Design and Implementation of a Mathematical Manipulative for Teaching Quadratic Number Fields

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Abstract: *We present the design and implementation of a mathematical manipulative aimed at enhancing the teaching of quadratic number fields. The tool visually represents algebraic integers as lattice points in a two-dimensional grid, using color-coded pegs to highlight special elements such as units, primes, and k-free integers. Constructed from laser-cut acrylic sheets and designed with modularity in mind, the manipulative helps bridge abstract algebraic concepts with tactile and spatial learning.*

1. Introduction

Mathematics is often perceived as more challenging than other subjects due to its inherently abstract nature. At the undergraduate level, particularly in pure mathematics courses, this difficulty is further amplified as students are introduced to increasingly abstract concepts, often for the first time. This poses a significant challenge in terms of comprehension and long-term retention. In such courses, the need for alternative and supplementary methods of grasping mathematical concepts becomes even more critical, as practical applications are frequently too complex to be explored in depth within the constraints of the curriculum. In the absence of immediate, tangible applications, students must rely heavily on textbook examples and external learning resources. However, these materials may not always be sufficient to ensure that students grasp the intended concepts effectively.

For instance, a survey conducted among 120 undergraduate students in Nepal [1] revealed that traditional learning methods, primarily lectures, board work, and written assessments, were less effective than intended in achieving learning outcomes in abstract algebra. Similarly, a case study [7] at a university in Indonesia identified a major source of difficulty in learning abstract algebra: students' heavy reliance on textbooks that are often too complex for undergraduate comprehension. These findings highlight the pressing need for alternative approaches and resources in the teaching and learning of mathematics even at the undergraduate level.

One recommended approach to enhance mathematics instruction involves the use of *mathematical manipulatives*–physical (e.g. mathematical toys and models), pictorial, or virtual (e.g. computer simulations) objects that students can interact with during their lessons [4]. These tools, along with the activities they support, offer students a more tangible way to engage with abstract concepts and promote active participation in classroom discussions. At the undergraduate level, pictorial and virtual manipulatives are more commonly used. However, physical manipulatives have the potential to foster more dynamic classroom interactions, as students can physically manipulate them, encouraging stronger connection between students and instructors.

For example, the use of physical manipulatives in pre-calculus and college geometry courses at the University of Puerto Rico helped bridge learning gaps for students with weaker mathematics backgrounds [11]. Another study [2] found that physical manipulatives at the undergraduate level not only increased student motivation and engagement but also encouraged students to ask more relevant questions and think more critically about the concepts being taught. Despite these benefits, physical manipulatives currently used in classrooms are geared toward introductory mathematics courses, such as pre-calculus and geometry. In contrast, for advanced subjects like abstract algebra, while virtual tools such as Group Explorer 3.0 [8] exist, there remains a notable lack of physical manipulatives designed specifically for these courses.

These challenges are also evident in the experiences of both students and instructors at Ateneo de Manila University, the authors' home institution, particularly in mathematics courses such as linear algebra, abstract algebra, and modern geometry. Student feedback from course and faculty evaluations indicates that manipulatives, when incorporated into lessons, are highly appreciated and contribute significantly to their learning. As students encounter increasingly abstract concepts for the first time in these courses, they often require various examples and visual aids to fully grasp the material. In response, the university's Department of Mathematics has initiated efforts to design and implement physical manipulatives in select courses. One such initiative involved constructing tiling models [10] and string art pieces using mathematics software and realizing them into laser-fabricated physical objects (see Figure 1.1).

(a) (b) (c) (d) (e)

Figure 1.1 Mathematical manipulatives developed at Ateneo de Manila University.

In this work, we present the design and fabrication of a mathematical manipulative that visually represents quadratic number fields using laser-cut acrylic sheets. Algebraic integers are depicted as labeled, color-coded pegs positioned at lattice points on a two-dimensional grid. These pegs highlight special elements of quadratic number fields, such as units, primes, and k-free points.

