Optimal Arrangement of Safety Net Problem on Sphere

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Abstract: *We devised and implemented the Spherical Game as mathematical teaching material for maritime students at a technical college, with the dual aim of fostering an understanding of spherical distances and enhancing students' motivation. Subsequently, we formulated the Safety Net Problem, which can be solved by employing a similar strategy. Both problems involve distances on a spherical surface such as the Earth, and are therefore familiar to students who are expected to become sailors in the future. The optimal arrangements for these problems can be expressed in terms of metric invariants, analogous to the strategy used in the Spherical Game. In this paper, we present the derivation of optimal arrangements and demonstrate them using Mathematica.*

1. Introduction

In this study, we consider the following *Safety Net Problem* as introductory material for mathematics: Assume that the Earth is a sphere of radius 1. A safety net system is constructed by placing k rescue bases on the surface of the Earth, where the bases may be located on the sea. The objective is to determine the optimal arrangement of these bases under the following conditions: (i) If an accident or natural disaster occurs at a point x on the Earth, the rescue team departs from the nearest of the k bases and travels to x along the shortest path. (ii) The entire Earth is uniformly covered, without geographical bias. (iii) The bases are placed so that the travel distance is as short as possible (see [6],[7]).

The *Spherical Game* was initially devised and implemented as teaching material for students in the Maritime Department of a technical college, with the aims of encouraging the study of spherical distances and enhancing the motivation of students. Building upon this, we subsequently developed the *Safety Net Problem*, which can be solved using a similar strategy. Both problems concern distances on a spherical surface such as the Earth, making them particularly meaningful for students preparing for maritime professions.

The optimal arrangements in both cases can be formulated in terms of metric invariants, in analogy with the strategy employed for the *Spherical Game*. In the following sections, we first describe the rules of the *Spherical Game* and introduce the concept of metric invariants.

2. Spherical Game and Metric Invariant

The *Spherical Game* is a tennis-like game played on the surface of a sphere, and its rules are defined as follows.

Two teams, each consisting of k players, compete on a sphere. The $2k$ players may freely move to decide their initial arrangement. A player from the first team throws a ball onto the sphere. From the moment the ball is thrown until it lands, the k players of the opposing team are not allowed to move. Once the ball lands, all k opposing players simultaneously run along the shortest path to the point where the ball has fallen, and the first player to arrive picks it up. That player then throws the ball back, and the same process is repeated, with the roles of the teams alternating.

The strategy for the throwing team is to send the ball as far away as possible from the positions of the opposing team. If a player from the opposing team reaches the ball, the distance traveled by that player becomes the score for the throwing team. Not all of the opposing players are required to move toward the ball. All players are assumed to move at the same speed. The sphere is taken to have radius 1, and is assumed to be colorless and transparent. Each player can determine whether the distance to the landing point of the ball is greater or less than $\pi/2$, the radius of a hemisphere.
The strategies that each team should adopt can be described using the metric invariant $m_k(X)$. On a surface X, the lower bound of the length of a curve connecting two points x and y is called the distance between them, denoted $dist(x, y)$. Such a curve is called a geodesic. In the case of a sphere of radius 1, when x and y are antipodal points, there are infinitely many geodesics connecting them, each of length π, which is half the circumference of a great circle. A great circle is obtained by intersecting the sphere with a plane passing through its center, and its circumference on a unit sphere is 2π.
We now define the metric invariant $m_k(X)$. Consider k points $x_1, x_2, \dots, x_k \in X$. For a point $x \in X$, define

$$p_x(x_1, x_2, \dots, x_k) \quad = max_{\{x \ in \ X\}} min_{\{l=1,2,\dots,k\}} dist(x, x_l).$$

That is, for each arrangement of k points, we take the minimum distance from x to the k points, and then consider the maximum of these values over all $x \in X$. Next, we minimize this maximum over all possible arrangements of the k points. The resulting value is defined as

$$m_k(X) = min_{\{(x_1, x_2, \dots, x_k) \ in \ X \times \dots \times X\}} max_{\{x \ in \ X\}} \ min_{\{l=1,2,\dots,k\}} dist(x, x_l).$$

The value $m_k(X)$ is a metric invariant, known as the *covering radius*. Several values of $m_k(X)$ are known for spaces such as spheres, hemispheres, and real projective spaces (see [1], [3], [8], [9]).

