Pedagogical Approaches to Visualizing Complex Roots of Polynomial Equations

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Abstract

With the realization of Japan's GIGA School Initiative, ICT environments have been established in schools throughout the country. The use of tablet devices has become commonplace in high schools, with an increasing number of students using them in their arithmetic and mathematics classes. In this paper, we examine a visualization of the cubic equation $x^3 - 3px - 2q = 0$ *(where* p *and* q *are real numbers) as a means for teaching a core mathematical concept using ICT. Utilizing GeoGebra and Python, we show the visualization of the complex solution to a cubic equation in three dimensions. Complex numbers, which must be considered in four dimensions, are visualized in three-dimensional space, allowing us to reinterpret complex numbers from a different perspective. We found that presenting both an algebraic approach and a visual approach using ICT can serve as an effective exploratory teaching methodology in high school mathematics.*

1 Introduction

With the implementation of the GIGA School Initiative, the development of ICT environments and the widespread use of tablet devices have led to an increase in mathematics classes utilizing these technologies. The "Curves on Planes and the Complex Plane" section of the "Course of Study Commentary: Discussion" (see [2], 120) points out that students can develop their thinking, judgment, and expressive abilities by "representing curves using information devices such as computers, solving problems using concepts such as parameters, polar coordinates, and the complex plane, reflecting on the problem-solving process, and considering the mathematical characteristics of the phenomenon in relation to other phenomena." In this regard, there has been active research into the visualization of complex solutions to algebraic equations and the complex roots of cubic equations using ICT tools such as GeoGebra and MATLAB (see [6], [7], [8]). Questions on shapes represented on the complex plane are sometimes included in Japanese university entrance exams (see [1], [3]). The author has previously presented research on the visualization of complex numbers at the Japan Society of Mathematics Education (see [4], [5]). The equation $x^3 - 15x - 4 = 0$ (known as the Bombelli equation) can be factorized as $(x - 4)(x^2 + 4x + 1) = 0$; however, when Cardano's solution method is applied, imaginary (complex) numbers appear in the process. In this paper, we call a cubic equation of the form

$x^3 - 3px - 2q = 0$ (where p and q are real numbers) a Bombelli-type equation and consider visualizing the complex solution to such an equation.
Numerous studies have been published on the visualization of complex solutions to cubic equations. In particular, a paper published in eJMT (see [7]) used GeoGebra to visualize the behavior of roots in the complex plane, providing valuable insights from a two-dimensional perspective. In this study, the visualization is performed as a spatial cross section of complex numbers ($v = 0$). While previous research in Japan has focused on complex functions and dynamical systems at the university level, this study is unique, in that it focuses on applications to high school mathematics. The 3D visualization of Bombelli-type equations offers a different approach from those in these previous studies and is considered useful as an educational and research tool. The approach proposed here builds on these earlier studies by incorporating GeoGebra's animation capabilities and extending the animation to a 3D representation of complex solutions and dynamic modeling using Python.

In line with Japan's GIGA School Initiative, this paper presents both an algebraic approach and a visual approach using ICT for solving Bombelli-type equations. Furthermore, by introducing algebraic methods such as Vieta's trigonometry, which are rarely taught in Japanese high schools, we aim to enrich inquiry-based mathematics teaching materials. In this study, GeoGebra and Python were used as the primary ICT tools. GeoGebra was adopted because it is "free to use, requires no programming, and features an interface that teachers and students can use intuitively. GeoGebra's 3D graphics display function can visualize solutions to cubic equations in three-dimensional space. Created teaching materials can be shared through GeoGebra Classroom." Python is considered "ideal for this learning format because it is free to use, can be applied to a variety of educational fields, including mathematics (numerical calculations and graph drawing), and can be implemented relatively easily as programming software. It allows for instruction tailored to the level and interests of students, and its compatibility with the GIGA School Initiative is highly regarded."

2 Visualization of Quadratic Equations

The solution to the quadratic equation $x^2 - a = 0$ (where a is a real number) is $x = \pm\sqrt{a}$ when $a \geq 0$. This corresponds to the x coordinate of the intersection $\mathrm{P}(x, y)$ of the quadratic function $y = x^2$ and the line $y = a$, and can be displayed as the x coordinate of the intersection $(\pm\sqrt{a}, a)$ on the xy plane. However, when $a < 0$, the solution to this quadratic equation, $x = \pm\sqrt{-a}i$, becomes imaginary and cannot be displayed as a point on the xy plane. Therefore, we will complexify the equation $x^2 = a$ with $z^2 = a$ and consider it as the solution to the simultaneous equations $f(z) = z^2$ and the plane $f(z) = a$.

If $z = a + bi$ (where a, b are real numbers) and $f(z) = u + vi$ (where u, v are real numbers), the solution to the equation $z^2 = a$ (where a is real) can be displayed as coordinates (x, y, u, v) in four-dimensional complex space. The solution to the equation $f(z) = a$ (where a is a real number) can be visualized as a three-dimensional point (x, y, u, v)=$(x, y, a, 0)$ $\leftrightarrow$ (x, y, a) on the cross section $v = 0$ in this complex space. Similarly, the solution to the equation $f(z) = bi$ (where b is a real number) can be visualized as a three-dimensional point (x, y, u, v)=$(x, y, 0, b)$ $\leftrightarrow$ (x, y, b) on the cross section $u = 0$ in this complex space.

