Geometrical reasoning through open-ended tasks in a Dynamic Geometry Environment: An analysis through the lens of Variation Theory

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Abstract

This article presents the Buried Treasure Problem as a carefully designed exploratory task in dynamic geometry, aimed at supporting students' transition from exploration to formal proof. Grounded in Marton's Theory of Variation, the study examines how grade 9 learners engaged with the task in a Dynamic Geometry Environment (DGE), GeoGebra. Variation theory is used as an epistemic lens to highlight how the four patterns of variation—contrast, separation, generalisation, and fusion —help interpret dragging activities and guide learners to notice invariants amidst variation. By systematically varying elements of the problem such as landmark positions, turn directions, and step lengths, students were able to discern critical mathematical features including midpoint constancy, congruency, and properties of quadrilaterals that remained unchanged despite alterations in the configuration. The task design thus employed key patterns of variation to focus attention on underlying structures essential for deductive reasoni ng. Findings illustrate how variation-informed design in DGE can support students in recognizing invariants, formulating conjectures, and constructing proofs, thereby bridging intuitive exploration with formal geometric argumentation. The study also extends the notion of variation -invariance duality as a theoretical basis for designing proof - oriented DGE tasks.

1.Introduction

The role of variation in mathematics education has long been recognized as central to supporting and developing conceptual understanding. Dienes' principle of multiple embodiment [1] suggests that exposing learners to the same mathematical concept through diverse representations, such as, concrete materials, visual models, or symbolic expressions helps them abstract essential ideas by recognizing what remains invariant across different contexts. "The perceptual variability principle states that to abstract a mathematical concept effectively one must meet it in a number of different situations (representations) to perceive its purely structural properties." ([1], p.158). For example, to understand the concept of a parallelogram, the student may be shown various representations which embody the concept – such as, a straw model, a paper cutout or an elastic band model on a geoboard. "The mathematical variability principle states that as every mathematical concept involves variables, all these mathematical variables need to be varied if the full generality of the mathematical concept is to be achieved." ([1], p.158). Thus to generalise the idea of a parallelogram, the student needs to see many parallelograms with varying side lengths and angle measures. Building upon this foundation, Marton and colleagues ([2], [3]) developed the Theory of Variation, which formalizes how carefully structured variation can direct learners' attention to critical aspects of a mathematical concept. By systematically varying certain features while keeping others constant, teachers can design tasks that help learners discern key properties, form generalizations, and deepen their understanding. This approach positions variation as a powerful pedagogical strategy to make mathematical structures and relationships visible to learners. DGEs such as GeoGebra [4] offer powerful tools for applying this approach, allowing learners to manipulate objects and instantly observe how specific variations affect, or leave unchanged, fundamental relationships between elements of a geometric configuration. DGE have therefore created new opportunities for enacting variation-based pedagogy in mathematics classrooms whereby learners can drag, manipulate, and dynamically vary geometric objects, and observe immediate changes to properties and relationships in real time. This interactivity enables students to engage directly with variation and invariance, fostering conjecture and paving the way toward formal reasoning and proof. Thus a DGE affords variational task design where critical aspects of a concept or problem can be explored via dragging.

In this article, the Buried Treasure Problem is presented as a variational task module, designed within a DGE, to illustrate how intentional variation can guide learners' attention to critical geometric relationships. This study aims to

1. Design and implement the Buried Treasure Problem as a DGE task, guided by Marton's Theory of Variation.
2. Examine how patterns of variation, when embedded in the DGE task can enable students to discern invariants in geometric configurations.
3. Analyze how students' engagement with structured variation supports their transition from conjecture-making to argumentation and proof.

The module was implemented with a group of 30 grade 9 students in the age group 14 to 15 years in a school in New Delhi. The school follows the curriculum prescribed by the Central Board of Secondary Education (CBSE) [5] a national curriculum board in India. The students had volunteered to participate in the study and were identified in consultation with an experienced teacher in the school who also volunteered to assist in conducting the study. Consent was taken from the students prior to the study and they were informed about the requirements of the study. Students' mathematical knowledge included basic geometrical ideas related to properties of lines, angles, triangles, quadrilaterals and circles. They attempted the tasks of the module on GeoGebra made available to them in a computer lab over three 90-minute sessions. A worksheet, posing the tasks in a sequential manner, was provided, in which students were required to record their observations as they explored the tasks on their computer screens. The researcher along with the teacher, acted as facilitators providing minimum intervention throughout the module. The tasks provided an opportunity to explore how students engaged with structured variation to develop conjectures and also to understand how they reasoned about invariance. This article reports on the task design process, as well as the detailed analysis of students' responses, highlighting how patterns of variation supported or constrained their understanding of the problem's underlying geometric structure and transitioned them from conjecture making to argumentation and proof. In section 2, we elaborate on dragging as a tool in a DGE and its relationship to the theory of variation. Section 3 describes the buried treasure problem module, explored by the grade 9 students. The sequence of tasks, right from setting up the problem in GeoGebra, to the systematic variation of different attributes of the problem to enable learners to experience variation is explained. Section 4 elucidates how the four functions of variation were incorporated in the tasks. Section 5 discusses a set of tasks which transitioned students from conjecture making to the proof of the problem. Finally, section 6 presents the conclusions drawn from the findings of the study, acknowledges the limitations and suggests directions for future research.