3. Relationship between Spherical Game and Safety Net Problem

We first clarify the relationship between the Spherical Game and the metric invariant.
Consider a k-versus-k match on a spherical surface X, consisting of team A ($A_1, A_2, \dots, A_k$) and team B ($B_1, B_2, \dots, B_k$). Suppose that A_1 throws the ball first. Among the opposing players, the one located closest to the landing point x of the ball will pick it up. For team A to obtain a higher score, A_1 should throw the ball as far away as possible from the positions of team B. This strategy corresponds to maximizing the shortest distance between x and the k points representing the positions of team B. This maximum value is expressed as

$$p_x(x_1, x_2, \dots, x_k),$$

which serves as the score of team A. Conversely, in order to minimize the score conceded, team B may choose their positions so as to minimize this value. The resulting quantity is

$$m_k(X),$$

which represents the optimal score of team A under the best strategy of team B. Hence, in the *Spherical Game*, both team A and team B can formulate their strategies through the metric invariant $m_k(X)$ (see [3]).

The *Safety Net Problem* can be interpreted in a similar manner. Condition (i), "rescue is dispatched from the nearest base by the shortest route," corresponds to the rule that "the nearest player goes to pick up the ball." Condition (ii), "the entire Earth is covered without bias," is equivalent to "considering the maximum value of the shortest distance to any point on the Earth," which parallels the strategy of throwing the ball as far as possible so that the minimum distance is maximized. Finally, condition (iii), "arrange the bases so that the travel distance is minimized," corresponds to "choosing an arrangement that minimizes the maximum value described in condition (ii)." In both cases, the mathematical structure reduces to the consideration of the metric invariant $m_k(X)$ (see [6]).
Thus, the conditions of the *Spherical Game* and those of the *Safety Net Problem* are in direct correspondence, and both can be analyzed and explained using the invariant $m_k(X)$.

4. Teaching Method, Results and Optimal Arrangements

The *Safety Net Problem* was employed as introductory teaching material in group mathematics classes at a technical college. In this setting, spherical trigonometry was deliberately avoided. Students were provided with worksheets containing several diagrams of spheres and were instructed to sketch possible arrangements of k points and record the reasoning process leading to their solutions. As homework, students were asked to practice the *Spherical Game*. After an explanation of the strategy involved, they were subsequently guided to consider the *Safety Net Problem*.

At first glance, the *Spherical Game* and the *Safety Net Problem* may appear unrelated. However, when asked to analyze the case of two points, most students recognized the similarity in strategies and were able to solve the *Safety Net Problem* accordingly. During problem-solving, students were encouraged to fix one arrangement of points and then modify it step by step while recording diagrams, which helped them compare configurations systematically. From their diagrams, it became evident that students were reasoning about the optimal arrangement by holding one configuration fixed and then examining how the minimum and maximum values changed under different variations. This suggests that they had discovered the underlying strategy common to both problems. Figures 4.1–4.3 illustrate representative stages of this reasoning process.
In these figures, boundary circles are considered to be great circles. Many students successfully identified the optimal arrangement by analyzing diagrams of this type. Specifically, the maximum of the minimum distances from two fixed points to x becomes smaller when they are farther apart (Figure 4.2) compared to when the two points are close to each other (Figure 4.1). Furthermore, the maximum shortest distance is minimized when the two points are placed at antipodal positions (Figure 4.3). These observations demonstrate the process of logical exploration by students.