When $a \geq 0$, the solution to the equation $z^2 = a$ can be expressed as complex space coordinates $(\sqrt{a}, 0, a, 0)$, since $z = \pm\sqrt{a}$.

When $a < 0$, the solution to the equation $z^2 = a$ can be expressed as complex space coordinates $(0, \pm\sqrt{-a}, a, 0)$, since $z = \pm\sqrt{-a}i$.

When we consider the xy plane as complex in this way, we are describing it in four dimensions; here, however, $v = 0$ is always the case, meaning that the solution can be visualized as three-dimensional coordinates (x, y, u) by excluding the coordinate v. This corresponds to viewing four-dimensional space as a three-dimensional cross section at $v = 0$.Therefore, the intersection of $f(z) = z^2$ and the plane $f(z) = a$ can be set as the solution in three-dimensional space.

For example, when $a = -4$, the coordinates in complex space of the equation $z^2 + 4 = 0$ are (x, y, u, v)=$(0, 2, -4, 0)$, $(0, -2, -4, 0)$, meaning that we can visualize it as two three-dimensional coordinates (x, y, u)=$(0, 2, -4)$,$(0, -2, -4)$ at the cross section $v = 0$ in this complex space. The GeoGebra file allows the user to change the values of a using sliders (see figure 1).

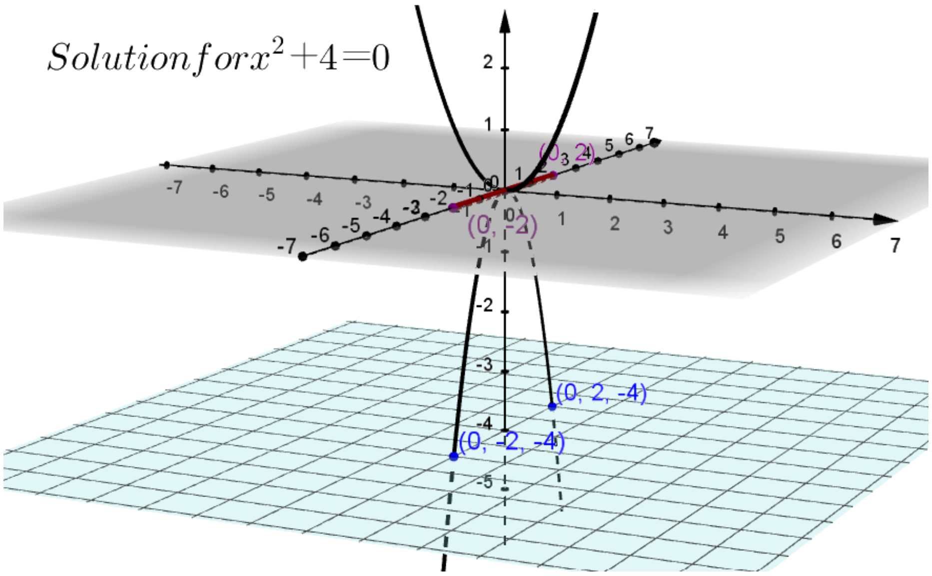

Figure 1: solution for $x^2 = -4$
(Note: Readers may wish to explore the interactive model at GeoGebra Interactive Figure.)

There is no loss of generality even if the quadratic coefficient of the quadratic equation is set to 1, making it the monic equation $X^2 + aX + b = 0$ (where a, b are real numbers). If $X = x - a/2$, then the quadratic equation can be expressed as $(x - a/2)^2 + a(x - a/2) + b = 0$, so $x^2 = (a^2 - 4b)/4$. If $D = (a^2 - 4b)/4$, then the quadratic equation can be expressed as $x^2 = D$, and the solution to this quadratic equation can be visualized in three dimensions. Therefore, the intersection of $f(z) = (z^2 - p)$ (where $2p = a$) and the plane $f(z) = a$ can be set as the solution in three-dimensional space. This GeoGebra file allows the user to change the values of a and p using sliders.

For example, the solution to $x^2 - 2x + 4 = 0$, $x = 1 \pm \sqrt{3}i$, can be transformed to $(x-1)^2 = -3$, making it possible to visualize (see figure 2).

