From Graphs for the Mind to Graphs for the Eye: Students' Coordination of Quantitative and Spatial Coordinate Systems

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Abstract:
Graphs are nearly ubiquitous in modern society and provide versatile representations of a wide range of real-world phenomena. Depending on their purpose, graphs may rely on different underlying coordinate systems, including quantitative and spatial coordinate systems. Many students, however, interpret graphs using coordinate systems that are inappropriate for the meanings those graphs are intended to convey, highlighting the importance of supporting them in coordinating these different coordinate systems. This study explores how students coordinate these two coordinate systems when constructing and justifying graphical representations of projectile motion. We present a case study of a ninth-grade student working on a task in which she constructed a ball's trajectory from graphs of its height and horizontal distance over time. The findings revealed two ways in which the student coordinated quantitative and spatial coordinate systems. First, she reasoned about the ball's position in the spatial coordinate system at particular moments in time. Second, she coordinated the two coordinate systems by engaging in multivariational reasoning to explain how the ball's height and horizontal distance varied with the common quantity of time as she justified the trajectory she had constructed. We discuss how these two forms of coordination supported the development of meanings for parametric points and meanings consistent with the idea of parametric functions, and conclude by considering implications for research and the design of instructional tasks.

1. Introduction

Graphs are nearly ubiquitous in modern society, appearing everywhere from mass media to scientific publications. They play a central role in communicating information about relationships among quantities and supporting reasoning about real-world phenomena. For example, in discussions of climate change, graphs are commonly used to represent changes in Earth's surface temperature and sea level over time [1–3]. In physics, however, graphs serve multiple representational purposes. Some graphs depict actual physical space, such as the trajectory of a projectile. In contrast, the same phenomenon can also be represented using graphs that do not depict physical space but instead express relationships among varying quantities in a new representational space, such as a projectile's height versus time or horizontal distance versus time. These two representational uses of graphs have been described as graphs for the eye and graphs for the mind, respectively [4].

The versatility of graphs enables them to represent a wide range of phenomena. This versatility, however, can also make graphs challenging to interpret, particularly when graphs for the mind are instead interpreted as graphs for the eye. Previous research has shown that students often treat graphs as literal representations of situations, such as pictures or maps, even when such interpretations are inappropriate [5]. One of the most widely cited examples comes from Clement [6]. When students were asked to construct a graph representing a bicycle's speed over time as it travelled across a hill, some drew a picture of the hill (ibid.). This phenomenon has been interpreted in different ways. Some have characterized it as a misconception about graphs, whereas others have argued that students were interpreting what teachers or researchers intended as a quantitative coordinate system (CS) as a spatial CS [7]. The present study adopts this latter perspective.

Distinguishing between spatial and quantitative CSs is important for understanding and supporting students' graph reasoning. A *spatial CS* refers to a CS mentally overlaid onto the physical space being represented, in which objects within that space are assigned coordinates, or equivalently, the spatial situation is mapped onto a CS [7]. In contrast, establishing a *quantitative CS* requires identifying quantities within a given phenomenon, disembodying those quantities from the original situation while maintaining their connection to it, and projecting them onto a new representational space that differs from the space in which the quantities were originally conceived (ibid.).

In representing some physical phenomena, such as the projectile motion introduced earlier, spatial and quantitative CSs are closely intertwined. Consequently, students must coordinate these CSs to make sense of the same phenomenon. Furthermore, projectile motion presents a particularly interesting case for two reasons. First, the projectile's trajectory in a spatial CS and the graph of the projectile's height over time in a quantitative CS are both parabolic. Despite their similar shapes, the two graphs represent fundamentally different aspects of the phenomenon. This similarity therefore presents an inherent representational challenge, as students must distinguish and coordinate their meanings for two representations. Second, projectile motion involves multiple quantities that vary simultaneously, e.g., time, the projectile's height, and its horizontal distance. Thus, this phenomenon provides a rich context for investigating how students reason about quantities that vary together. Furthermore, this context can provide students with opportunities to develop meanings for parametric points and parametric functions.

