Generalized Stirling Numbers

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Abstract: *Stirling numbers are implemented in several major computational systems, but they are restricted to arguments that are positive integers. Numerous authors have, however, generalized Stirling's original definitions*[1]*. There have been definitions for negative arguments, for rational arguments, and for complex arguments. In addition, since Stirling numbers solve combinatorial problems, other numbers that solve related combinatorial problems have been named Stirling numbers also. For example, there are* associated *Stirling numbers, and r-Stirling numbers. This review considers these generalizations and discusses the relations between them, with a view to extending the current implementations to more general cases, for the convenience of users of symbolic-computation systems.*

1 Introduction

Stirling introduced the numbers which are now named after him (by Niels Nielsen [12]) in order to transform between polynomials expressed in monomial powers and their equivalent in factorial powers. The importance of his numbers, however, derives from the many additional applications that have been found for them. In combinatorial calculations, the numbers are well established. In computations of series expansions, using, for example, Lagrange inversion, Stirling numbers and their generating functions are invaluable. The importance of the numbers has been recognized by Mathematica and Maple, both offering built-in functions, but only for the basic case in which the arguments are positive integers.

Inevitably, the original definitions have been generalized in various directions, not all generalizations being mutually compatible. This review intends to discuss these generalizations, and to suggest avenues for the computational systems to generalize their implementations.

2 Basic Stirling numbers

We start by recalling Stirling's original applications. For $k \in \mathbb{N}$ and $z \in \mathbb{C}$, falling factorial powers $z^{\underline{k}}$ are defined by

$$z^{\underline{k}} = z(z-1)(z-2)\dots(z-k+1) = \frac{z!}{(z-k)!} = \frac{\Gamma(z+1)}{\Gamma(z-k+1)} \ . \tag{1}$$

Similarly, rising factorials $z^{\overline{k}}$ are

$$z^{\overline{k}} = z(z+1)(z+2)\cdots(z+k-1) = \frac{(z+k-1)!}{(z-1)!} = \frac{\Gamma(z+k)}{\Gamma(z)} \ . \tag{2}$$

[1]From the very useful Wikipedia article Stirling numbers we learn that Stirling first wrote about them in 1730, and indeed seems to have been the first to do so.

These functions have algebraic properties analogous to those of the factorial function, such as

$$z^{\underline{k}} = z(z-1)^{\underline{k-1}} = \frac{(z+1)^{\underline{k+1}}}{z+1} \tag{3}$$

Then Stirling partition numbers[2] are defined to be those numbers appearing when monomial powers are recast as factorial powers:

$$z^n = \sum_k \left\{ {n \atop k} \right\} z^{\underline{k}} = \sum_k (-1)^{k+n} \left\{ {n \atop k} \right\} z^{\overline{k}} \,. \tag{4}$$

Inversely, Stirling cycle numbers[3] take factorial powers to monomials.

$$z^{\underline{n}} = \sum_k (-1)^{n+k} \left[{n \atop k} \right] z^k \,, \tag{5}$$

$$z^{\overline{n}} = \sum_k \left[{n \atop k} \right] z^k \,. \tag{6}$$

Rather than following Stirling, many authors define Stirling numbers through their combinatorial properties. Thus the partition numbers $\left\{ {n \atop k} \right\}$ give the number of ways in which a set of n distinct objects can be partitioned into k non-empty subsets; the cycle numbers $\left[{n \atop k} \right]$ give the number of ways k cycles can be formed from n distinct objects. Both definitions assume positive integers n, k.

2.1 Negative integers

As a first generalization, we consider negative integers. From the definitions, we obtain the following recurrence relations.

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \,, \text{ and } \left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = \delta_{n0} \,, \tag{7}$$

$$\left[{n+1 \atop k} \right] = n \left[{n \atop k} \right] + \left[{n \atop k-1} \right] \,, \text{ and } \left[{n \atop 0} \right] = \left[{0 \atop n} \right] = \delta_{n0} \,, \tag{8}$$

where δ_{ij} is the Kronecker delta function [5]. If these are assumed to continue to apply for negative integers, we are led to the well-known reciprocal relation

$$\left\{ {n \atop k} \right\} = \left[{-k \atop -n} \right] \,. \tag{9}$$

We thus have a simple extension to negative integers, and at the same time we see that the two types of numbers are simply different aspects of a single set of Stirling numbers.

[2] Also called subset numbers [10], numbers of the second kind [3].

[3] Also Stirling numbers of the first kind [3].

