Decreasing nature of a certain function related to the Steiner's inellipse

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Abstract

This paper answers an open problem proposed by Jean-Jacques Dahan in his ATCM 2024 paper, asking for a formal proof that the function $f(x) = \sum_{k=0}^{2} \sqrt{1 - c^2 \sin^2\left(x + \frac{2k\pi}{3}\right)}$, *where c is a constant in* $[0, 1]$, *is decreasing on the interval* $\left[0, \frac{\pi}{6}\right]$. *Crucially, we employ a CAS (Computer Algebra System) and some basic inequalities to yield the proof. In passing, we take note of some salient pedagogical lessons gleaned from this cross-generational collaborative research.*

1 Introduction

1.1 Mathematical background: Steiner inellipse, director circle and trigonometric formulation

The open problem addressed in this paper is rooted in a rich geometric framework involving ellipses, triangle constructions, and classical theorems such as Marden's Theorem. To appreciate the significance of the particular function f – our central object of study defined in the abstract – and the conjecture surrounding its monotonicity, we begin by recalling the key geometric constructs that motivated its appearance in [2].

Steiner ellipse and Marden's Theorem. Given any non-degenerate triangle $\triangle ABC$, the *Steiner inellipse* is the unique ellipse tangent to the triangle at the midpoints of its sides. It is also the ellipse of maximum area that can be inscribed in $\triangle ABC$. When the triangle is viewed in the complex plane as defined by the roots of a cubic polynomial $p(z) = (z - z_1)(z - z_1)(z - z_3)$, the foci of the Steiner ellipse are precisely the roots of the derivative polynomial $p'(z)$, as per Marden's Theorem ([4]).

This classical result, first known in complex analysis and later popularized in dynamic geometry explorations, establishes a beautiful correspondence between algebraic properties of polynomials and geometric configurations in the Argand plane.

The director circle and bifocal definition of an ellipse. Dahan's investigations [2] employed Dynamic Geometry Software (DGS) to experimentally generate ellipses using their bifocal definition: an ellipse is the locus of points M such that the sum of the distances MF_1 and $MF_2 = 2a$ is constant. In this approach, one fixes a point F_1 and draws a *director circle*[1] of radius $2a$ centred at F_1. For each point V on the circle, the corresponding point M on the ellipse is determined as the intersection of the ray $\overrightarrow{F_1V}$ and the perpendicular bisector of the segment $\overrightarrow{F_2V}$. As the point V varies, the set of all such M traces out an ellipse (see Figure 1).

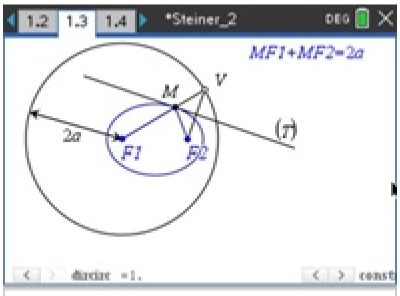

Figure 1: Ellipse (bifocal definition)

The function f: tracing the triangle's height. For convenience of the readers, we keep to the symbols originally used by Dahan in [2]. Figure 2 shows an ellipse E whose foci are f_1 and f_2. In Dahan's dynamic explorations, he considered the set of equilateral triangles circumscribed about a fixed ellipse, whose sides were constructed via the perpendicular bisectors of lines joining the second foci (f_2) and a rotating point m_1 on the director circle (see Figure 2).

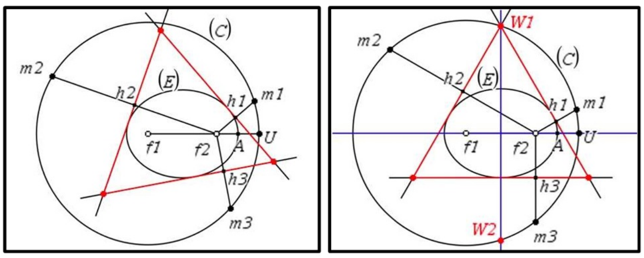

Figure 2: Equilateral triangles circumscribed around a given ellipse.