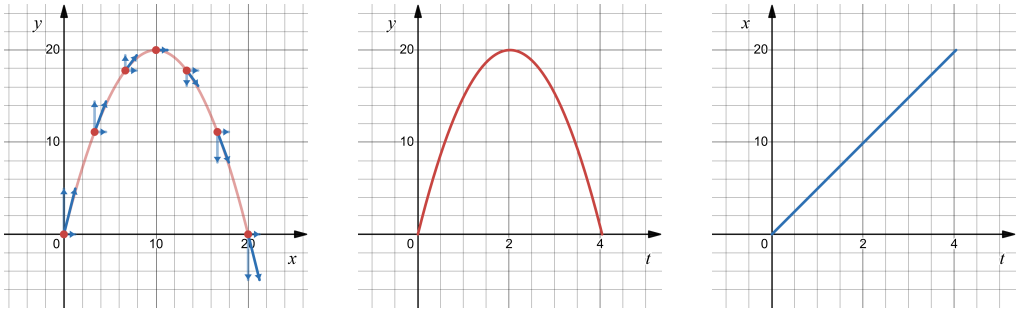
In this paper, informed by research on students' conceptions of points [8, 9], we conceptualize a *parametric point* as a conception of a point whose position in a CS is constrained by corresponding values of quantities that vary with a common quantity. For example, a student may conceive a point representing a projectile's position as a parametric point whose position is constrained by corresponding values of the projectile's horizontal distance and height at a particular time. Informed by research on students' conceptions of functions and parametric functions [9–11], we conceptualize a *parametric function* as a conception of quantities varying together with a common quantity, such that each value of the common quantity corresponds to exactly one value of each of the other quantities.

Given the representational richness of projectile motion, this study aims to explore how students coordinate quantitative and spatial CSs when constructing and justifying graphical representations of the phenomenon. Accordingly, this study is guided by the following research question: How do students construct and justify a projectile's trajectory by coordinating quantitative and spatial CSs?

2. Theoretical Framework

In many introductory physics textbooks, e.g., [12], projectile motion is represented by plotting the projectile's trajectory on a coordinate plane, typically together with selected positions of the projectile and the corresponding velocity vector and its horizontal and vertical components (Figure 1a). The trajectory can be viewed as a graph for the eye because it represents the locus of the projectile's actual physical positions, which is intended to be interpreted through a spatial CS.

Projectile motion can be decomposed into two components: vertical motion and horizontal motion. The vertical component describes the relationship between the projectile's height and time, which follows a quadratic pattern and is represented by the graph in Figure 1b. In contrast, the horizontal component describes the relationship between the projectile's horizontal distance and time, which follows a linear pattern and is represented by the graph in Figure 1c. Both graphs are intended to be interpreted through a quantitative CS.



(a) (b) (c)

Figure 1. Representing the Same Projectile Motion in Spatial and Quantitative CSs

Because projectile motion can be represented in both spatial and quantitative CSs, it provides a productive context for investigating how students coordinate these two CSs. To examine this coordination, the present study asks students to construct a projectile's trajectory from graphs of the projectile's height over time and horizontal distance over time. We anticipate that this task provides opportunities for students to engage in multivariational reasoning [13], particularly *partial dependent multivariational reasoning* [14], as they reason about how changes in time are simultaneously associated with changes in the projectile's height and horizontal distance. Through interpreting quantitative graphs to construct the projectile's trajectory in a spatial CS, students may also develop meanings for the parametric nature of the projectile's position over time. Our conceptualization of how students coordinate quantitative and spatial CSs is presented in Figure 2. In Figure 2, the quantitative and spatial CSs encompass representations of relationships among quantities, not the quantities themselves.

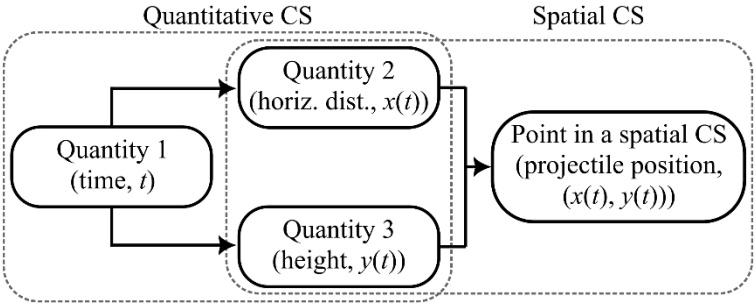


Figure 2. A Conceptual Model of Students' Coordination of Quantitative and Spatial CSs

