

On a Systematic Construction of a Knot with Prescribed Symmetry Group

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Abstract: *This paper presents a method for constructing a mathematical knot with a prescribed symmetry group. In particular, we focus on the groups $[q]$, $[q]^+$, $[2, q]^+$, $[2^+, 2q]$, and $[2^+, 2q^+]$, described in Grünbaum and Shephard's paper entitled "Symmetry Groups of Knots", as possible symmetry groups of knots. We detail the procedure for generating a knot for each symmetry group type. Using GeoGebra's dynamic geometry tools, we construct knots from single-curved motifs and then extend these ideas to motifs with multiple curves. In each process we apply three dimensional isometries to a fundamental region of the group in order to generate a knot. This systematic approach provides a constructive framework for visualizing and analyzing knots and offers new avenues for integrating symmetry considerations into knot theory via dynamic visualization tools. We also illustrate how Mathematica's AI-assisted tools can automate the modeling of knots, showing its potential as a complementary platform for visualization and experimentation.*

1. Introduction

A *knot* is a closed, non-self-intersecting curve embedded in three-dimensional space \mathbb{R}^3 . The study of knots has been of interest to mathematicians not only for their underlying algebraic and geometric theories, but also due to their applications in different fields. In biology, for instance, knots can be used to describe the topology of certain molecular structures, such as the conformations of protein chains [1]. Meanwhile, in physics, knots can be utilized to model null electromagnetic fields, where the electric and magnetic field lines can form knotted structures that are preserved over time [2].

One notable development in the mathematical understanding of knots, specifically with regard to their symmetry groups is presented in a paper of Grünbaum and Shephard [4]. In their work, the authors revealed that the only possible symmetry groups of knots (expressed in Coxeter notation) are $[q]$, $[q]^+$, $[2, q]^+$, $[2^+, 2q]$, $[2^+, 2q^+]$, $[2, q]$, $[2, q^+]$, and $[2, \infty]$. The symmetry groups $[2, q]$, $[2, q^+]$, and $[2, \infty]$ are only possible for plane knots (i.e. knots lying entirely on a plane). Yet, there remains limited discussion in the literature regarding the methods by which a knot exhibiting a

specific symmetry group can be systematically constructed. This paper aims to address this gap by discussing a systematic method for constructing a knot with a particular symmetry group type. This method is carried out using the dynamic geometry software *GeoGebra* [5], which is equipped with the tools necessary for constructing knots and visualizing its properties. The process of being able to create knots by experimentation using technology can support the teaching and learning of the concepts of knots and their symmetries, since properties of three-dimensional objects like knots are often difficult to visualize. With the help of its interactive tools, *GeoGebra* also allows knots to be manipulated more easily, which can help students gain a better understanding of their symmetries. This paper will only focus on constructing knots with symmetry groups $[q]$, $[q]^+$, $[2, q]^+$, $[2^+, 2q]$, and $[2^+, 2q^+]$.

This paper is organized as follows. Section 2 discusses the preliminaries, establishing foundational terms and concepts used throughout the paper. Section 3 presents a detailed construction of a knot with a particular symmetry group using a single-curved knot motif. Section 4 extends the approach to knot motifs consisting of multiple curves. Section 5 demonstrates how AI-assisted tools in Mathematica can be used to automate the construction and visualization of knots. Section 6 presents concluding remarks and future directions. Finally, Section 7 provides the *GeoGebra* files used in the construction of knots and knot motifs presented in this paper.

2. Preliminaries

The symmetry group of a knot K , denoted by $Sym(K)$, is the group of isometries of \mathbb{R}^3 that leave K invariant. To better distinguish the five symmetry groups considered in this paper, it is helpful to describe a set of generators each group possesses. The group $[q]$ has generators q reflections and a q -fold rotation. In contrast, the group $[q]^+$ only has a q -fold rotation. The group $[2, q]^+$, has a q -fold rotation and q 2-fold rotations. Meanwhile, the group $[2^+, 2q]$ has a q -fold rotation, q reflections and q 2-fold rotations. Lastly, the group $[2^+, 2q^+]$ has a roto-reflection of period $2q$.

It is important to note that all symmetries of a knot leave one point invariant, namely its centroid [4]. For simplicity, we assume that the centroid of the knots considered in this paper lies at the origin. All axes of rotation and planes of reflection of a symmetric knot pass through the centroid.

In order to construct a knot K with a particular symmetry group G , it is necessary to introduce the concepts of knot motif and fundamental region for G . A *knot motif* m is a curve or set of curves in \mathbb{R}^3 which, when transformed under the action of all elements of G , produces a knot K with $Sym(K) = G$. To achieve this, the knot motif is constructed within a *fundamental region* for G . Adopting the definition of Klein [3], a fundamental region for a group of transformations is a system or set of points which contains one and only one point of every conjugate set.

The concept of a fundamental region has been extensively used in the study of tilings and other geometrical structures. In the context of plane tilings, a fundamental region is the smallest region which, when acted upon by the symmetries of the tiling, generates the entire pattern [6, 7, 8]. This concept also plays a central role in the analysis of the mathematical symmetries of Philippine indigenous textiles [9, 10] where the fundamental region and symmetries of the textile can be used to recover the whole textile. This was also used in [11] where different tilings were constructed by considering different vertex figure in the fundamental region and using symmetries to obtain the whole tiling. By analogy, our construction of knots proceeds by first placing the knot motif within a fundamental region of the prescribed symmetry group, so that the action of the group on this region produces the entire knot.

For any of the five groups presented, a fundamental region is the region bounded by two planes, P_1 and P_2 , intersecting along the z-axis at a certain angle. Specifically, for the groups $[q]$, $[2, q]^+$,

and $[2^+, 2q^+]$ the planes that bound the region must intersect at an angle of π/q . On the other hand, for $[q]^+$, which involves only rotations, a fundamental region spans twice the angular width of that for $[q]$, requiring the planes that bound the region to intersect at an angle of $2\pi/q$. Lastly, for $[2^+, 2q]$, the bounding planes of the region must intersect at an angle of $\pi/2q$.