While Marden's Theorem identifies the foci of the Steiner inellipse, Dahan's investigations turned the classical picture inside out. Instead of seeking an ellipse inscribed in a triangle, he considered the dual problem: fixing an ellipse and constructing all possible equilateral triangles

[1]We follow Dahan's terminology of director circle, which differs from the classical definition of director circle being the geometric locus of points through which passes a pair of orthogonal tangents.

that circumscribe it. The goal was to determine which configuration leads to a triangle of minimum area. Let us now dive into the details.

Let the distance f_1f_2 between f_1 and f_2 be $2c$, where $0 \leq c \leq 1$. The director circle C of E, associated with f_1 has radius $f_1U = 2$, where U is the point on C that lies on the extension of f_1f_2. In fact, the eccentricity of E is c. The reader may refer to Figure 2 for the constructions of all the equilateral triangles (in red) circumsribed around E. Note that each triangle is determined by the varying position of the point m_1 on C, and

$$\angle m_1f_2m_2 = \angle m_2f_2m_3 = \angle m_3f_2m_1 = \frac{2\pi}{3}.$$

Also note that the sides of each circumscribing triangle are defined by the perpendicular bisectors of the line segments $[f_2m_1]$, $[f_2m_2]$ and $[f_2m_3]$.

The problem posed by Dahan in [2] is to determine the positions of m_1 on C that result in the minimum area of a circumscribing triangle. Let the position of m_1 on C be specified by $\angle Uf_2m_1 = x$. Since the area of an equilateral triangle of height h is given by $\frac{h^2}{\sqrt{3}}$ and because

$$h = f_2h_1 + f_2h_2 + f_2h_3 = \frac{1}{2}(f_2m_1 + f_2m_2 + f_2m_3),$$

it suffices to examine the variation of the function

$$f(x) = f_2m_1 + f_2m_2 + f_2m_3.$$

By elementary trigonometry, one immediately has the following.

$$f_2h_1 = -c\cos(x) + \sqrt{1 - c^2\sin^2(x)} \tag{1}$$

$$f_2h_2 = -c\cos\left(x + \frac{2\pi}{3}\right) + \sqrt{1 - c^2\sin^2\left(x + \frac{2\pi}{3}\right)} \tag{2}$$

$$f_2h_3 = -c\cos\left(x + \frac{4\pi}{3}\right) + \sqrt{1 - c^2\sin^2\left(x + \frac{4\pi}{3}\right)} \tag{3}$$

Since

$$\cos(x) + \cos\left(x + \frac{2\pi}{3}\right) + \cos\left(x + \frac{4\pi}{3}\right) = 0$$

for all x, it follows that

$$f(x) = \sqrt{1 - c^2\sin^2(x)} + \sqrt{1 - c^2\sin^2\left(x + \frac{2\pi}{3}\right)} + \sqrt{1 - c^2\sin^2\left(x - \frac{2\pi}{3}\right)}.$$

It turns out that f is a periodic function of period $\frac{\pi}{3}$ and, furthermore, is an even function ([2]). Whence it suffices to study the behavior of f on the closed bounded interval $\left[0, \frac{\pi}{6}\right]$. In summary, the triangle with the minimum area corresponds to the minimum value of this function f, hence motivating Dahan's conjecture that f is decreasing over the said interval.

Experiments using graphing software yield compelling empirical evidence that suggests the following conjecture; for example, the graph of $y = f(x)$ in Figure 3 appears to be decreasing over the interval $\left[0, \frac{\pi}{6}\right]$.