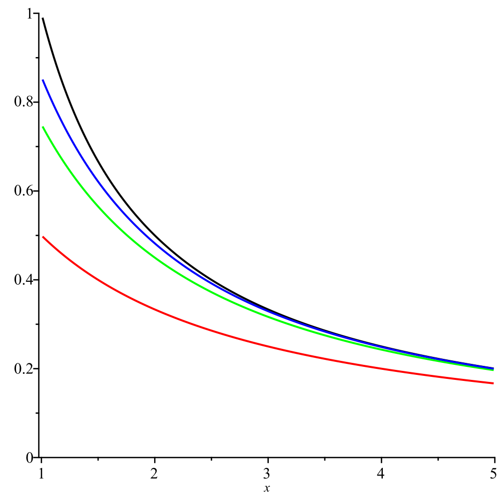

Figure 1: A plot of $1/x$ and approximations based on (12). Black is $1/x$; red is one term; green is 3 terms and blue is 6 terms.

We should not end the discussion with (9), however, but ask what benefits it brings elsewhere. This requires first generalizing the factorial powers. Using the factorial form of the definition in (1), we define for $k \in \mathbb{N}$

$$z^{\underline{-k}} = \frac{z!}{(z+k)!} = \frac{1}{(z+1)(z+2)\dots(z+k)} \tag{10}$$

$$= \frac{1}{(z+1)^{\overline{k}}} = \frac{z}{z^{\overline{k+1}}} \; . \tag{11}$$

Substituting $n = -p$, with $p > 0$, into (4) gives

$$z^{-p} = \sum_k \left\{ {-p \atop -k} \right\} z^{\underline{-k}} = \sum_k \left\{ {-p \atop -k} \right\} \frac{1}{(z+1)^{\overline{k}}} \; . \tag{12}$$

Some explanation is needed. A consequence of the recurrence (7) is that $\left\{ {-p \atop m} \right\}$ is zero for all $m > 0$ and nonzero only for $m < 0$. This is consistent with our expectation that negative monomial powers cannot be expanded using positive factorial powers. Thus (12) has been written assuming both arguments are negative. For the particular case $p = 1$, we have

$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{(x+1)(x+2)} + \frac{2}{(x+1)(x+2)(x+3)} + \dots$$

This is the formal result, and it can be proved that the infinite series converges to $1/x$, so (10) and (9) work together consistently and give correct results. It is also of practical interest to plot the approximation and see how good it is. This is done in Figure 12, where $1/x$ is compared with the series summed to 1, 3, and 6 terms. The fact that the approximation is better for large x is consistent with Stirling's

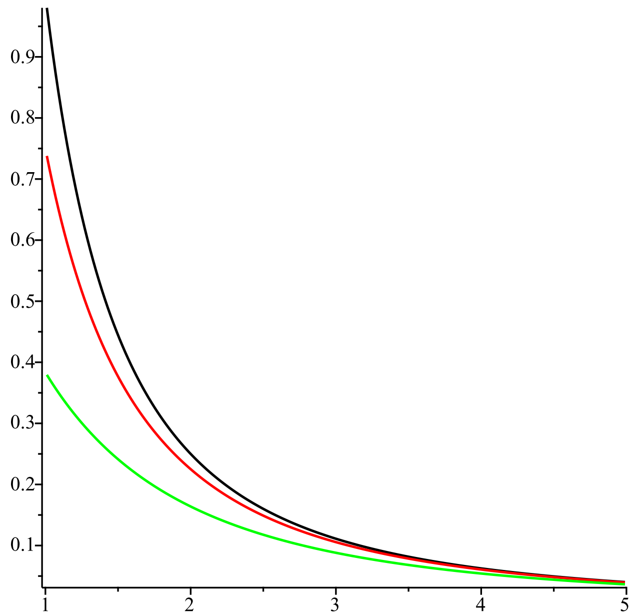

Figure 2: Plots of $1/x^2$ in black, (15) in red and (14) in green.

interest in summing the tails of infinite series. The case $p = 2$ has some further interest. Nielsen [13, §30(6)] gives a variation on (12).

$$\frac{1}{x^p} = \sum_k \left\{ {-(p-1) \atop -k} \right\} \frac{1}{x^{\overline{k+1}}} \tag{13}$$

The first 3 terms of the two series are $1/x^2 =$

$$\frac{1}{(x+1)(x+2)} + \frac{3}{(x+1)(x+2)(x+3)} + \frac{11}{(x+1)(x+2)(x+3)(x+4)} \tag{14}$$

$$= \frac{1}{x\,(x+1)} + \frac{1}{x\,(x+1)\,(x+2)} + \frac{2}{x\,(x+1)\,(x+2)\,(x+3)} \tag{15}$$

In Figure 2, the two curves are plotted, and Nielsen's form converges faster. The conclusion is that the extension to negative factorials is consistent with the behaviour we desire.