The manipulative serves as an accessible and engaging tool for exploring both elementary and algebraic number theory, enabling students to visualize and better understand classroom discussion on algebraic structures. In addition to promoting active participation, it enhances students' pattern recognition and geometric intuition by revealing rich geometric structures underlying these mathematical objects. The manipulative is inspired by recent studies, including our own research (see [3, 5] and references therein), on the symmetries of dynamical shift spaces derived from k-free points of number fields.

2. Quadratic Number Fields

The *quadratic (number) field* associated with a square-free integer $d \neq 0, 1$ is defined as the set $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$. An element of $\mathbb{Q}(\sqrt{d})$ is called an *algebraic integer* (or simply an *integer*) if it is a root of a monic quadratic polynomial with coefficients in $\mathbb{Z}$. The set of all such algebraic integers forms the *ring of integers* of the field.

A fundamental result in algebraic number theory (see Proposition 2.34 of [6]) states that the ring of integers of $\mathbb{Q}(\sqrt{d})$ is:

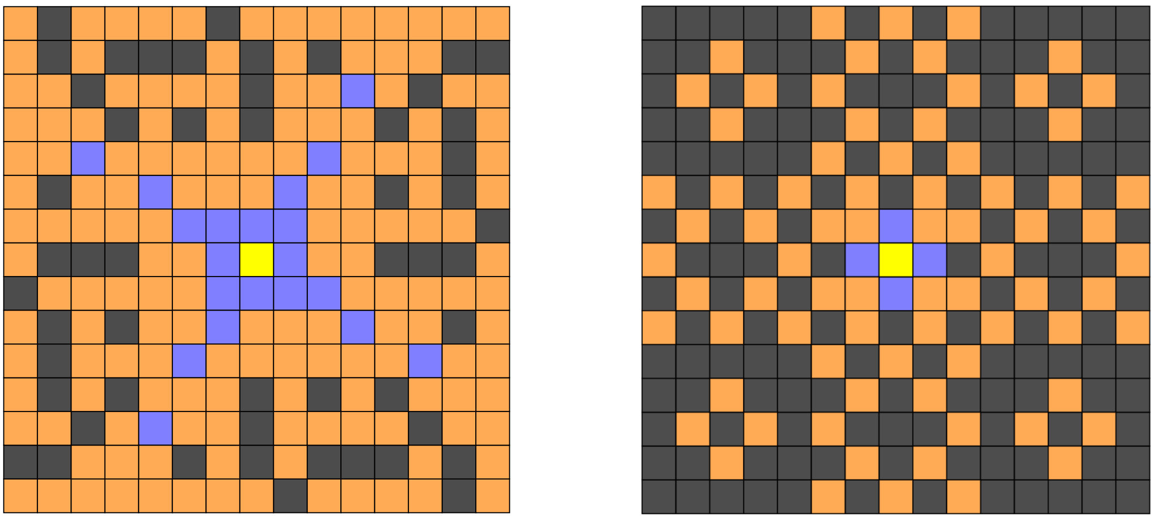
$$\mathbb{Z}[\omega_d] = \{a + b\omega_d \mid a, b \in \mathbb{Z}\}, \quad \text{where } \omega_d = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4, \\ \sqrt{d} & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

The ring of integers of a quadratic field (or more generally, a *number field*) extends the familiar notion of the integers $\mathbb{Z}$, often referred to as the ring of *rational integers* to avoid confusion. It generalizes key arithmetic concepts, such as primality and factorization into primes, which are central concepts in number theory.

Below are some important terms used to classify *special integers* in the ring $\mathbb{Z}[\omega_d]$:

- *Unit*: An integer whose multiplicative inverse in $\mathbb{Q}(\sqrt{d})$ is also an integer in $\mathbb{Z}[\omega_d]$.
- *Prime*: A nonzero, non-unit integer that can only be divided in $\mathbb{Z}[\omega_d]$ by itself or by a unit.
- *k-Free*: A nonzero, non-unit integer that is not divisible by the kth power of any prime in $\mathbb{Z}[\omega_d]$.