3. Single-Curved Motifs

The construction of a knot with a particular symmetry group begins by constructing a fundamental region of the symmetry group. Using *GeoGebra*, a fundamental region for a knot with a symmetry group $[q]$, $[2, q]^+$, or $[2^+, 2q^+]$ can be constructed by first creating an initial plane P_1 that passes through the z -axis. Using the “Rotate Around Line” tool, P_1 is then rotated counterclockwise by an angle of π/q about the z -axis to get a second plane P_2 . The region bounded by P_1 and P_2 serves as a fundamental region for a knot with a symmetry group $[q]$, $[2, q]^+$, or $[2^+, 2q^+]$. In the case of $[q]^+$, P_1 is rotated by $2\pi/q$ instead to obtain P_2 . For $[2^+, 2q]$, P_1 must be rotated by $\pi/2q$ to get P_2 .

The simplest knot motif m that can generate a symmetric non-planar knot is an open, asymmetric curve that does not have any crossings, an example of which is illustrated in Figure 3.1. This curve is constructed in *GeoGebra* (see Section 7) using the “Point”, “Circumcircular Arc”, and “Line Segment” tools, and positioned within the fundamental domain so that its two endpoints, e and e^* , lie separately at specific locations in the interior of the bounding planes P_1 and P_2 , respectively. To simplify the application of transformations on m , the circumcircular arcs and segments that “make up” m are grouped into a list. During the construction, it must also be ensured that m does not lie entirely in the interior of a plane perpendicular to the z -axis. Otherwise, the motif will form a plane knot under the method presented in the succeeding paragraphs.

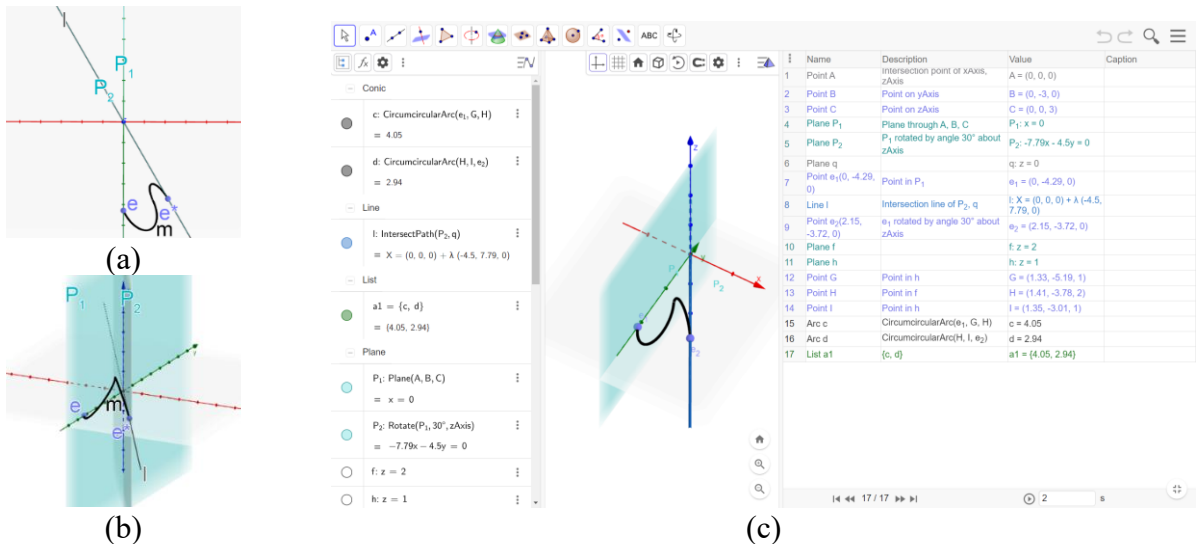
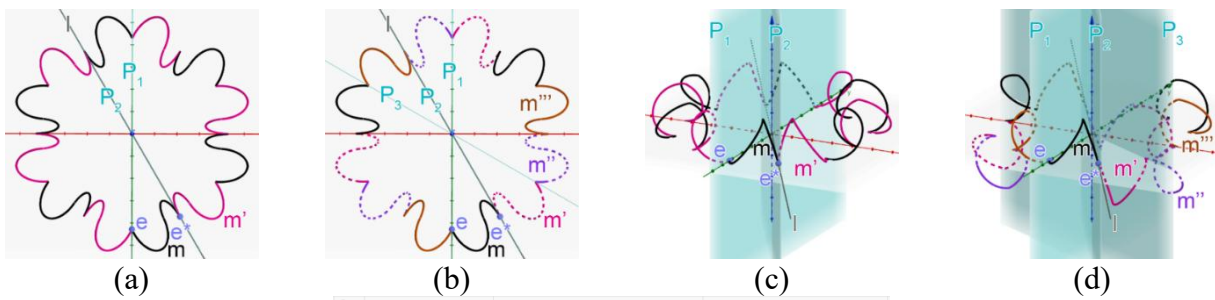


Figure 3.1: (a) Top view; and (b) side view of a single-curved, open, asymmetric, crossing-free knot motif m constructed in *GeoGebra*; (c) the entire *GeoGebra* interface with the Algebra and Construction Protocol windows.

For $[q]$ and $[2^+, 2q]$, the endpoint e of m can be placed arbitrarily in the interior of P_1 . Meanwhile, the endpoint e^* of m must lie on line l , defined as the intersection of the bounding plane P_2 and the z -plane. The line l can be constructed in *GeoGebra* using the “Intersect Two Surfaces” tool. Moreover, neither endpoint must coincide with the origin. To produce a knot with a symmetry group

$[q]$, the motif m is reflected across P_2 to obtain its mirror image m' . The list $\{m, m'\}$ is subsequently rotated counterclockwise about the z -axis by $2\pi j/q$ for $j = 1, 2, \dots, (q - 1)$. When this method is applied to the knot motif shown in Figure 3.1, a knot with a symmetry group $[6]$ is obtained, as shown in Figure 3.2(a).

