Incorporating digital figures in linear algebra coursework: Utilizing Markov chains as a context to facilitate mathematical observation and conjecture

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Abstract: *This paper presents a new interactive assignment in introductory linear algebra that uses digital figures and an applied context of population migration modeled by Markov chains to support conceptual exploration of eigentheory. Students interact with stochastic and non-stochastic matrices to observe properties of their eigenvalues and eigenvectors, and examine the convergence behavior of Markov chains. The assignment emphasizes authentic mathematical activity, guided by our Observation–Conjecture–Proof–Theorem (OCPT) framework. The figures automate computation, allowing students to focus on observing patterns and forming conjectures. Future directions include analyzing student data on the effectiveness of this activity and refining and developing further interactive figure assignments modalities.*

1. Introduction

Linear algebra is a foundational area of mathematics with wide-reaching applications across numerous disciplines, serving as a core component in fields such as data science, artificial intelligence, and various industrial technologies. Befitting these varied applied contexts, the role of technology in the application and utilization of linear algebra is clear, mirrored by the specific recommendations from the first and second Linear Algebra Curriculum Study Group (LACSG, LACSG 2.0; [1] and [7]) to increase curricular attention to abstract, formal topics and to improve technology integration in linear algebra education. Yet there is still a gap in studies of the deployment effects of technology usage in linear algebra classrooms [6].

For many students, especially those early in their mathematical studies, linear algebra introduces an unfamiliar landscape of mathematical objects and operations. Linear algebra is rife with dense notational complexity, as students contend with matrix, vector, and scalar quantities that may appear symbolically identical but can carry very different algebraic and computational nuances. Eigentheory in particular provides an example of this notational complexity, as in the fundamental eigenvalue-eigenvector equation, $Ax = \lambda x$, each variable denotes a separate object type, combined via two distinct operations that must be understood separately yet still equate. While the topic of eigentheory in linear algebra education has received some attention ([8], [9]), educational studies involving Markov chains are light in the literature (e.g., [2]), possibly due to the topic being reserved for advanced or topics courses in linear algebra.

In a prior study [4], we described a series of digital interactive figures designed to help students engage with the abstract structures of eigentheory through observation and pattern recognition. That earlier work focused on building intuition around eigenvalues and eigenvectors using randomly generated matrices and matrix powers, with minimal emphasis on context or application. The pedagogical approach emphasized reducing computational demands in favor of fostering conceptual insight and mathematical experimentation via structured digital environments.

In this subsequent paper, we present a new assignment built on the same interactive framework but extended to incorporate an applied context: population migration modeled by Markov chains. Markov chains are employed in a wide array of domains such as physics and engineering, the life sciences, and business to predict the behavior of iterated experiments or measurements when each state relies only on the preceding state. By embedding eigentheory in a realistic iterative system and prompting students to explore the long-term behavior of matrix iteration, we aim to ground abstract mathematical phenomena in a setting that is both interpretable and motivating.

2. The Objectives and Design of Interactive Figures

2.1 Introductory Markov Chain Content

The mathematics content to be delivered by this assignment can be decomposed into three inquiries: First, the transition matrix system $x_{n+1} = Ax_n$ as it defines some iterative time process of interest; second, the properties of a stochastic transition matrix A (furthermore, this assignment provides regular stochastic matrices only, though the notion of regularity is not explored); and lastly, connecting the limiting behavior of the Markov chain to the probability eigenvector corresponding to eigenvalue 1 of the matrix A, namely that the Markov chain will converge to the eigenvalue-1 eigenvector scaled to a probability vector.

The first of the interactive figures we will discuss targets the properties of stochastic matrices. The activity introduces stochastic matrices with the following textbook [3] definition:

Definition: A stochastic matrix is an $n \times n$ matrix whose columns are probability vectors, those being vectors whose entries are non-negative and sum to 1.

The resultant properties of stochastic matrices to explore include that powers of stochastic matrices are also stochastic, 1 always occurs as an eigenvalue, the remaining eigenvalues are strictly less than 1 in magnitude, and eigenvectors corresponding to eigenvalues other than 1 must sum to 0.