Since an integer $a + b\omega_d$ in the quadratic field $\mathbb{Q}(\sqrt{d})$ is a $\mathbb{Z}$-linear combination of 1 and ω_d, it can be visually represented as a unit square centered at the lattice point $(a, b) \in \mathbb{Z}^2$.



(a) (b)

Figure 2.1 Visual representation of selected integers in the quadratic fields (a) $\mathbb{Q}(\sqrt{5})$, (b) $\mathbb{Q}(i)$, highlighting special elements: yellow $= 0$, blue $=$ unit, orange $=$ prime.

Consequently, the ring $\mathbb{Z}[\omega_d]$ can be depicted as a grid of squares in the Cartesian coordinate plane, reflecting the structure of the underlying lattice formed by algebraic integers. Figure 2.1 shows a visual representation of the integers $a + b\omega_d$ in the quadratic fields $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(i)$, where $a, b \in [-7,7]$. In the diagram, the yellow square represents 0, the blue squares represent units, and the orange squares represent primes.

3. Design and Construction of the Manipulative

The primary objective of this work is to present the design and construction of a mathematical manipulative that visually illustrates the two-dimensional lattice structure of a quadratic field, as illustrated in Figure 2.1. This tool is designed to assist students in exploring and identifying integers within a number field that exhibits special properties.

3.1. 3D Model and Digital Design

We transform the abstract visual representation illustrated in Figure 2.1(b) into the 3D model shown in Figure 3.1. In this model, each integer $a + bi$ in the quadratic field $\mathbb{Q}(i)$ is represented by a removable, colored cylindrical peg labeled on top with the pair "a, b". The pegs are placed into shallow holes on a base composed of two layers, with the top layer specifically designed to accommodate the insertion and removal of pegs. The pegs vary in both color and height to convey additional information: *special* integers are highlighted using taller, brightly colored pegs, while non-special integers are represented by shorter, darker ones. This visual contrast enhances the visibility of special integers, making them easier to identify and analyze within the ring $\mathbb{Z}[i]$.