On the other hand, to produce a knot with a symmetry group $[2^+, 2q]$, the motif m is first rotated counterclockwise about l by an angle π using the “Rotate Around Line” tool, producing the rotated image m' . Subsequently, P_1 is rotated counterclockwise by an angle of π/q about the z -axis to create a third plane P_3 . Both m and m' are reflected across P_3 to obtain their mirror images m''' and m'' , respectively. The list $\{m, m', m'', m'''\}$ is then rotated counterclockwise about the z -axis by $2\pi j/q$ for $j = 1, 2, \dots, (q - 1)$. Applying this method to the knot motif in Figure 3.1 yields a knot with a symmetry group $[2^+, 6]$, as shown in Figure 3.2(b).



i	Name	Description	Value
1	Point A	Intersection point of xAxis, zAxis	A = (0, 0, 0)
2	Point B	Point on yAxis	B = (0, -3, 0)
3	Point C	Point on zAxis	C = (0, 0, 3)
4	Plane P1	Plane through A, B, C	P1: x = 0
5	Plane P2	P1 rotated by angle 30° about zAxis	P2: -7.79x - 4.5y = 0
6	Plane q		q: z = 0
7	Point e1(0, -4.29, 0)	Point in P1	e1 = (0, -4.29, 0)
8	Line l	Intersection line of P2, q	l: X = (0, 0, 0) + λ(-4.5, 7.79, 0)
9	Point e2(2.15, -3.72, 0)	Point on l	e2 = (2.15, -3.72, 0)
10	Plane f		f: z = 2
11	Plane h		h: z = 1
12	Point G	Point in h	G = (1.33, -5.19, 1)
13	Point H	Point in f	H = (1.41, -3.78, 2)
14	Point I	Point in h	I = (1.35, -3.01, 1)
15	Arc c	CircumcircularArc(e1, G, H)	c = 4.05
16	Arc d	CircumcircularArc(H, I, e2)	d = 2.94
17	List a1	{c, d}	a1 = {4.05, 2.94}
18	List a1'	a1 mirrored at P2	a1' = {4.05, 2.94}
19	List m1	{a1, a1'}	m1 = {{4.05, 2.94}, {4.05, 2.94}}
20	List m1'	m1 rotated by angle 60° about zAxis	m1' = {{4.05, 2.94}, {4.05, 2.94}}
21	List m1'_1	m1 rotated by angle 120° about zAxis	m1'_1 = {{4.05, 2.94}, {4.05, 2.94}}
22	List m1'_2	m1 rotated by angle 180° about zAxis	m1'_2 = {{4.05, 2.94}, {4.05, 2.94}}
23	List m1'_3	m1 rotated by angle 240° about zAxis	m1'_3 = {{4.05, 2.94}, {4.05, 2.94}}
24	List m1'_4	m1 rotated by angle 300° about zAxis	m1'_4 = {{4.05, 2.94}, {4.05, 2.94}}

(e)

Figure 3.2: (a) A knot with symmetry group $[6]$ and (b) a knot with symmetry group $[2^+, 6]$ constructed in *GeoGebra* (see Section 7) using the motif in Figure 3.1 and the method detailed above. In (a) and (b), portions of the knot that are above the z -plane are shown as solid curves, while those below appear as dotted curves. (c) and (d) show the side views of (a) and (b), respectively. Meanwhile, (e) shows the construction protocol for (a).

In contrast, for the symmetry groups $[q]^+$, $[2, q]^+$, and $[2^+, 2q]^+$, the endpoint e of m must lie on line l' , defined as the intersection of the bounding plane P_1 and the z -plane. The line l' can be constructed in *GeoGebra* using the “Intersect Two Surfaces” tool. Meanwhile, the endpoint e^* must lie on l such that it is the image of e under a rotation about the z -axis by an angle equal to that of the intersection angle of P_1 and P_2 determined by the symmetry group. Additionally, no endpoint must coincide with the origin.

To produce a knot with a symmetry group $[q]^+$, the motif m must be rotated about the z -axis by $2\pi j/q$ for $j = 1, 2, \dots, (q - 1)$. When this method is applied to the knot motif shown in Figure 3.1, a knot with a symmetry group $[12]^+$ is formed, as shown in Figure 3.3(a).

In the case of $[2, q]^+$, the motif m is first rotated counterclockwise about l by an angle π using the “Rotate Around Line” tool, producing the rotated image m' . Afterwards, the list $\{m, m'\}$ is rotated counterclockwise about the z -axis by $2\pi j/q$ for $j = 1, 2, \dots, (q - 1)$. Applying this method to the knot motif in Figure 3.1 generates a knot with a symmetry group $[2, 6]^+$, as shown in Figure 3.3(b).

