Designing a Three-Phase Cyclical Flipped Applied Calculus Classroom

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Abstract: *This paper addresses a common gap in flipped classroom implementations within higher education mathematics: a lack of explicit grounding in learning theories. To bridge this theory-practice divide, we introduce the three-phase cyclical flipped classroom (3PCFC) model, which organizes learning into a repeating cycle: (1) pre-class multimedia and quizzing, (2) in-class collaborative problem-solving, and (3) post-class reflection. Each phase is intentionally designed based on a synthesis of established learning theories. The pre-class phase is guided by the cognitive theory of multimedia learning (CTML) and cognitive load theory (CLT) to create cognitively efficient instructional materials. The in-class phase leverages constructivism and CLT to foster productive struggle and social knowledge construction and refinement. The post-class phase uses principles of self-regulated learning (SRL) and to promote metacognition. This paper details each phase, providing practical guidelines for implementation and demonstrating how the interplay of these theories offers a robust, coherent blueprint for designing effective, technology-enhanced mathematics instruction.*

1. Introduction

The flipped classroom emerged as a response to calls for more active, student-centered learning environments and switches the time and place of teaching and learning activities in a traditional (i.e., lecture-based) classroom [1]. In a flipped classroom, students first engage with new topics independently through digital resources before going to the in-class period. This frees up in-class time for more dynamic, collaborative activities such as collaborative problem-solving for mathematics courses. In mathematics learning, where there is an interplay between conceptual understanding and procedural fluency, the flipped classroom allows students to more carefully study foundational concepts and procedures at their own pace while dedicating in-class time with the teacher to the complex processes of mathematical reasoning and problem-solving [2].

Despite its widespread adoption and documented benefits such as improved student engagement and learning outcomes [3, 4], the design and implementation of flipped classrooms often lack solid theoretical grounding and are driven more by pragmatic choices about content delivery and the use of technology—as a collection of adopted “best practices” without an explicit theoretical basis that explain why these practices are effective or how they should be designed to optimize learning [5]. Without strong theoretical grounding, teachers turn to anecdotal evidence and personal preferences to make design decisions and to respond to challenges in implementation. This approach may fail to address small but critical dimensions in learning mathematics, such as cognitive and social considerations, which may potentially lead to poorly designed materials that adversely affect student learning [4].

This paper introduces the three-phase cyclical flipped classroom (3PCFC) model, developed to bridge the gap between theory and practice in mathematics education. The 3PCFC model is intentionally grounded in several learning theories to provide a rationale for its every element. The model is primarily informed by two frameworks: the cognitive theory of multimedia learning (CTML), providing design principles for effective multimedia instructional materials that align with the limitations of the human cognitive architecture [6], and the PICRAT framework for technology integration, guiding the deliberate integration of technology away from passive consumption [7]. These two are complemented by principles from cognitive load theory (CLT) [8] and self-regulated learning (SRL) [9].

Therefore, to bridge the theory-practice gap, this paper offers examples of activities to illustrate how the 3PCFC model's theoretical principles translate into concrete pedagogical design in applied calculus. Thus, this paper does not seek to present an empirical validation of a teaching method; instead, it offers a theoretically grounded flipped classroom model that mathematics educators and instructional designers can use to develop and implement flipped classrooms.

2. Theoretical Foundations

The design of multimedia learning materials must be guided by an understanding of how the human mind processes information. Cognitive load theory (CLT) posits that because working memory is limited, any form of instruction must manage the cognitive demands placed upon it [8]. The cognitive theory of multimedia learning (CTML) specializes CLT in the context of multimedia learning [6]. CLT and CTML have the same primary goal: to manage cognitive load during the learning process. Intrinsic cognitive load is the inherent difficulty of the content, extraneous cognitive load is the additional cognitive load due to the way information is presented, and germane cognitive load is the productive mental effort that is dedicated to creating a schema [10].