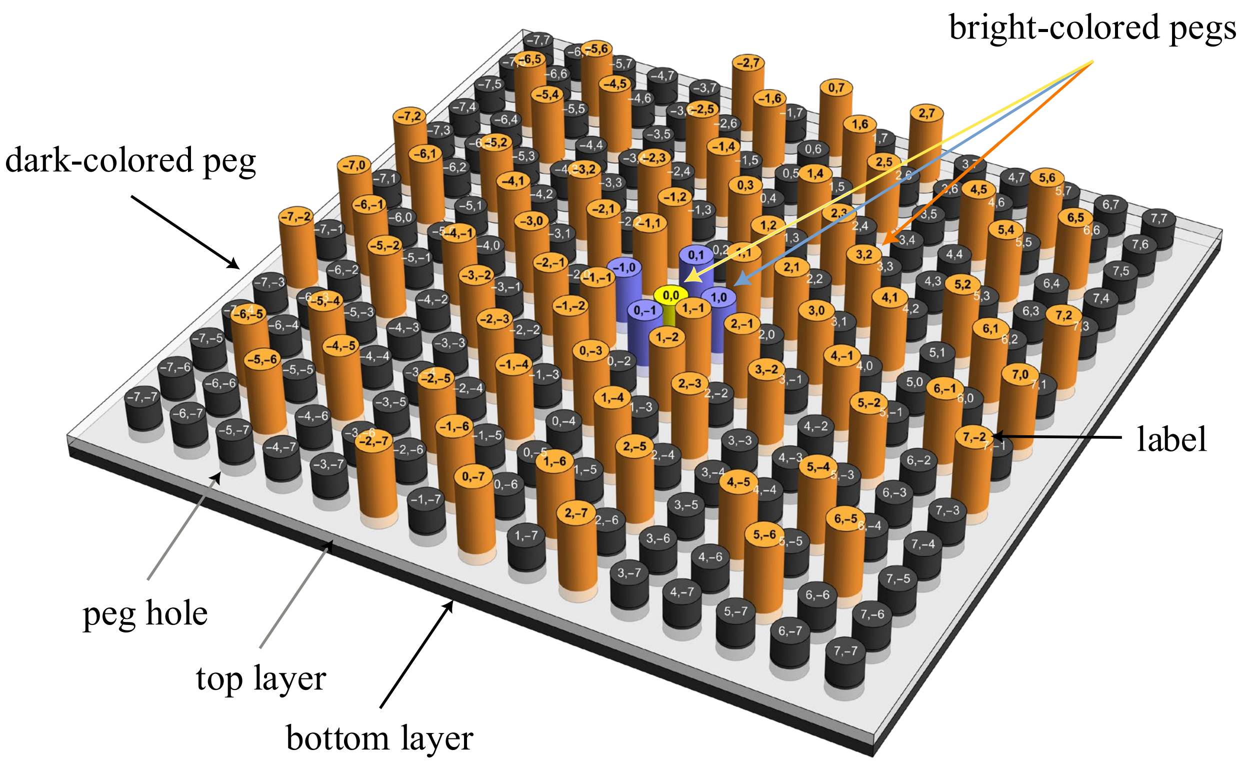
