# Technology-assisted Investigations of Identification Colorings of Circulant Graphs

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**Abstract**

*In graph theory, the four color theorem became widely popular not only for its long and interesting history but also for being the first major mathematical theorem to have been proved with the assistance of a computer. While the four color theorem pertains to proper vertex colorings of graphs, several other graph colorings with various interesting properties have been introduced and studied in the literature. One example is the notion of an identification coloring, or ID-coloring, which has been introduced as an approach for uniquely identifying the vertices of a graph. An ID-coloring of a graph $G$ is a coloring of its vertices, using only the colors red or white, such that the multiset codes of the vertices are distinct. Here, the multiset code of a vertex refers to the multiset containing its distances to each of the red vertices in the graph. If a graph has at least one ID-coloring, it is called an ID-graph. The study of ID-colorings is particularly interesting for graphs with nontrivial automorphisms. One such family of graphs is that of the circulant graphs, which are known to be vertex-transitive. In this work, we again harness the power of technology by developing a Python notebook that can assist in verifying, visualizing, and searching for ID-colorings of circulant graphs. We then demonstrate how we have used this Python notebook to obtain a characterization of circulant graphs $C_n(1,2)$ that are ID-graphs.*

## 1 Introduction

Students often mistake mathematics as something that is solely focused on computations and numbers. However, as Devlin [1, p.3] aptly puts it, "...mathematics is the science of patterns. What the mathematician does is examine abstract 'patterns'... [that] can be either real or imagined, visual or mental, static or dynamic, qualitative or quantitative, purely utilitarian or of little more than recreational interest. They can arise from the world around us, from the depths of space and time, or from the inner workings of the human mind." One way for students to have a genuine experience of mathematics is through mathematical research, which is oriented towards finding unknown patterns and which is characterized by abstract reasoning and logical proofs. However, many areas in mathematics require advanced knowledge and skills before one can understand and attempt open research problems. In this case, an area like graph theory offers an advantage since some of the problems therein can be understood and appreciated by students even with minimal prerequisites.

Graph theory is the area of mathematics focused on the study of mathematical objects called graphs. Here, a *graph* is defined to be a collection of *vertices* that may be connected using *edges*. Two vertices that are connected directly by an edge are called *adjacent* vertices. A classical and popular topic in graph theory is *proper vertex coloring*: how many colors are needed so that the vertices of a graph can be colored in such a way that no two adjacent vertices are colored the same? Consider, for example, the graph $C_7$ (i.e., the cycle graph of seven vertices) shown in Figure 1. The figure on the left shows a proper vertex coloring of $C_7$ (i.e., no adjacent vertices are colored the same) using four colors (blue, yellow, green, and red). It is not difficult to notice, of course, that the red vertex may be colored yellow instead to obtain the coloring on the right. Thus, three colors are enough to construct a proper vertex coloring of $C_7$; moreover, it should be easy to realize that using only two colors cannot work.

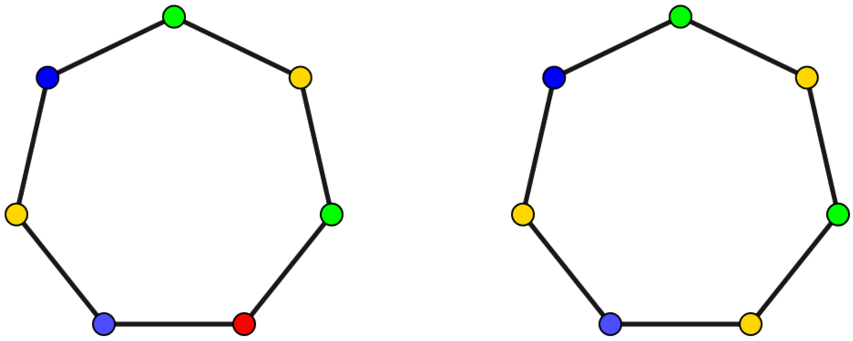


Figure 1: The graph $C_7$ (i.e., the cycle graph with 7 vertices) that has been colored using four colors (left) and three colors (right).

Proper vertex colorings have been studied both from the theoretical and applied perspectives. Some of the most-well known theorems (e.g., the four color theorem, and Brook's theorem) in graph theory pertain to this topic. On the other hand, proper vertex colorings have found applications in map coloring, scheduling, and resource allocation.

In proper vertex colorings, if two vertices are not adjacent, then they may be colored the same; that is, only adjacent vertices are distinguished from each other. What if we want to impose that all vertices are distinguished from one another? Clearly, if we do this on the basis of vertex colors, then it becomes largely uninteresting as we will be forced to use one color per vertex (i.e., no color cannot be repeated anymore). Thus, graph theorists have conceptualized other notions that allow vertices to be distinguished from each other.

In this paper, we focus on one such notion called *identification colorings*. In an identification coloring, vertices may only be colored red or white. Every time a vertex is colored red, each vertex $v$ in the graph receives a number equal to the distance between $v$ and the vertex colored red. When multiple vertices are colored red, the numbers received by each vertex are collected in a multiset. Recall that a multiset is a collection of objects such that the ordering of the objects does matter but multiple copies of each object are allowed. An identification coloring is completed when all vertices have received distinct multisets.

To illustrate, consider again the cycle graph $C_7$ shown in Figure 2. In the leftmost figure, only the top vertex is colored red; hence, the red vertex has the multiset $\{0\}$ (because its distance to itself is 0) while vertices adjacent to the red vertex has multiset $\{1\}$ (because their distance to the red vertex is 1), and so on. In the middle figure, two vertices are now colored red; each vertex now has a multiset with exactly two elements. However, at this point, some vertices still have the same multisets; thus, an identification coloring of $C_7$ has not been completed yet. In the rightmost figure, three vertices have been colored red; and based on the figure, all vertices now have distinct multisets. Therefore, we have successfully completed an identification coloring of $C_7$.