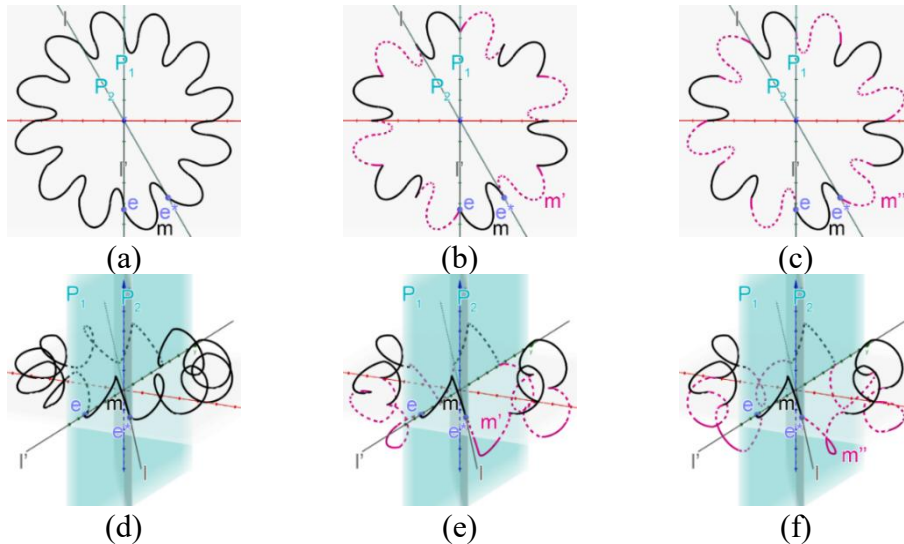


Figure 3.3: (a) A knot with symmetry group $[12]^+$; (b) a knot with symmetry group $[2, 6]^+$; and (c) a knot with symmetry group $[2^+, 12]^+$ constructed in *GeoGebra* (see Section 7) using the motif in Figure 3.1 and the method detailed above. In (a), (b), and (c), portions of the knot that are above the z -plane are shown as solid curves, while those below appear as dotted curves. (d), (e), and (f) show the side views of (a), (b), and (c), respectively.

Finally, for $[2^+, 2q]^+$, the motif m is first rotated counterclockwise by an angle of π/q about the z -axis to form the rotated image m' . The rotated image m' is reflected across the z -plane to produce m'' . The list $\{m, m''\}$ is rotated counterclockwise about the z -axis by $2\pi j/q$ for $j = 1, 2, \dots, (q - 1)$. Using this method, the motif in Figure 3.1 gives rise to a knot exhibiting a symmetry group $[2^+, 12]^+$, as shown in Figure 3.3(c).

The single-curved motif m may also be an open, asymmetric curve that contains crossings solely due to twisting. However, both this and the previously described crossing-free, single-curved motif will result in a non-planar knot that can be deformed to a trivial knot. To construct a symmetric knot that is not equivalent to a trivial knot using a single-curved motif, one can consider a known non-trivial knot, remove a small arc outside of its projection, and deform it to make it asymmetric. The resulting open, asymmetric curve then serves as the motif m , and a knot with a specific symmetry group can be made using the same method detailed earlier.

As an example, consider the trefoil knot shown in Figure 3.4(a). A single-curved motif m is derived from this knot shown in Figure 3.4(b) by removing a small arc outside its projection, resulting to endpoints e and e^* . The open curve is deformed to make it asymmetric and is then constructed in *GeoGebra* (see Section 7) within the fundamental region bounded by P_1 and P_2 , which intersect at the z -axis at an angle $\pi/2$. The endpoints e and e^* are placed on P_1 and P_2 , respectively, depending on the symmetry group of the knot one is trying to create. To construct a knot with a given symmetry

group, one may apply the method described above. For instance, to create a knot with a symmetry group $[2]$, the endpoint e is constructed in the interior of the bounding plane P_1 , while the endpoint e^* is constructed on line l . Moreover, both endpoints do not coincide with the origin. The motif m is reflected across P_2 to obtain its mirror image m' . The list $\{m, m'\}$ is then rotated counterclockwise about the z -axis by an angle of π to obtain the knot shown in Figure 3.4(c).

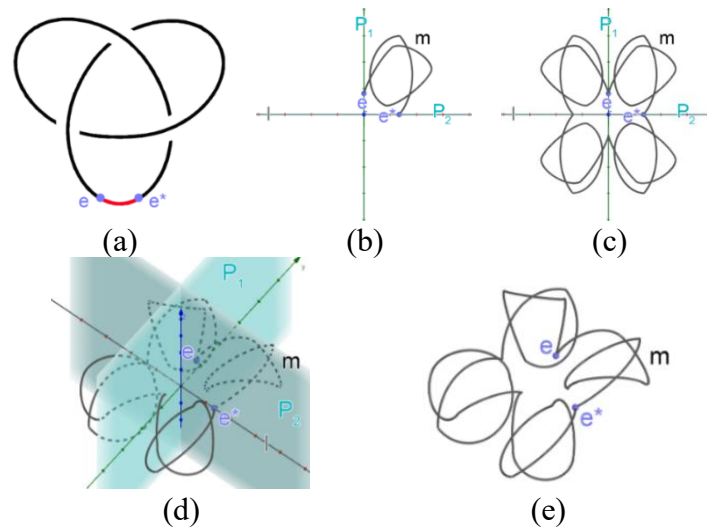


Figure 3.4: Example construction of a symmetric knot using a single-curved motif derived from nontrivial knots. (a) A projection of a trefoil knot; (b) a motif derived from the trefoil knot; (c) top view of a knot with symmetry group $[2]$; (d) and (e) side view of a knot with symmetry group $[2]$.

One way to verify whether the knot created has the prescribed symmetry group is to construct the knot using a different set of generators than what was originally used. For instance, since rotations about an angle 2θ is equal to the product of two reflections where the reflection planes intersect at an angle of θ , the symmetry groups $[q]^+$ could instead be generated by the product of those two reflections. Figure 3.5(a) shows a knot motif m bounded by P_1 and P_2 intersecting at an angle of 30° . If this knot motif is to be rotated at the same angle, the plane P_3 can be constructed where P_3 is separated from P_1 and P_2 at angle of 15° . The image of m under a composition of reflections about P_1 followed by P_3 is m' , which is equivalent to the image of m under a rotation by an angle of 30° . Similar sets of generators, where a generator is either a reflection or a product of reflections, could be found for the other symmetry groups [12].

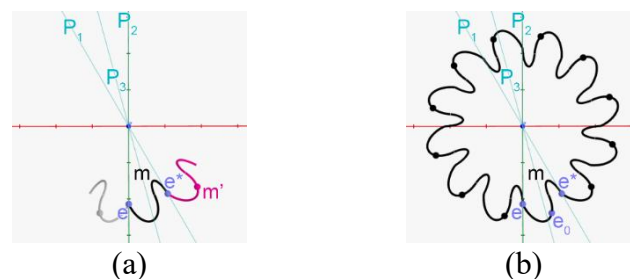


Figure 3.5: (a) Rotating a knot motif m using a composition of reflections and (b) positions of e_0 under transformations.

