), which sets the educational goals, content, and outcomes for different school levels. For primary and secondary schools, specific programs are RVP ZV (for primary education) and RVP G (for grammar schools), detailing the expected outcomes and curriculum for mathematics. Based on the RVP, each school creates its own School Educational Program (ŠVP), which concretizes and adapts the educational content to the needs and conditions of the school. The ŠVP provides a detailed description of how mathematics education will be implemented in a specific school.

Additionally, the ministries of education issue various methodological guidelines and recommendations that offer teachers support and instructions for effective mathematics teaching. These documents may include recommendations for using modern technologies in education or working with heterogeneous student groups. The primary sources of information on educational programs and methodological guidelines are the official websites of the MŠMT and the National Pedagogical Institute of the Czech Republic, where these documents are publicly available.

In the next sections, we describe how 3D modelling and printing can contribute to fulfilling these aims.

3.3. Individual Assignments: Using 3D Printing in Geometry Education – Activity on Cross Sections of Solids

Three-dimensional geometry is rich in problems that can be "completed" using 3D printing. These problems mostly belong to the 1:1 category, as specified above, because no other school subject needs to be involved for their complete solution. Such tasks can be done individually or by pairs of learners because they test narrow knowledge: mathematics and 3D printing skills.

Teaching the concept of cross sections of solids can be challenging, as students often struggle to visualize and understand 3D objects and what a planar section of these objects means. Traditionally, this topic begins with simple sketches of a cube in a notebook using free parallel projection to represent a 3D object on a 2D plane. However, this method can leave students without a clear understanding of the 3D object and the practical implications of a planar section.

To enhance students' comprehension of cross sections of solids, we implemented an innovative activity using 3D modelling and printing to provide a hands-on, interactive learning experience. This pilot study was conducted during the winter term of the 2023/2024 academic year, involving seventeen second-year bachelor's degree students specializing in teaching for the second grade of elementary schools at the University of Ostrava.

The activity began with an introduction to 3D modelling, where students were guided through creating a solid, such as a cube, using GeoGebra 3D software. Each student ensured their model was accurate and then exported it as a .stl file for further processing. This initial step aimed to bridge the gap between theoretical understanding and practical visualization, as students often find it challenging to conceptualize 3D objects from 2D drawings.

Following the modelling phase, students were introduced to the PrusaSlicer software. They learned how to import their .stl files into PrusaSlicer and slice the solid with a plane. This step allowed students to set the slicing plane at any angle and position, making it simple to adjust the plane and observe the resulting cross sections. The ease of this process, requiring just a few clicks, encouraged students' motivation to explore and experiment.

After preparing the sliced models, students exported the files as .gcode and printed them using a 3D printer. This hands-on approach enabled students to physically observe and manipulate the sliced solids, providing a tangible understanding of how a plane intersects a 3D object. By manipulating the printed sliced models of the cube, students can discover basic principles that will later help them solve tasks in parallel projection. For example, they will understand that the intersection of two parallel planes with a third plane that is not parallel to them is the intersection of two parallel lines. This experiential learning method significantly enhanced their spatial reasoning and visualization skills.

Students worked in pairs to further deepen their understanding and explore different slicing angles and positions. They were provided with research questions to guide their exploration, such as determining the shapes that may result from slicing a cube with a plane, creating slices that form regular polygons, and identifying special cases. Through practical experimentation, students discovered that slicing a cube could result in various polygons, including triangles, quadrilaterals, pentagons, and hexagons. They documented their findings and shared their results with the class, fostering collaborative learning and discussion.

This activity aligns with the STEAM approach by integrating technology and engineering principles with traditional mathematics education. 3D modelling software and 3D printing technology encourage students to engage in engineering design processes, fostering creativity and problem-solving skills [22]. By incorporating artistic elements in the design and visualization of geometric shapes, the activity also touches on the art component of STEAM, promoting a holistic understanding of complex concepts [23].

The benefits of this activity extend beyond understanding cross sections. It supports students in grasping geometric relationships and spatial configurations, which are essential for solving positional tasks in space [24]. It also improves their comprehension of 3D objects' representation in 2D, which is fundamental for fields such as engineering, architecture, and computer graphics [25]. Furthermore, developing spatial reasoning and visualization skills is critical for success in STEM fields and is linked to enhanced problem-solving abilities and academic performance in mathematics and science [26].