Given that phenomena resembling projectile motion are familiar to many students (e.g., a basketball thrown toward the hoop), it is likely that students already have an intuitive sense of the characteristic shape of a projectile's trajectory, i.e., a parabola, even if they are not yet familiar with the mathematical term "parabola". Nevertheless, constructing an accurate trajectory, including correctly locating the apex and the point at which the projectile reaches the ground, may require students to identify and plot key points. Previous research has provided evidence that students often plot key points when constructing graphs [11, 15]. In the present study, we interpret these key points as *constraints* on the imagined trajectory. That is, the trajectory students envision must pass through the plotted points. Identifying these key points accurately, however, requires students to simultaneously interpret what is represented by the graphs of the projectile's height and horizontal distance over time. In this way, students coordinate the two CSs by extracting information from graphs represented in quantitative CS and using that information to construct points in a spatial CS. Furthermore, because these points are constrained by the projectile's horizontal distance and height, both of which vary with time, students may conceive of them as parametric points.

The task also has the potential to engage students in partial dependent multivariational reasoning. Constructing or interpreting a projectile's trajectory in coordination with the graphs of the projectile's height and horizontal distance over time may support students in reasoning about how time simultaneously influences both quantities. For example, when explaining the projectile's position from immediately after launch until it reaches its apex (Figure 1a), students may draw on the graphs in Figure 1b and Figure 1c to explain that, over this interval, both the projectile's height and horizontal distance increase as time increases. Such reasoning reflects students coordinating the simultaneous variation of multiple quantities. It also highlights the parametric nature of their reasoning about the projectile's position over time (cf. [10]). Such parametric reasoning can support students in developing meanings toward parametric functions.

Despite its potential to support students' multivariational reasoning and the development of meanings for the parametric nature of a projectile's position over time, projectile motion also presents representational challenges. In particular, graphs expressed in different CSs may share the same parabolic shape (Figure 1a and Figure 1b). As suggested by previous research [5], students may therefore interpret the graph of the projectile's height over time (Figure 1b) as a literal picture of its trajectory (Figure 1a). Consequently, students need support in attending to the CS in which a graph is represented when interpreting its meaning. To address this issue, several researchers [16, 17] have proposed breaking the convention by interchanging the quantities typically represented on the horizontal and vertical axes. In the context of projectile motion, for example, the graph in Figure 1b can be modified so that the horizontal axis represents the projectile's height and the vertical axis represents time. First, this modification eliminates the visual similarity between the trajectory and the height-time graph, thereby reducing the representational ambiguity associated with determining the appropriate CS. Second, the unconventional graph encourages students to focus on interpreting the relationships among quantities rather than on the graph's shape [16].

3. Methods

We present a case study [18] to investigate how students coordinate quantitative and spatial CSs when constructing and interpreting graphs representing projectile motion. The following sections describe the participants, the focal task, and the procedures for data collection and analysis.

3.1. Participants

The participants in this study were two ninth-grade students, referred to by the pseudonyms Fania and Bianca. They were selected based on their willingness to participate and their ability to verbally articulate their reasoning. Before engaging in the focal task, they completed preceding tasks in which they developed meanings for graphs representing a projectile's height and horizontal distance over time [19, 20]. Throughout the instructional sequence, the first author served as the teacher-researcher (TR), while the third author served as the witness-researcher. In this report, we focus on Fania's activity in the focal task as she provided rich data allowing us to offer a clear and authentic portrayal of a student coordinating spatial and quantitative CSs to make sense of projectile motion.

3.2. The Focal Task

The goal of the focal task was for students to construct the trajectory of a ball launched with a given initial angle and initial velocity, assuming ideal projectile motion. To support this construction, students were provided with graphs representing the relationships between the ball's height and time and between its horizontal distance and time. The task was presented on a single screen in a digital environment. The screen included task instructions, a left panel displaying a coordinate plane together with the two graphs, and a right panel displaying a coordinate plane on which students constructed the ball's trajectory (Figure 3).