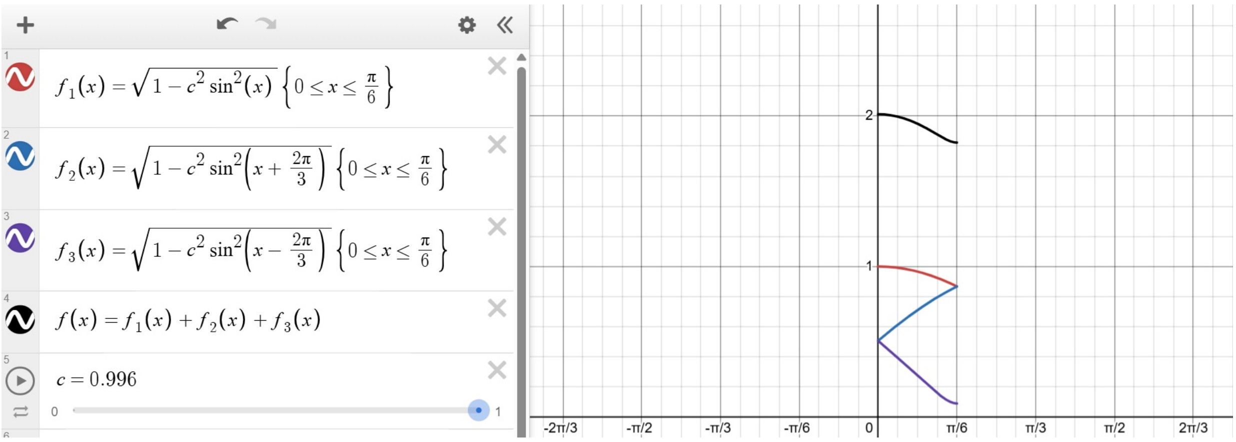

Figure 3: Decreasing nature of f on $\left[0, \frac{\pi}{6}\right]$

This transition – from geometric construction to analytical function – marks the central theme of our present study: translating a problem in ellipse-triangle configurations into a question about the monotonicity of a trigonometric sum involving square roots. Let us state the conjecture at hand formally as follows.

Conjecture 1 (Dahan's conjecture) *Let $c \in [0, 1]$. Then the function*

$$f(x) = \sqrt{1 - c^2 \sin^2(x)} + \sqrt{1 - c^2 \sin^2\left(x + \frac{2\pi}{3}\right)} + \sqrt{1 - c^2 \sin^2\left(x - \frac{2\pi}{3}\right)}$$

is strictly decreasing on $\left[0, \frac{\pi}{6}\right]$.

In a separate ATCM 2024 paper ([3]), Dahan wrote about the above problem:

> "After a hiatus of several years, I decided to resume the search for a proof of what seemed to be a simple problem. However, after several months of unsuccessful efforts, I decided to stop focusing on this single problem."

The rest of this paper is to provide a formal proof of Conjecture 1, which is an essential result that finally fills the gap in the original paper [2]. The main tool we used in cracking this stubborn problem is CAS (Computer Algebra System), but a nifty trick involving simple inequalities is a much-needed ingredient. For the benefits of CAS in mathematical exploration, the reader is referred to [1].

2 Proof of conjecture

2.1 Decomposition and differentiation

We begin by expressing f as the sum of three component functions:

$$f(x) = f_1(x) + f_2(x) + f_3(x),$$

where

$$
\begin{aligned}
f_1(x) &= \sqrt{1 - c^2 \sin^2(x)}, \\
f_2(x) &= \sqrt{1 - c^2 \sin^2\left(x + \frac{2\pi}{3}\right)}, \\
f_3(x) &= \sqrt{1 - c^2 \sin^2\left(x - \frac{2\pi}{3}\right)}.
\end{aligned}
$$

Clearly, $f' = f_1' + f_2' + f_3'$. Introducing two fresh variables

$$p = \cos(x),\ \ q = \sin(x)$$

and invoking the compound angle formulae, the derivatives of the above functions f_i $(i = 1, 2, 3)$ are calculated below:

$$
\begin{aligned}
f_1'(x) &= -\frac{c^2 pq}{\sqrt{1 - c^2 q^2}}, \\
f_2'(x) &= \frac{\frac{1}{4}c^2(\sqrt{3}p - q)(p + \sqrt{3}q)}{\sqrt{1 - \frac{1}{4}c^2(\sqrt{3}p - q)^2}}, \\
f_3'(x) &= -\frac{\frac{1}{4}c^2(\sqrt{3}p + q)(p - \sqrt{3}q)}{\sqrt{1 - \frac{1}{4}c^2(\sqrt{3}p + q)^2}}.
\end{aligned}
$$