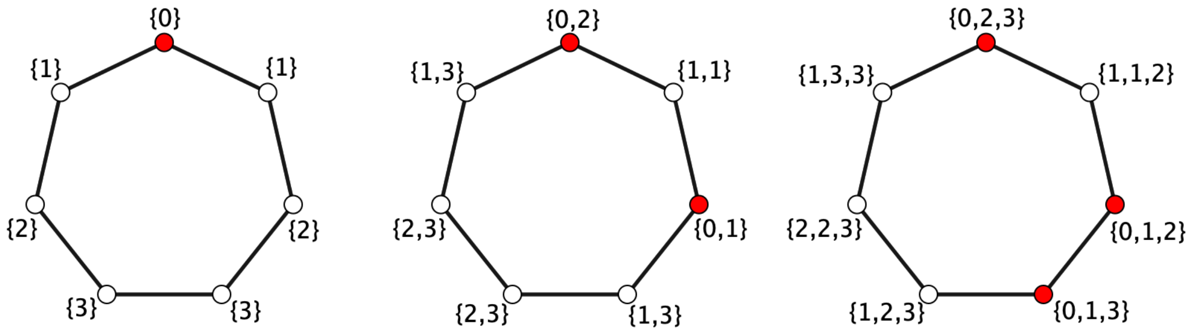



Figure 2: Constructing an identification coloring of the cycle graph $C_7$.

The notion of identification colorings (or ID-colorings) was introduced by Chartrand et al. in [2]. In [3], Hakanen and Yero established that ID-colorings are, in fact, equivalent to the older notion of multiset resolving sets independently introduced by Saenpholphat [4] and Simanjuntak et al. [5].

Combining aspects from [2, 4, 5], we now present a precise definition of ID-colorings. Suppose $c$ is a coloring of a simple, connected, nontrivial graph $G$ such that each vertex is colored red or white, and at least one vertex is colored red. For each $v \in V(G)$, define its multiset code $M(v)$ to be the multiset of all distances between $v$ and each of the red vertices. Then $c$ is called an **identification coloring** (or **ID-coloring**) [2] of $G$ if $M(u) \neq M(v)$ for any pair of distinct vertices $u, v$ in $G$. Equivalently, by [3], the set of red vertices is called a **multiset resolving set** [4, 5]. If $G$ possesses an ID-coloring, then $G$ is called an ID-graph and, in this case, its ID-coloring can be used to distinguish its vertices from each other. This is particularly relevant when $G$ has nontrivial automorphisms [2].

One such family of graphs is that of the circulant graphs [6], which are vertex-transitive. In this paper, we restrict our attention to circulant graphs $C_n(1,2)$, where $n \geq 3$, whose vertex and edge sets are given by

$$V(C_n(1,2)) = \mathbb{Z}_n = \text{the group of integers modulo } n,$$
$$E(C_n(1,2)) = \{ij : i - j = \pm 1 \text{ or } \pm 2 \text{ in } \mathbb{Z}_n\},$$

respectively. Intuitively, the circulant graph $C_n(1,2)$ can be constructed by arranging the $n$ vertices $0, 1, ..., n-1$, in this order, in a circle, and drawing all edges of the form $\{i, (i+1) \bmod n\}$ and $\{i, (i+2) \bmod n\}$.

The circulant graphs $C_3(1,2), C_4(1,2)$, and $C_5(1,2)$ are isomorphic to the complete graphs $K_3, K_4$, and $K_5$, respectively, and they are known to be not ID-graphs [2]. In Figure 3, we present an ID-coloring of the circulant graph $C_{12}(1,2)$.

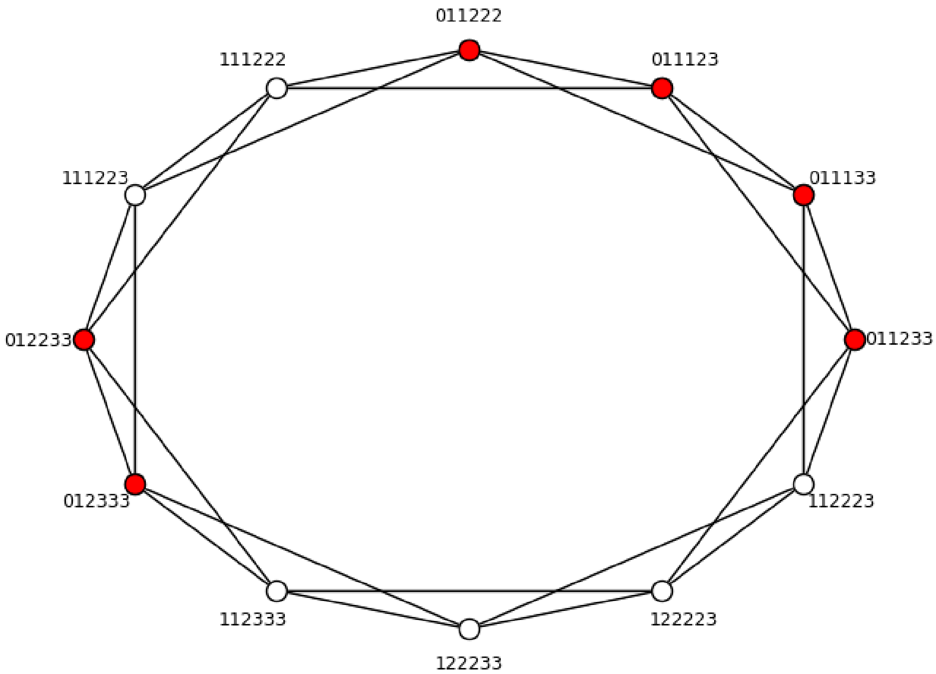



Figure 3: An ID-coloring of the circulant graph $C_{12}(1,2)$. Listed next to each vertex are the elements of its multiset code.

The objective of this paper is to characterize circulant graphs $C_n(1,2)$ that are ID-graphs. This characterization is a new result in graph theory and illustrates mathematical research that is accessible even to undergraduate students and to new mathematicians with minimal mathematical background. Moreover, we illustrate how technology can be used to aid original mathematical research. It is worth noting that the four color theorem, which states that any planar graph can be properly colored using only at most four colors, is considered the first major theorem in mathematics with a computer-assisted proof.