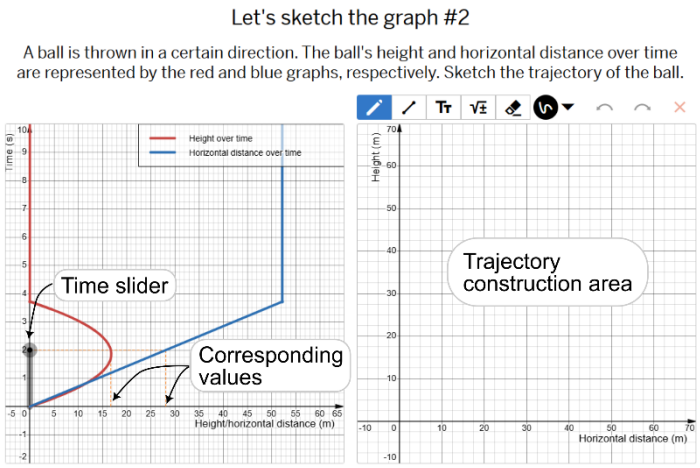


Figure 3. The Focal Task Presented in the Digital Environment. *Note:* This task is presented as Screen 8 in a digital task sequence available at https://classroom.amplify.com/activity/6a5b501520d7c296792eca8e.

The focal task was designed to support students in coordinating quantitative and spatial CSs while engaging in multivariational reasoning. During the task, students were encouraged to interpret the graphs in the left panel as represented in a quantitative CS and to use the information represented in these graphs to construct the ball's trajectory in the spatial CS displayed in the right panel. The axes of the graphs in the left panel were intentionally reversed from their conventional orientation to encourage students to interpret the graphs as representations of relationships between quantities rather than to focus on their shapes. To support multivariational reasoning, the left panel included a vertical slider representing time. As the slider was moved up and down, orange dashed lines highlighted the corresponding values of the ball's height and horizontal distance on the two graphs. In addition, the digital environment allowed students to click any point on the graphs to display its coordinates.

3.3. Data Collection and Analysis

We used data from a teaching experiment [21] involving the two students. During data collection, the students worked collaboratively on a single laptop to complete the task. The TR sat facing them to observe and interpret their activity and to respond when appropriate. Throughout the session, the students' screen was recorded to capture their interactions with the task, while their verbal explanations and gestures were also video-recorded. These recordings were subsequently synchronized into a single video to support the analysis.

Consistent with the teaching experiment methodology [21], we modelled one student's mathematics, focusing on Fania's engagement with the task. In doing so, we were guided by the student's coordination of quantitative and spatial CSs presented in the previous section. In particular, we identified instances in her activity that provided evidence of how she reasoned as she interpreted graphs in quantitative CSs and used those interpretations to construct a graph in a spatial CS. We also looked for evidence suggesting that she engaged in multivariational reasoning, such as by coordinating changes in the projectile's height, horizontal distance, and elapsed time. Through this process, we constructed models of how the student coordinated quantitative and spatial CSs while coordinating multiple quantities related to projectile motion.

4. Findings

Overall, Fania coordinated quantitative and spatial CSs in two ways. First, she reasoned about the ball's position in the spatial CS at particular moments in time as she identified and plotted key points, which she conceived as parametric points, to construct the trajectory. Second, she engaged in partial dependent multivariational reasoning as she justified the trajectory by explaining how variations in time simultaneously influenced variations in the ball's height and horizontal distance. Table 1 summarizes the mathematical episodes illustrating these two forms of coordination, which we elaborate on in Sections 4.1 and 4.2.

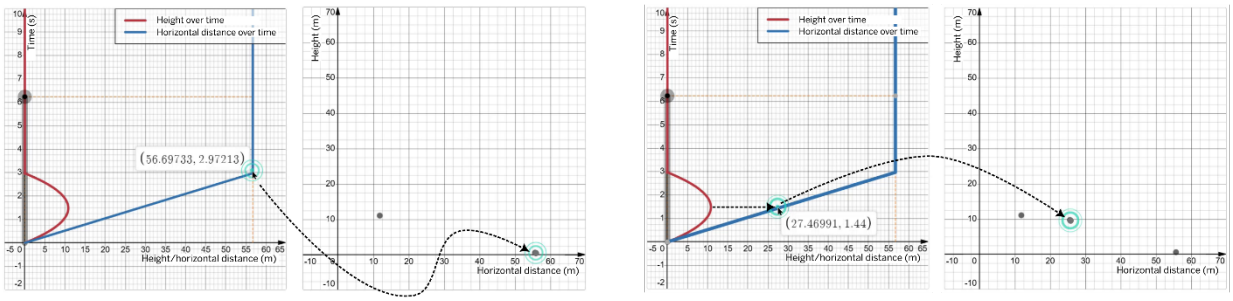
Table 1. Summary of the Episodes Illustrating Fania's Coordination of Quantitative and Spatial CSs