CTML is built upon three assumptions: (1) that humans process visual and auditory information through separate channels (dual-channel assumption), (2) that humans can handle and process only a few pieces of information at any given time (limited-capacity assumption), and (3) that meaningful learning and schema-building happen through active engagement with new information (active-processing assumption) [6]. From these assumptions, CTML offers principles for designing multimedia learning materials to reduce extraneous cognitive load; we discuss four of these principles. The segmenting principle posits that students learn better when content is segmented (i.e., complex information broken into chunks) and when they have control over learning pace. For example, chunking the presentation of a multi-step solution into individual steps helps manage intrinsic load by allowing students to consolidate understanding of one step before continuing. The coherence principle reduces extraneous cognitive load by eliminating unnecessary information or details, such as distracting elements that do not contribute to the mathematical discussion—this includes unnecessary visuals, filler words, and long pauses that possibly distract or break students' train of thought while watching a video. The personalization principle advises the use of a conversational tone to maintain a personal connection with the student. The signaling principle posits that visual cues to draw attention to information enhance cognitive processing. When there is much visual information presented, highlighting portions that are being talked about in the audio becomes crucial to avoid cognitive overload. This supports the active-processing assumption by drawing the student to what is important or currently being discussed, allowing them to select and process the information more efficiently. In mathematics learning, this may be implemented via live digital ink annotations in videos, which complements the worked-example effect posited by CLT [11]. Using different colors in live digital ink annotations allows students to track different steps in a solution, reducing the mental effort needed to follow the presentation of the solution, and thus more of the student's cognitive resources can be used for germane processing.

While CTML guides the design of multimedia instructional materials, the PICRAT framework for technology integration offers a lens to evaluate the use of technology with respect to the learning process. The PICRAT framework has two dimensions, each of which has three levels. The first dimension (PIC) concerns the question "What are students doing with the technology?" The three levels in this dimension are passive (students passively receive content through technology), interactive (students interact with technology), and creative (students use technology to create artifacts that demonstrate learning) [7]. The second dimension (RAT) concerns the question "How does this use of technology impact the teacher's pedagogy?" This dimension adopts the RAT model proposed by [12]. The three levels in this dimension are replacement (technology acts as a direct substitute for an analog tool), amplification (technology makes a traditional task more efficient or powerful, improving learning outcomes), and transformation (technology allows new pedagogical tasks that cannot be done without it). Effective use of technology ideally falls on the "higher" levels of both axes, but this is not to say that technology which falls under the interactive or amplification levels are ineffective in teaching and learning.

While CTML (and by extension, CLT) and PICRAT are the primary frameworks grounding the 3PCFC model, the 3PCFC model is also explicitly designed to foster self-regulated learning (SRL), which is "the self-directive process by which learners transform their mental abilities into academic skills" [9]. Self-regulated learning views learning as an activity that students must do proactively rather than as a reaction to teaching. Zimmerman's model of SRL views self-regulated learning as a cycle with three phases: forethought (characterized by task analysis behaviors and influenced by self-motivation beliefs), performance (characterized by self-control and observation behaviors), and self-reflection (characterized by self-judgment and self-reaction behaviors).

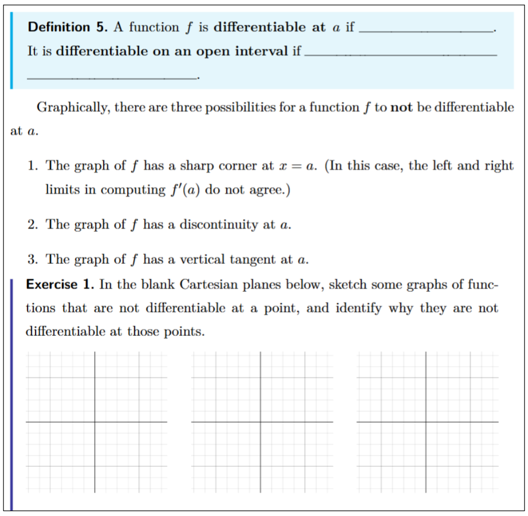
3 The Three-Phase Cyclical Flipped Classroom (3PCFC) Model

The three-phase cyclical flipped classroom (3PCFC) model organizes learning into a repeating, three-phase sequence—pre-class, where students gain exposure to fundamental concepts and procedural skill; in-class, where students apply and deepen understanding; and post-class, where students consolidate knowledge and engage in metacognitive reflection—reflecting the view that learning is a recursive process. Each cycle is a self-contained learning cycle for a given topic, preparing the student for the next topic [13] and reinforcing mathematical learning through opportunities for repeated applications. The 3PCFC model also develops students' capacity for self-regulated learning by aligning its phases to Zimmerman's three-phase model of SRL.