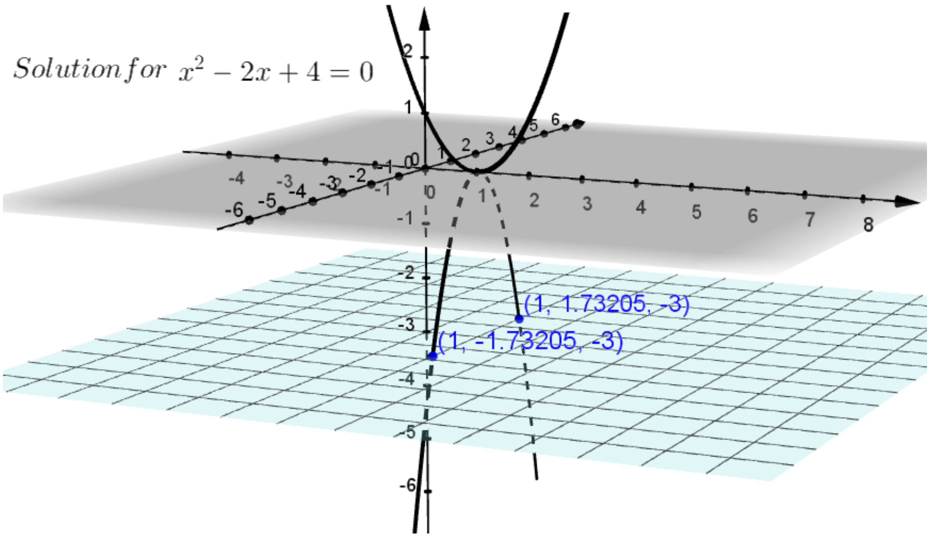

Figure 2: solution for $(x-1)^2 = -3$
(Note: Readers may wish to explore the interactive model at GeoGebra Interactive Figure.)

3 Algebraic and Historical Background of the Bombelli Equation

The cubic equation $x^3 = 15x + 4$, called the Bombelli equation (named for Rafael Bombelli (1526–1572), an Italian mathematician born in Bologna), has three real solutions: $x = 4, -2 \pm \sqrt{3}$.

The fact that this equation can be factorized as $(x-4)(x^2+4x+1) = 0$ means that three real solutions can be found. However, although there are three real solutions, when Cardano's formula is used, imaginary (complex) numbers appear along the way to the solution.

Usefully, even in such cases, by creating a simple program using "NumPy" (Numerical Python), it is possible to show the three real solutions on a complex plane (see figure 3).

By changing the four coefficient values in a cubic equation, the solution to the Bombelli equation $x^3 - 15x - 4 = 0$ can be plotted on a given plane. Figure 4 shows the three real roots of the Bombelli equation plotted on the complex plane using Python. The symmetry of the roots is visually evident.

3.1 Using the Formula for Factoring (Cubic Expressions)

If the complex solution to $z^3 = 1$ is ω, then $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$ hold. We can transform the cubic expression $x^3 + y^3 + z^3 - 3xyz$ into the product of three linear expressions:

```
import numpy as np
import matplotlib.pyplot as plt

# Coefficient of a cubic equation (e.g. x^3 + x^2 - 2x - 2 = 0)
coefficients = [1, 0, 3, -14]
roots = np.roots(coefficients)
real_parts = [root.real for root in roots]
imag_parts = [root.imag for root in roots]
print(real_parts)
print(imag_parts)

plt.figure(figsize=(6, 6))
plt.axhline(0, color='black', linewidth=0.5)
plt.axvline(0, color='black', linewidth=0.5)
plt.grid(color='gray', linestyle='--', linewidth=0.5)
plt.scatter(real_parts, imag_parts, color='red', label='Roots')
# Axis Labels and Titles
plt.title(”Complex Roots of Cubic Equation”)
plt.xlabel(”Real Part”)
plt.ylabel(”Imaginary Part”)
plt.legend()
# Displaying the Solution
plt.show()
```

Figure 3: Python program

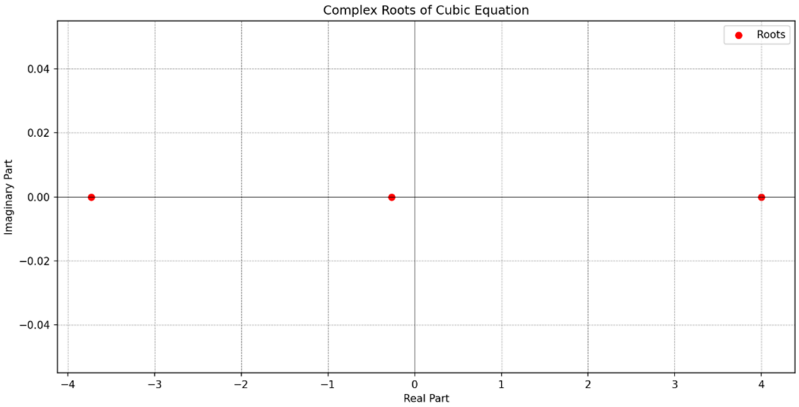

Figure 4: solution for $x^3 - 15x + 4 = 0$ (from Python)

$$\begin{aligned}x^3+y^3+z^3-3xyz &= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)\\ &= (x+y+z)\{x^2+\omega^3y^2+\omega^3z^2+(\omega+\omega^2)xy+(\omega+\omega^2)yz+(\omega+\omega^2)zx\}\\ &= (x+y+z)(x+\omega y+\omega^2z)(x+\omega^2y+\omega z).\end{aligned}$$

The solution to $x^3+y^3+z^3-3xyz=0$ is thus

$$x=\begin{cases}-(y+z)\\ -(\omega y+\omega^2z)\\ -(\omega^2y+\omega z)\end{cases}$$

Using the above results, the simultaneous equations $y^3+z^3=-4$ and $yz=5$ hold for the Bombelli equation $x^3-15x-4=0$. If we set $y=-(2+i)$ and $z=-(2-i)$, then

$$x=\begin{cases}(2+i)+(2-i)=4\\ \omega(2+i)+\omega^2(2-i)=-2-\sqrt{3}\\ \omega^2(2+i)+\omega(2-i)=-2+\sqrt{3}\end{cases}$$