2.Dragging as a tool and the theory of variation

Among all the features offered by a Dynamic Geometry Environment (DGE), the ability to drag parts of a figure is perhaps the most powerful. Being able to discern what varies and what remains invariant during a dragging episode is key to experiencing a mathematical concept or property. This feature can be effectively exploited to enable students to make conjectures while performing geometrical explorations.

As Leung [6] points out, what makes a DGE a powerful environment for knowledge acquisition is its capacity to make explicit the "implicit dynamism" of thinking about mathematical concepts. When reasoning mathematically, learners often use a form of *mental animation*—visualizing variations of objects in order to detect patterns or invariants. A DGE helps to externalize this process, making such variation immediately visible and accessible to students.

Dragging has been widely studied, with research highlighting its potential to support students in constructing figures, exploring problems, formulating conjectures, and generating proofs ([7], [8], [9], [10], [11], [12]). Different strategies or modalities of dragging have been investigated to understand how this action contributes to different DGE experiences. Leung [6] highlights that variation is the epistemic essence of dragging: the purpose of most dragging strategies is to uncover invariant properties amidst varying components. Additional DGE functionalities, such as measurement, hide/show, and trace can enhance this process by making patterns of variation more salient. Thus, variation provides a framework for interpreting the pedagogical and epistemological significance of dragging in DGE.

Marton [3], in his work on learning and awareness, further emphasized the role of variation and simultaneity in discernment. According to him, to perceive a particular aspect of a situation, learners must experience variation in that aspect as "there is no discernment without variation." Effective mathematical action, therefore, depends on recognizing critical aspects and considering them simultaneously.

He further proposed four inter-related functions (or patterns) of variation. They are described as follows:

> Contrast: ''... in order to experience something, a person must experience something else to compare it with.''
> Generalisation: ''... in order to fully understand what ''three'' is, we must also experience varying appearances of ''three''.
> Separation: ''In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant.''
> Fusion: ''If there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously.''(p. 16).

These four functions form the theoretical foundation for analyzing learning in a DGE. In this study, the tasks of the Buried Treasure module were designed explicitly around these functions, ensuring that students' explorations and conjectures were supported by structured opportunities to notice variation and invariance.

4. DGE Variational Task Design – The Buried Treasure Problem

Researchers such as Leung [14] have attempted to conceptualize and design geometry reasoning tasks in DGE using the elements of variational thinking where the variation-invariant duality is leveraged to frame the task sequence. According to Leung [14], "the variation-invariant acts "design" the tasks, whilst the students construct knowledge by activating the variation-invariant interplay design under the drag-mode." In this section we shall describe how the lens of variation was used to design and sequence the buried treasure task and how this enabled the grade 9 students to transition from experimentation, conjecture making, and argumentation to proof. The task design implicitly prompted students to experience the variation-invariant duality using the drag mode and other functionalities of DGE at various stages of the task.

3.1.The task

The grade 9 students were presented with a worksheet which stated the buried treasure problem as follows:

> *There was a young and adventurous man who found among his great-grandfather's papers a piece of parchment that revealed the location of a hidden treasure. The instructions read: "Sail to North latitude and West longitude where thou wilt find a deserted island. There lieth a large meadow, not pent, on the north shore of the island where standeth a lonely oak and a lonely pine. There thou wilt see also an old gallows on which we once were wont to hang traitors. Start thou from the gallows and walk to the oak counting thy steps. At the oak thou must turn right by a right angle and take the same number of steps. Put here a spike in the ground. Now must thou return to the gallows and walk to the pine counting thy steps. At the pine thou must turn left by a right angle and see that thou takest the same number of steps, and put another spike into the ground. Dig halfway between the spikes; the treasure is there.*
>
> *However when the young man reached the meadow, the gallows were nowhere to be found! Time had destroyed the gallows and there was no trace left. Since the gallows was the starting point, the young man could not follow the instructions which would lead him to the treasure and came back disappointed.*

Deliberate on the problem. Can you find a way to locate the treasure without knowing the position of the gallows? Explain your reasoning, and show your construction using GeoGebra.

Students were given access to GeoGebra as the DGE platform to explore the problem. The initial 15 minutes of the first session was spent in reflecting about what was given in the problem. One student reflected "we can fix the positions of the oak and pine trees obviously because trees are fixed, but we don't know where the gallows are. So how to begin the exploration?" Another suggested that the position of the gallows can be "any random point on the screen". At this point the teacher prompted the students to set up the exploration on GeoGebra.

3.2.Setting up the problem

The following steps, identified to simulate the problem on GeoGebra, were presented by the researcher on slides to the entire class so that students could try them out on their respective computer screens.

Step 1: Fix the landmarks. Open a GeoGebra screen. Using the **Point** tool select two fixed points - O for the Oak tree and P for the Pine tree. For convenience these may be chosen on the x-axis as O(2,0) and P(10,0) and may be entered using the **Input Bar**. Then select the **Point Tool** and click anywhere on the screen. Label this point as G (for the gallows).