## Some Notations and Known Results

We now introduce some notations and known results that will be used throughout the paper. First, suppose we are given the multiset code $M(v)$ of a vertex $v$ of some graph. We can order the elements of $M(v)$ as $m_1, m_2, ..., m_{|M(v)|}$ such that $m_1 \leq m_2 \leq \cdots \leq m_{|M(v)|}$. Then the element $m_1$ and $m_{|M(v)|}$ (i.e., the smallest and largest elements) will be denoted by $M^{\min}(v)$ and $M^{\max}(v)$, respectively. We also denote $m_2$ by $M^{\min 2}(v)$; note that $M^{\min}(v)$ and $M^{\min 2}(v)$ are not necessarily distinct. The following is an easy observation: given two distinct vertices $v$ and $w$, if $M^{\min}(v) \neq M^{\min}(w)$ or $M^{\max}(v) \neq M^{\max}(w)$ or $M^{\min 2}(v) \neq M^{\min 2}(w)$, then $M(v) \neq M(w)$.

In the following theorem, we summarize some results from [2] that will be used in this paper.

**Theorem 1** ([2]) (a) *There is no ID-coloring of a connected graph with exactly two red vertices.*

(b) *Suppose $G$ is a connected graph with at least two vertices. Then $G$ has an ID-coloring with only one red vertex if and only if $G$ is a path.*

(c) *Suppose $G$ is vertex-transitive, connected, and $|V(G)| = n \geq 2$. If $G$ has an ID-coloring in which exactly $r$ vertices are colored red, then $G$ also has an ID-coloring in which exactly $n - r$ vertices are colored red.*

## 2 Python Notebook: ID-colorings of Circulant Graphs

We now present a Python notebook that we have developed to assist in verifying, visualizing, and searching for ID-colorings of circulant graphs. We have set up this Python notebook online using Google Colab[1] and is accessible via the following link: `https://bit.ly/ID-col_Circulant`.

The first set of features of the Python notebook is intended for checking if a given red-white coloring of a circulant graph is an ID-coloring and in visualizing the graph, with the ID-coloring and the vertices' multiset codes. In Figure 4(a), we present a code block containing the function `check_if_ID` that takes as inputs a graph $G$ and a red-white coloring $c$ of $G$. If $c$ is an ID-coloring of $G$, the function outputs a Python dictionary containing the code of each vertex of $G$; otherwise, the function simply outputs `False`. Example 1 (see Figure 4(b)) of the Python notebook then illustrates how a circulant graph and a red-white coloring can be instantiated and how the function `check_if_ID` is used. The example then proceeds towards visualizing the outputs; it is actually through this example that Figure 3 has been constructed.

**Verifying and Visualizing ID-colorings**

```python
def check_if_ID(G,c):
#Input: graph G, red-white coloring c
#Output: v if c is an ID coloring, where v is the dictionary with
#        keys = vertices and values = vector codes;
#        False if c is not an ID coloring; can print a pair of conflict vertices
    v = {}
    dist_gen=nx.all_pairs_shortest_path_length(G)
    distances = {x[0]:x[1] for x in dist_gen}
    diam = nx.algorithms.distance_measures.diameter(G)
    for node in G.nodes:
        temp = []
        for i in range(diam):
            temp.append(0)
        for node2 in G.nodes:
            if node2 != node and c[node2] == 1:
                temp[distances[node][node2]-1]+=1
        v[node] = temp
    for pair in itertools.combinations(G.nodes,2):
        if v[pair[0]] == v[pair[1]]:
            #print("Coloring is not ID.", pair, v[pair[0]], v[pair[1]])
            #Uncomment to print a pair of conflict vertices.
            return False
    return v
```

(a)

**Example 1**

```python
#Begin by defining a circulant graph.
G = circulant(12,1,2)
#
#Define a red-white coloring c of G.
c = {}
for v in G.nodes():
    c[v] = 0
red_vertices = [0,1,2,3,6,7]
r = len(red_vertices)
for v in red_vertices:
    c[v] = 1
#
#Check if c is an ID-coloring.
L2 = [check_if_ID(G,c)] #If c is not an ID-coloring, L2 will be [False].
L3 = get_m_codes(L2) #This line will return an error if c is not an ID-coloring.
```

```python
#This code block sets up some requisite components for drawing the graph.
#
color = {}
for v in G.nodes():
    if c[v] == 1:
        color[v] = 'red'
    else:
        color[v] = 'white'
```

(b)

Figure 4: Code blocks from the Python notebook for (a) checking if a given red-white coloring is an ID-coloring and (b) an example illustrating the use of (a).

The next set of features of our Python notebook is intended for performing exhaustive searches for ID-colorings of a given graph. In Figure 5(a), we present the function `search_for_ID`,

[1] `https://colab.research.google.com/`

which takes as input a graph $G$ and the desired number $r$ of red vertices. The function has two modes, one in which it tries to find only one ID-coloring and a second mode in which it tries to find as many ID-colorings as possible, up to a user-defined maximum. In Figure 5(b), we have the function `solve_ID`, which is intended to determine the graph's **ID number**, which is least number of red vertices needed for an ID-coloring of the given graph. The ID number has also been of interest in previous works on this topic; see [7, 8, 9] for examples. For convenience, the function `print_coloring_to_file` (Figure 5(b)) is also provided for saving the obtained ID-colorings to local text files (Figure 5(c)).