2.2 Interactive Figure Assignment Pedagogy

The primary aim of this collection of interactive figures is to provide students with an accessible environment in which to encounter key ideas and patterns in linear algebra, specifically those related to Markov chains, while scaffolding their exploration with conceptual questions and simple proof problems. While our previous digital worksheets were designed as brief, introductory experiences that required approximately 15 to 20 minutes of engagement, this interactive figure assignment blends our interactive figure model with more traditional written homework styles. The particular assignment discussed herein would be given at or near the end of the content section on eigentheory, after all of the fundamental conceptual and computational ideas of eigenvalues and eigenvectors have been delivered.

The design process of the figures was shaped by two core student-centered principles: lowering the barrier to entry in terms of both time and computational complexity, and ensuring that students can interact meaningfully without needing prior technology or software expertise. On the instructional side, we prioritized automated example generation and built-in opportunities for students to engage in informal conjecturing based on their observations.

These design philosophies mean that students should be able to begin exploring with minimal setup or prerequisite knowledge. Operations such as matrix scaling, multiplication, and, in the present figures, eigenvalue/eigenvector computation are performed by the worksheet itself. This allows students to focus on analyzing patterns and relationships rather than carrying out routine calculations. By removing these procedural burdens, the figures seek to make it possible for students to quickly examine a wide variety of examples and more easily identify recurring behaviors or structures. As such, the functions available to students were limited to easy-to-use interfaces such as buttons and sliders that are immediately understandable to digital native students [5].

Although some students may wish to test their ideas beyond the worksheet's constraints, the restricted input format is intentional. The goal is not open-ended experimentation, but structured interaction that promotes pattern recognition and supports the formulation of general principles. For students interested in deeper or alternative exploration, we subsequently refer them to tools such as Wolfram Alpha and MATLAB.

2.3 Encouraging Mathematical Activity through Observation and Conjecture

Our design builds upon a conceptual discovery cycle common in mathematical practice: Observation, Conjecture, Proof, and Theorem (OCPT). This cycle is in opposition to the classical Definition-Lemma-Proof-Theorem-Proof-Corollary (DLPTPC) model of textbook mathematics, wherein concise mathematical statements are presented, proved, and sequenced in a polished trajectory. The OCPT model of discovery aims to engage students in the process of identifying and explicating the properties of mathematical phenomena, enabling them to gain greater ownership, insight, and understanding of the proof logic to follow and the conceptual connections between the objects being studied. It is also worth mentioning that the OCPT model is not meant to prescribe a perfectly linear process, but instead to invite a cyclic revision of ideas where, broadly, observations of mathematical phenomena lead to rules and statements about the structures or relationships involved (conjectures) which then demand some validation or rejection by proof to be accepted as fact.

The figures we have designed for this assignment are thus intended to provide space for students to encounter mathematical relationships with which they are not already familiar, and to construct their own perspective on the structures at play. At the introductory linear algebra level, where many students have likely not encountered the techniques of formal mathematical proof, we intend to provide students with an arena to engage primarily in observation and conjecture. This work can be followed up on in class or subsequent assignments to develop the formalized proof justifying their conjectures. As such, each figure was constructed with one or more conjectures that we would like students to pick up on, which we will discuss in Section 3. In our experience, students often miss patterns we intentionally designed the figures to reveal, while also noticing unexpected features that were not part of our instructional focus. These occurrences further necessitate the role of the instructor as mediator, with the figures and assignments being actively attended to in other class discussions.

2.4 Technology and Software Requirements

The interactive figures were developed in Mathematica using its ‘Manipulate‘ function, allowing dynamic, self-contained modules that respond to sliders, buttons, and menus with no coding experience required. This streamlined design makes the tools accessible to students with varying technological proficiency. Mathematica’s support for combining text, visuals, and code enables the creation of narrative-style worksheets, with interactivity scaffolded by explanatory guidance. While developing figures requires a Mathematica license, students can access and use them freely via the Wolfram Player, ensuring broad usability regardless of institutional software access.