Step 2: Walk to the Oak tree and Turn Left
Use the **Segment tool** to join G to O. Then select the **Rotate around point** tool, select segment GO and rotate it anticlockwise by 90° about O. Rename the end point as S_1. S_1 is the position of the first spike. Note that angle GOS_1 is 90° and $GO = OS_1$ as shown in Figure 1.

Step 3: Walk to the Pine tree and Turn Right
Now use the **Segment tool** to join G to P. Then use the **Rotate around point** tool, select segment GP and rotate it clockwise by 90° about P. Rename the end point as S_2. S_2 is the position of the second spike. Note that angle GPS_2 is 90° and $GP = PS_2$.

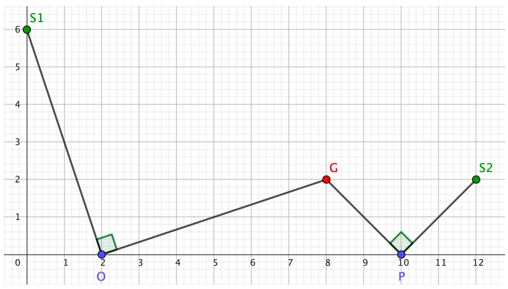

Figure 1. Setting up the buried treasure problem on GeoGebra

Students were prompted to drag G to various positions on the screen (Figure 2) and note their observations on their respective worksheets. They identified $GOS_1 = GPS_2 = 90°$ and $GO = OS_1$ and $GP = PS_2$ as invariants. One student reflected "no matter where I take G on the screen, the angles remain 90°and their arm lengths also remain equal". Another student felt that this was obvious because of "the way it was constructed" in GeoGebra.

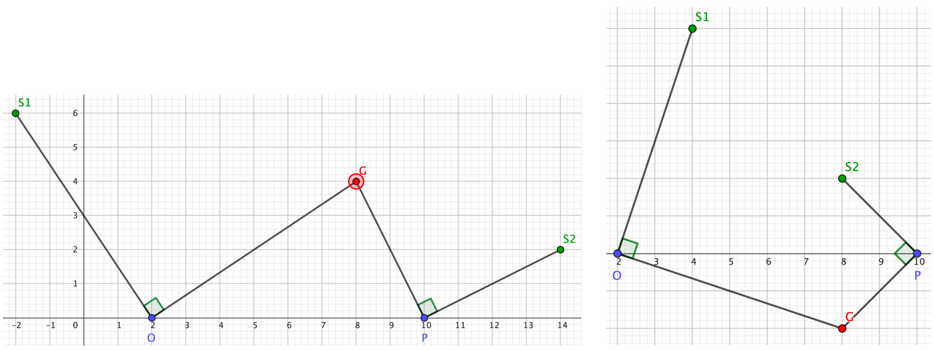

Figure 2. As G was dragged to various positions on the screen, invariances became evident.

Step 4: Find the Treasure
Join S_1S_2 using the **Segment** tool. Then use the **Midpoint or Center** tool to find the midpoint of S_1S_2. Label this point as T, which is the position of the treasure (Figure 3).

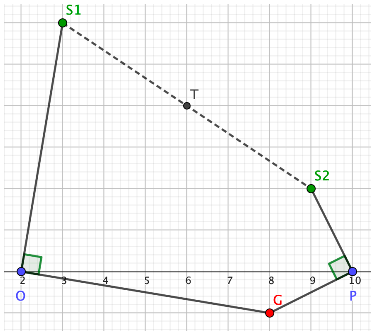

Figure 3. Locating the treasure (T) as the midpoint of S_1S_2

Students were encouraged to use wandering dragging to drag O, P and G, one at a time, in search of patterns. In each case, they tried to predict the position of T. The researcher posed further questions to steer their exploration.

3.3. Performing dragging tasks for exploration

In this section we shall elaborate upon the tasks which were designed keeping in mind the four functions of variation. The aim was to enable the students to vary different aspects of the buried treasure problem using the dragging tool and observe the impact on the position of the point T (representing the treasure).

Varying the positions of O and P

In order to see the effect of dragging G on T, it was important to first do so by keeping O and P fixed and then by varying them. The tasks were posed as follows:

(i) Keeping O and P fixed, drag G to different positions on the screen and record your observation regarding T.

(ii) Drag O and P individually to different positions on the x-axis and record your observation regarding T.

Can you make a conjecture about the position of T based on the positions of O and P?

Students observed that as they dragged G, keeping O and P fixed (Figure 4), the coordinates of T remained fixed at (6,4)! This was quite unexpected for many. Comments like "whatever we do to G, the figure changes entirely but T remains fixed" came from students. However, when O or P or both were shifted (Figure 5), the position of T changed. This led students to conjecture that T was independent of G as long as O and P were fixed.