**Searching for ID-colorings**

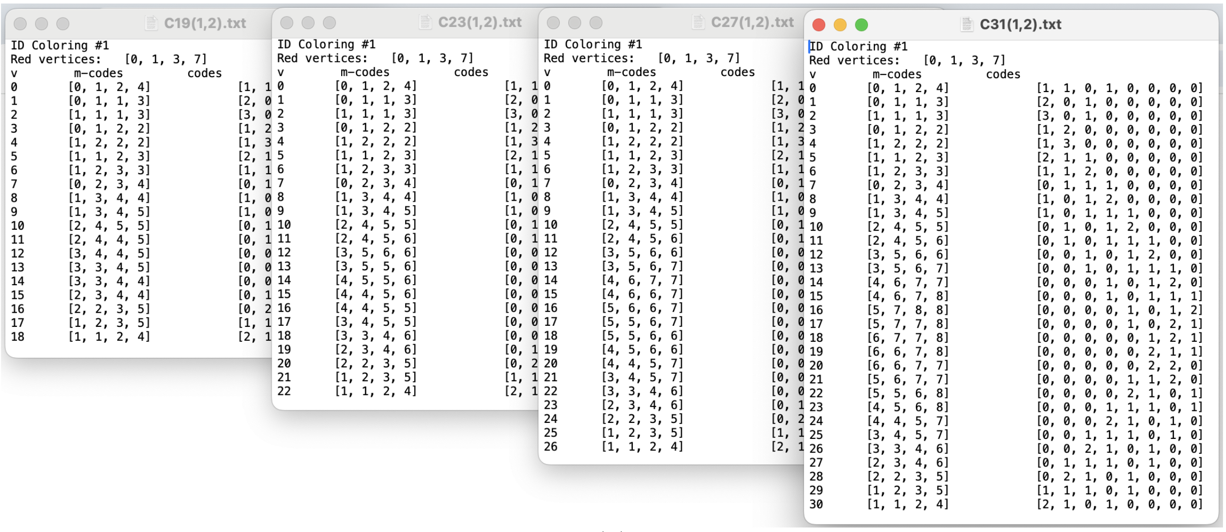
```
def search_for_ID(G,r,mode="find_all"):
#Input: graph G, r = no. of red vertices to use
#Output: dicts L, L2, where L = ID colorings of G with r red vertices;
#        L2 = corresponding vector codes
#        returns False if no coloring is found
#        if mode is find_one, code will only return one coloring, if it exists;
#        otherwise, it will return multiple possible colorings
    L = []
    L2 = []
    S = distinct_combinations(G.nodes,r)
    ctr = 0
    for combi in S:
        c = {}
        for node in G.nodes:
            c[node] = 0
        for node in combi:
            c[node] = 1
        s = sum(c.values())
        if s == r:
            v = check_if_ID(G,c)
        else:
            v = False
        if v != False:
            L.append(c)
            L2.append(v)
            ctr += 1
            if mode == "find_one" or ctr > 29: #this limits number of colorings
            #                                   to search for to at most 30;
            #                                   for runtime considerations
                return L, L2
    return L,L2
```

(a)

```
def solve_ID(G):
    #Input: graph G
    #Output: ID = ID number of G; cols = optimal ID colorings of G;
    #        ds = corresponding codes; spec = ID spectrum of G
        N = len(G.nodes)
        ID = N
        spec = []
        for r in range(1,int(N/2)+1):
            L,L2 = search_for_ID(G,r,mode="find_one")
            if len(L) > 0:
                if len(spec) == 0:
                    ID = r
                spec.append(r)
        if ID == N:
            return False, False, False, False
        else:
            L,L2 = search_for_ID(G,ID,mode="find_all")
            return ID, L, L2, spec
```

```
#This function is for printing the outputs to files that will be saved locally.
#This requires running this notebook in your local device.
def print_coloring_to_file(filename,m,col,ms,ds,x):
    with open(filename, 'a') as f:
        f.write("ID Coloring #" + str(x+1) + "\n")
        f.write("Red vertices: \t")
        red = [i for i, x in enumerate(col) if x == 1]
        f.write(str(red))
        f.write("\n")
        f.write("v \t m-codes \t codes \n")
        for k in range(0,m):
            f.write(str(k) + "\t" + str(ms[k]) + "\t \t" + str(ds[k]) + "\n")
        f.write("\n")
    f.close()
```

(b)




(c)

Figure 5: Code blocks from the Python notebook for (a) searching for ID-colorings of a given graph and for (b) determining the least number of red vertices needed for an ID-coloring of the graph, and then saving the outputs to local text files. In (c), we present some screenshots of these text files.

In the next section, we demonstrate further, particularly through Examples 2-4 of the

Python notebook, how we have used the above functions to investigate ID-colorings of circulant graphs $C_n(1,2)$.

# 3 ID-colorings of $C_n$(1,2)

In [9], identification colorings of antiprism graphs have been investigated. An antiprism graph is a graph corresponding to the skeleton of an antiprism. It can be observed that any antiprism graph is isomorphic to a circulant graph $C_n(1,2)$, where $n$ is even and $n \geq 6$.

One of the main results in [9] is a characterization of all ID-antiprisms. Stated in terms of circulant graphs, this chracterization is as follows.

**Theorem 2 ([9])** *For any even integer $n \geq 6$, the circulant graph $C_n(1,2)$ is an ID-graph if and only if $n \geq 12$.*

A natural extension of the preceding result is a characterization, for odd integers $n$, of circulant graphs $C_n(1,2)$ that are ID-graphs. We accomplish this objective assisted by our Python notebook discussed in Section 2.

We begin by considering small values of $n$, where $n$ is odd and $n \geq 7$. Example 2 in our Python notebook provides a verification, through exhaustive searches, that $C_7(1,2)$, $C_9(1,2)$, and $C_{11}(1,2)$ are not ID-graphs. Below, we state this result and, for brevity, we present a proof only for $C_7(1,2)$.

**Proposition 3** *The circulant graph $C_n(1,2)$ is not an ID-graph for $n \in \{7,9,11\}$.*

**Proof.** We will prove that $C_7(1,2)$ is not an ID-graph; the proofs for $C_9(1,2)$ and $C_{11}(1,2)$ are similar albeit longer.

Let $G = C_7(1,2)$. Since $G$ is vertex-transitive and is not isomorphic to a path, Theorem 1 implies that $G$ has no ID-coloring with 1, 2, 5, 6, or 7 red vertices. We now show that $G$ cannot have an ID-coloring with 3 red vertices. By Theorem 1(c), it will also follow that $G$ cannot have an ID-coloring with 4 red vertices.