3.1 Pre-Class Phase

The primary goal of the pre-class phase is to provide students with self-paced initial exposure to new mathematical content primarily through carefully crafted multimedia resources designed according to the principles of CTML. The core component of this phase is lecture videos. To align with the segmenting principle of CTML, mathematical content is delivered through at most four videos (maximum 15 minutes each), each focusing on a single concept or procedure. This acknowledges students' limited working memory capacity and allows them to pause, rewind, or rewatch concepts or procedures that they find particularly difficult, managing the intrinsic cognitive load of the material [14]. The videos are narrated in a less formal (but not too casual) style, adhering to the personalization principle. Live digital ink annotations enhance the videos by the signaling principle. For example, the presentation of worked solutions, as in the videos in this playlist, involves different ink colors to make visible the teacher's often-unspoken thought processes and to provide explicit cues to guide the students' attention. The presentation of worked solutions using live digital ink in itself reduces the extraneous cognitive load required to decipher a static, pre-written solution [15].

To promote active engagement with instructional videos, students are also provided fill-in-the-blank handouts, such as that in Figure 3.1, which provide the structure of the instructional videos (definitions, theorems, examples) but omit definitions, notation, and derivations. These fill-in-the-blank handouts are presented as optional material that may be completed while watching the instructional videos. By providing the main structure of the lesson, the fill-in-the-blank handout reduces the extraneous cognitive load associated with students having to organize their own notes, thereby allowing them to focus their cognitive resources on understanding the lesson. This process transforms a passive viewing experience into an active learning task.



Definition 5. A function f is **differentiable at** a if _________________.
It is **differentiable on an open interval** if ______________________
____________________.

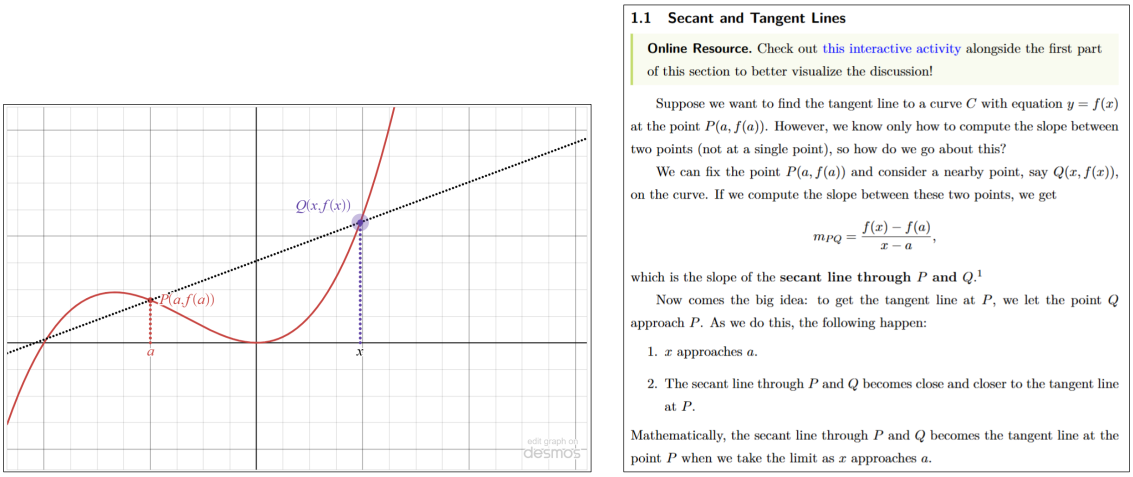
Graphically, there are three possibilities for a function f to **not** be differentiable at a.

1. The graph of f has a sharp corner at $x = a$. (In this case, the left and right limits in computing $f'(a)$ do not agree.)
2. The graph of f has a discontinuity at a.
3. The graph of f has a vertical tangent at a.

Exercise 1. In the blank Cartesian planes below, sketch some graphs of functions that are not differentiable at a point, and identify why they are not differentiable at those points.

Figure 3.1. A portion of the fill-in-the-blank handouts accompanying the video lecture on the derivative. This also includes exercises that are left to the students to answer independently.

For some topics, the 3PCFC model incorporates interactive graphical elements to deepen conceptual understanding. Using interactive Desmos graphs, students can manipulate mathematical objects and observe transformations in real time.



1.1 Secant and Tangent Lines

Online Resource. Check out this interactive activity alongside the first part of this section to better visualize the discussion!

Suppose we want to find the tangent line to a curve C with equation $y = f(x)$ at the point $P(a, f(a))$. However, we know only how to compute the slope between two points (not at a single point), so how do we go about this?