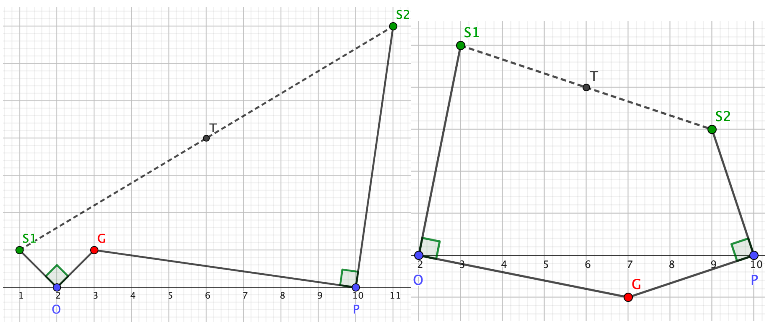

Figure 4. Students observed the invariance of T by dragging G.

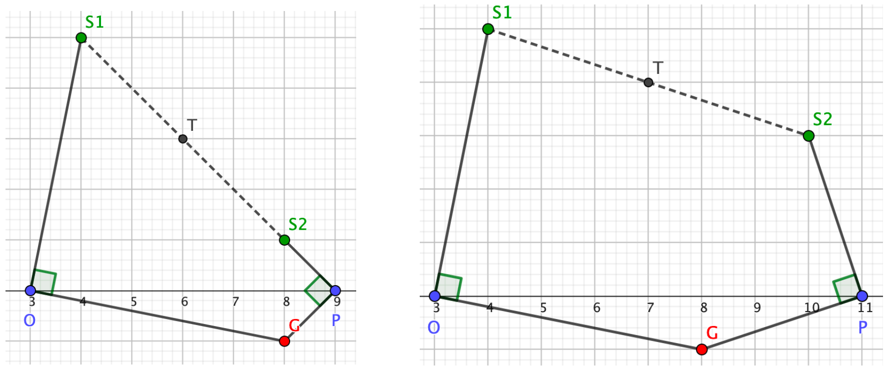

Figure 5. The dependence of T on O and P.

Varying the distances

In the next exercise students explored the relationship between G and T when the lengths of the arms of the angles GOS_1 and GPS_2 were unequal. The task was posed as follows:

(iii) Suppose $GO \neq OS_1$ and $GP \neq PS_2$, that is, the young man does not walk the same number of steps after taking a 90° turn at the oak tree and the pine tree. Create a figure and make a conjecture about the position of T with regard to the distances.

Students attempted this exercise by choosing arbitrary lengths for the distances while ensuring $GO \neq OS_1$ and $GP \neq PS_2$. Figure 6 shows two such instances. After dragging G, they conjectured that T varied and was not fixed. It was interesting to note students' responses such as "if the arm lengths of the two angles are not equal, then the diagram is not symmetrical like before. So, T changes as we drag G".

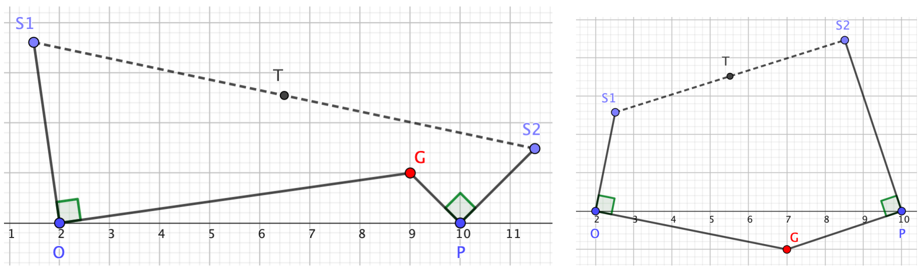

Figure 6. Students conjectured that T varied if the distances were unequal.

For the researcher, this was a high point as the students began to notice various aspects of the figure, while focusing their attention on the position of T.

Varying turn directions

The next exercise prompted students to explore the impact on T if the 90° turns were both in the same direction, that is, both clockwise or both anticlockwise.

(iv) Suppose you make an anticlockwise 90° turn both at O and P. How does this affect the position of T? Try the same with a 90° turn clockwise both at O and P.

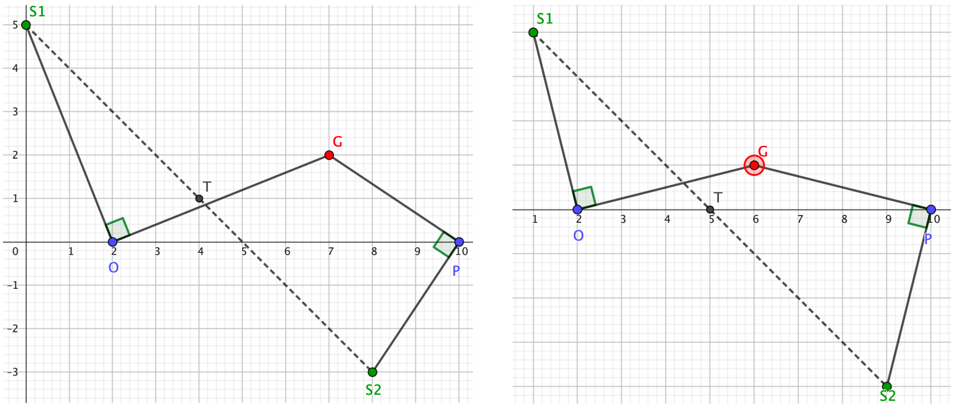

Figure 7. A 90° anticlockwise turn at both O and P revealed that T was dependent on G.