3. The Markov Chain Assignment and Figures

In this section, we will show the two Markov chain interactive figures, as well as the associated assignment questions that prompt students to investigate these figures and apply their findings to solve problems.

3.1 Assignment Introduction: Partitioned Population Dynamics

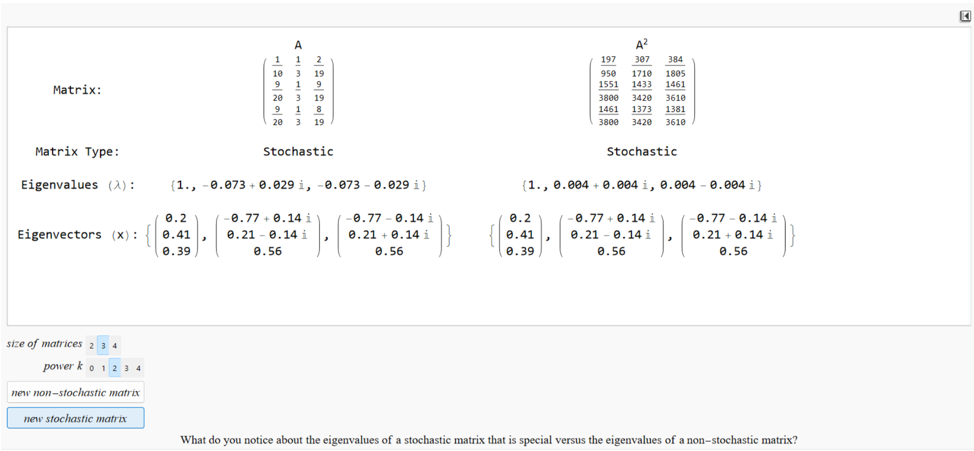
This assignment seeks to introduce fundamental Markov chain ideas through a contextualized problem scenario. One of the most readily accessible real-world contexts utilizing Markov chains is population dynamics in a partitioned population. Thus, the problem statement was adapted from [3] to prompt a two-population Markov relation, which students are instructed to create the following annual transition matrix and initial state vector:

$$A = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}, x_0 = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

Once the matrix in question is identified, students are asked to consider how the population changes across several years, at 2 years, 5 years, and a general k years. This problem is intended to get students to think about the iterative nature of Markov chains and potentially its connection to matrix exponentiation, which will be called upon in the next section.

3.2 Interactive Figure 1: Investigating the Properties of Stochastic Matrices

Consider the two images below of the first interactive figure focused on the eigenvalue properties of stochastic matrices (Figure 1). As standard for our matrix interactive figures, the student is able to select the size of the matrix (up to window display constraints) and the scalar exponent via button controls. Two “new matrix” buttons then allow the student to decide to generate new examples of either stochastic or non-stochastic matrices. Figure 1.1 depicts a typical result of generating a new stochastic matrix, with dimensions set to 3×3 and exponent set to 2. Figure 1.2 then depicts a typical result of generating a non-stochastic matrix, this time with dimensions 4×4 and exponent 3.



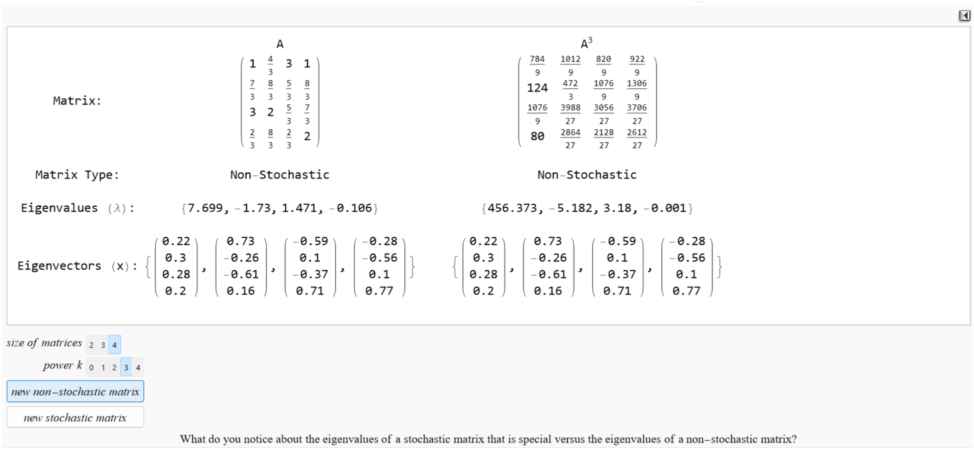


Figure 1: Powers of Stochastic (top) and Non-Stochastic (bottom) Matrices Interactive Figure