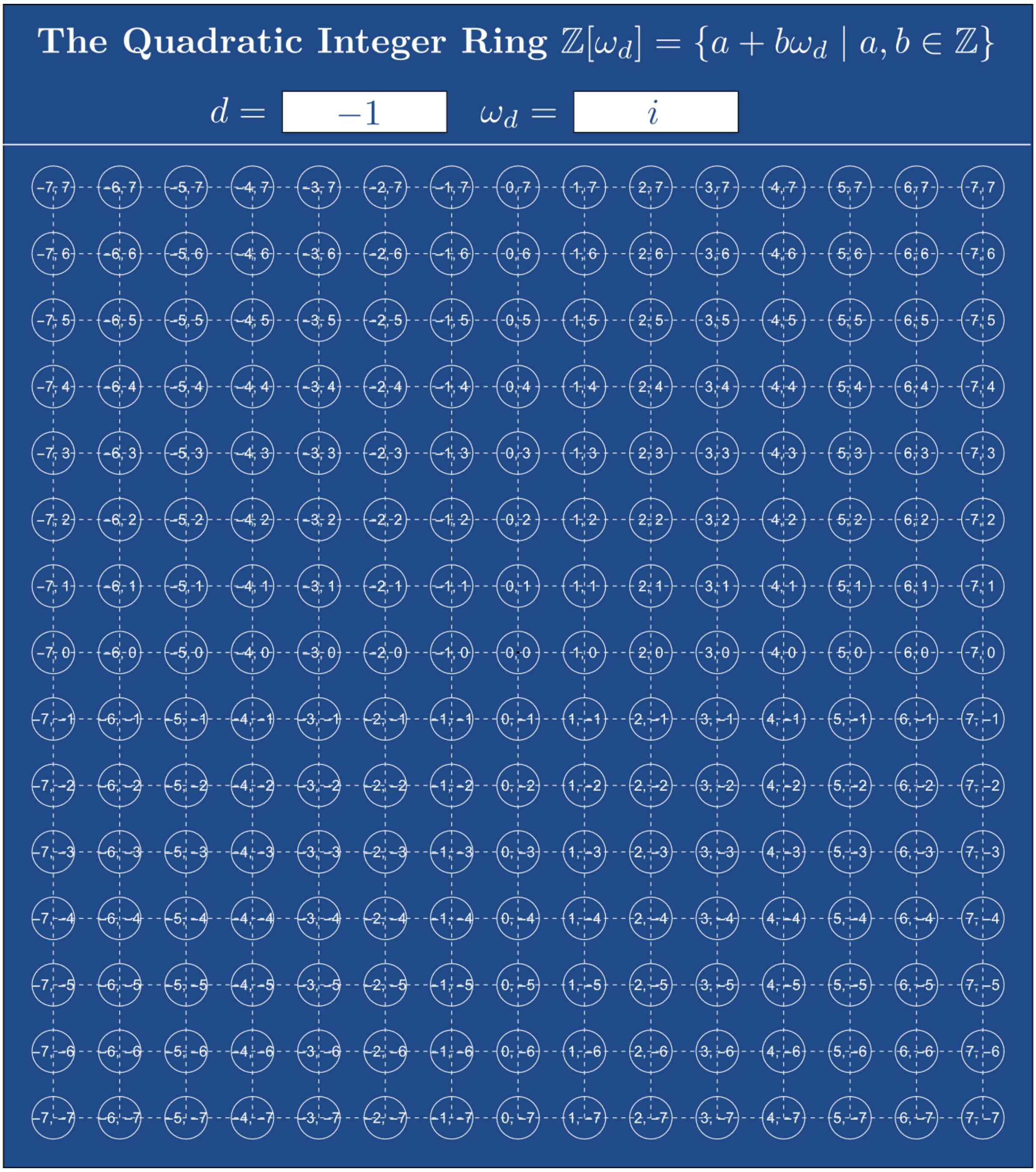