Once again dragging G helped to verify that T varied as G was dragged across the screen.

Varying the angles of rotation
Finally, the researcher prompted students to explore the impact of changing the angle of rotation from 90° to other angle measures.

(v)Modify the figure so that that GOS_1 and GPS_2 are not 90°. You may consider the turn angle to be 60°, 80°, 120° etc.

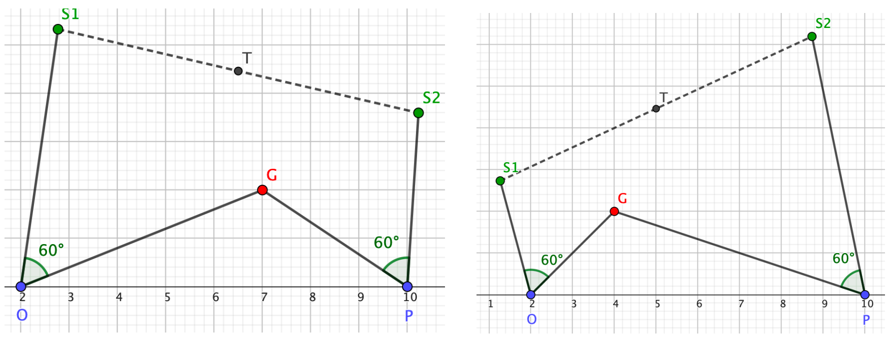

Figure 8. The buried treasure problem exploration with turn angle 60°.

Students observed that if the turn angle was anything other than 90°, then T varied with G. Figure 8 illustrates the case when both turn angles are 60°.

In designing the above tasks, the researcher identified four dimensions of variation to enable the participants to discern the relationship between the points T, G, O and P. The purpose of the task was to highlight the critical aspects of the problem by allowing them to vary these aspects, one at a time. These are summarized in Table 1 which also mentions the dragging modalities required to experience the dimensions of variation.

4.Analysing the buried treasure task using the Theory of Variation

In this section we analyse the 9th grade students' responses to the tasks, using the four functions of variation, namely – contrast, separation, generalization and fusion. The students' worksheets were collected at the end of the activity and analysed qualitatively. Each worksheet contained written explanations, diagrams, and recorded observations corresponding to the different tasks of the module. These responses were examined to identify instances where students explicitly noticed invariance, articulated relationships, or highlighted changes in outcomes. The analysis was then coded in relation to the four functions of variation—contrast, separation, generalisation, and fusion.

Contrast
Students experienced contrast in tasks (i) and (ii). As G was dragged, in (i), T remained invariant for fixed O and P, whereas in (ii) T was dependent on O and P. When students varied the turn directions at O and P (e.g., both clockwise or both anticlockwise), they experienced contrast in the outcomes. It helped them notice that only the specific combination of an anticlockwise rotation at O and a clockwise rotation at P produced a unique and invariant treasure location T. Similarly, if the angles of rotation were changed from 90° to 60° or 120°, T was no longer invariant. For example, in one worksheet a student wrote: "When I changed the angles from 90° to 60°, the treasure point shifted and was not the same." This statement was coded under *contrast*, as it reflected the student's awareness of how altering one variable (angle measure) produced a different outcome, thereby drawing attention to the critical role of right angles. Students discerned the critical role of right angles in preserving the treasure location.

Separation
Students discerned a specific critical aspect of the problem by varying only that aspect while keeping other critical aspects constant or fixed. Hence by dragging G, while keeping O and P fixed, the invariance of T came to the fore. Further, keeping the turn angles and directions constant, while changing the distances GO, GS_1, GP, GS_2 helped students separate distance as a critical aspect and conclude that T remains invariant only when GO = GS_1 and GP = GS_2. Also, dragging G while keeping O, P, and all turn/distance rules the same, students separated the role of the G from the rest of the construction to see that it did not affect the treasure's position.

Generalisation
Working through the tasks (i) to (v) enabled the students to generalize the construction process. By repeatedly performing the task with different positions of O and P (while maintaining the same rules with regard to turn angles, distances and turn directions), led students to write their conclusions in their worksheets. Here are two examples.
"the treasure's location T, is fixed and does not depend on the position of G, as long as the directions, turn angles, and distances used in the construction remain the same."
"When G is moved, the positions of the spikes, S_1 and S_2, change because they are constructed based on the directions and distances from G. So, the segment S_1S_2 does not remain the same when G is moved. But the midpoint of segment S_1S_2, which is the treasure's location remains the same."
These highlight the fact that working through the tasks led to geometric reasoning and insight on the part of the students.

Fusion
Students coordinated multiple critical aspects simultaneously to fully grasp the concept. In recognizing that equal distances must be walked, the turn directions must be opposite and must be 90°, and the midpoint of the constructed segment S_1S_2 is invariant, they fused all these critical aspects into a single, coherent understanding of the geometric structure of the problem. This fusion is what ultimately enabled the grade 9 students to develop or recreate the generalized construction using only O and P.