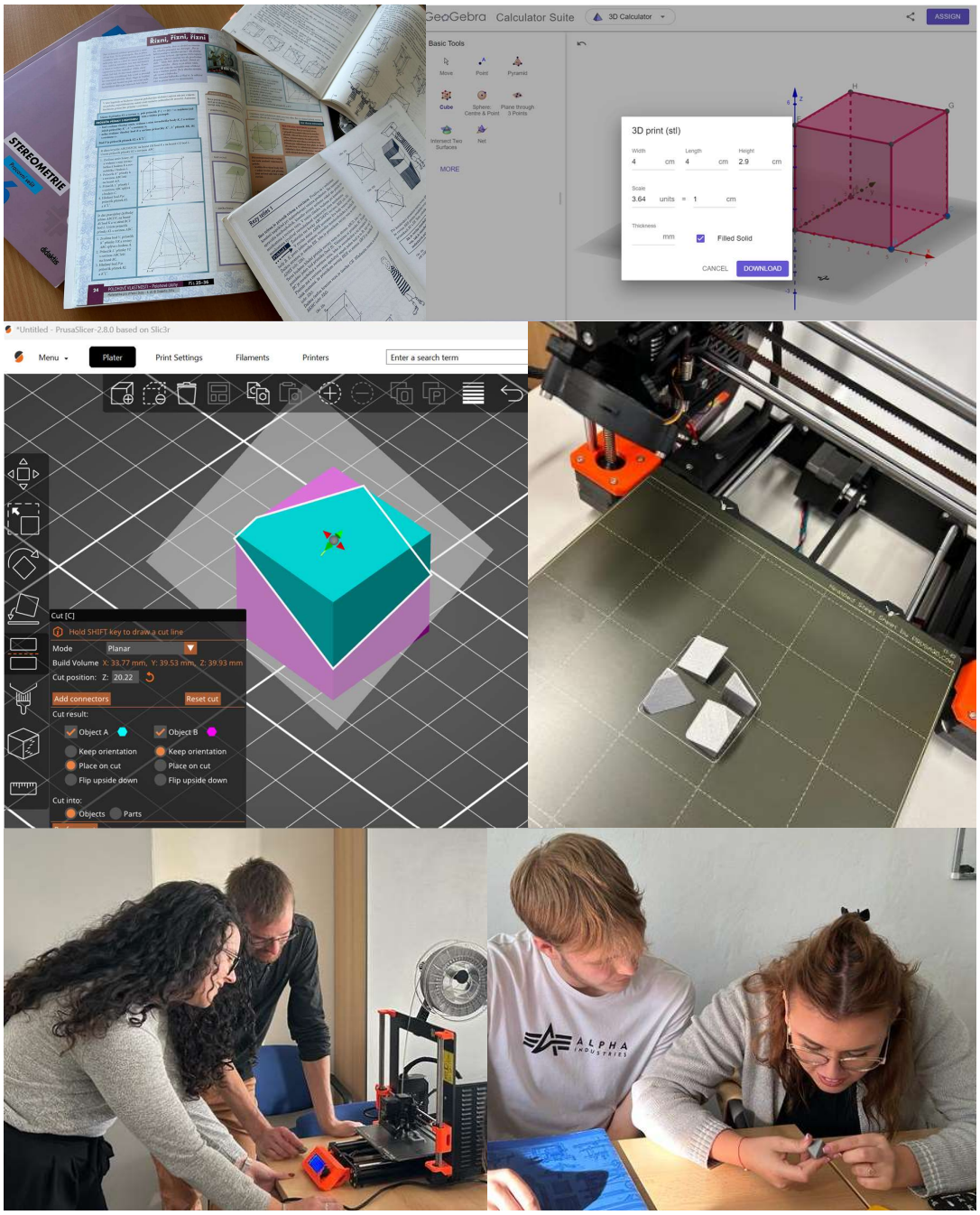

Figure 1 3D modelling and printing activities on Cross Sections of Solids

3.4. Assignments for Pairs: Creative Tangram Design Using 3D Printing in Geometry Education

Tangram is an ancient Chinese puzzle with seven geometric shapes known as "tans." These shapes – triangles, squares, and parallelograms – can be arranged into various patterns and images. Using Tangrams in education has been widely recognized for its ability to develop spatial imagination, geometric reasoning, and creative problem-solving skills. Research has shown that working with Tangram pieces helps students improve their understanding of geometric shapes, relationships, and patterns while also fostering their ability to solve geometric problems through hands-on activities [27].

In the field of geometry education, Tangrams are a valuable tool for teaching geometric transformations and helping students discover the relationships between polygons. By manipulating the pieces, students enhance their logical thinking and strengthen their understanding of geometric concepts [28].

In this activity designed for future mathematics teachers, students are tasked with creating their own variation of a Tangram puzzle using TinkerCAD software and 3D printing technology. This process combines creativity with technology, allowing students to gain experience in 3D modelling while also developing their digital competencies [29]. Students begin by designing unique Tangrams in shapes such as hearts, eggs, or other creative forms. TinkerCAD, an intuitive software, enables students to model various polygonal shapes that can later serve as educational tools in their teaching practice.