2.2 Reformulation: the inequality goal

Let $A = -f_1'(x)$, $B = f_2'(x)$, and $C = -f_3'(x)$. Then $f'(x) \leq 0$ is equivalent to $f_1'(x) + f_2'(x) + f_3'(x) \leq 0$, i.e.,

$$A + C \geq B. \tag{4}$$

Since $x \in \left[0, \frac{\pi}{6}\right]$, we have $p,\ q \geq 0$. So, it follows immediately that

$$A,\ C \geq 0.$$

However, B may be negative or positive, so we need a strategy that handles all cases.

2.3 A failed approach and a clever detour

A natural first attempt is to square both sides of (4) and try to prove:

$$(A + C)^2 \geq B^2. \tag{5}$$

But this approach yields a complicated expression for $(A + C)^2 - B^2$ that resists simplification.

So instead, we take a detour inspired by the following inequality:

Proposition 2 *Let $x,\ y \in \mathbb{R}$ with $x \geq 0$. If $x^2 \geq y^2$ then $x \geq y$.*

Proof. Since $x \geq 0$ and $|y| \geq 0$, it follows from the increasing nature of the positive square-root function that $x \geq |y|$. Now, $|y| \geq y$ and thus $x \geq y$. ∎

This leads us to a refined inequality:

$$(2AC)^2 \geq (B^2 - (A^2 + C^2))^2 \tag{6}$$

which implies

$$2AC \geq B^2 - (A^2 + C^2) \implies A^2 + 2AC + C^2 \geq B^2,$$

i.e., $(A + C)^2 - B^2$ – which recovers (5), but now through a *factored expression* that we can hope to evaluate.

2.4 The role of CAS

At this point, a symbolic manipulation of the expression

$$4A^2C^2 - (B^2 - A^2 - C^2)^2$$

becomes crucial. This is far more complex to handle manually – here, we employ a Computer Algebra System (CAS), specifically SAGE on a CoCalc[2] platform, to expand and factor this expression in terms of $p = \cos(x)$ and $q = \sin(x)$, and c. The code snippet is given in Figure 4 below:

```
# Define variables
c, p, q = var('c p q')

A = (c^2*p*q)/sqrt(1-c^2*q^2)
B = 1/4*c^2*(sqrt(3)*p-q)*(p+sqrt(3)*q)
    /sqrt(1-1/4*c^2*(sqrt(3)*p-q)^2)
C = 1/4*c^2*(sqrt(3)*p+q)*(p-sqrt(3)*q)
    /sqrt(1-1/4*c^2*(sqrt(3)*p+q)^2)
result = factor(4*A^2*C^2 - (B^2 - A^2 - C^2)^2)
print(result)
```

Figure 4: Codes written in CoCalc platform of SAGE

Running these codes yields the following output:

```
-9*(c^2*p^2 + c^2*q^2 - 1)*(3*p^2 - q^2)^2*
(p^2 - 3*q^2)^2*c^12*p^2*q^2
/((2*sqrt(3)*c^2*p*q + 3*c^2*p^2 + c^2*q^2 - 4)^2*
(2*sqrt(3)*c^2*p*q - 3*c^2*p^2 - c^2*q^2 + 4)^2*
(c*q + 1)^2*(c*q - 1)^2)
```

[2]Visit `https://cocalc.com/`.