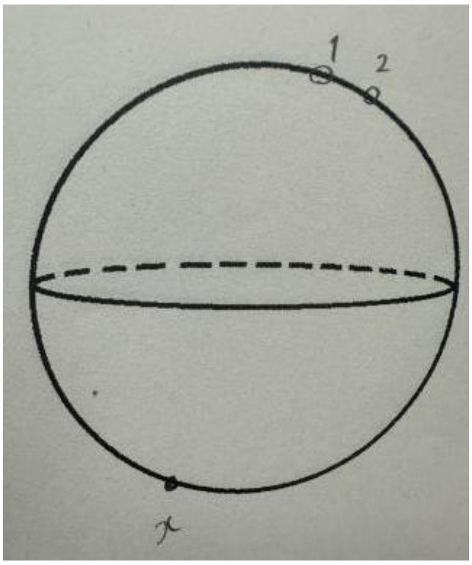

Figure 4.1 Two closely spaced points

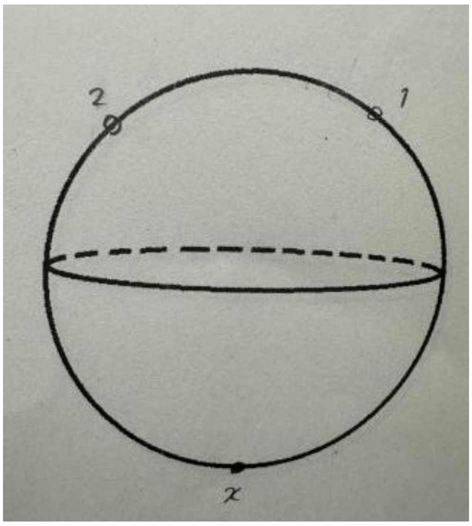

Figure 4.2 Two points a little far apart

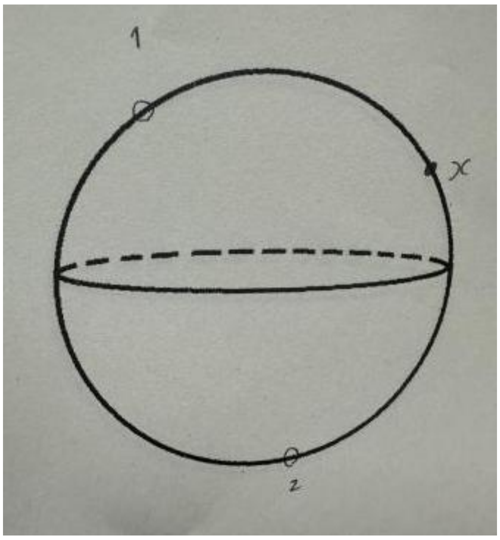

Figure 4.3 The two most distant points

The following summarizes optimal arrangements of rescue bases on the sphere:
(1) *k*=1: the location is arbitrary.
(2) *k*=2: the two points should be placed at antipodal positions.
(3) *k*=3: The three points should lie on a great circle such that the sum of the pairwise distances is 2π. (We call this unbiased.) In particular, if two bases are fixed at antipodal positions, the remaining point may be placed arbitrarily.

Figures 4.4 and 4.5 show *Mathematica* visualizations of the cases *k*=2 and *k*=3. In both cases, the value of $m_k(X)$ is π/2. In Figure 4.4, the two bases are placed at antipodal position, so that the distance from any point on the equator to those two points is π/2. In Figure 4.5, the three bases are positioned on a great circle in an unbiased manner, and the sphere is covered by disks defined by great circles at distance π/2 from each base.