Let $c$ be a red-white coloring of $G$ such that exactly 3 vertices are colored red. Up to rotations and reflections, we only need to consider the following cases.

**Case 1.** Suppose the red vertices are 0, 1, and 2. Then $M(0) = \{0,1,1\} = M(2)$.

**Case 2.** Suppose the red vertices are 0, 2, and 3. Then $M(0) = \{0,1,2\} = M(3)$.

**Case 3.** Suppose the red vertices are 0, 3, and 4. Then $M(3) = \{0,1,2\} = M(4)$.

**Case 4.** Suppose the red vertices are 0, 2, and 4. Then $M(0) = \{0,1,2\} = M(4)$.

In any of the cases above, there are two distinct vertices with equal multiset codes. Therefore, any red-white coloring $c$ of $G$ cannot be an ID-coloring of $G$. ■

We continue our investigations of ID-colorings of $C_n(1,2)$ for small values of odd $n$. Based on the outputs from Example 4 of our Python notebook, we observe that the circulant graphs $C_{13}(1,2)$, $C_{15}(1,2)$, $C_{17}(1,2)$, and $C_{21}(1,2)$ are ID-graphs. In Figure 6, we present an example of an ID-coloring of each of these four circulant graphs. Thus, we have the following observation.

**Observation 4** *The circulant graph $C_n(1,2)$ is an ID-graph for $n \in \{13,15,17,21\}$.*

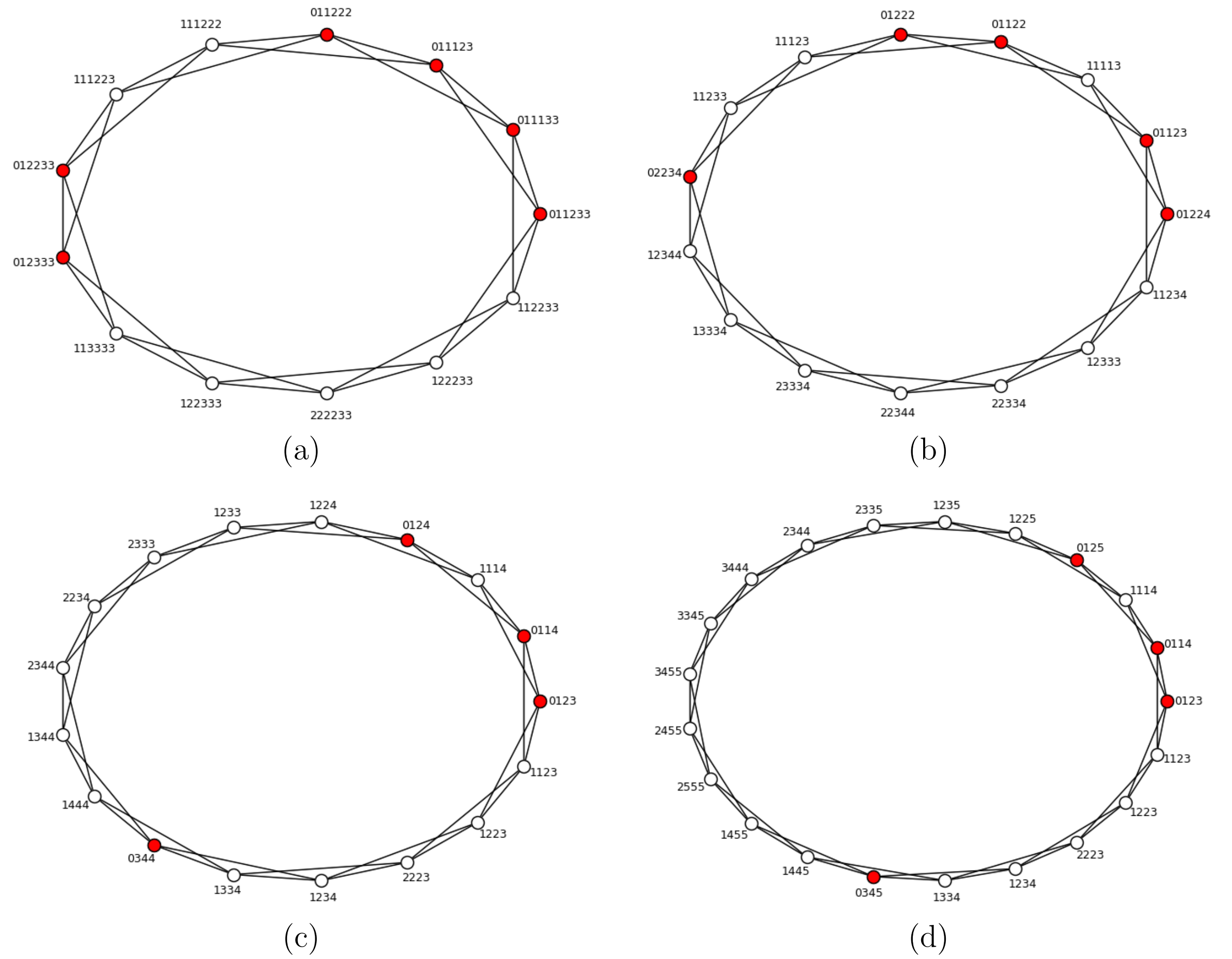


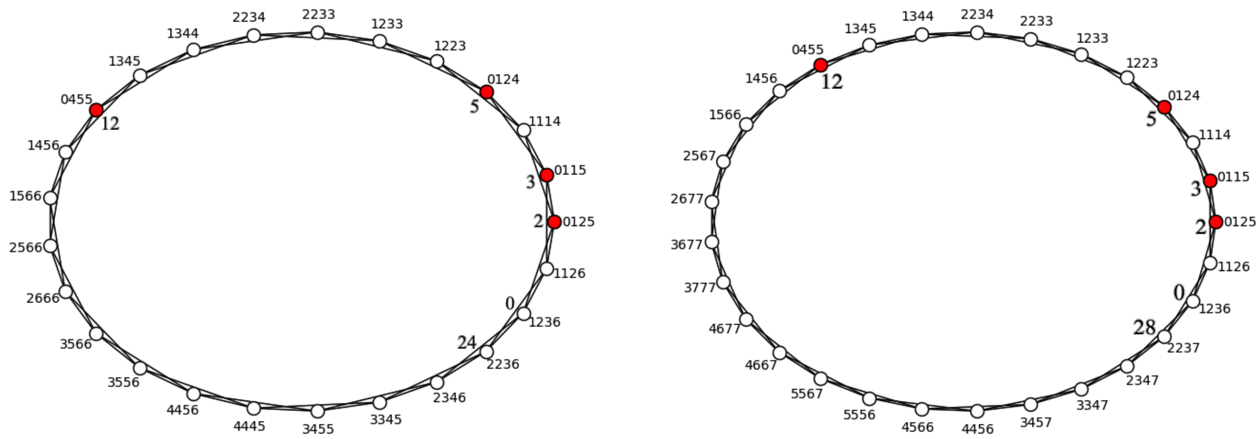

Figure 6: ID-colorings of (a) $C_{13}(1,2)$, (b) $C_{15}(1,2)$, (c) $C_{17}(1,2)$, and (d) $C_{21}(1,2)$. Listed next to each vertex are the elements of its multiset code.

Still guided by the outputs from Example 4 of our Python notebook (see also Figure 5(c)), we now turn to investigate ID-colorings of $C_n(1,2)$ for general values of odd $n$. We observe the emergence of a pattern for the ID-colorings; particularly, we see that the pattern depends on the congruence class of $n$ modulo 4. When $n \equiv 1 \pmod 4$ and $n \geq 25$, we find that the red-white coloring $c_1$ in which only the vertices $2, 3, 5$, and 12 are colored red is an ID-coloring of $C_n(1,2)$. On the other hand, when $n \equiv 3 \pmod 4$ and $n \geq 19$, we find that the red-white coloring $c_2$ in which only the vertices $2, 3, 5$, and 9 are colored red is an ID-coloring of $C_n(1,2)$.

**Case I:** $n \equiv 1 \pmod 4$ **and** $n \geq 25$

Let us focus first on the case where $n \equiv 1 \pmod 4$ and $n \geq 25$. Using the codes in Example 1 of our Python notebook, we visualize the red-white coloring $c_1$ of $C_{25}(1,2)$ and $C_{29}(1,2)$. We present the outputs in Figure 7, where the corresponding multiset code of each vertex is also indicated. From these two examples, it is evident that the multiset codes in $C_{29}(1,2)$ can be obtained from those in $C_{25}(1,2)$.