Invoking the fact that $p^2+q^2=\cos^2(x)+\sin^2(x)=1$, the expression $4A^2C^2-(B^2-A^2-C^2)^2$ is *magically* re-written as

$$\frac{9(1-c^2)(3p^2-q^2)^2(p^2-3q^2)^2c^{12}p^2q^2}{256(1-c^2q^2)^2\left[1-c^2\left(\frac{3}{4}p^2-\frac{\sqrt{3}}{2}pq+\frac{1}{4}q^2\right)\right]^2\left[1-c^2\left(\frac{1}{4}q^2+\frac{\sqrt{3}}{2}pq+\frac{3}{4}p^2\right)\right]^2}.$$

Note that $0\le c\le 1$ and so $1-c^2\ge 0$. Furthermore, each of the remaining terms has an even exponent. Consequently, we have proven that

$$4A^2C^2-(B^2-A^2-C^2)^2\ge 0.$$

2.5 Conclusion of the proof

Since $4A^2C^2-(B^2-A^2-C^2)^2\ge 0$, we can conclude via Proposition 2 that

$$(2AC)^2\ge (B^2-(A^2+C^2))^2 \implies A+C\ge B.$$

Therefore,

$$f'(x)=f_1'(x)+f_2'(x)+f_3'(x)\le 0,$$

with equality only at isolated points, confirming that f is strictly decreasing over $\left[0,\frac{\pi}{6}\right]$.

Thus, we have obtained the long-awaited proof of the following theorem:

Theorem 3 *Let $c\in[0,1]$. Then the function*

$$f(x)=\sqrt{1-c^2\sin^2(x)}+\sqrt{1-c^2\sin^2\left(x+\frac{2\pi}{3}\right)}+\sqrt{1-c^2\sin^2\left(x-\frac{2\pi}{3}\right)}$$

is strictly decreasing on $\left[0,\frac{\pi}{6}\right]$.

3 Pedagogical reflection: research collaboration across generations

This paper is more than a resolution of an open problem – it is also a quiet testament to the power of *mentorship*, *curiosity* and *mathematical empowerment*. The first author, 17-year old senior high school student, began this project with no experience in mathematical research; in particular, he has never encountered the use of CAS in his school mathematics. Yet through guided exploration, regular discussions with the other (more senior) authors, and a collaborative research progress grounded in real mathematical purpose of solving a challenging problem suitably chosen, he contributed meaningfully to the formulation and execution of a proof that had eluded others for years.

Such collaborations challenge the conventional view of mathematics learning as a linear, curriculum-bound journey. Instead, they invite (and entice!) learners – regardless of age and formal background – into the authentic world of *mathematical inquiry*, where conjectures arise not from textbook exercises but from geometric intuition, visual experimentation, and symbolic pattern recognition.

3.1 From classroom geometry to research mathematics

The first author's exposure to the problem began with a geometric sketch provided by the third author: the elegant movement of the equilateral triangles circumscribing a fixed ellipse. While the original construction involved notions from advanced geometry (e.g., director circle, midline tangency, and ellipse eccentricity), the curiosity it triggered by visual means was accessible and intuitive.

From this geometric seed, the student was guided to formulate the key function $f(x)$ – reading diligently the work by the third author – explore its behavior using graphical software, and ask the crucial question: *Can we prove that this function is decreasing?*

Each step in the research process – breaking the function into summands, differentiating each term, introducing substitutions, attempting inequalities, identifying dead ends, and using a CAS to assist algebraic factorization – was not just a means to a proof, but a structured learning journey into higher mathematics. Importantly, mistakes and failures were not treated as setbacks, but as natural terrain in the problem-solving landscape.

3.2 The role of CAS in accessible research

One pivotal enabler of this collaboration was the use of a Computer Algebra System (CAS). While the algebraic expression

$$4A^2C^2 - (B^2 - A^2 - C^2)$$

would have been daunting even for seasoned researchers, the use of SAGE on the CoCalc platform allowed us to experiment symbolically, test conjectures, and factor expressions that would otherwise be intractable by hand. By presenting mathematics as both human and computational, we are able to engage with complexity in a way that was not just feasible, but meaningful.

This CAS-supported approach mirrors modern mathematical practice, where computer assistance does not replace reasoning but *augments insight* – especially for students beginning to discover what mathematics can be.