On the other hand, in order to verify whether the knot is symmetrical, it is possible to ‘track’ a specific point in the knot motif as it undergoes the transformations. If the point eventually returns to

itself, the knot is symmetrical. Figure 3.5(b) shows a point e_0 and its rotations. Note that the point eventually ‘returns’ to itself.

4. Motifs with Multiple Curves

In this section, we discuss the construction of knots having motifs with multiple curves. The construction is similar to the one described in the previous section. However, special attention must be given to the symmetries to ensure that the resulting structure is a knot and not a link. By a link, we mean a collection of two or more disjoint knots.

Consider the symmetry group $[q]$. Using this symmetry group to create a knot from a motif with multiple curves will ‘connect’ each curve to its reflection in the next segment. Figure 4.1 shows that this process creates a link instead of a knot. As such, it is not possible for knots whose motifs are composed of multiple strands to have a symmetry group $[q]$.

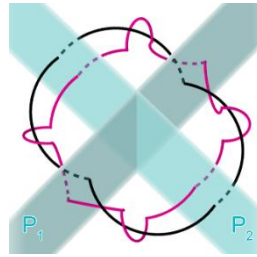


Figure 4.1: Multi-stranded link with symmetry group $[2]$ arising from a motif containing two curves.

Suppose that the endpoints of each individual curve are in the interior of different bounding planes. With this restriction, it is necessary that if e_1 in the interior of P_1 is an endpoint of a curve in a knot motif, then a curve in the knot motif should have e_1' as an endpoint in the interior of P_2 which is the image of e_1 under a rotation about the z -axis. For creating knots with symmetry group $[2, q]^+$, $[2^+, 2q]$, or $[2^+, 2q^+]$, then it is necessary that if e_1 in the interior of P_1 is an endpoint of a curve in a knot motif, its mirror image across a reflection on the z -plane, say e_d , must be also be an endpoint of a curve in the knot motif.

The shape of a curve within the fundamental region does not affect whether the knot motif will produce a knot or a link; only the connections to the rotated endpoints matter. Thus, the important consideration is how the curves will permute the endpoints and how those permutations are affected by a transformation. In Figure 4.2(a), the images of the points e_1, e_2, e_3 and e_4 under a rotation about the z -axis are e_1', e_2', e_3' and e_4' , respectively. Furthermore, e_1 and e_4 are images of each other under a reflection across the z -plane. Since the curves of the knot motif connect e_1 to e_2' , e_2 to e_3' , e_3 to e_4' , and e_4 to e_1' , this knot motif is associated with the permutation (e_1, e_2, e_3, e_4) , a 4-cycle.

Each isometry has a different effect on the permutation. Figure 4.2(b) shows that rotations about the z -axis preserve the permutation $((e_1, e_2, e_3, e_4) \rightarrow (e_1, e_2, e_3, e_4))$ while Figure 4.2(c) shows reflections invert the permutation $((e_1, e_2, e_3, e_4) \rightarrow (e_4, e_3, e_2, e_1))$. Figure 4.2(d) shows 2-fold rotations about an axis lying on the z -plane will invert the permutation and send mirror points to each other $(e_1, e_2, e_3, e_4) \rightarrow (e_4, e_3, e_2, e_1)$. Figure 4.3(e) shows roto-reflections will send mirror points to each other but otherwise preserve the permutation $(e_1, e_2, e_3, e_4) \rightarrow (e_4, e_2, e_3, e_1)$.

With this information, it becomes easier to see whether a knot or link will be created. Consider the same knot motif as before under a rotation. The resulting product is $(e_1, e_2, e_3, e_4)(e_1, e_2, e_3, e_4) = (e_1, e_3)(e_2, e_4)$. This implies that a rotation will send curves that send

e_1 to e_3 and vice versa, the same for e_2 and e_4 . Extending this result to a full revolution, if the product of the permutation and its necessary transformations is a p -cycle, where p is the number of curves in the knot motif, then the isometries result in a knot, otherwise there would be a link with n knots, where n is the number of disjoint cycles in the product.

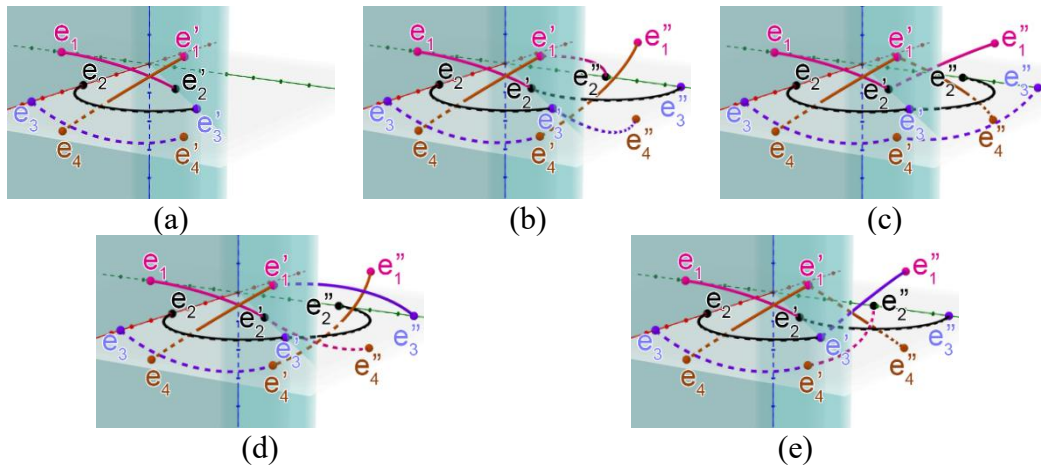


Figure 4.2: (a) a knot motif associated with the permutation (e_1, e_2, e_3, e_4) and the corresponding changes under the following transformations: (b) rotation: $(e_1, e_2, e_3, e_4) \rightarrow (e_1, e_2, e_3, e_4)$; (c) reflection: $(e_1, e_2, e_3, e_4) \rightarrow (e_4, e_3, e_2, e_1)$; (d) 2-fold rotation: $(e_1, e_2, e_3, e_4) \rightarrow (e_4, e_3, e_2, e_1)$; and (e) rotoreflection: $(e_1, e_2, e_3, e_4) \rightarrow (e_4, e_2, e_3, e_1)$