Using our Python code, we have observed that the property continues further; that is, with the same red vertices, the multiset codes of the vertices in $C_{n+4}(1,2)$ can be obtained




(a) (b)

Figure 7: ID-colorings for (a) $C_{25}(1,2)$ and (b) $C_{29}(1,2)$ in which the vertices $2, 3, 5$, and $12$ are colored red. Listed next to each vertex are the elements of its multiset code.

from those in $C_n(1,2)$. In the following lemma, we precisely state and establish the generality of this observation. For a multiset $M$ and an integer $k$, we denote by $k + M$ the multiset $\{k + m : m \in M\}$.

**Lemma 5** *Suppose each $C_n(1,2)$, where $n \geq 25$ with $n \equiv 1 \pmod 4$, has been colored such that $2, 3, 5, 12$ are red while the other vertices are white. Denote by $M_n(v)$ the multiset code of $v$ as a vertex of $C_n(1,2)$. Then*

$$M_{n+4}(i) = \begin{cases} M_n(i), & i \in \{0, 1, ..., \frac{n+3}{2}\}, \\ 1 + M_n(i-2), & i \in \{\frac{n+3}{2} + 1, \frac{n+3}{2} + 2, ..., n+3\}. \end{cases}$$

**Proof.** Let $d_k(v, w)$ be the distance between vertices $v$ and $w$ in $C_k(1,2)$. Note that the diameter of $C_n(1,2)$ is $\frac{n-1}{4}$ and that this is exactly equal to both $d_n(0, \frac{n-1}{2})$ and $d_n(2, \frac{n+3}{2})$. This implies that for any vertex $i \in \{0, 1, ..., \frac{n+3}{2}\}$, the shortest path, in $C_n(1,2)$, between $i$ and any of the four red vertices involves only vertices in $\{0, 1, ..., \frac{n+3}{2}\}$ as well. Thus, any such shortest path does not change in $C_{n+4}(1,2)$. This gives $M_{n+4}(i) = M_n(i)$ for $i \in \{0, 1, ..., \frac{n+3}{2}\}$, as desired.

We are left to prove that $M_{n+4}(i) = 1 + M_n(i-2)$ for $i \in \{\frac{n+3}{2} + 1, \frac{n+3}{2} + 2, ..., n+3\}$. Note that the argument $i - 2$ of $M_n(\cdot)$ is treated as an element of $\mathbb{Z}_n$; that is, when $i = n+2$ or $n+3$, the corresponding vertex $i-2$ in $C_n(1,2)$ is 0 or 1, respectively.

Now, to complete the proof, it is sufficient to prove that for any red vertex $r \in \{2, 3, 5, 12\}$, we have $d_{n+4}(i, r) = 1 + d_n(i-2, r)$ for any $i \in \{\frac{n+3}{2} + 1, \frac{n+3}{2} + 2, ..., n+3\}$. Fix such a vertex $i$ and a red vertex $r$; then

$$d_{n+4}(i, r) = \min\left\{\left\lceil \frac{i-r}{2} \right\rceil, \left\lceil \frac{n+4+r-i}{2} \right\rceil\right\} = \begin{cases} \left\lceil \frac{i-r}{2} \right\rceil, & \frac{n+3}{2} + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + r + 2, \\ \left\lceil \frac{n+4+r-i}{2} \right\rceil, & \lfloor \frac{n}{2} \rfloor + r + 3 \leq i \leq n+3. \end{cases}$$