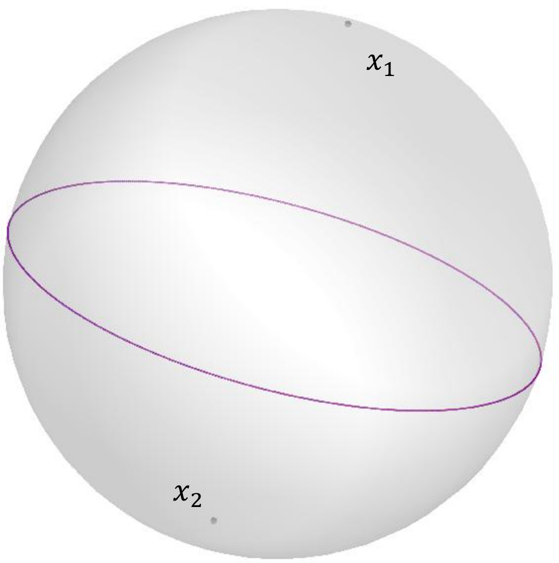

Figure 4.4 Optimal arrangement for *k*=2, $m_2(X) = \pi/2$

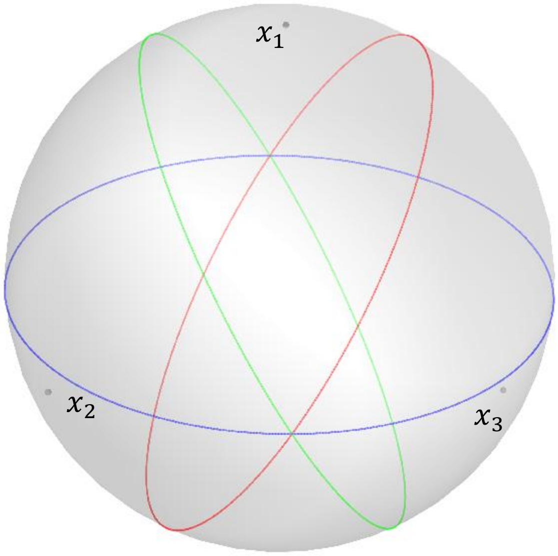

Figure 4.5 Optimal arrangement for *k=3*, $m_3(X) = \pi/2$

We also found that when *X* is a sphere, the vertices of various polyhedra appear as optimal arrangements of *k* points for large values of *k*.
k=4: vertices of a regular tetrahedron
k=5: vertices of a hexahedron
k=6: vertices of a regular octahedron
k=7: vertices of a decahedron
Since it is difficult to consider when the value of *k* is large, the optimal arrangement diagram for a sphere drawn in *Mathematica* is shown below.
The computed values include

$m_4(X) = arccos\frac{1}{3}$, $m_5(X) = arccos\frac{1}{\sqrt{5}}$, $m_6(X) = arccos\frac{1}{\sqrt{3}}$, $m_7(X) = arcsin\sqrt{\frac{8}{11+\sqrt{5}}}$, etc. (see [3],[8],[9])

In the corresponding *Mathematica* diagrams, the circles centered at the vertices of the polyhedra have radii equal to $m_k(X)$. The diagrams clearly show that multiple circles intersect precisely at common points, and that the entire sphere is covered without gaps. If the radius were smaller, coverage would become incomplete, and shifting the vertices could not resolve the deficiency.