Figure 3.1 3D model of Figure 2.1(b), featuring removal pegs of varying colors and heights.

To ensure the resulting manipulative is adaptable across all quadratic fields, not just limited to $\mathbb{Q}(i)$, we incorporate replaceable label tiles that uniquely identify a field and, thereby, its ring of integers. Additionally, gridlines and lattice point labels are introduced to serve as visual guides. The digital sketch in Figure 3.2(a) illustrates these added features, along with a brief title at the top describing the manipulative.

The 3D model and accompanying sketch were created using the computer algebra system *Wolfram Mathematica* [14] and refined using the vector graphics software *Inkscape* [9].



(a)

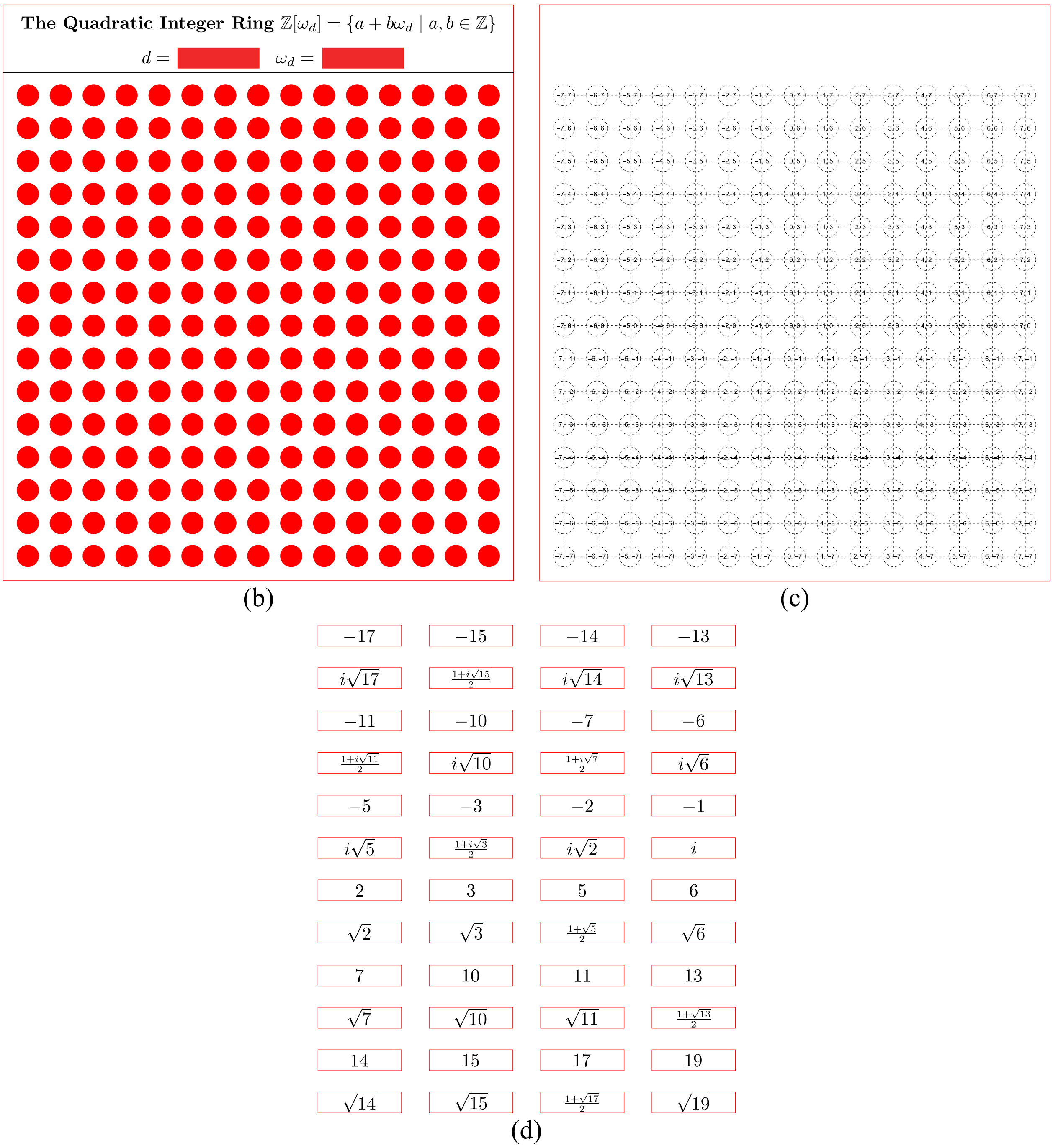


Figure 3.2 (a) Digital sketch of the manipulative, along with laser-ready files for the (b) top acrylic layer, (c) bottom acrylic layer, and (d) acrylic label tiles.

3.2. Laser Fabrication

To construct the manipulative, we bonded two acrylic sheets together to form the base. The cutting of holes, tiles, and pegs was carried out using the Universal Laser Systems (ULS) VLS 6.75 laser platform [13] (see Figure 3.3), located at the Eugenio Lopez Jr. Makerspace, our university's dedicated fabrication facility [12]. Labels were laser-engraved directly onto the acrylic for durability.

(a) (b) (c)

Figure 3.3 The ULS VLS 6.75 laser platform located at the Eugenio Lopez Jr. Makerspace.

The top layer is a 30 cm x 25 cm x 3mm clear acrylic sheet, where fixed labels, 1-cm diameter peg holes, and tile slots were engraved and cut. The bottom layer is made from a black acrylic sheet of identical dimensions, engraved with gridlines and lattice point labels. Each peg is constructed from stacked colored acrylic discs with a diameter slightly smaller than the peg holes to allow easy insertion and removal. The replaceable tiles are similarly made from stacked, laser-engraved black acrylic sheets.

Derived from the computer-generated sketch in Figure 3.2(a), we prepared three laser-ready vector graphics files–one each for the top layer (Figure 3.2(b)), the bottom layer (Figure 3.2(c)), and the replaceable tiles (Figure 3.2(c)). These files were designed in accordance with the specific guidelines of the laser platform's user interface for marking areas to be engraved or cut. In our configuration, regions for engraving are marked in black, while areas to be cut are marked in red. After laser fabrication, we assembled the final manipulative, as shown in Figure 3.4.