3.2 Cardano's Solution Formula

There is no loss of generality even if the coefficient of the cube term of the cubic equation is set to 1 (i.e., the equation is monic); that is,

$$X^3+aX^2+bX+c=0 \tag{1}$$

If $X=x-\frac{a}{3}$, equation (1) is $(x-\frac{a}{3})^3+a(x-\frac{a}{3})^2+b(x-\frac{a}{3})+c=0$, so we can write $x^3-(\frac{a^2}{3}-b)x-(\frac{2a^3}{27}-\frac{ab}{3}+c)=0$. If we now set $3p=\frac{a^2}{3}-b$, $2q=\frac{2a^3}{27}-\frac{ab}{3}+c$, equation (1) becomes

$$x^3=3px+2q \tag{2}$$

Thus, we just need to find the solution to this cubic equation. In equation (2), if $x=s+t$, then
(left side) $=(s+t)^3=s^3+3s^2t+3st^2+t^3=3st(s+t)+(s^3+t^3)=3stx+(s^3+t^3)$, and
(right side) $=3px+2q$.

Since

$$st=p \tag{3}$$

then

$$s^3+t^3=2q \tag{4}$$

holds.

The solution to (2), $x=s+t$, can thus be found by finding the values of s and t that satisfy (3) and (4).

If $t = \frac{p}{s}$ from (3) is substituted into (4), $s^3 + (\frac{p}{s})^3 = 2q$, so $s^6 - 2qs^3 + p^3 = 0$ is obtained.

If we set $s^3 = u$, then $u^2 - 2qu + p^3 = 0$, and the solution to this quadratic equation is $s^3 = q \pm \sqrt{q^2 - p^3}$. One of the solutions to this cubic equation is $s{=}\sqrt[3]{q \pm \sqrt{q^2 - p^3}}$.
In this case, from (4), $t^3 = 2q{-}(q{\pm}\sqrt{q^2 - p^3}) = q{\mp}\sqrt{q^2 - p^3}$, so we can find $t = \sqrt[3]{q \mp \sqrt{q^2 - p^3}}$ (decoding in the same order).

By construction, s and t are symmetric, so

$$x = s + t = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}} \tag{5}$$

is one of the solutions to this cubic equation.

Therefore, $x^3 = 3px + 2q$.

If $\omega = \frac{-1+\sqrt{3}i}{2}$ or $\omega = \frac{-1-\sqrt{3}i}{2}$, the three solutions can be expressed as follows:

$$x = \begin{cases} s + t = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}} \\ \omega s + \omega^2 t = \omega\sqrt[3]{q + \sqrt{q^2 - p^3}} + \omega^2\sqrt[3]{q - \sqrt{q^2 - p^3}} \\ \omega^2 s + \omega t = \omega^2\sqrt[3]{q + \sqrt{q^2 - p^3}} + \omega\sqrt[3]{q - \sqrt{q^2 - p^3}} \end{cases} \tag{6}$$

For example, using Cardano's solution formula, the solution to the cubic equation $x^3 + 3x - 14 = 0$ is when $p = -1, q = 7$. If the discriminant of the cubic equation is $D = q^2 - p^3$, then $D = 7^2 - (-1)^3 = 50 > 0$, so the number inside the radical is positive. Since $s^3 = 7 + 5\sqrt{2} = (1 + \sqrt{2})^3$ and $t^3 = 7 - 5\sqrt{2} = (1 - \sqrt{2})^3$,
we can say that $s = 1 + \sqrt{2}$ and $t = 1 - \sqrt{2}$. Substituting into (6),

$$x = \begin{cases} s + t = (1 + \sqrt{2}) + (1 - \sqrt{2}) = 2 \\ \omega s + \omega^2 t = \omega(1 + \sqrt{2}) + \omega^2(1 - \sqrt{2}) = -1 + \sqrt{6}i \\ \omega^2 s + \omega t = \omega^2(1 + \sqrt{2}) + \omega(1 - \sqrt{2}) = -1 - \sqrt{6}i \end{cases}$$

and, therefore,
$x = 2, \ -1 \pm \sqrt{6}i.$

Figure 5 shows how the complex solution to this cubic equation can be expressed in Python.

Using Cardano's solution formula, the solution to the Bombelli equation $x^3 = 15x + 4$ is obtainable when $p = 5$, $q = 2$. Since $D = q^2 - p^3 = 2^2 - 5^3 = -11^2 < 0$, there is a negative number inside the radical. However, the Bombelli equation has a solution of $x = 4$; thus, if we set $\sqrt{-1} = i$, $4 = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$.