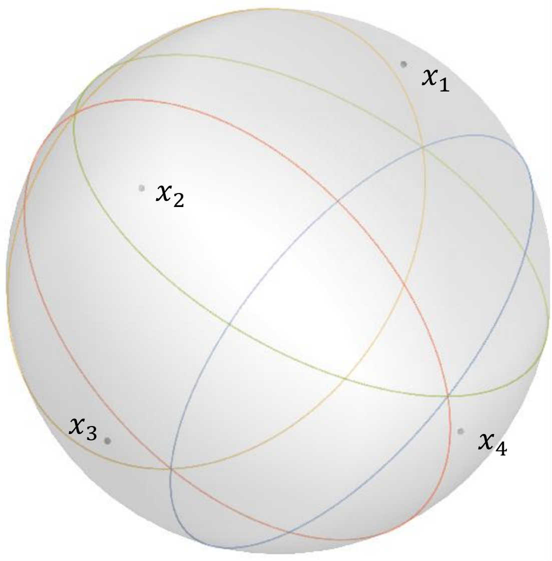

Figure 4.6 Optimal arrangement for *k*=4, $m_4(X) = arccos\frac{1}{3}$

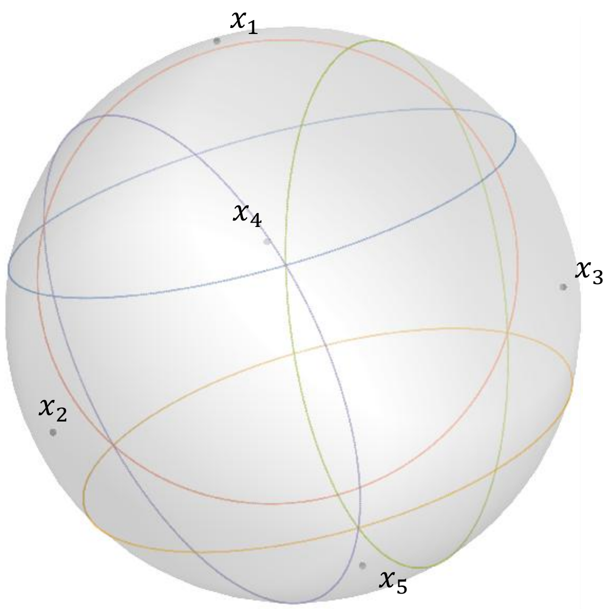

Figure 4.7 Optimal arrangement for *k*=5, $m_5(X) = arccos\frac{1}{\sqrt{5}}$

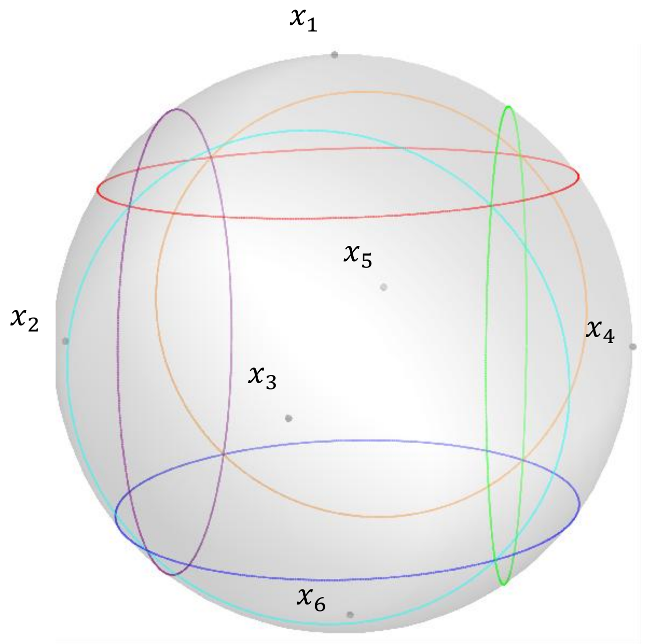

Figure 4.8 Optimal arrangement for k=6, $m_6(X) = arccos\frac{1}{\sqrt{3}}$

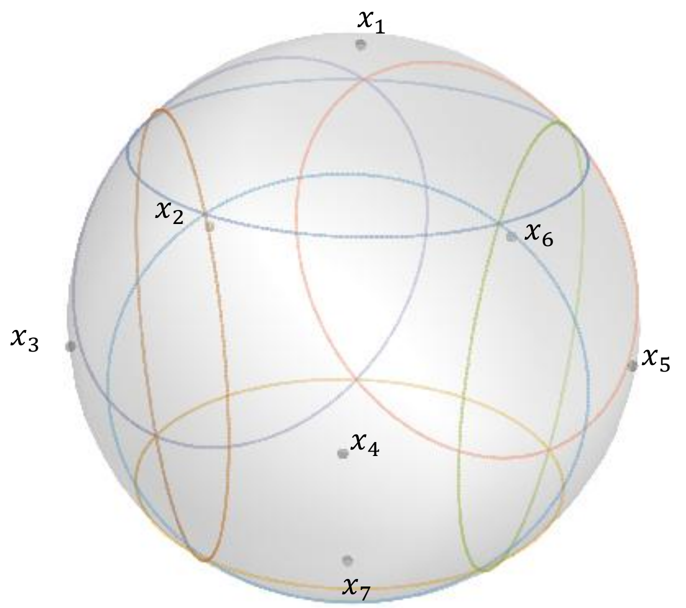

Figure 4. 9 Optimal arrangement for k=7, $m_7(X) = arcsin\sqrt{\frac{8}{11+\sqrt{5}}}$

5. Conclusion

In the *Safety Net Problem*, students approached the determination of optimal arrangements by fixing certain configurations and systematically comparing alternatives. From this process, it can be inferred that they employed reasoning analogous to the strategy used in the *Spherical Game.*

The case of two points on a sphere proved relatively accessible for students. Building on this foundation, we intend to introduce further problems, such as the cases of three points on a sphere and two or three points on a hemisphere, as new teaching materials.
In addition, the use of *Mathematica* revealed that the vertices of various polyhedra emerge as optimal arrangements of k points on the sphere. This provides both a mathematical insight into the problem structure and a valuable visualization tool for educational purposes.

References

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