We can fix the point $P(a, f(a))$ and consider a nearby point, say $Q(x, f(x))$, on the curve. If we compute the slope between these two points, we get

$$m_{PQ} = \frac{f(x) - f(a)}{x - a},$$

which is the slope of the **secant line through** P **and** Q.[1]

Now comes the big idea: to get the tangent line at P, we let the point Q approach P. As we do this, the following happen:

1. x approaches a.
2. The secant line through P and Q becomes close and closer to the tangent line at P.

Mathematically, the secant line through P and Q becomes the tangent line at the point P when we take the limit as x approaches a.

(a) (b)

Figure 3.2. (a) An interactive Desmos graph embedded in the course LMS to motivate the definition of the derivative, and (b) the corresponding explanation included in the handout.

For example, teaching the definition of the derivative at a point usually begins with approximating the slope of the tangent line to a function at a point where it is continuous using a secant line through two points on the graph of the function and gradually moving one point closer to the other. In the interactive graph in Figure 3.2a, students, guided by the discussion in the handout in Figure 3.2b, can drag Point Q to see how the secant line PQ approximates the tangent line at P as x approaches a. This interactivity allows for exploration that static images cannot, helping students build more robust mental models of abstract concepts [16].

Each pre-class phase ends with a short, low-stakes self-check quiz—typically composed of multiple-choice or short-answer questions—in the class learning management system (LMS). These formative assessments take advantage of the "testing effect," where the act of retrieving information strengthens long-term memory [17]. They also provide immediate, customized feedback (especially for multiple-choice questions), as in the example question in Figure 3.3, which allows students to self-assess their comprehension and identify areas of confusion. This instant feedback reduces transactional distance and empowers students to take ownership of their learning, prompting them to review a topic they do not correctly understand. Finally, the results of the self-check quizzes can also inform the teacher about possible misconceptions or particular details of the lesson that students find difficult to understand.

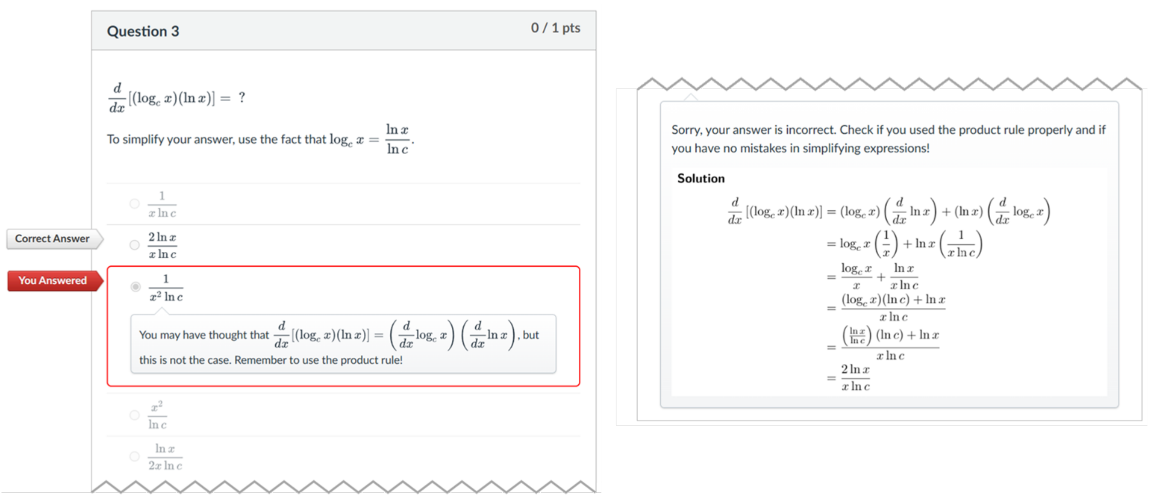


Figure 3.3. One question out of the three which form the self-check quiz on the topic of derivatives of exponential and logarithmic functions, including the adaptive feedback based on their answer.

3.2 In-Class Phase

After acquiring foundational knowledge in the pre-class phase, students go to the physical classroom to engage in the most critical part of the 3PCFC model: collaborative problem solving. The goal of the in-class phase is to move students beyond simple, routine exercises toward more complex problems requiring multi-step, higher-order thinking. The in-class phase typically begins with a 10-to-15-minute warm-up activity. This could be a "mini-lecture," structured as a whole-class discussion in which students can ask clarificatory questions about the pre-class materials, allowing the teacher to correct any misconceptions. The results of the pre-class self-check quiz allows the teacher to tailor the discussion to reinforce parts of the lesson which students find difficult to understand. Alternatively, the warm-up activity could be game based, using clicker questions (e.g., Kahoot!, as in Figure 3.4) to quickly reinforce foundational knowledge in an engaging manner. This initial segment bridges the individual pre-class learning to the more collaborative in-class activities and ensures that students are ready for in-class problem-solving.