Now, consider a knot motif which permutes the points e_1, e_2 , and e_3 , where e_1 and e_3 are mirror images of each other about the z -plane. Since the knot motif connects e_1 to e_2' , e_2 to e_1 and e_3 to e_3 , we obtain the permutation (e_1, e_2) . In order to construct a knot with symmetry group $[2, 4]^+$, a 2-fold rotation followed by three rotations about the z -axis is needed to complete the knot (note that a 2-fold rotation will transform (e_1, e_2) to (e_2, e_3)). This corresponds to the product: $((e_1, e_2)(e_2, e_3))^{1+3} = (e_1, e_3, e_2)^4 = (e_1, e_3, e_2)$. The product implies that in a “revolution”, a curve originating from e_1 will go to e_3 , a curve originating from e_3 will go to e_2 , while a curve originating from e_2 will go to e_1 . Since all the curves connect each starting point to a different starting point, then the knot motif produces a knot under the symmetry group. Figure 4.3(a) confirms this result.

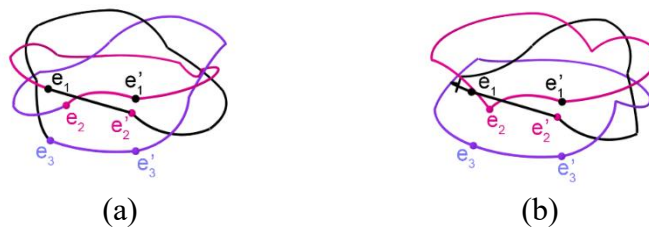


Figure 4.3: Knots whose motif has points e_1, e_2 , and e_3 , where e_1 and e_3 are mirrors images of each other on the z -plane and having symmetry group (a) $[2, 4]^+$; and (b) $[2^+, 8]$

A consequence of this result is that it is impossible to produce knots with symmetry group $[2^+, 2q]$ under the restriction. Suppose a motif with k curves with associated permutation (e_1, e_2, \dots, e_r) is used to create a knot with symmetry group $[2^+, 2q]$. Suppose the permutation transforms into (e_r, \dots, e_2, e_1) under a 2-fold rotation. Constructing the knot would require a 2-fold rotation followed by a reflection, and finally $q - 1$ rotations about the z -axis. This corresponds to the product:

$\left(((e_1, e_2, \dots, e_r)(e_r, \dots, e_2, e_1))((e_1, e_2, \dots, e_r)(e_r, \dots, e_2, e_1))^{-1} \right)^{1+q-1} = (e_1)$. Since the product is the identity permutation, it is a link with k knots. Figure 4.3(b) shows the resulting link if the motif of the knot in Figure 4.3(a) was used to create a knot with symmetry group $[2^+, 8]$ instead.

Figures of these knots and links could be constructed in *GeoGebra* (see Section 7). Suppose you want to create a knot from a motif containing t curves. First, the endpoints e_1, e_2, \dots, e_t of the curves are placed in the first bounding plane P_1 (Figure 4.4(a)). These points are then rotated about the z -axis such that the rotated points e_1', e_2', \dots, e_t' lie in the interior of the second bounding plane P_2 (Figure 4.4(b)). ‘‘Circumcircular Arcs’’ and ‘‘Line Segments’’ could then be used to construct the curves that make the knot motif (Figure 4.4(c)). Each separate curve is then put into separate lists $\{m_1, m_2, \dots\}$. The processes described in Section 3 are then applied to these lists in order to complete the knot with the desired symmetry group (Figure 4.4(d)). Figure 4.4(e) shows another perspective of the completed knot without the bounding planes.