At the time this assignment would be given, students should be familiar with basic algebraic properties of eigenvalues and eigenvectors, for instance, that the eigenvalues of A^k are the eigenvalues of A individually raised to the k power, or that the eigenvectors of A^k are the same for all positive integer values of k. Nevertheless, these properties are observable in this figure.

However, the intended focus of this figure is on the properties of stochastic matrices and their powers. Students are prompted to experiment with this figure in Question 3 of the assignment, which asks them to note anything they "notice about the eigenvalues and eigenvectors of stochastic matrices compared to non-stochastic matrices?"

Through example generation, students could notice that 1 always occurs as an eigenvalue of a stochastic matrix, and furthermore that this eigenvalue is the largest in magnitude. However, this observation becomes trickier for novice students when imaginary eigenvalues occur. This potential observation is in contrast to the non-stochastic matrices, which are capable of generating leading eigenvalues greater or less than 1.

A deeper observation, utilizing the notion that matrix exponentiation applies to the eigenvalues of the original matrix, is that, since all eigenvalues of a stochastic matrix besides eigenvalue 1 are then strictly less than one, larger exponents will cause the tailing eigenvalues to shrink. Even as the figures currently bound the largest exponent to 4, within the necessary rounding of the figure's eigenvalue display, it can be seen that the tailing eigenvalues of A^k are quite close to 0. Such an observation can later be connected to the notion that the Markov chain iteration of a regular stochastic matrix will become dominated by the eigenvector corresponding to the eigenvalue 1, as the remaining eigenvalues tend to 0 for large exponents k.

Furthermore, given the probability vector definition of stochastic matrices provided, students may be inclined to investigate further the eigenvectors of stochastic matrices. As such, it can be observed that the eigenvector corresponding to eigenvalue 1 of a stochastic matrix is a probability vector, whereas the remaining eigenvectors' entries must sum to 0. Indeed, this is guaranteed as proof justifying that all stochastic matrices have 1 as an eigenvalue also demonstrates that the vector with all 1 entries is always a left eigenvector corresponding to eigenvalue 1. Therefore, since left and right eigenvectors of distinct eigenvalues must be orthogonal, all right eigenvectors of eigenvalue less than 1 must be orthogonal to the all-ones vector and thus have entries that sum to 0. As with observations about the eigenvalues of stochastic matrices, the non-stochastic matrices defy these patterns and serve to identify stochastic matrices as specially possessing these properties.

3.3 Subsequent Conceptual Questions

Following the above directed observational work with the first interactive figure, the students are tasked with a few introductory-accessible conceptual and simple algebraic proof problems. These problems are meant to reinforce the conjectures students may have made earlier (or point them towards these ideas, if not), as well as prime students for future observations in the final interactive figure.

4a. If $\lambda = 1$ is an eigenvalue of A, then 1 is an eigenvalue of A^k.

4b. What do you notice about the size of the other eigenvalues of A^k? Explain why you expect this to happen.

5a. Verify $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to eigenvalue 1.

5b. Find an eigenvector of your matrix A from (1.) corresponding to eigenvalue 1 that is also a probability vector.

The problems shown above are intended to extend from the observations students made in working with the interactive figures. For instance, after observing that 1 is an eigenvalue of a stochastic matrix (and perhaps as well an eigenvalue of A^k), Question 4.a. above codifies this relationship via an introductory-level proof problem requiring only basic manipulation of the fundamental eigenvalue-eigenvector equation.