Episode	Coordination of the CSs	Form of reasoning
Plotting the apex	Identifying the ball's maximum height and corresponding time from the graph in the quantitative CS, then locating its position as a parametric point in the spatial CS.	Reasoning about the values of quantities at particular moments in time.
Plotting the landing position	Identifying when the ball reached height 0 and the corresponding horizontal distance from the graph in the quantitative CS, then locating its position as a parametric point in the spatial CS.	Reasoning about the values of quantities at particular moments in time.
Justifying the trajectory	Coordinating changes in height and horizontal distance as time increased using the graphs in the quantitative CS to justify the constructed trajectory in the spatial CS.	Partial dependent multivariational reasoning.

4.1. Constructing the Trajectory Through Coordinating Quantitative and Spatial CSs

Fania constructed the ball's trajectory by plotting two key points: the point where the ball hits the ground and the trajectory's apex. Constructing these points involved two sequential steps. First, she interpreted the graphs of the ball's height and horizontal distance over time to extract pointwise information relevant to each key moment. She then used this information to determine the corresponding coordinates of the points in the spatial CS.

The first point whose coordinates Fania identified was the point where the ball hits the ground. To identify this point, she clicked on the corner of the graph of the ball's horizontal distance over time, prompting the digital environment to display its coordinates (56.69733, 2.97213). Using this information, she then plotted the point where the ball hits the ground at approximately (56, 0) in the spatial CS, see Figure 4(a).



(a) (b)

Figure 4. Fania's Sequence of Actions in Plotting the Trajectory's Key Points

This episode suggests that Fania engaged in two forms of reasoning across quantitative and spatial CSs. First, we interpret her as situationally interpreting the graph of the ball's horizontal distance over time. Specifically, she interpreted the corner (marking the transition between the ball's

motion, during which its horizontal distance increases over time, and its stationary state, during which its horizontal distance remains constant as time continues) as the moment when the ball hits the ground. Second, she interpreted the graph as we intended by taking the first coordinate of the corner, which she approximated as 56, to represent the ball's horizontal distance at the moment it hits the ground. She then used this value in the spatial CS to mark a point at (56, 0).

To determine the coordinates of the trajectory's apex, Fania first directed the pointer to the vertex of the graph of the ball's height over time, estimating the maximum height of 10 (Figure 4b). She then moved the pointer horizontally to the right until it reached the corresponding point on the graph of the ball's horizontal distance over time and clicked on that point, prompting the digital environment to display its coordinates (27.46991, 1.44). Using this information, she plotted a point at approximately (27,10) in the spatial CS to represent the trajectory's apex. Throughout this sequence of actions, she also communicated her reasoning to Bianca, as shown in the following excerpt.

> If it's here [*pointing to the vertex of the red graph, then moving the pointer horizontally to the right*], it's at … [*clicking the blue graph to display the coordinates of a point*] twenty-seven. So, the highest point is here [*plotting the ball's highest point at approximately (27, 10)*]. (Fania)

We interpret Fania's reasoning in determining the trajectory's apex as similar to the reasoning she used to identify the point where the ball hits the ground. In this case, however, she extracted two pieces of spatial information by interpreting two different graphs in a quantitative CS. She understood that the apex of the situation would occur at the vertex of the height-time graph, obtaining the second coordinate of the apex from the height-time graph. She also used the vertex of the height-time graph to determine the first coordinate of the trajectory by moving the pointer horizontally from that point to the corresponding point on the graph of horizontal distance over time. Hence, we infer she developed a meaning for the ball's position at any particular moment as being characterized simultaneously by its height and horizontal distance. Accordingly, she conceived the points representing the ball's positions as parametric points in the spatial CS.