Table 1

Dimension of Variation	**Dragging modalities**	**Purpose (highlighting critical aspects of the problem)**
Position of the gallows (G)	Students used wandering dragging to see the impact of G on the position of T, first with O and P fixed and later by varying O and P.	To enable students to see the invariance of T as G is dragged (with O and P fixed). The positions of O and P are a critical aspect of the problem.
Unequal distances (steps taken by the young man) $GO \neq OS_1$ and $GP \neq PS_2$	Modify the construction to vary the distances GO, GS_1, GP, GS_2 and then using wandering dragging on G.	Highlighting equal distances as a critical aspect of the problem. T does not remain invariant if the distances are unequal.
Turn directions (both clockwise or anticlockwise) at O and P	Modify the construction to ensure that both 90° turns are in the same direction, followed by wandering dragging on G.	Highlighting that turn directions, that is, anticlockwise at O and clockwise at P are a critical aspect of the problem. T does not remain invariant if both turn directions are same.
Measure of the turn angles at O and P ($\neq$ 90°)	Modify the construction to make the turn angle anything other than 90°, followed by wandering dragging on G.	Highlighting that the measure of the turn angle is a critical aspect of the problem.

5.Towards proof

The previous section focused on tasks aimed to lead students to explore the buried treasure problem and arrive at conjectures. The task design was based on the theory of Variation. In this section we present a set of tasks which aimed to transition students from conjecture making to proof. The tasks were presented to students in a stepwise manner and students were required to write their observations on their worksheets while trying out the problems on their GeoGebra screens. They were as follows:

Exploring special cases

Students were encouraged to observe special cases of the problem by choosing G at specific positions. This was facilitated by the following task.

Let us return to the situation where G is dragged keeping O (2,0) and P (10,0) fixed. Here we shall explore some special cases. What happens to T if

(i) G is the midpoint of OP? Note the coordinates of T. What kind of a figure is OS_1S_2P?

(ii) G is equidistant from O and P. Note the coordinates of T.

(iii) G is moved horizontally on OP. Once again note the coordinates of T. What kind of a figure is OS_1S_2P?

In (i) students noted that OS_1S_2P is a rectangle. When asked to provide a reasoning, some argued that the measures of the opposite sides are 8 and 4 units respectively while all angles are 90°. The researcher asked the students to remove the grid from the background and unselect the algebra view so that measures of sides and angles were no longer visible. They were then asked to provide an explanation as to why OS_1S_2P was a rectangle. Students arrived at the conclusion that $OP = S_1S_2$, $OS_1 = PS_2$ and that T, the midpoint of S_1S_2 was fixed. In (ii) students used maintaining dragging to ensure that GO = GP. Dragging G on the perpendicular bisector of OP, ensured the equality of OS_1, GO, GP, PS_2 even though the shape of the figure changed. Also, as S_1S_2 remained fixed, T was also fixed. In (iii) as G was dragged on OP, the quadrilateral OS_1S_2P became a trapezium with $OP = OS_1 + PS_2$. While the trapeziums varied, T remained invariant. Figure 9 illustrates these three cases.

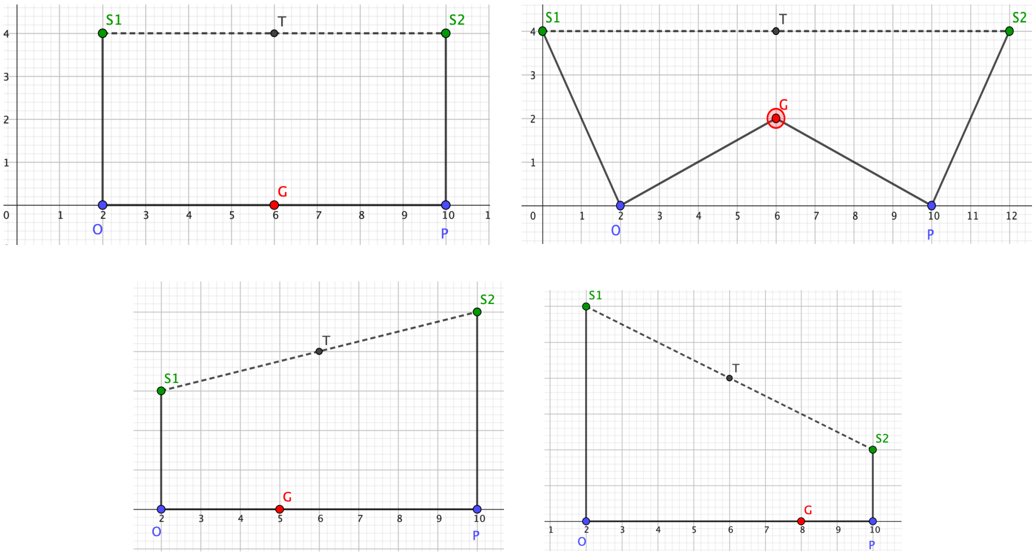

Figure 9. Quadrilateral OS_1S_2P becomes a trapezium as G is moved along OP.