On the other hand, we have

$$\begin{aligned} d_n(i-2,r) &= \min\left\{\left\lceil\frac{i-2-r}{2}\right\rceil, \left\lceil\frac{n+r-i+2}{2}\right\rceil\right\} \\ &= \begin{cases} \lceil\frac{i-r}{2}\rceil - 1, & \frac{n+3}{2}+1 \le i \le \lfloor\frac{n}{2}\rfloor + r + 2, \\ \lceil\frac{n+4+r-i}{2}\rceil - 1, & \lfloor\frac{n}{2}\rfloor + r + 3 \le i \le n+1. \end{cases} \\ &= d_{n+4}(i,r) - 1 \end{aligned}$$

for $i \in \{\frac{n+3}{2}+1, \frac{n+3}{2}+2, ..., n+1\}$. Hence, we are left to consider the cases when $i = n+2$ and $i = n+3$. For these, we have the following: $d_{n+4}(n+2,r) = \lceil\frac{r+2}{2}\rceil = 1 + \lceil\frac{r}{2}\rceil = 1 + d_n(0,r)$, and $d_{n+4}(n+3,r) = \lceil\frac{r+1}{2}\rceil = 1 + \lceil\frac{r-1}{2}\rceil = 1 + d_n(1,r)$. This completes the proof. ■

We are now ready to prove that the coloring $c_1$ is indeed an ID-coloring of $C_n(1,2)$ for any $n \ge 25$ and $n \equiv 1 \pmod 4$.

**Theorem 6** *Let $n \ge 25$ with $n \equiv 1 \pmod 4$. Then the red-white coloring $c_1$ where only the vertices $2,3,5,12$ are colored red is an ID-coloring of the circulant graph $C_n(1,2)$. Thus, these circulant graphs are ID-graphs.*

**Proof.** The cases $n = 25$ and $n = 29$ can be verified easily from the examples in Figure 7. Now, suppose the result is true for some $n \ge 29$ with $n \equiv 1 \pmod 4$. We proceed inductively and prove that $c_1$ is an ID-coloring of the circulant graph $C_{n+4}(1,2)$.

We will use the same notations $M_n$ and $d_n$ from Lemma 5. Let $i, j \in \{0, 1, ..., n+3\}$, where $i < j$. We need to prove that $M_{n+4}(i) \ne M_{n+4}(j)$.

**Case 1.** Suppose $0 \le i < j \le \frac{n+3}{2}$.

Then by Lemma 5 and the inductive hypothesis, we have $M_{n+4}(i) = M_n(i) \ne M_n(j) = M_{n+4}(j)$.

**Case 2.** Suppose $\frac{n+3}{2} + 1 \le i < j \le n+3$.

Then by Lemma 5 and the inductive hypothesis, we have $M_{n+4}(i) = 1 + M_n(i-2) \ne 1 + M_n(j-2) = M_{n+4}(j)$.

**Case 3.** Suppose $0 \le i \le \frac{n+3}{2}$ and $\frac{n+3}{2} + 1 \le j \le n+3$.

*Case 3.1.* Suppose $i \in \{0, 1, ..., 12\} - \{8, 9\}$.

By Lemma 5, we have $M_{n+4}(i) = M_{25}(i)$; from Figure 7(a), it is clear that $M^{\min}_{n+4}(i) \le 1$. Now, since $\frac{n+3}{2}+1 \le j \le n+3$, Lemma 5 implies that $M_{n+4}(j) = 1+M_n(j-2)$. Since $j-2 > 12$, the vertex $j-2$ in $C_{n+4}(1,2)$ is not colored red and we must have $M^{\min}_{n+4}(j) \ge 1+M^{\min}_n(j-2) \ge 2$. Hence, $M^{\min}_{n+4}(i) \ne M^{\min}_{n+4}(j)$, as desired.

*Case 3.2.* Suppose $i \in \{8, 9\}$.

Again by Lemma 5 and from Figure 7(a), we have $M^{\min}_{n+4}(i) = M^{\min 2}_{n+4}(i) = 2$. Now, for $\frac{n+3}{2} + 1 \le j \le n+3$, note that $M^{\min}_{n+4}(j) = M^{\min 2}_{n+4}(j) = 2$ if and only if $j = n+3$; thus we are left to consider only this value of $j$. Given that

$$M^{\max}_{n+4}(8) = 3, \quad M^{\max}_{n+4}(9) = 4, \quad \text{while} \quad M^{\max}_{n+4}(n+3) = 7,$$

the desired result follows.

*Case 3.3.* Suppose $i \in \{13, 14, ..., \frac{n+3}{2} - 1, \frac{n+3}{2}\}$.

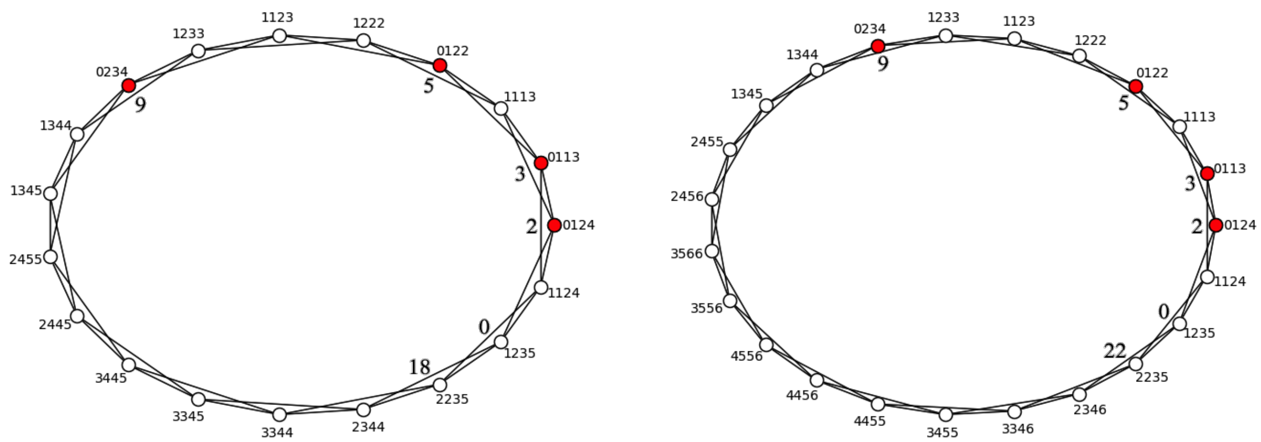
Then $i = 12 + k$, where $k \in \{1, 2, ..., \frac{n-1}{2} - 10\}$. For any red vertex $r \in \{2, 3, 5, 12\}$, note that $d_{n+4}(i, r) = \min\{\lceil \frac{i-r}{2} \rceil, \lceil \frac{n+4+r-i}{2} \rceil\} = \lceil \frac{i-r}{2} \rceil$. Thus,