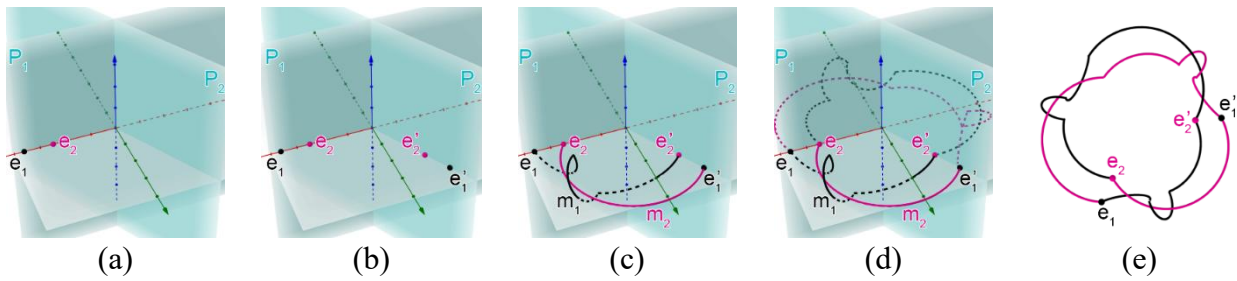


Figure 4.4: Construction of a knot from a knot motif with two curves

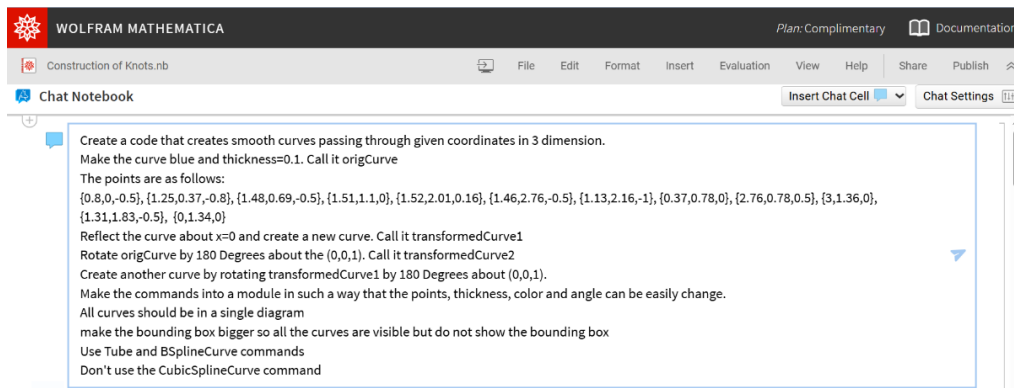
5. AI-Assisted Knot Construction in Mathematica

Recent advancements in AI-assisted computation within mathematical software have introduced powerful tools for automating and visualizing complex structures like knots. One such software is *Wolfram Mathematica* [13], which now offers AI-assisted notebook generation. These tools allow users to input natural language queries and receive code that models intricate mathematical structures with minimal manual effort.

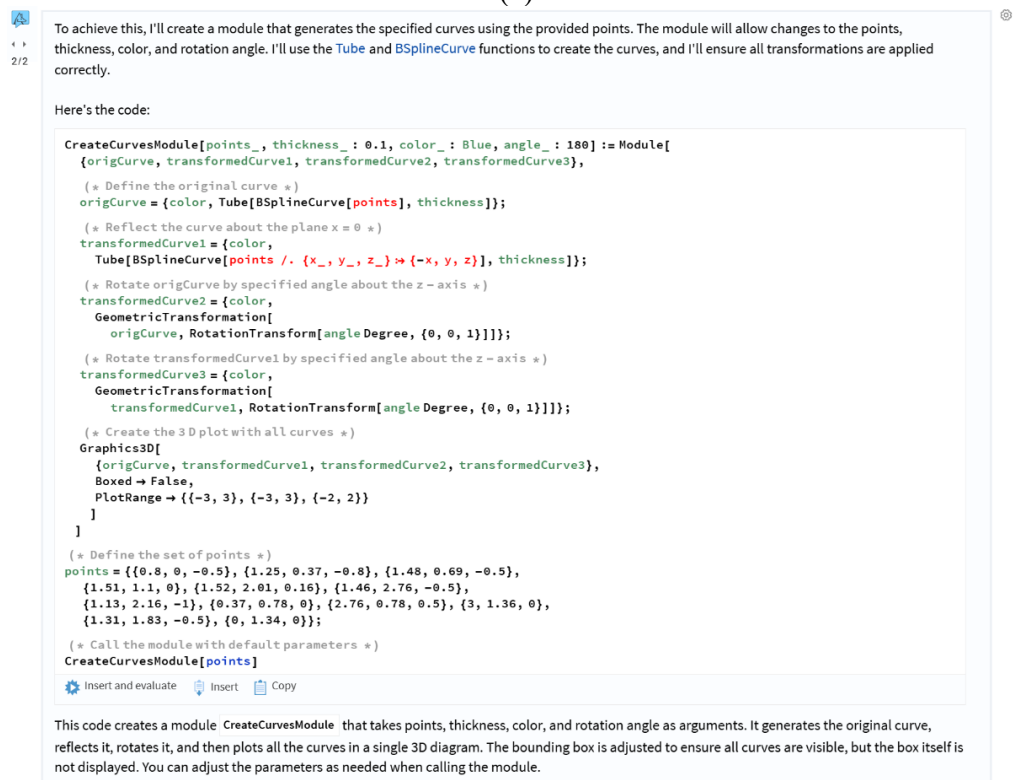
To illustrate this new feature, we recreated the knot shown in Figure 3.4, originally constructed in *GeoGebra*, using *Mathematica*’s AI Chat Notebook environment. The prompt used to initiate the construction is shown in Figure 5.1(a). *Mathematica* responded by generating a code that implements the instructions in the prompt (Figure 5.1(b)). When executed, the code produced a 3D model of the knot with symmetry group $[2]$ and has single-curved knot motif derived from the trefoil knot. The knot generated is shown in Figure 5.1(c) and its top view perspective in Figure 5.1(d)

To verify the correctness of the knots produced in *Mathematica*, we adopted a consistency check based on alternative generator sets. Specifically, for a given symmetry group, we constructed the same knot motif by providing the same initial set of points, but then applied a different generating set of isometries corresponding to the group. The resulting knot was compared against those obtained from the original generating set. The set of points in the resulting knots coincides with each other confirming that the *Mathematica*-generated knot is consistent with construction. This procedure ensures that the AI-assisted code is not merely producing a visually plausible structure but is mathematically accurate with respect to the intended symmetry group. For example, the knot in Figure 5.1 generated by a reflection and a 2-fold rotation can also be generated by two reflections. Figure 5.2(a) shows the *Mathematica* code used construct a second knot using two reflections as generators. The points in the resulting knots were then compared using the code in Figure 5.2(b).

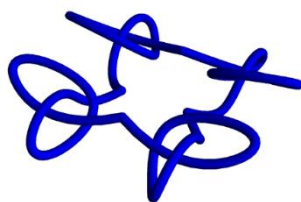
This example illustrates the growing role of AI in mathematical experimentation and communication. Tools like *Mathematica* can now serve not only as computational tools but also as collaborators, capable of interpreting natural language, generating mathematically meaningful code, and offering immediate visual feedback.