Students noticed that in all three cases T remains fixed at (6,4). Further, they observed that there are many line segments S_1S_2 for which the midpoint T is (6,4). This was an important observation as it helped to generalize the problem and arrive at the proof.

The next exercise was designed to lead students to prove the invariance of T for all positions of G on OP.

(iv) Use the **Rotate about point** option to rotate the trapezium OS_1S_2P about T by 180°. What can you conclude about the resulting figure? As you drag G what happens to T? Justify your reasoning.

Students observed that the figure formed by rotating the trapezium OS_1S_2P about T was a square (Figure 10). The equality of the adjacent sides OP and OP_1' was established using the argument $OP = OG + GP = OS_2' + S_2'P_1$. This argument was extended to prove the equality of all four sides. Also, it was evident that all angles were right angles. One student remarked "dragging G on OP does not change the position and size of the square". The invariance of the square was an important observation and others concurred remarking "the square remains immobile since OP is fixed." The researcher pointed out that since the square was immobile, so was its centre. "But where does the centre of a square lie?" asked the teacher. While many students immediately responded by saying that the centre of a square must be the midpoint of its diagonals, one student remarked "there can be many line segments which pass through the centre of the square and whose end points lie on the sides." This revealing insight was yet another high point for the researcher. She scaffolded the students into observing that the segment S_1S_2 divides the square into two congruent halves (identical trapeziums). Therefore, it passed through the centre of the square. The centre of the square, (also the mid-point of S_1S_2) remained fixed as the square itself was fixed.

The sequence of arguments leading to the conclusion that T was fixed and independent of G was reinforced as students dragged G on OP and saw the invariance of T amidst variation of the segment S_1S_2.

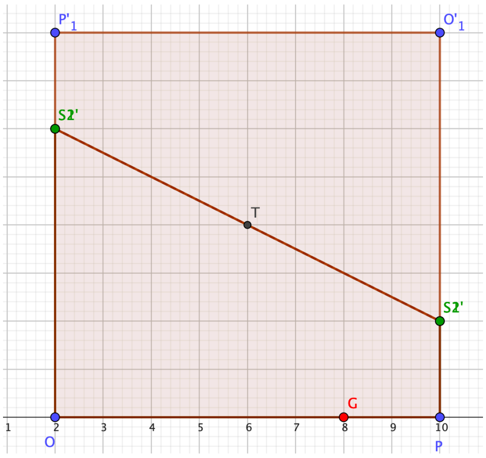

Figure 10. Proof of the fact that T is independent of G when it lies on OP.

Towards a more general proof

A query arose amongst the students – "we know T is independent of G when G is on OP. But what happens when G is somewhere else?" Clearly it had become necessary to consider a more general case when G was any point on the screen. To scaffold students towards the proof the following task was posed.

(i) In the initial diagram, with O, P, S_1, S_2 and T marked, construct a square with side OP using the **Regular Polygon** tool. What happens to the square as G is dragged?

(ii) Join S_1 to B and S_2 to A.

(iii) Using the **Polygon tool** highlight the triangles OBS_1, GOP and PAS_2. What is your observation with regard to these triangles?

(iv) What can you conclude about the figure OS_1AS_2?

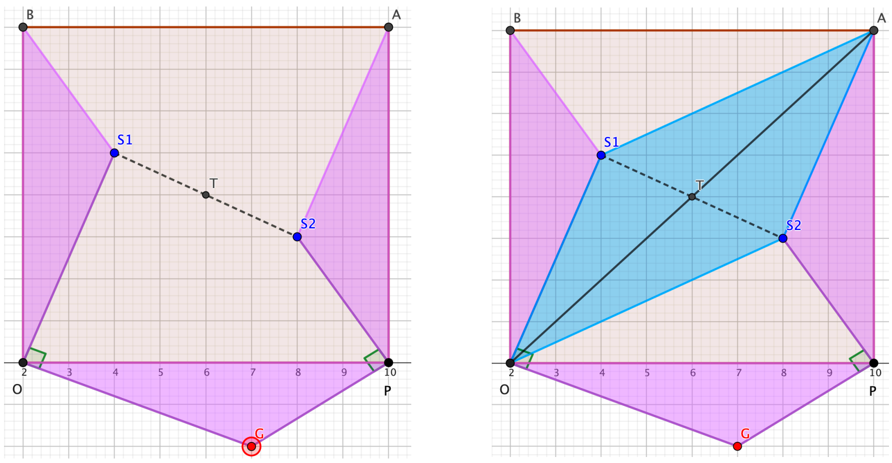

Figure 11. The proof of the invariance of the location of the treasure T.