Since the reciprocal relation allows negative arguments to be computed using $\left\{ {-n \atop -k} \right\} = \left[{k \atop n} \right]$, the question arises whether implementing negative arguments is necessary. The situation can be compared to the binomial coefficient. Although $\binom{-n}{k}$ can be computed as $(-1)^k \binom{n+k-1}{k}$, computational systems still implement negative arguments for their users. Thus it would be a convenience for systems to implement negative arguments of Stirling numbers. At present, Maple returns `stirling2(-2,-3)=0`, which, in view of the above development, can be argued to be an error.

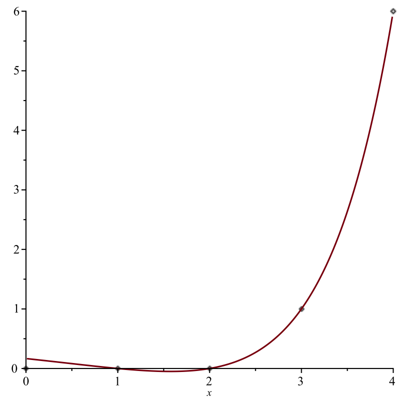

Figure 3: Semi-real Stirling numbers interpolate Stirling Partition numbers for positive arguments. Here $\left\{ {n \atop 3} \right\}_{SR}$ is plotted against n, with the standard definition marked as circles.

2.2 Rational arguments

Users of computer algebra systems value exact results. Thus we should search for cases in which Stirling numbers can be evaluated in closed form. One such case is supplied by semireal Stirling numbers, defined by [16]

$$\left\{ {x \atop k} \right\} = \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} j^x \,, \tag{16}$$

where k is a positive integer and x is a real number. The background to this definition is that (16) is known to agree with the earlier definition when $x \in \mathbb{N}$, and the authors proposed to generalize x to $x \in \mathbb{R}$. Note that the argument k remains a positive integer so that the summation is defined.

It is simple to define (16) as a function in a computational system and plot the function. This shows that it interpolates the case of integer arguments (mostly), as shown in Figure 3.

There are several incompatibilities, however. In Figure 3, we see that the semireal definition disagrees with the standard definition at the origin. Further, it was stated above that $\left\{ {n \atop m} \right\}$ is zero if $n < 0$ and $m > 0$. Definition (16), however, gives $\left\{ {-1 \atop 3} \right\} =? \frac{11}{36}$.

Given that the definition interpolates successfully for $x \geq 1$, we can provisionally accept this as a definition for that case, and otherwise the notation should be decorated somehow if authors find future applications for other values of x. Here we use $\left\{ {x \atop k} \right\}_{SR}$.

The important case for this subsection is x being rational, when this definition will return an algebraic value, for example,

$$\left\{ {3/2 \atop 3} \right\}_{SR} = \frac{1}{2} - \sqrt{2} + \frac{\sqrt{3}}{2} \,, \tag{17}$$

$$\left\{ {11/2 \atop 3} \right\}_{SR} = \frac{1}{2} - 16\sqrt{2} + \tfrac{81}{2}\sqrt{3} \,, \tag{18}$$

$$\left\{ {10/3 \atop 2} \right\}_{SR} = -1 + 4\sqrt[3]{2} \,. \tag{19}$$

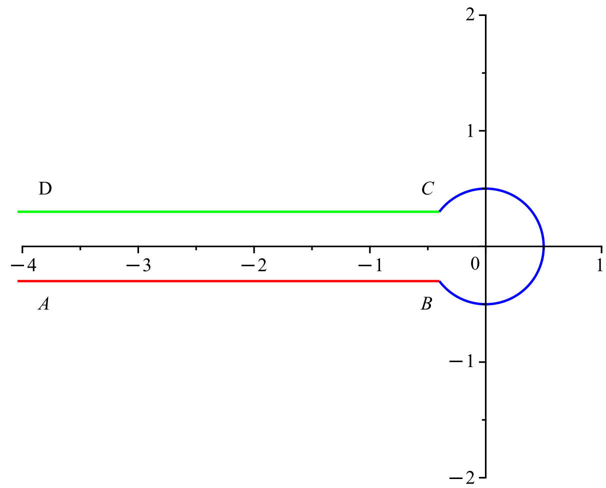

Figure 4: A Hankel contour for evaluation of the integral3 (21) or (22). The contour runs ABCD so that the origin is circled anti-clockwise.