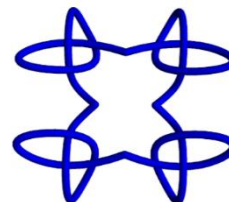
(a)



(b)



(c)



(d)

Figure 5.1: Construction of a knot using Mathematica. (a) Chat notebook prompt; (b) Mathematica code generated; and (c)-(d) the knot generated

```

CreateCurvesModule[points_, thickness_ : 0.1, color_ : Blue] := Module[
{origCurve, transformedCurve1, transformedCurve2, transformedCurve3},
(* Define the original curve *)
origCurve = {color, Tube[BSPLineCurve[points], thickness]};

(* Reflect the curve about the plane x = 0 *)
transformedCurve1 = {color, Tube[BSPLineCurve[points /. {x_., y_., z_} >> {-x, y, z}], thickness]};
(* Reflect origCurve about the xz plane *)
transformedCurve2 = {color, Tube[BSPLineCurve[points /. {x_., y_., z_} >> {x, -y, z}], thickness]};
(* Reflect transformedCurve1 about the xz plane *)
transformedCurve3 = {color, Tube[BSPLineCurve[points /. {x_., y_., z_} >> {-x, -y, z}], thickness]};

(* Create the 3D plot with all curves *)
Graphics3D[
{origCurve, transformedCurve1, transformedCurve2, transformedCurve3},
Boxed -> False,
PlotRange -> {{-3, 3}, {-3, 3}, {-2, 2}}
]
]

(* Define the set of points *)
points = {{0.8, 0, -0.5}, {1.25, 0.37, -0.8}, {1.48, 0.69, -0.5},
{1.51, 1.1, 0}, {1.52, 2.01, 0.16}, {1.46, 2.76, -0.5},
{1.13, 2.16, -1}, {0.37, 0.78, 0}, {2.76, 0.78, 0.5}, {3, 1.36, 0},
{1.31, 1.83, -0.5}, {0, 1.34, 0}};

(* Call the module with default parameters *)
CreateCurvesModule[points]

```

(a)

```

(* Original set of points *)
points = {{0.8, 0, -0.5}, {1.25, 0.37, -0.8}, {1.48, 0.69, -0.5},
{1.51, 1.1, 0}, {1.52, 2.01, 0.16}, {1.46, 2.76, -0.5},
{1.13, 2.16, -1}, {0.37, 0.78, 0}, {2.76, 0.78, 0.5}, {3, 1.36, 0},
{1.31, 1.83, -0.5}, {0, 1.34, 0}};

(* Method 1: Reflection about x = 0 and 180-degree rotation about the z-axis *)
reflectedPoints = points /. {x_., y_., z_} >> {-x, y, z};
rotatedPoints = points /. {x_., y_., z_} >> {-x, -y, z};

(* Method 2: Two reflections (about x = 0 and then xz plane) *)
reflectedPoints1 = points /. {x_., y_., z_} >> {-x, y, z};
reflectedPoints2 = reflectedPoints1 /. {x_., y_., z_} >> {x, -y, z};

(* Print enumerated points for both methods *)
Print["Points from Method 1: ", rotatedPoints];
Print["Points from Method 2: ", reflectedPoints2];

(* Compare the two sets of points *)
samePointsQ = Sort[rotatedPoints] == Sort[reflectedPoints2];
Print["Do the two methods generate the same points? ", samePointsQ];

```

(b)

Figure 5.2: Mathematica code used in (a) constructing a knot generated by two reflections; and (b) comparing the two resulting knots.

6. Conclusion and Outlook

We have demonstrated methods for constructing knots with prescribed symmetry groups using carefully chosen motifs and transformations in *GeoGebra*. For single-curved motifs, our procedure ensures that desired symmetry groups are realized, while for motifs with multiple curves, we have identified conditions under which the construction results in a knot rather than a link. These methods not only give concrete ways to model knots but also deepen our understanding of how symmetry constrains knot structures.

A future direction of study in this subject could involve removing the restriction imposed on Section 4 and letting both endpoints of a curve lie on a single bounding plane.

In addition to theoretical implications, there is significant potential for educational applications. Interactive tools and visualizations built on this framework can be integrated into curricula for geometry, topology, and mathematical art. By enabling students to manipulate knot motifs and immediately observe the resulting symmetries, this approach can foster a deeper understanding of spatial reasoning and group actions.

Moreover, the inclusion of AI-assisted mathematical software such as Wolfram Mathematica offers a new approach to knot construction. Its ability to interpret natural language prompts and generate accurate models makes it a powerful tool for both researchers and educators. This technology enhances experimentation, facilitates visualization, and bridges the gap between abstract mathematical concepts and their tangible representations. Future work may explore further integration of AI tools in interactive knot modeling platforms, opening up new avenues for discovery and instruction.

7. Supplementary Electronic Materials

The following files provide the *GeoGebra* constructions of the motifs and knots discussed in this paper. These materials are accessible online through the links listed below.

1. Figure 3.1: <https://www.geogebra.org/m/gcy2e4bj>

2. Figure 3.2(a): <https://www.geogebra.org/m/xyybhk3>
3. Figure 3.2(b): <https://www.geogebra.org/m/zy9ygkjc>
4. Figure 3.3(a): <https://www.geogebra.org/m/jxzvmycz>
5. Figure 3.3(b): <https://www.geogebra.org/m/syjbmax>
6. Figure 3.3(c): <https://www.geogebra.org/m/cfct6ew3>
7. Figure 3.4: <https://www.geogebra.org/m/szra2rb9>
8. Figure 3.5: <https://www.geogebra.org/m/p9av4bkr>
9. Figure 4.1: <https://www.geogebra.org/m/heghx67k>
10. Figure 4.2: <https://www.geogebra.org/m/gkwd6hku>
11. Figure 4.3(a): <https://www.geogebra.org/m/m5qfvtwe>
12. Figure 4.3(b): <https://www.geogebra.org/m/tk6q6w8t>
13. Figure 4.4: <https://www.geogebra.org/m/hm2xwqht>

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