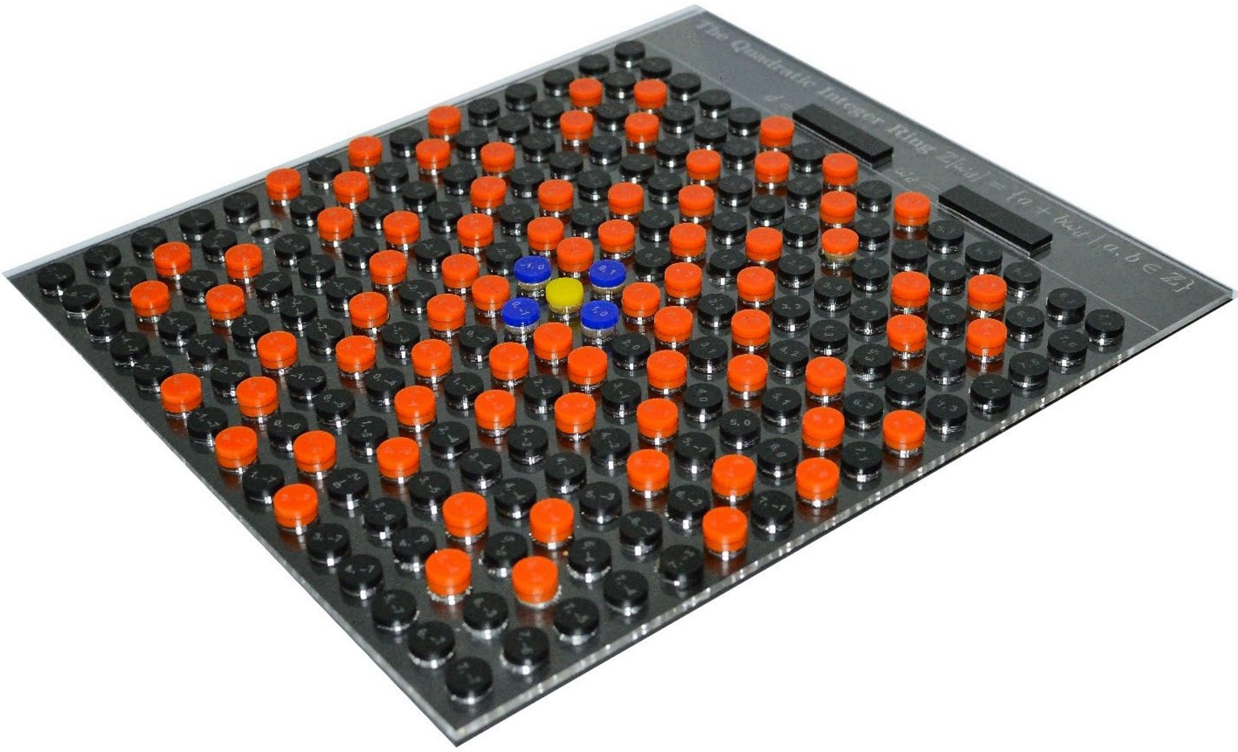


Figure 3.4 The assembled manipulative, reflecting the layout of the 3D model in Figure 3.1.

4. Integration into a Learning Module

Designed for integration into an algebraic number theory course, the manipulative described above serves as a hands-on educational tool to deepen students' conceptual understanding of quadratic fields through visual and spatial representation.

We present below a sample collaborative activity that can be integrated into a learning module on divisibility in a quadratic field. This activity combines traditional paper-and-pencil computation or computer-aided symbolic computations with the use of the manipulative.