After plotting the two key points, Fania connected them, together with the origin, using a smooth curve resembling a parabola. We interpret this activity as suggesting that the two key points, together with the origin, served as constraints on the trajectory she envisioned. Although Fania coordinated quantitative and spatial CSs in determining and plotting the two key points, she did not yet provide evidence of coordinating them when constructing the curve itself. Instead, we interpret the shape of the curve as relying primarily on her prior knowledge of the spatial form of a projectile's trajectory. To probe her reasoning further, the TR asked her to justify why the trajectory had the shape that she had constructed. Her justification is presented in the following subsection.

4.2. Justifying the Trajectory Through Multivariational Reasoning

Providing an opportunity to justify the trajectory prompted Fania to coordinate quantitative and spatial CSs. Similar to the previous subsection, she first coordinated the quantitative and spatial CSs by determining the ball's height and horizontal distance at particular moments in time. She also engaged in partial dependent multivariational reasoning to explain how three quantities, namely time, the ball's height, and its horizontal distance, changed simultaneously. In particular, she justified the covariation between the ball's height and horizontal distance by reasoning that both quantities vary with the same quantity, time.

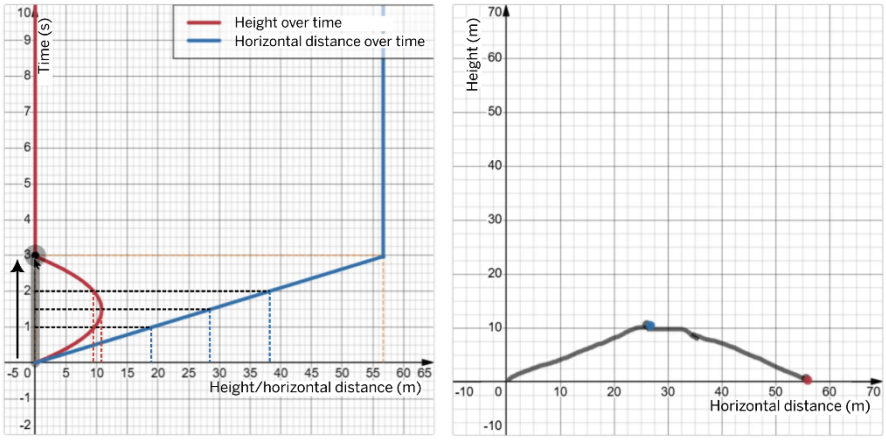


Figure 5. The Task Interface Illustrating the Values Referenced by Fania, as Indicated by the Dashed Segments

As Fania explained how the three quantities changed simultaneously, she used the time slider to support her reasoning. Table 2 presents an excerpt from the discussion between Fania and the TR during this process. Figure 5 illustrates the task interface with the values referenced by Fania during her explanation.

Table 2. Excerpt From the Discussion Between Fania and the TR

Line	Speaker	Utterance and Action
1	Fania	Initially, the time is zero. Then, as time passes [*moves the time slider upward, from 0 to 1*], at the first second, the ball's height, as shown here, is almost 10 meters, and its horizontal distance is almost 20 meters. Then, at [*moves the time slider upward, from 1 to 1.5*] about one and a half seconds, the ball reaches its highest point, which is about 10 meters high, while the horizontal distance continues to increase from before and is now almost 30 meters.
2	TR	30?
3	Fania	Yes, almost 30. Then, if we move it further [*moves the time slider upward, from 1.5 to 2*] to the second, because the ball has already reached its highest point, it starts to fall. This downward motion only affects its height, but its horizontal distance keeps moving.
4	TR	The horizontal distance keeps ...
5	Fania	Moving.
6	TR	By "keeps moving," you mean it continues to increase?
7	Fania	Yes, it continues to increase. Only the height decreases.
8	TR	Okay.
9	Fania	Here we can see that the height decreases again to just under 10 meters, but the horizontal distance continues to increase to about 35 meters. Then, at the third second [*moves the time slider upward, from 2 to 3*], this is the final moment before the ball stops, or, in this context, when it lands back at the same height as its starting position. At the third second, we can see that the ball's height has returned to zero meters, while its horizontal distance has continued to increase until it finally stops.
10	TR	Okay.
11	Fania	But time continues.
12	TR	It's time that continues.
13	Fania	Yes. So after that, the height remains constant because the ball is already on the ground, and the horizontal distance also remains constant because the ball is already on the ground. However, time continues, so both graphs continue as straight upward lines.