Figure 3.4. (a) A Kahoot! session during the in-class phase for the topic on local extrema of a function. (b) The Kahoot! interface on screen. (c) The Kahoot! interface on students' devices.

After the warm-up, the bulk of in-class time is dedicated to students working in small groups of three to five on a printed set of in-class problems prepared by the teacher. These problems are intentionally varied, ranging from routine exercises to more complex, concept-heavy, or exploratory problems that demand deeper thinking or justification. This focus on active, collaborative work is the essence of flipped mathematics classrooms, transforming the classroom from a place of knowledge transmission to a dynamic space for social knowledge construction, application, and reinforcement. Communication required by group work (i.e., explaining one's thinking to peers) is a powerful learning tool which forces students to clarify their own understanding and confront inconsistencies—the "self-explanation effect" [18]. During this activity, the teacher becomes a facilitator, moving around groups to probe their thinking, to provide just-in-time scaffolding, and to occasionally call a group to the board to present their solution to the class.

The in-class phase is deeply rooted in social constructivism [19] and CLT. The focus on peer-to-peer interaction is an avenue for co-constructing and refining understanding through dialogue, acting as a scaffolding mechanism. Moreover, the cognitive load of solving a complex problem is distributed among the members of a group, effectively creating a "collective working memory" that is greater than the working memory capacity of any single student and allowing the group to tackle more complex problems than any individual student could alone [20].

3.3 Post-Class Phase

One learning cycle ends with the post-class phase which provides a structured, asynchronous reflective task. After the in-class phase, students complete a journal log, submitted through the class LMS. This journal log is structured, that is, guided by a set of carefully crafted prompts that encourage students to look back on the entire learning cycle, to move students beyond simply learning and doing mathematics to thinking about *how* they learned and did mathematics, what they understood, and where they struggled. This structured reflection promotes metacognition—"thinking about thinking"—which is a critical component of self-regulated learning [9, 21]. This includes questions such as *"Describe the effort you put into learning (and mastering) this lesson on optimization, both in pre-class and in-class activities. Do you believe this effort will contribute to your overall understanding and success in the course?"* and *"In general, do you find yourself investing more time in learning for the course than if it were delivered in a traditional (i.e., lecture-based) setting? If yes, what are your thoughts on having to spend more time in a flipped classroom? If not, what could be the reason/s why you have not felt the need to spend more time to learn in a flipped classroom setting?"* By making reflection an explicit and required part of the learning process, the 3PCFC model helps students develop the habit of monitoring and evaluating their own learning strategies.

This reflective task is grounded primarily in self-regulated learning. The prompts in the journal logs are explicitly designed to trigger the self-reflection stage of SRL, which involves self-judgment and self-reaction behaviors. For example, prompts that ask students to reflect and identify learning habits that are not effective encourage self-judgment about whether their approaches to studying work for them or not. Prompts that ask what they would do differently in the next learning cycle explicitly encourage them to think about how their previous learning experience can inform future learning behaviors, thus informing the forethought stage of self-regulation for the next learning cycle. Consistent engagement with this structured reflection can help students internalize self-regulated learning behaviors.

Beyond the reflective journal log, the post-class phase can also include homework assignments to further deepen conceptual understanding and strengthen procedural fluency. Continued practice after class need not be formal (e.g., due to a graded homework submission); in fact, students may engage in deliberate practice individually or collaboratively. This allows them to encounter exercise problems of various difficulties, from routine exercises to more challenging problems that require synthesis of concepts [22].

4 Considerations and Guidelines for Implementation

For teachers who wish to adopt or adapt the 3PCFC model, its theoretical basis translates into a set of actionable design principles that serve as a practical starting point for designing a theoretically grounded and pedagogically effective flipped mathematics course.