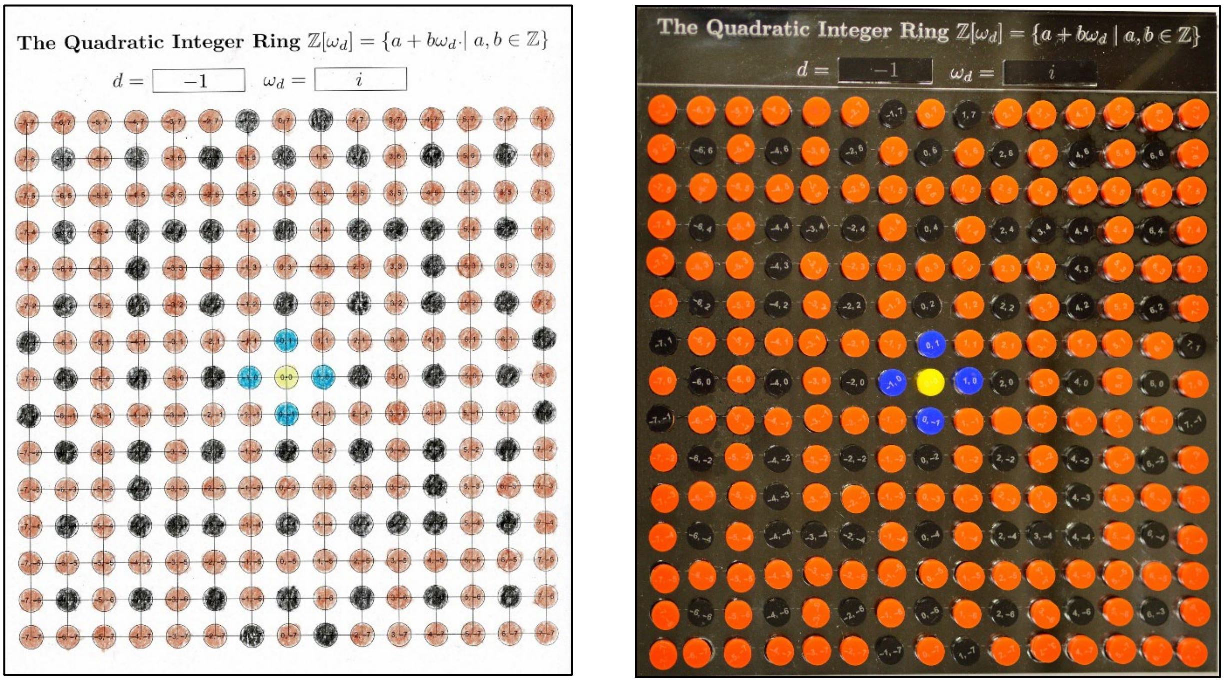
Student Group Activity: This assessment is worth **20 points**:

- Computation – 10 points
- Usage of Manipulative – 5 points
- Observations and Hypotheses – 5 points

Instructions: You are required to work closely with your group and will be given 30 minutes to complete the activity. For the quadratic field $\mathbb{Q}(i)$, complete the following tasks:

1. **Identify:**
   - The **units** in $\mathbb{Z}[i]$
   - The 2-**free** integers $a + bi$, where $a, b \in [-7,7]$
2. **Usage of Manipulative:**
   - Use one set of **brightly colored** pegs to mark the units.
   - Use another set of **brightly colored** pegs to mark the 2-free integers.
   - Use **dark-colored** pegs to mark the remaining integers on the manipulative.
3. **Observations and Hypotheses:**
   - Write three **observations** about the 2-free integers in $\mathbb{Z}[i]$.
   - Write two **hypotheses** about the 2-free integers in $\mathbb{Z}[i]$.
4. **Submission Requirements:**
   - A **list** of the units and 2-free integers, with clear and well-explained computations. If you use a computer algebra system, attach the **code** used in your submission.
   - A **photo** of your manipulative setup.
   - Your **written observations and hypotheses**.

A version of this activity was conducted by the second author in a previous Galois Theory course. At that time, prior to the development of this manipulative, students used colored pens to highlight special integers in a quadratic integer ring represented as a square grid printed on a worksheet (see Figure 4.1(a)). While this method proved effective, the newly introduced version, featuring the manipulative in Figure 4.1(b), offers a more dynamic and engaging learning experience. It allows students to explore mathematical objects more efficiently and facilitate quicker recognition of patterns, even within abstract structures.



(a) (b)

Figure 4.1 2-free integers in $\mathbb{Q}(i)$, as shown in (a) a student worksheet and (b) the manipulative.

5. Conclusion

This paper presented the design and implementation of a mathematical manipulative developed to support the teaching of quadratic number fields. Constructed from laser-cut acrylic sheets, the manipulative visually represents algebraic integers in these fields. Its design makes it a versatile resource for exploring core algebraic number theory concepts. By transforming abstract concepts into tangible, visual representations, the manipulative aims to foster deeper understanding and engagement among students.

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