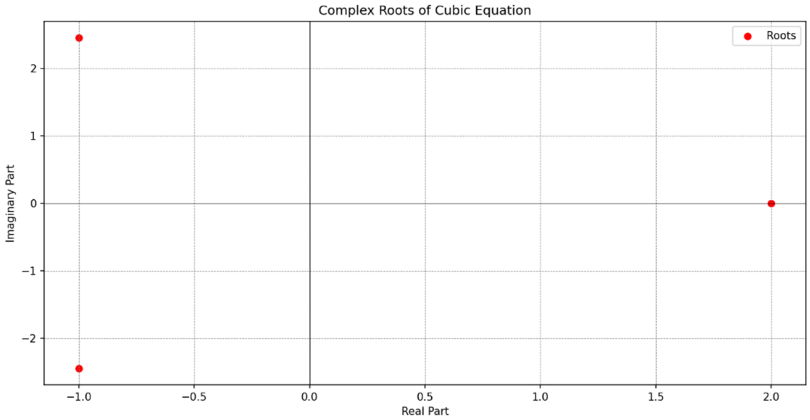

Figure 5: solution for $x^3 + 3x - 14 = 0$ (from Python)

Since
$(2+i)^3 = (2+i)(3+4i) = 2+11i$ and $(2-i)^3 = (2-i)(3-4i) = 2-11i$, if we set $s = 2+i$, $t = 2-i$, then

$$x = \begin{cases} s + t = \sqrt[3]{2+11i} + \sqrt[3]{2-11i} = (2+i) + (2-i) = 4 \\ \omega s + \omega^2 t = \omega(2+i) + \omega^2(2-i) = -2 - \sqrt{3} \\ \omega^2 s + \omega t = \omega^2(2+i) + \omega(2-i) = -2 + \sqrt{3} \end{cases}$$

and we can confirm that the solution to the Bombelli equation is $x = 4,\ -2 \pm \sqrt{3}$.

4 Visualization of Cubic Equations Using ICT

As with quadratic equations, we can make the Bombelli-type equation $x^3 - 3px - 2p = 0$ complex as $z^3 - 3pz = 2p$. We can then consider the simultaneous equations $f(z) = z^3 - 3pz$ and the plane $f(z) = h$, where $h = 2q$ (h is a real number), and visually represent it as a three-dimensional point.

4.1 $f(x) = x^3 + 3x - h$ (h is real)

The cubic function $f(x) = x^3 + 3x - h$ (where h is a real number) is a function that has no extreme values, since $f'(x) = 3x^2 + 3 > 0$. The solution to $x^3 + 3x - h = 0$ can be easily visualized in three-dimensional space. Let $f(x) = x^3 + 3x$. If $x = a + bi$ (where a, b are real numbers), then $f(x) = (a+bi)^3 + 3(a+bi) = (a^3 - 3ab^2 + 3a) + b(3a^2 - b^2 + 3)i$. $Imf(a+bi) = 0$ when $b(3a^2 - b^2 + 3) = 0$,

1. When $b = 0$, $f(x) = f(a+bi) = 3a^3 + 3a$, so $(a, 0, a^3 + 3a)$.
2. When $b \neq 0$, $b^2 = 3a^2 + 3$, $f(x) = -2a(4a^2 + 3)$, so $(a, \pm\sqrt{3(a^2+1)}, -2a(4a^2+3))$.

The solution to the cubic equation $x^3 = -3x + h$ (where h is real) can be found when the first and constant terms of the Bombelli-type equation are $3p = -3$ and $2q = h$. When, for discriminant D, $D = (h/2)^2 + 1 > 0$, there is always one real solution and a pair of

complex conjugates . This can be shown as $f(x) = x^3 + 3x$ in three dimensions and three points on the plane $f(x) = h$: $P_1 = (s+t, 0, h)$, $P_2 = (-(s+t)/2, \sqrt{3}(s-t)/2), h)$, $P_3 = (-(s+t)/2, -\sqrt{3}(s-t)/2, h)$. By changing the value of h, we can see how the solution to this equation changes in three-dimensional space. Furthermore, the solution to this equation can be found using equation (6).
This GeoGebra file allows you to change the values of h using sliders.
When $h = 14$, the equation becomes $x^3 + 3x - 14 = 0$, and the solution is $x = 2, -1 \pm \sqrt{6}i$. Figure 6 shows how this can be visualized in three-dimensional space.