Prior to this task, students had observed that the square with side OP was fixed and independent of G. With some scaffolding from the researcher, they were able to prove that triangles OBS_1 and GOP were congruent using SAS criteria (OB = OP, OS_1 = OG, equality of angles BOS_1 and POG). Similarly triangles PAS_2 and GOP were also shown to be congruent. This led to the conclusion that triangles OBS_1 and PAS_2 were congruent and that $OS_1 = AS_2$. Students were then asked if they could identify another pair of congruent triangles in the diagram. After some exploration triangles OPS_2 and ABS_1 were proven to be congruent, leading to the conclusion that $OS_2 = AS_1$. The researcher asked students to focus on the properties of the quadrilateral OS_1AS_2. They used their prior knowledge of properties of parallelograms to conclude that OS_1AS_2 was a parallelogram as its opposite sides were equal. Some students commented "it looks like a parallelogram" but were asked to provide a reasoning. The equality of opposite sides remained invariant as G was dragged and thus the parallelogram OS_1AS_2 was an invariance. One student commented "OS_1AS_2 remains a parallelogram though the side lengths and angles vary. However, T remains fixed."

The teacher scaffolded students toward the final step of the proof by showing that S_1S_2 was a diagonal of a parallelogram with T as its midpoint. Since diagonals of a parallelogram bisect each other, OA must also pass through T. Moreover, OA was also the diagonal of the fixed square OPAB, making T its fixed midpoint. As OA was independent of G, T also remained independent of G and therefore invariant. Throughout the task, students were encouraged to write their observations, conjectures and reasoning on their worksheets. This provided the researcher with insights into their thinking that led to the development of the proof.

6.Discussion and conclusion

This article describes a study in which 30 grade 9 students explored the *Buried Treasure Problem* using GeoGebra. The study set out to (i) design and implement the Buried Treasure Problem as a DGE task guided by Marton's Theory of Variation, (ii) examine how patterns of variation can help students discern invariants in geometric configurations, and (iii) analyze how engagement with structured variation supports the transition from conjecture-making to argumentation and proof.

The findings provide compelling evidence that carefully sequenced variational tasks in GeoGebra can scaffold students' transition from intuitive exploration to formal reasoning. Through the four functions of variation - contrast, separation, generalisation, and fusion, students learned to distinguish critical aspects of the problem, such as the role of equal distances, right angles, rotation directions, and to recognize invariant properties of quadrilaterals and midpoints. The design of deliberate special cases further supported their transition from dynamic observation to deductive argumentation.

First, the task focusing on dragging G while keeping O and P fixed exemplified the pattern of separation, helping students isolate the effect of G and recognize that the treasure location T remains invariant under this variation. Conversely, tasks where O and P were moved highlighted contrast, enabling students to see that T's invariance is dependent on the fixed positions of these landmarks. Similarly, variations in turn directions and angles created sharp contrasts that drew students' attention to the necessity of having specific rotation directions (anticlockwise at O and clockwise at P) and right angles to ensure T's invariance. Moreover, the structured exploration of unequal step distances illustrated the principle of fusion, requiring students to coordinate multiple critical aspects, namely, equal distances, fixed landmarks, turn angles of 90°, and specific rotation directions—to fully grasp the underlying geometric structure of the problem. This fusion of critical features allowed many students to generalize their insights, as reflected in their written conjectures describing conditions under which T remains fixed.

Further, while developing the formal proof, the deliberate design of special cases, such as, placing G on the segment OP and exploring congruences and invariant shapes helped students extend their conjectures towards proof. By investigating these cases dynamically in GeoGebra, students observed invariant properties of squares, parallelograms, and congruent triangles, scaffolding them towards deductive arguments that explained why T's position is independent of G's location. This aligns with Leung's ([6], [14]) assertion that dragging as a variational tool can reveal invariances that are foundational to constructing proofs.

In this study, the researcher adopted a minimum-intervention strategy while students explored the tasks on GeoGebra, making a conscious effort not to provide direct answers. Together with the teacher, she moved around the classroom, encouraging students to record their observations on the worksheets as they worked. At the end of each task, whole-class discussions were facilitated to help students summarize their findings. Finally, the researcher scaffolded the process that led students toward formulating a proof of the invariance of T.

These outcomes highlight important implications for teaching with DGE. Variationally designed tasks can serve as powerful pedagogical tools for bridging intuition and proof, making non-routine geometry problems accessible, and cultivating mathematical thinking practices such as conjecturing, generalizing, and justifying. In the study, the structured experiences of variation and invariance enabled students not only to see and describe geometric relationships but also to begin articulating why these relationships hold. Thus, designing tasks with explicit patterns of variation can create fertile ground for learners to progress from conjecture to proof, fulfilling key goals of learning geometry. The findings of the study provide insight into the nature of tasks that are appropriate for a DGE environment, which can be "triggers" for proof and extends the notion of variation-invariant duality as a theoretical basis for DGE task design. For classroom practice, the findings suggest that teachers can use dragging strategically, not only as an exploratory tool but also as a trigger for proof-oriented discussions.

Despite the encouraging results, the study has limitations. It was conducted with a relatively small group of Grade 9 students in a single school context, and the scope was restricted to one problem. Further research is needed to explore how variational design can be applied across different mathematical domains, with larger and more diverse student populations, and over longer periods of instruction. Overall, the study contributes to understanding how the variation–invariance duality can inform task design in DGE, showing how structured variation can support students in progressing from exploration to proof.

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