$$M^{\min}_{n+4}(i) = \left\lceil \frac{i-12}{2} \right\rceil = \left\lceil \frac{k}{2} \right\rceil \qquad \text{and} \qquad M^{\min 2}_{n+4}(i) = \left\lceil \frac{i-5}{2} \right\rceil = \left\lceil \frac{k+7}{2} \right\rceil \geq \left\lceil \frac{k}{2} \right\rceil + 3.$$

Now, for $\frac{n+3}{2} + 1 \leq j \leq n + 3$, it can be verified that $M^{\min}_{n+4}(j) = \lceil \frac{k}{2} \rceil = M^{\min}_{n+4}(i)$ if and only if $j = n + 6 - 2\lceil \frac{k}{2} \rceil$ or $j = n + 6 - 2\lceil \frac{k}{2} \rceil + 1$. Hence, we are left to consider only these two values of $j$. Note that for these two values of $j$, we have $M^{\min}_{n+4}(j) = d_{n+4}(j, 2)$. Finally, the desired result follows from the following computations: $M^{\min 2}_{n+4}\left(n + 6 - 2\lceil \frac{k}{2} \rceil\right) \leq d_{n+4}\left(3, n + 6 - 2\lceil \frac{k}{2} \rceil\right) = \min\left\{\left\lceil \lceil \frac{k}{2} \rceil + \frac{1}{2} \right\rceil, \left\lceil \frac{n+3}{2} - \lceil \frac{k}{2} \rceil \right\rceil\right\} = \lceil \frac{k}{2} \rceil + 1 < \lceil \frac{k}{2} \rceil + 3 \leq M^{\min 2}_{n+4}(i)$. Similarly, $M^{\min 2}_{n+4}\left(n + 6 - 2\lceil \frac{k}{2} \rceil + 1\right) \leq d_{n+4}\left(3, n + 6 - 2\lceil \frac{k}{2} \rceil + 1\right) = \left\lceil \frac{n+7-(n+6-2\lceil \frac{k}{2} \rceil + 1)}{2} \right\rceil < \lceil \frac{k}{2} \rceil + 3 \leq M^{\min 2}_{n+4}(i)$. This completes the proof. ■

**Case II:** $n \equiv 3\,(\textbf{mod}\,4)$ **and** $n \geq 19$

We now turn to the case where $n \equiv 3\,(\mathrm{mod}\,4)$ and $n \geq 19$. Recall the red-white coloring $c_2$ of $C_n(1, 2)$ in which the red vertices are 2,3,5, and 9. As has been done in the previous case, we have visualized in Figure 8 the coloring $c_2$ for $C_{19}(1, 2)$ and $C_{23}(1, 2)$.




(a) (b)

Figure 8: ID-colorings for (a) $C_{19}(1, 2)$ and (b) $C_{23}(1, 2)$ in which the vertices 2, 3, 5, and 9 are colored red. Listed next to each vertex are the elements of its multiset code.

As evident from the figure, there is again a strong relationship between the multiset codes in $C_{19}(1, 2)$ and $C_{23}(1, 2)$. This relationship is precisely stated in the following lemma, whose proof is similar to that of Lemma 5.

**Lemma 7** *Suppose each* $C_n(1, 2)$*, where* $n \geq 19$ *with* $n \equiv 3\,(\mathrm{mod}\,4)$*, has been colored such that* $2, 3, 5, 9$ *are red while the other vertices are white. Denote by* $M_n(v)$ *the multiset code of* $v$ *as a vertex of* $C_n(1, 2)$*. Then*

$$M_{n+4}(i) = \begin{cases} M_n(i), & i \in \{0, 1, ..., \frac{n+5}{2}\}, \\ 1 + M_n(i-2), & i \in \{\frac{n+5}{2} + 1, \frac{n+5}{2} + 2, ..., n + 3\}. \end{cases}$$

Using the preceding lemma and applying a similar approach as in Theorem 6, we obtain the following theorem.

**Theorem 8** *Let $n \geq 19$ with $n \equiv 3 \pmod 4$. Then the red-white coloring $c_2$ where only the vertices $2, 3, 5, 9$ are colored red is an ID-coloring of the circulant graph $C_n(1, 2)$. Thus, these circulant graphs are ID-graphs.*

As previously mentioned, the circulant graphs $C_3(1, 2)$, $C_4(1, 2)$, and $C_5(1, 2)$ are not ID-graphs. Combining this with Theorem 2, Proposition 3, Observation 4, Theorem 6, and Theorem 8, we obtain a complete characterization of circulant graphs $C_n(1, 2)$ that are ID-graphs.

**Theorem 9** *For $n \geq 3$, the circulant graph $C_n(1, 2)$ is an ID-graph if and only if $n \geq 12$.*

# 4 Conclusion

In this work, we developed a Python notebook (`https://bit.ly/ID-col_Circulant`) that can assist with the investigations and research works on ID-colorings of circulant graphs. The Python notebook can be used to verify, to visualize, and to search for ID-colorings. With the assistance of the Python notebook, we were able to investigate ID-colorings of $C_n(1, 2)$ for odd $n$, beginning with small values of $n$ (Proposition 3 and Observation 4) and then for general values of $n$ (Theorem 6 and Theorem 8). Through our results, we were able to conclude that for $n \geq 3$, the circulant graph $C_n(1, 2)$ is an ID-graph if and only if $n \geq 12$ (Theorem 9).

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