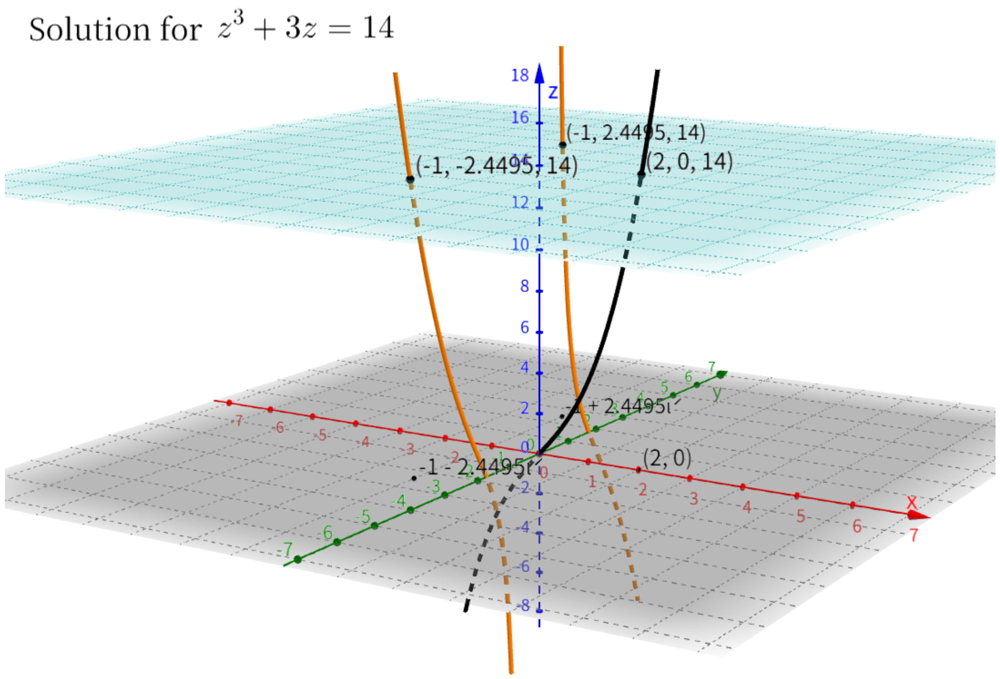

Figure 6: solution for $x^3 + 3x = 14$
Readers may explore the interactive model at the website GeoGebra Interactive Figure.

4.2 $f(x) = x^3 - 15x - h$ (h is a real number)

The cubic function $f(x) = x^3 - 15x - h$ (h is a real number) has two distinct extreme values, since $f'(x) = 3x^2 - 15 > 0$. Let $f(x) = x^3 - 15x$. Substituting $x = a + bi$ (a and b are real), we get $f(x) = (a+bi)^3 - 15(a+bi) = (a^3 - 3ab^2 - 15a) + b(3a^2 - b^2 - 15)i$. $Imf(a+bi) = 0$ when $b(3a^2 - b^2 + 15) = 0$,

1. When b=0, $f(x) = f(a+bi) = 3a^3 - 15a$, so $(a, 0, a^3 - 15a)$.
2. When $b \neq 0$, $b^2 = 3a^2 - 15$, $f(x) = -2a(4a^3 + 3a + 7)$,
   so $(a, \pm\sqrt{3a^2 - 15}, -2a(4a^2 + 3a + 7))$.

The solutions to the cubic equation $x^3 = 15x + h$ are $3p = 15, 2q = h$ (h is real). If the discriminant of this equation is D, then when $-10\sqrt{5} < h < 10\sqrt{5}$, $D = q^2 - p^3 = (h/2)^2 - p^3 < 0$, indicating that the equation has three different real solutions.
If $s = \dfrac{\sqrt[3]{h - \sqrt{h^2 - 500}}}{\sqrt[3]{2}}$, $t = \dfrac{\sqrt[3]{h + \sqrt{h^2 - 500}}}{\sqrt[3]{2}}$, then according to Cardano's solution formula,

$$x = \begin{cases} s+t \\ \omega s + \omega^2 t \\ \omega^2 s + \omega t \end{cases}$$

To visually represent the solution in three-dimensional space, it is necessary to distinguish between the cases based on the sign of the discriminant D.
The solution to the cubic equation $x^3 = 15x+h$ is $3p = 15, 2q = h$ (where h is a real number). If the discriminant of this equation is D, then when $-10\sqrt{5}h < 10\sqrt{5}$, $D = q^2-p^3 = (h/2)^2-p^3 < 0$, indicating that the equation has three different real solutions.
When $s = \dfrac{\sqrt[3]{h-\sqrt{h^2-500}}}{\sqrt[3]{2}}$ and $t = \dfrac{\sqrt[3]{h+\sqrt{h^2-500}}}{\sqrt[3]{2}}$, according to Cardano's solution formula, we need to distinguish between cases based on the sign of the discriminant D in (6) in order to visually represent the solution in three-dimensional space. When there are three real solutions, implementing the solution for this type of cubic equation in this three-dimensional space is not easy. Since our purpose is to use this as teaching material for high school students, we decided to use Python to find the specific solution to the cubic equation and paste that solution into this three-dimensional space. Below is an example when $h = 24, 10\sqrt{5}, 4$:

4.3 When D>0 $(h < -10\sqrt{5}$ or $10\sqrt{5} < h)$

Example: When $h = 24$, the equation is $x^3 - 15x - 24 = 0$.
In this case, $p = 5$ and $q = 12$. The discriminant D is $D = 12^2 - 5^3 = 19 > 0$,
indicating that this cubic equation has both real and complex conjugate solutions.
From equation (6),

$$x = \begin{cases} \sqrt[3]{12-\sqrt{19}} + \sqrt[3]{12+\sqrt{19}} \\ \omega\sqrt[3]{12-\sqrt{19}} + \omega^2\sqrt[3]{12+\sqrt{19}} \\ \omega\sqrt[3]{12+\sqrt{19}} + \omega^2\sqrt[3]{12-\sqrt{19}} \end{cases}$$

Therefore, the three solutions are $x \simeq 4.5082$, $-2.2541 + 0.49269i$, and $-2.2541 - 0.49269i$ (see figure 7).

4.4 When D=0 $(h = -10\sqrt{5}$ or $10\sqrt{5} = h)$

Example: When $h = 10\sqrt{5}$, The equation is $x^3 - 15x - 24 = 0$.
In this case, $p = 5, q = 5\sqrt{5}$. The discriminant D is $D = (5\sqrt{5})^2 - 5^3 = 0$,
indicating that the cubic equation has two real solutions.
From equation (6),
$x = 2\sqrt{5}, -\sqrt{5}$ (multiple solutions) (see figure 8).