Table 2 illustrates how Fania interpreted the graphs quantitatively in the situational CS situation. Based on these interpretations, she was able to determine the values of the ball's horizontal distance and height at particular moments in time, as well as explain how changes in time affected changes in both quantities. Line 1 provides evidence of her determining the values of these two quantities at specific times, namely 1 s and 1.5 s. Lines 3–13 indicate that she engaged in partial dependent multivariational reasoning as she explained how changes in time affected changes in both the ball's horizontal distance and height. For example, in lines 3–9, she explained that as time increased from 1.5 s to 2 s, the ball's horizontal distance increased while its height decreased.

Because Fania justified the trajectory she had constructed by coordinating the quantitative and spatial CSs while interpreting the graphs situationally, we infer that the meanings she developed for the trajectory extended beyond viewing it as a representation of the ball's possible physical positions. Rather, she construed it as a graph representing the locus of a moving point whose coordinates were determined by the values of two quantities, namely the ball's horizontal distance and height, both of which varied simultaneously with the common quantity of time. In this sense, the meanings she developed for the trajectory were consistent with the idea of parametric functions describing the ball's position over time. Specifically, she recognized that each value of time, t, simultaneously determined a horizontal distance, $x(t)$, and a height, $y(t)$, which together specified the ball's position, $(x(t), y(t))$, although she was not aware of the mathematical term "parametric function" or its conventional symbolic representation at the time.

5. Discussion and Conclusion

In this paper, we presented Fania's coordination of quantitative and spatial CSs as she constructed a ball's trajectory from graphs of the ball's height over time and horizontal distance over time. We first discuss how the findings contribute to research on students' graph reasoning and the coordination of quantitative and spatial CSs. We then outline the implications of the study, acknowledge its limitations, and identify opportunities for future research.

5.1. Students' Coordination of Quantitative and Spatial CSs

The findings revealed two ways in which Fania coordinated quantitative and spatial CSs. First, she did so by reasoning about the ball's position in the spatial CS at particular moments in time represented in a quantitative CS. When this reasoning produced points in the spatial CS, these points eventually came to represent more than the ball's position. Rather, they represented the coordination of the quantities involved in the ball's motion (i.e., time, height, and horizontal distance) as interpreted from the graphs in the quantitative CSs. Her reasoning when plotting the key points representing the ball trajectory's apex and the ball's position when it hit the ground provides evidence of this mental process. This way of constructing points gives rise to meanings for parametric points that involve not only quantitative and spatial attributes but also the coordination of the two. We hypothesize that these meanings for parametric points are productive for analyzing projectile motion and, more broadly, other forms of two-dimensional kinematics in which an object's position is determined by the coordinated values of multiple quantities at a given moment in time.

Second, Fania coordinated quantitative and spatial CSs by engaging in partial dependent multivariational reasoning as she justified the trajectory she had constructed. Her justification of the trajectory in the spatial CS was grounded in her interpretation of the graphs in the quantitative CS, which was supported by her use of the time slider. For example, as she moved the slider from 1.5 s to 2 s, she provided evidence of partial dependent multivariational reasoning by explaining that, as time increased over this interval, the ball's horizontal distance increased while its height decreased. Grounded in this reasoning, together with her reasoning about points at particular moments in time

discussed earlier, she developed meanings for the trajectory that we argue are productive. These meanings are consistent with the meanings for graphs proposed in previous research [22, 23].

These two ways of coordinating quantitative and spatial CSs supported the development of meanings toward parametric functions. Fania recognized how time simultaneously determined both the horizontal distance and the height of the ball through her interpretation of the graphs in the quantitative CS. Furthermore, she was able to identify corresponding pairs of horizontal distance and height from those graphs to construct the ball's trajectory in the spatial CS. Thus, the meanings she developed for constructing the trajectory were consistent with the idea of parametrically defined functions, a concept that commonly appears in advanced mathematics. As such, the empirical findings of this study contribute to the literature on the processes through which students develop meanings for parametric functions [10, 11].