2.3 Contour integral definition

Several proposals have been made to generalize Stirling numbers using contour integrals. The most elegant is proposed in [6]. Starting from the exponential generating function

$$(e^z - 1)^k = k! \sum_{n=k} \left\{ {n \atop k} \right\} \frac{z^n}{n!} \,, \tag{20}$$

one can extract a particular Stirling number using the Cauchy coefficient formula.

$$\left\{ {n \atop k} \right\} = \frac{n!}{k!} \frac{1}{2\pi i} \int_\gamma \frac{(e^z - 1)^k}{z^{n+1}} \, \mathrm{d}z \,, \tag{21}$$

where γ is a closed contour around the origin. We now consider whether n, k in (21) can be replaced by complex constants $x, y \in \mathbb{C}$. The problem is that the replacement induces a branch cut along the negative real axis of z, which invalidates the closed contour γ. Proposals to correct this have been made by [14] and [6]. The latter proposal replaces γ with a Hankel contour $\mathcal{H}$, following [17]. The contour is shown in Figure 4. For theoretical purposes, the contours AB and CD are considered to be separated from the negative real axis by a distance $\epsilon \ll 1$, although for numerical purposes other contours are possible. The integral converges for $\Re(n) > 0$, where $\Re$ denotes the real part of n, which is now regarded as a complex quantity.

A final step is to integrate by parts to arrive at

$$\left\{ {x \atop y} \right\} = \frac{(x-1)!}{(y-1)!} \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{e^z (e^z - 1)^{y-1}}{z^x} \, \mathrm{d}z \,. \tag{22}$$

Here we have changed the notation to x, y to avoid confusion with n, k being integers. The benefit of the integration by parts is to increase the rate of decay as $\Re z \to -\infty$ along the contour, which in turn ensures the integral now converges for all x, y complex.

This definition obeys the generalization of (7), but its practicality for numerical evaluation is largely untested. A preliminary computation [4] obtains numerical agreement between this definition and other definitions for a selection of arguments.

3 Associated Stirling numbers

The associated Stirling numbers modify the combinatorial definition of the basic Stirling numbers. The number of partitions of a set of n elements into k subsets, all with cardinality $\geq s$, is denoted $\left\{{n \atop k}\right\}_{\geq s}$, with $s = 1$ being the basic case. Comtet [3, pp 221-222] calls these numbers[4] "s-associated Stirling numbers of the second kind", but this lengthy name can be conveniently abbreviated to Stirling s-partition numbers. Similarly, Stirling s-cycle numbers $\left[{n \atop k}\right]_{\geq s}$ count cycles of minimum length s.

In addition to the combinatorial applications which were used to define them, associated numbers appear in other contexts. For example, Stirling's approximation for $n!$ can be written as [1][9][11]

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sum_{k\geq 0} \frac{a_k}{(2n)^k}\,, \tag{23}$$

$$a_k = \sum_{\ell=0}^{2k} \frac{(-1)^\ell}{2^\ell\,(\ell+k)!}\left[{2k+2\ell \atop \ell}\right]_{\geq 3}. \tag{24}$$

There are similarities between properties of associated numbers and basic Stirling numbers. For example, the recurrence relations are similar:

$$\left\{{n+1 \atop k}\right\}_{\geq s} = k\left\{{n \atop k}\right\}_{\geq s} + \binom{n}{s-1}\left\{{n-s+1 \atop k-1}\right\}_{\geq s}, \tag{25}$$

$$\left[{n+1 \atop k}\right]_{\geq s} = n\left[{n \atop k}\right]_{\geq s} + n^{\underline{s-1}}\left[{n-s+1 \atop k-1}\right]_{\geq s}. \tag{26}$$

Some properties, for example, (9) do not have known counterparts for $s > 1$. As more applications of these functions are found, the few studies of them [7][8] will be added to.

4 r-Stirling numbers

The r-Stirling numbers are similar to the associated numbers in that the number of possible partitions or cycles are subject to an additional constraint relative to the basic Stirling numbers. They give the number of partitions of the set $\{1,\ldots,n\}$ into k subsets subject to the constraint that numbers $1,\ldots,r$ are in distinct subsets. These have been denoted [2] $\left\{{n \atop k}\right\}_r$. A similar requirement of numbers $1,\ldots,r$ being in distinct cycles can be applied to cycle numbers, with the notation $\left[{n \atop k}\right]_r$. We note that $r = 1$ returns us to the basic Stirling numbers.