4.5 When D<0 $(-10\sqrt{5} < h < 10\sqrt{5})$

Example: When $h = 4$, the equation is $x^3 - 15x - 4 = 0$.
In this case, $p = 5, q = 2$. Here, the discriminant D is $D = 2^2 - 5^3 = -121 < 0$,
indicating that this cubic equation has three real solutions.

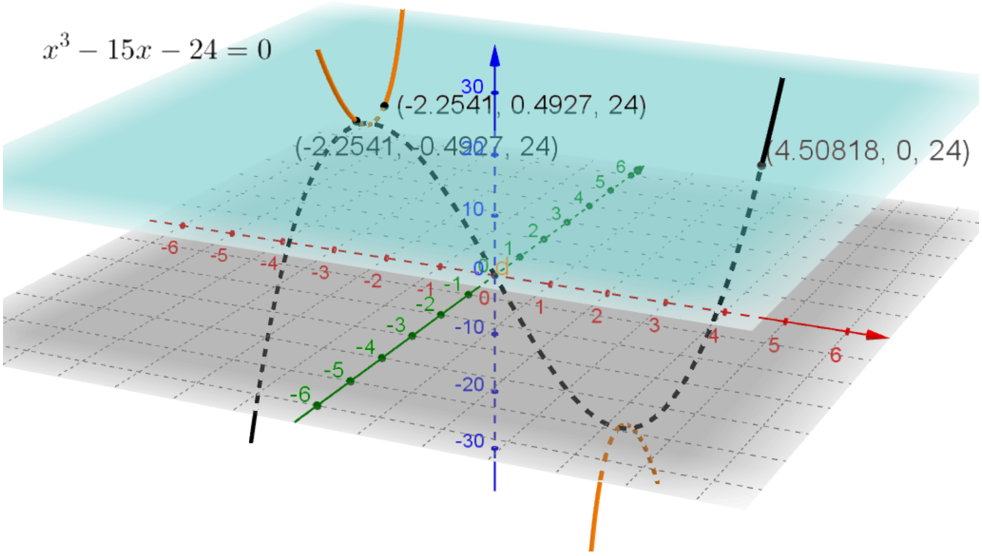

Figure 7: solution for $x^3 - 15x - 24 = 0$
Readers may explore the interactive model at the website GeoGebra Interactive Figure.

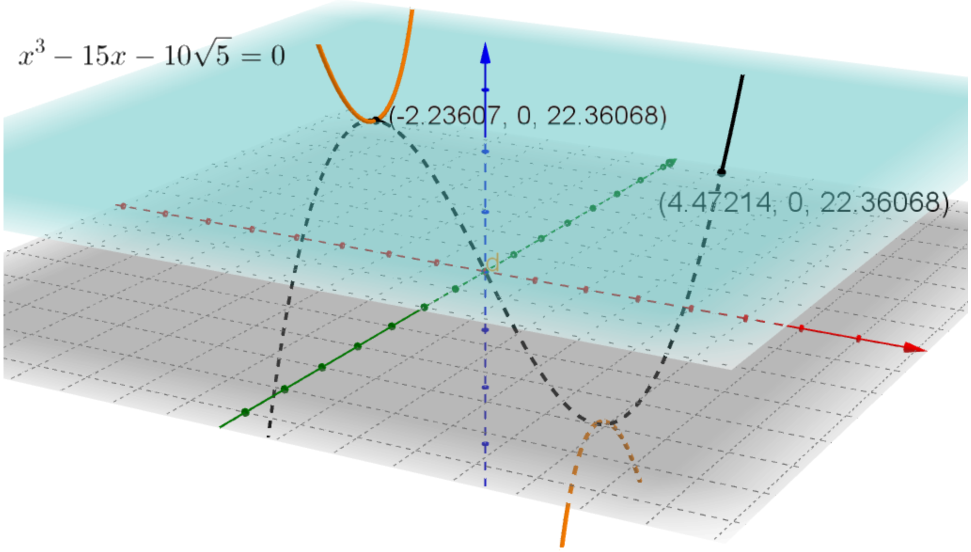

Figure 8: solution for $x^3 - 15x - 10\sqrt{5} = 0$
Readers may explore the interactive model at the website GeoGebra Interactive Figure.

From equation (6),
$x = 4, -2 \pm \sqrt{3}$ (see figure 9).