As a general rule, pre-class content must not be treated as a standalone homework assignment, as it directly prepares students for in-class problem-solving. Ensuring the quality and alignment of pre-class material with in-class activities is crucial for a smooth transition between the phases. The pre-class phase requires the teacher to invest a significant amount of time in preparing learning materials, especially for video production. Creating the slides for, recording, and editing a single 10-minute high-quality instruction video can take at least two hours, depending on the teacher's familiarity with the technology. One common pitfall to be avoided is having filler words (such as "uhhh" and "hmmm") or long pauses that may distract a student's train of thought and create extraneous cognitive load. To mitigate this increased workload, a teacher can build a repository of reusable instructional videos over several semesters or collaborate with colleagues to develop materials for shared courses. However, in spite of this initial increase in effort and time commitment, a well-designed set of pre-class learning materials provides a robust resource that enhances learning opportunities during class time.

For the in-class phase, successful implementation requires careful design of the physical classroom space as well as the in-class problems. The composition of groups for collaborative problem-solving may influence the success of the activity; teachers can form groups based on students' pre-assessment data, prior exposure to the topic, or random selection. During this phase, the instructor must also resist the urge to lecture and instead be a "guide on the side" [23], asking probing questions to guide students in their problem-solving without giving away the answer—"telling by asking" [24]. The adjustment to being a facilitator in class entails being able to monitor multiple groups simultaneously and provide differentiated support; this becomes difficult for large classes. Another challenge during the in-class phase is ensuring equitable participation within groups, avoiding only one or two doing most of the work while the rest become spectators. The in-class problems must lead to productive struggle, that is, they should be challenging enough that students cannot immediately solve them but not too difficult as to lead to frustration and disengagement. Productive struggle in this milieu encourages collaboration necessary for problem-solving success. During collaborative problem-solving, students must have opportunities to articulate their reasoning and make their thinking visible through board work or some other method. This allows students to develop skills in mathematical communication and move away from the belief that only the final answer is important.

The post-class phase, with its reflective nature, requires a careful design of prompts to ensure that students reflect on their learning. Such prompts are specific and process-oriented, such as "Describe one concept or procedure from the previous cycle that became clearer during the in-class problem-solving. What helped clarify it?" and "What learning habits do you think helped or did not help you understand this topic?" Student responses to these prompts need to be collated and summarized by the teacher to tailor the delivery of the remaining course lessons.

Finally, because flipped classrooms, in whatever form it is implemented, relies heavily on technology, teachers must be aware of and proactively address issues of equity and accessibility. Teachers should survey students at the beginning of the course to check their access to appropriate devices and reliable internet connections. Providing offline alternatives for students with limited internet access is crucial. This may include making learning materials downloadable for offline use. Furthermore, all learning materials should be designed in accordance with principles of Universal Design for Learning to accommodate diverse student backgrounds, abilities, and learning styles. This includes providing captions for videos and ensuring web-based content is screen-reader friendly whenever possible. These adjustments ensure that the 3PCFC model provides an equitable and effective learning experience for all students.

5 Limitations and Future Research Directions

As an expository framework, the primary contribution of this paper is the 3PCFC model's solid theoretical grounding drawn from the connections between existing learning theories and frameworks. However, this paper does not present an empirical validation of the model's effectiveness. As such, future research should point toward validating, refining, and extending the 3PCFC model, perhaps conducting quasi-experimental studies to compare it against traditional lecture-based classrooms or other implementations of flipped classrooms. Future studies should also measure both immediate learning gains and long-term retention of mathematical concepts in order to provide further evidence for the model's effectiveness.

The scalability of the 3PCFC model to large classes may also be investigated as a starting point for its possible implementation in large-enrollment courses or massive open online courses. This could involve injecting AI-powered tools to provide automated, targeted feedback to make the model's most time-consuming tasks for the teacher more manageable.

There is also a major opportunity to explore adaptive versions of the learning that happens in the pre-class phase. Further studies could design web video players that can capture micro-level analytics for each student's video-watching behavior (e.g., to identify parts of the videos that a student had to watch multiple times). This, together with tracking a student's performance in the pre-class self-check quizzes, could personalize their learning path. For example, a student who struggled to answer a particular question on the self-check quiz or had to rewatch a portion of a video multiple times may automatically be given supplementary material with additional discussions and worked examples. Research in this area may examine whether adaptive learning tools can improve learning in a flipped mathematics classroom.

Finally, the transferability of the 3PCFC model warrants investigation. Although the context for which the 3PCFC model was designed was university mathematics, its core principles may be applicable to other domains. Research could adapt the model other STEM fields to see which components of the model are mathematics-specific and would require significant modification. This could lead to the operationalization of a more general flipped classroom model that could be used across disciplines.

Acknowledgements: The authors would like to thank the Department of Science and Technology – Science Education Institute (DOST-SEI) for making this study possible.

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