[4] In his book, Comtet uses the letter r, rather than s, but since we also discuss r-Stirling numbers below, Comtet's choice is here changed to s to emphasize the distinction between the definitions.

In addition to the specific combinatorial problem used for their definition, the numbers appear in other contexts. For example, the derivatives of the Lambert W function obey

$$\frac{d^n W}{dz^n} = \frac{e^{-nW} p_n(W)}{(1+W)^{2n+1}} \ , \tag{27}$$

where $p_n(W)$ is a polynomial with coefficients

$$\beta_{n,k} = \sum_{m=0}^{k} (-1)^m \binom{2n-1}{k-m} \left\{ {2n-1+m \atop n+m} \right\}_n \quad . \tag{28}$$

The recurrences

$$\left\{ {n+1 \atop k} \right\}_r = k \left\{ {n \atop k} \right\}_r + \left\{ {n \atop k-1} \right\}_r \ , \tag{29}$$

$$\left[{n+1 \atop k} \right]_r = n \left[{n \atop k} \right]_r + \left[{n \atop k-1} \right]_r \ , \tag{30}$$

continue to apply, but the initial conditions are altered to

$$\left\{ {n \atop k} \right\}_r = 0 \text{ for } n < r \ , \tag{31}$$

$$\left\{ {n \atop k} \right\}_r = \delta_{mr} \text{ for } n = r \ , \tag{32}$$

$$\left[{n \atop k} \right]_r = 0 \ , \text{ for } n < r \ , \tag{33}$$

$$\left[{n \atop k} \right]_r = \delta_{mr} \ , \text{ for } n = r \ . \tag{34}$$

5 Other generalizations

A final generalization to discuss differs from those above in that there is no special case in which the new numbers specialize to the basic Stirling numbers. The numbers $\left\{ {n+\frac{1}{2} \atop k} \right\}_H$ obey the Stirling recurrence law (7)

$$\left\{ {n+\frac{1}{2} \atop k} \right\}_H = k \left\{ {n-\frac{1}{2} \atop k} \right\}_H + \left\{ {n-\frac{1}{2} \atop k-1} \right\}_H \ , \tag{35}$$

but now obey new boundary conditions [15].

$$\left\{ {n+\frac{1}{2} \atop n} \right\}_H = n \ . \tag{36}$$

Special cases, according to these definitions are

$$\left\{ {n+\frac{1}{2} \atop 2} \right\}_H = 3(2^n) - 1 \,, \quad \left\{ {n+\frac{1}{2} \atop 3} \right\}_H = \frac{17}{2}\, 3^n - 6(2^n) + \frac{1}{2} \,, \tag{37}$$

$$\left\{ {n+\frac{1}{2} \atop 4} \right\}_H = \frac{71}{3}\, 4^n - \frac{51}{2}\, 3^n + 6(2^n) - \frac{1}{6} \,. \tag{38}$$

Another important comparison is to compute $\left\{ {5/2 \atop 2} \right\}$ using the current definition and the semireal definition (16). We see [15]

$$\left\{ {5/2 \atop 2} \right\}_{SR} = 2\sqrt{2} - 1 \approx 1.8284 \,, \tag{39}$$

$$\left\{ {5/2 \atop 2} \right\}_H = 2 \,. \tag{40}$$

These numbers found one application in [10], but have not been identified in other contexts.

6 Conclusions

The basic Stirling numbers have been implemented in several computational systems, but although they agree for positive integer arguments, their response to negative arguments differ. Mathematica returns an error for negative arguments, which may disappoint users who know of the gerneralizations, but does not mislead. Maple returns 0 for negative arguments which might be considered misleading. In all cases, it would be simple to program (9), and it would not be difficult to regard this as recognition of a well-established convention.

Associated Stirling numbers require a third argument relative to basic ones. It would thus be straightforward to offer associated numbers to users be returning basic numbers for 2 arguments and associates numbers when a third argument is present. One implementation is available in [8].

Stirling numbers for complex arguments could be left for the moment until efficient algorithms have been developed. Also there is the question of whether sufficiently many applications will be found to justify the time of implementors at the commercial computational systems. The r-Stirling numbers would have to be given a separate name for their implementation, since they share with associated Stirling numbers the need for 3 arguments.

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