5.2. Implications, Limitations, and Future Studies

Our conceptualization of students' coordination of quantitative and spatial CSs guided us in interpreting the student's activity as she constructed the trajectory based on their interpretations of the graphs in the quantitative CS. We argue that this conceptualization can also serve as a lens for researchers seeking to attend to and interpret students' reasoning about graphs that require coordination between these two CSs. In addition, it may serve as a design lens to develop tasks that support students in coordinating both CSs and developing meanings toward parametric functions.

Conceptually, Fania's meanings for the parametric points she plotted as constraints on the trajectory she constructed extend the conceptualization of students' meanings for points proposed by Tasova and Moore [8]. These parametric points were not merely the result of Fania pairing the ball's horizontal distance and height. Rather, she conceived the two quantities that constrained the positions of these points as varying with the common quantity of time. We argue that the construct introduced in this paper, i.e., the parametric point, may be useful for future studies investigating not only how students coordinate quantitative and spatial CSs but also how they develop meanings for parametric functions.

We note that the conflation of quantitative and spatial CSs occurs across many topics, such as the study of two-dimensional motion in physics. In such contexts, the spatial CS is used to represent the actual physical space of the phenomenon under investigation, whereas the quantitative CS represents relationships among the quantities involved. This conflation may create challenges for students in determining which CS is appropriate for interpreting a given graph. However, when provided with appropriate support, students can successfully coordinate these different CSs. Therefore, echoing the recommendations of previous research [7, 8], we suggest paying greater attention to the distinct roles of quantitative and spatial CSs while recognizing that they complement one another in supporting students' reasoning.

We also note that Fania's parametric reasoning, as evidenced in this study, was sufficient to account for the overall direction of the trajectory but not its curvature. This limitation may reflect the design of our task. We conjecture that extending the task could support students in developing reasoning that also accounts for the trajectory's curvature. For example, rather than asking students only to construct the projectile's trajectory from graphs of its horizontal distance and height over time, the task could also ask them to interpret an accurate representation of the projectile's trajectory. In doing so, students could be given opportunities to observe how a parametric point, whose position is constrained by the projectile's horizontal distance and height, moves along the trajectory over several short intervals of time, as suggested by previous research [11, 23]. Projectile motion is particularly suitable for this purpose because time and horizontal distance are linearly related. Thus, the task could be further simplified by prompting students to analyse how the height of a parametric point changes

over a small interval of horizontal distance and then over progressively smaller intervals, allowing them to infer that the trajectory has a parabolic shape.

We acknowledge several limitations of this study. First, our analysis focused on the activity of one student as she coordinated quantitative and spatial CSs. Consistent with Yin [18], however, the implications of this case are intended to be "generalizable to theoretical propositions and not to populations" (p. 20). Second, our task sequence did not extend to the formal study of parametric functions. Nevertheless, based on what Fania learned in this study, we hypothesize that her learning experiences provided an important foundation for developing meanings for parametric functions. Thus, the opportunities to learn afforded by the projectile motion task may make advanced mathematical ideas accessible to young students like Fania. Future studies could extend our work by exploring ways to make advanced mathematical ideas accessible to young students before they are formally introduced to them.

5.3. Conclusion

We have demonstrated how a student coordinated quantitative and spatial CSs. She coordinated these two distinct yet complementary CSs by interpreting graphs for the mind to construct the corresponding graph for the eye. In particular, she reasoned about the ball's position in the spatial CS at particular moments in time, recognizing the corresponding values of the ball's horizontal distance and height. She also recognized that as time increased, the ball's horizontal distance and height varied. In doing so, she connected the quantitative relationships represented in the graphs to the ball's position in the spatial CS. This mathematical activity supported her in reasoning about relationships among multiple quantities and, in turn, in developing meanings for parametric points and meanings consistent with the idea of parametric functions. Future studies could extend this work by exploring a broader range of contexts. In particular, we suggest investigating other topics involving two-dimensional motion in physics where students likewise need to coordinate quantitative and spatial CSs. In addition, future studies could explore how students coordinate these CSs not only in rectangular CSs but also in polar CSs. Such work would help students take fuller advantage of the versatility of graphs, representations that are nearly ubiquitous in everyday life, for understanding real-world phenomena.

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