After completing the design, students export their models as .stl files and prepare them for 3D printing using PrusaSlicer software. This hands-on approach allows students to experience the full process from design to final realization. Once the models are printed, students can work in pairs, exchanging Tangrams and attempting to assemble the provided puzzle, or they may collaborate from the very beginning by jointly designing their Tangram. This approach promotes early-stage teamwork, encouraging idea-sharing and cooperative problem-solving, which enhances collaboration, analytical thinking, and creativity throughout the process.

We suggest using this activity as a motivational task before beginning lessons on planimetry. When students assemble Tangram puzzles, they engage with various types of polygons in a fun and interactive way. They can explore metrical relationships, such as internal angle measurements and positional relationships, by connecting different types of polygons to form larger shapes. The complexity of the task can be gradually increased, for example, by asking students to create Tangrams in the shape of a heart or an egg, where part of the polygon transforms into a circular sector.

An important aspect of this activity is its alignment with the STEAM approach. By integrating creativity, technology, and geometry, this activity encourages students to engage in artistic design while solving mathematical problems. 3D printing technology fosters engineering skills, while the Tangram pieces' design and assembly engage students in scientific inquiry and creative exploration. This interdisciplinary approach promotes a holistic understanding of complex concepts, helping students see the connections between different fields of study. The focus on hands-on learning, problem-solving, and creative expression makes this activity an excellent example of the STEAM framework in action.

The intuitive connection between 2D and 3D space is another essential aspect of this activity. While modelling, students work with different types of polygons in a 3D environment, such as polyhedra (prisms with polygonal bases). This process gives them practical experience that enhances their understanding of more complex geometric concepts, preparing them for applying these skills in their future teaching practice.

This activity not only encourages creative problem-solving but also supports the development of digital literacy and geometric thinking – key skills for modern mathematics

teachers. Using 3D printing and modelling in creating Tangrams allows students to better understand the relationship between 2D and 3D space while deepening their geometric knowledge and skills. By aligning with the STEAM approach, the activity allows students to integrate creative design, technological tools, and mathematical reasoning, making it an effective and engaging educational strategy. This approach is suitable for implementation at various educational levels, where students can discover new geometric concepts through hands-on exploration.

3.5. Group Assignments: 3D Modelling and Printing – a Base for Collaboration

Contemporary educational methodologies underline the need for multifaceted tasks that an individual can hardly complete. 3D modelling and printing offer an excellent solution for such problems. In the above mathematical problems, the printouts have been directly specified by their mathematical properties. At the same time, real-life objects are rarely regular, and formulas do not define their shapes. As a result, several specialists must be invited to complete them, and the solution is made in stages [21].

Our example task is to create medals for a large school sports competition. First, it is transformed into a group project. The student team(s) are asked to resolve it using 3D printers. The medal's look and size are exclusively upon the group's decision. The outcome requires role distribution, planning, and collaboration. At least five roles have to be established: Mathematician (all needed numerical calculations), Scientist (selection of material with required qualities), Economist (financial planning and cost calculations), Artist (design of medals), Engineer (3D printing).

There is an optional plan of work:

i. The manager must learn the number of medals and other requirements of event organizers.
ii. The medal designer then prepares and proposes several sketches for approval by the sports event's organizing committee.
iii. The technologist analyzes the feasibility of producing the selected pattern and specifies the most appropriate material, medal dimensions, weight, durability, and other technology factors.
iv. They choose the filament and its amount for each of the three medal types, and the manager must ensure its delivery.
v. The medal design is transformed to be prepared for 3D printing.
vi. The first samples are shown as the pattern to the organizers for final approval. After their consent, the requested number of medals is printed out.

Task division is part of the problem solution and should be done by the students. Notice that the mathematician is not directly mentioned in any of the above subtasks. Nevertheless, without him/her, the whole problem would not be solvable.

4. Conclusion

The above examples of individual, pair, and group tasks indicate a high potential for 3D modelling and printing in education. These experiments were done at the university level, but other experiences indicate that STEM-oriented teaching can be successfully completed at elementary schools, too [30].

Our observations, as well as positive feedback from the questionnaires, suggest that actively incorporating 3D modelling and printing into the classroom increased student motivation and engagement in exploring spatial relationships. This practical experience of solving real problems in a tangible space prevented potential misunderstandings in the interpretation of 3D objects, which are often represented in 2D tasks. Thanks to the printed

models of positional problems, students could verify their existing theoretical knowledge while also discovering new concepts. Using tools such as GeoGebra, TinkerCAD, and PrusaSlicer, they could increase their digital competence, deepen their understanding of geometric relationships, and improve their spatial imagination and reasoning. Such an approach, rooted in the STEAM framework, proved beneficial in fostering creativity, problem-solving skills, and collaborative learning, indicating its wider applicability in other fields of education.

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