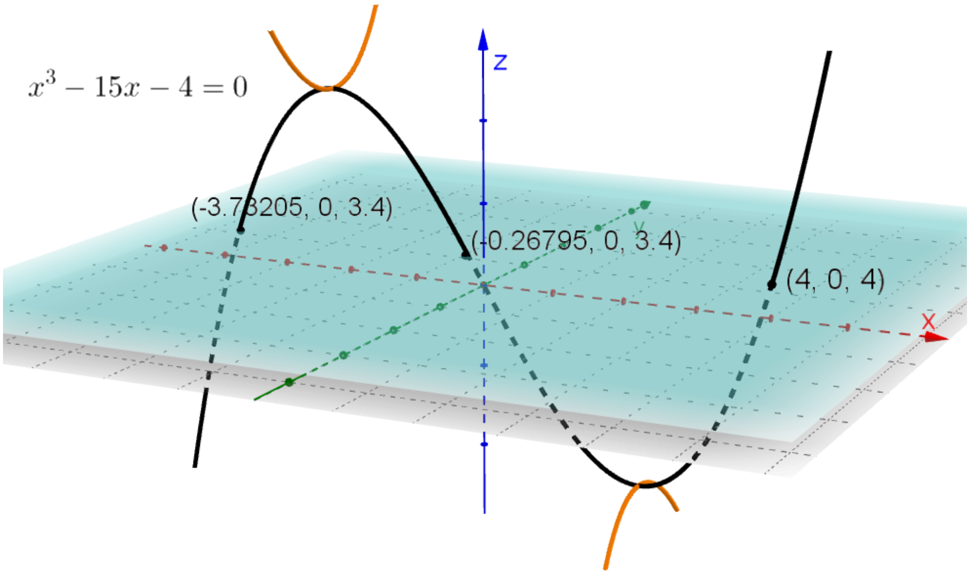

Figure 9: solution for $x^3 - 15x - 4 = 0$
Readers may explore the interactive model at the website GeoGebra Interactive Figure.

Finally, a cubic equation with all real coefficients has at least one real solution. When using Cardano's formula to solve a cubic equation, imaginary (complex) numbers may appear during the solution process, as in the case of the Bombelli equation ($x^3 - 15x - 4 = 0$). This makes it difficult to visualize the solution in three-dimensional space. Although rarely taught in Japanese high schools, Vieta's method of solving cubic equations combines algebra and trigonometric functions to find real solutions. When combined with visualization using GeoGebra or Python, this method can be an effective teaching tool for intuitively understanding the "geometric meaning of algebraic structures." Here, we introduce an approach to three real solutions using trigonometric functions (see [8]).

The discriminant D of the solution to the Bombelli equation $x^3 - 15x - 4 = 0$ has three real solutions, since $D = 2^2 - 5^3 = -121 < 0$. One of the solutions to this equation is 4, so if we set $2\sqrt{5}\cos\frac{\theta}{3} = 4$, we get $\cos\frac{\theta}{3} = \frac{2}{\sqrt{5}}$ and $\sin\frac{\theta}{3} = \frac{1}{\sqrt{5}}$. At this point, we can confirm that the remaining two solutions are the following equations.

$$\omega s + \omega^2 t = 2\sqrt{5}\cos(\tfrac{\theta}{3} + \tfrac{2\pi}{3}) = 2\sqrt{5}(\cos\tfrac{\theta}{3}\cos\tfrac{2\pi}{3} - \sin\tfrac{\theta}{3}\sin\tfrac{2\pi}{3}) = -2 - \sqrt{3}$$

$$\omega^2 s + \omega t = 2\sqrt{5}\cos(\tfrac{\theta}{3} + \tfrac{4\pi}{3}) = 2\sqrt{5}(\cos\tfrac{\theta}{3}\cos\tfrac{4\pi}{3} - \sin\tfrac{\theta}{3}\sin\tfrac{4\pi}{3}) = -2 + \sqrt{3}$$

This Bombelli-type equation always has a real solution; if this real solution can somehow be found, it can be solved using Vieta's trigonometry without having to deal with complex numbers.

5 Pedagogical Implications and Summary

By presenting this Bombelli-type equation using ICT (GeoGebra, Python) in both an algebraic and visual approach, we have created teaching materials that allow students to experience the multifaceted nature of mathematical structures, demonstrating the potential for cultivating mathematical perspectives and thinking skills. In high school mathematics, higher-order equations with complex solutions can be easily solved by factoring, using factor theorem formulas, or following the instructions in the problem statement. However, some students do not properly understand the meaning of the calculations. Deepening their understanding of complex numbers is essential for students aiming to enter science universities and those engaged in inquiry-based learning. Visualizing complex numbers in three-dimensional space rather than four-dimensional space allows students to view them from a different perspective. We believe this will be an effective teaching method for presenting teaching materials and developing lessons.

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References

[1] Detailed Explanations of Mathematics Entrance Exam Questions for National Universities, Vol. 1 (2004). Seibunshinsha, pp.208–210.

[2] GAKUSHUUSHIDOUYOURYOU-KAISETU (Guidelines Commentary for Senior High School): RISUUHEN (Mathematics and Science Section) (2019), MEXT.

[3] Gentaro Sasaki (1975). Monograph 17. Theory and Solutions of Equations. Kagaku Shinkosha, p.71.

[4] Jun Yamada (2015). Creating learning materials to promote mathematical application and teaching them. Journal of the Japan Society for Mathematical Education, Vol. 97, Special issue, special issue of the 97th conference (Hokkaido conference), p.510.

[5] Jun Yamada (2024). Creating visualization materials: The trajectory of solutions of complex numbers. Journal of the Japan Society for Mathematical Education, Abstracts of the 106th conference. p.494.

[6] KOKOROWOYURASU TANOSHIIJYUGYOU-WADAIGENSUUGAKU (Fun and heart shaking lessons): Mathematics, Vol. 1 (1989). Toho, p.83.

[7] Páll-Szabó Ágnes Orsolya (2017).Visualizing the roots of a cubic equation. eJMT.Vol 11-1, pp.35–49. https://ejmt.mathandtech.org/Contents/eJMT_v11n1n1.pdf

[8] Stewart, J. (2015). Calculus: Early Transcendentals (8th ed.). Cengage Learning.