3.3 Reflections on mentorship and mathematical growth

From the perspective of the senior co-authors, this collaboration was a rare privilege: to witness a young learner's first steps into the mathematical creation. From the student's perspective, the project opened a door into a living mathematical world – where problems are communicated among mathematicians (this problem was asked by the third author to the second, and in turn handed to the first by the second), and ideas matter.

This experience underscores a broader pedagogical claim that students, when given intellectually honest problems and thoughtful mentorship, can engage in research-level thinking. Tools like DGS, CAS, and visual experiments can serve as cognitive scaffolds for mathematical discovery, even in secondary school and pre-university settings.

We hope this project serves as a modest example of how meaningful mathematical inquiry can be cultivated across generations – and how today's students, when entrusted with genuine problems and guided with care, can become tomorrow's problem solvers.

4 Conclusion and future directions

This paper has established the first complete proof of Dahan's conjecture for the case $n = 3$, answering a question originally posed in his 2019 ATCM paper ([2]). Our proof hinged on two essential ingredients: a strategic reformulation of the inequality using elementary but nontrivial algebra, and the decisive application of a Computer Algebra (CAS) to handle expressions beyond the reach of manual symbolic manipulation.

While the core result concerns the monotonicity of a specific trigonometric sum involving square roots, the proof process has broader implications – for geometry, symbolic computation, and the pedagogy of mathematical inquiry.

4.1 Reflections on method and insight

A highlight our our method is the deliberate transformation of the inequality $A + B \geq C$ into a form amenable to algebraic factoring:

$$(2AC)^2 \geq (B^2 - (A^2 + C^2))^2.$$

This move, while seemingly modest, turned an opaque inequality into a verifiable expression once aided by CAS. The successful factorization revealed a rich structure buried within the function f, suggesting that similar strategies may apply to generalizations of the conjecture.

The use of CAS here was not merely supportive – it is essential. It allowed us to inspect and confirm factorization patterns with precision and confidence. In the growing field of computer-aided proofs, our result adds a modest case study affirming the synergy between human intuition and computational tools.

4.2 The general case and the challenge ahead

In a follow-up paper presented at ATCM 2024 ([3]), Dahan proposed a bold generalization: for all odd integers $n \geq 3$, the function

$$B(n, x) = \sum_{k=0}^{n-1} \sqrt{1 - c^2 \sin^2\left(x + \frac{2k\pi}{n}\right)}$$

is strictly decreasing on the interval $\left[0, \frac{\pi}{2n}\right]$.

Initial attempts by the first and second authors to extend our method to $n = 5$ were met with serious computational obstacles. The expressions for the derivatives quickly grow in size and complexity, with CAS systems running into memory and simplification limits.

This "combinatorial explosion" suggests that a direct generalization of our method may be infeasible. Future work may require:

- Identifying *symmetry-based* reductions to cut down the number of independent terms,
- Developing *recursive relations* between summands for different n,
- Introducing Schur-convexity techniques or other tools from inequality theory,

- Exploring the geometric and analytic structure of $B(n, x)$ through group actions or Fourier analysis.

A particularly intriguing ideas is the possibility of a block structure in the sign patterns of the derivative summands. If a consistent alternating behaviour or grouping emerges, it may offer a path to bounding the full derivative from below.

4.3 Broader implications and final thoughts

Beyond the specific conjecture, this work offers a model for integrating geometric intuition, analytic reformulation, and computational exploration in mathematical problem solving. It exemplifies how a visually driven problem in dynamic geometry can lead to a deep analytic result—and how young learners, when mentored thoughtfully, can take part in genuine mathematical discovery.

The case $n = 3$ is now settled. But in settling it, we have lit the way toward a deeper understanding of trigonometric sum inequalities, geometric optimization, and the computational boundaries of modern symbolic reasoning.

We hope this paper inspires further efforts toward a general proof – and opens the door for others, including young learners, to join in the continuing dialogue between geometry, analysis, and computation.

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