Question 5 then seeks to prepare students for the main observation of the final interactive figure, that being that the steady-state vector of a Markov chain is the eigenvector of the matrix corresponding to eigenvalue 1, expressed as a probability vector. Providing the eigenvalues of the matrix and asking students to confirm a given eigenvector follows our principle of minimizing computation in favor of conceptual emphasis, while still encouraging students to bring to mind and engage with the fundamental equation $Ax = \lambda x$. Furthermore, scaling the eigenvector to be a probability vector prepares students to recognize this vector as the Markov chain limit, rather than the integer-valued eigenvector provided in part (5a) that would likely first arise from standard row reduction.

3.4 Culminating Interactive Figure and Observations

The final section of this assignment seeks to unite the observed properties of stochastic matrices with the long-term behavior of Markov chains. The question prompt asks students to consider the second interactive figure (Figure 2 below), and to adjust the initial population vector and observe what happens when the Markov chain is iterated, and how this behavior changes when the initial population is varied.

In the interactive figure, the transition matrix defined in question (1.) is provided, along with an adjustable x_0 probability vector. Users may apply

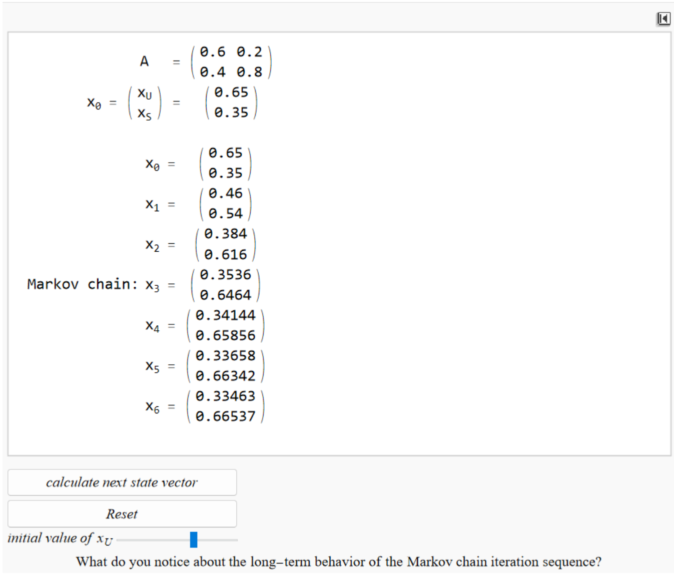


Figure 2: Markov Chain Iteration Interactive Figure

A iteratively, computing $x_n = Ax_{n-1}$, with the ability to reset the chain to test new initial vectors. Two main observations are intended to be made, directed by the question prompt: first, the sequence of iterates approaches the eigenvector calculated in question (5.b); and second, that this convergence occurs for any chosen vector x_0.

While the ideas in this figure are more complex than we could reasonably expect students to work through algebraically, the observations made in the figures still provide useful introductory leverage for students first engaging with Markov chains. Furthermore, as discussed in the pedagogy of the Observation and Conjecturing levels of activity, these observations can then be followed up upon in subsequent class time, where an instructor can demonstrate further or more complex ideas.

4. Concluding Remarks and Future Works

This assignment represents our continued efforts to refine the role of digital interactive figures in supporting conceptual, theoretical understanding in linear algebra via a system of observation and conjecture as authentic mathematical experience. As instructors, we are still actively exploring how best to balance freeform student observation with more targeted conceptual questions, and how to effectively frame these explorations within meaningful applied contexts. The integration of an application, here a population-based Markov chain, offers a promising way to make abstract concepts more accessible and explanatory, but we continue to examine how such contexts can most effectively motivate and structure student thinking.

Our design process remains iterative. Each implementation of these materials provides us with new insights into how students interpret, navigate, and reason with the mathematical structures we present. We view this work as part of a broader pedagogical exploration: how can technology support not just visualization or calculation, but as a genuine mathematical activity? Looking ahead, we are analyzing task-based interview data observing students perform this assignment individually and in small groups. Based upon our pilot experimentation, we are in the process of implementing these figures in an introductory course assignment. Furthermore, we are considering extensions to this assignment, including a possible precursor activity that introduces the power method which connects to Markov chain convergence.

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