Geometric Proofs in Education: Integrating Digital Tools for Innovative Teaching Practices

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Abstract

Mathematical proof constitutes a fundamental logical structure for validating propositions and holds a central place in mathematics education. In the field of geometry, it takes on particular importance by fostering the development of deductive reasoning, argumentation, and conceptual understanding. Recognised as a key area of the curriculum, geometry significantly contributes to the development of logical-spatial thinking and the ability to solve real-world problems.

Within this framework, dynamic geometry systems enable the exploration of geometric properties in a visual and interactive way, facilitating the formulation and validation of conjectures by students. As an example, an activity is proposed in which students use digital tools to construct figures, identify patterns, formulate conjectures, and apply inference rules with the aim of demonstrating geometric properties - thus promoting an active, meaningful approach to learning geometric proof, in line with mathematical reasoning principles.

1 Introduction

Mathematical proof, as a logical structure that supports the validity of propositions, occupies a central place in the tradition of school mathematics. In geometry in particular, proofs are not only tools for justifying properties, but also opportunities for developing deductive thinking, argumentation, and a deep understanding of concepts. Geometry is recognized as a fundamental subject in the mathematics curriculum, as it develops logical-spatial thinking and real-world problem-solving skills.

In Portugal, primary and secondary education curricula place considerable emphasis on geometric demonstrations, especially from the 7 th grade onward. Many students view demonstrations as formal activities, distant from their concrete mathematical experience. Teachers, in turn, face challenges related to teaching time, professional training, and access to effective teaching resources.

The Portuguese mathematics curriculum, in the 3rd cycle of primary education (7th, 8th and 9th grades) and secondary education (10th, 11th and 12th grades), progressively integrates geometry. Adopting a spiral approach, the various topics are revisited throughout the school career, with increasing levels of depth, complexity, and formalization, promoting a gradual and integrated consolidation of knowledge [2].

The Mathematics curriculum aims to prepare students to form critical judgments and make informed decisions, fostering the development of reflective and participatory citizens. It also seeks to emphasize the role of Mathematics as a constantly evolving science, with an aesthetic dimension and historical relevance. It aligns with the Students' Profile by the End of Compulsory Schooling, promoting critical thinking, problem-solving, creativity, communication, and collaborative work [3].

In the 3rd cycle of primary education, Essential Math Learning emphasizes geometric exploration through problem-solving, the development of logical thinking, and spatial visualization. Key topics include:

- **Plane figures and figures in space**: identification, classification and representation of two-dimensional and three-dimensional geometric figures.
- **Operations with plane figures**: decomposition, composition and calculation of perimeters and areas.
- **Geometry in the plane and in space**: Study of plane and spatial geometric figures, including polygons, geometric solids and their properties.
- **Geometric transformations**: Analysis of symmetries, translations, rotations, reflections and homotheties.
- **Geometric places**: Identification and construction of geometric places, such as bisectors and circles.
- **Metric relations**: Application of concepts such as perimeter, area, volume and Pythagorean theorem.
- **Geometric representation**: Use of tools such as compasses, rulers and set squares to construct and analyze geometric figures.

In secondary education, the 10th grade Mathematics classes cover Geometry in greater depth, focusing on analyzing and solving complex problems. Key topics include:

- **Synthetic geometry in the plane**: study of geometric relationships and properties through constructions and deductive reasoning, without resorting to measurement.
- **Analytical geometry in the plane and in space**: Study of lines, planes, distances, angles and relative positions using Cartesian coordinates.
- **Vectors in the plane and in space**: Vector analysis, its operations and applications in geometric problems.

This article aims to propose an activity using a dynamic geometry system, in order to show the functionalities that allow approaching geometric proofs in the teaching of Mathematics, allowing "doing (new) things better than doing (old) things better" [1].

2 Technological Tools in Teaching Geometry

The need to develop educational tools to support the learning of mathematics with understanding is highlighted by the National Council of Teachers of Mathematics [11]. According to the United Nations Educational, Scientific, and Cultural Organization [19], quality education for all today cannot be achieved without considering technological factors that can facilitate universal access to education, reduce learning inequalities, support teacher development, improve the quality and relevance of learning, strengthen inclusion, and improve educational administration and governance [20]. The use of computers in teaching represents a technological support for the visualization of abstract concepts, enabling the production of mental models of the concept. With computer programs, students interact with educational material to develop the skills necessary to solve problems, using mathematics as a foundation. Digital technologies are now plentiful, accessible, and can be used to learn anywhere, anytime, and in a variety of ways.

The digital revolution has transformed pedagogical practices and demands new approaches to geometry teaching. Several authors point out that traditional methods alone are no longer enough to engage students [18]. Therefore, investing in active strategies with technological support has been seen as a promising strategy for making mathematics classes more dynamic and motivating.

Various digital technologies have been applied to geometry teaching, offering new ways to explore concepts previously confined to pencil and paper. Dynamic Geometry Systems (DGS) are a distinguished class: programs like GeoGebra, Cabri Géométrie and C.a.R. (Compass and Ruler) allow you to build and manipulate geometric figures in real time, streamlining the learning process [16].

New times demand new teaching methodologies that can further influence students' thinking. The importance of DGS in geometry teaching is highlighted. DGS facilitate:

- **Building figures**: Enabling the interactive creation of geometric figures for deeper understanding.
- **Conjecture Formulation**: Enabling students to formulate and test conjectures, promoting logical reasoning and mathematical inquiry.
- **Visual verification**: Providing opportunities for students to visually verify their conjectures, improving spatial intuition and understanding of geometric properties.
- **Supporting geometry teaching**: DGS are valuable resources for teachers, simplifying the presentation of complex concepts in an accessible and engaging way.

Mathematics is fundamentally based on deductive reasoning and the ability to construct logical proofs. For students to develop a deeper understanding of the subject, it is essential that deductive reasoning becomes more accessible and naturally integrated into their learning practices. In this sense, the use of automated tools can play an important role, supporting not only students in developing logical thinking but also teachers in planning and explaining complex concepts.

3 Proof in Geometry

Proof, especially in geometry, is seen as an instrument for constructing and validating mathematical knowledge [17]. Since the time of Euclid, deductive reasoning has been the central axis of mathematical activity.

Hanna and Barbeau [6] identify several functions of demonstration in teaching: explanatory, verification, exploratory, and aesthetic. In a school context, the explanatory function is the most valued, but it is often not sufficiently understood by students, who tend to memorize steps without understanding why.

Ponte et al. [12] emphasize that curricula promote proof as an essential skill, but actual practice in schools reveals gaps. Initial teacher training does not always adequately prepare students for teaching mathematical proof, and there is a tendency to emphasize formalization over understanding.

Several studies indicate that the use of dynamic geometry system, such as GeoGebra, can facilitate the construction of conjectures and the visualization of properties, supporting the transition from empirical to deductive reasoning [7]. The secondary education states that students should use "dynamic geometry to solve problems, understand concepts, formulate conjectures, visualize, and test properties" [3, p. 33]. However, the integration of these resources into classrooms remains limited.

The introduction of automated deduction systems in schools is important for proving conjectures. These systems serve as valuable tools for verifying and proving mathematical conjectures. In essence, the combination of DGS and automated deduction systems fosters a comprehensive and interactive mathematical learning environment where students can confidently explore, reason, and verify their mathematical ideas [15].

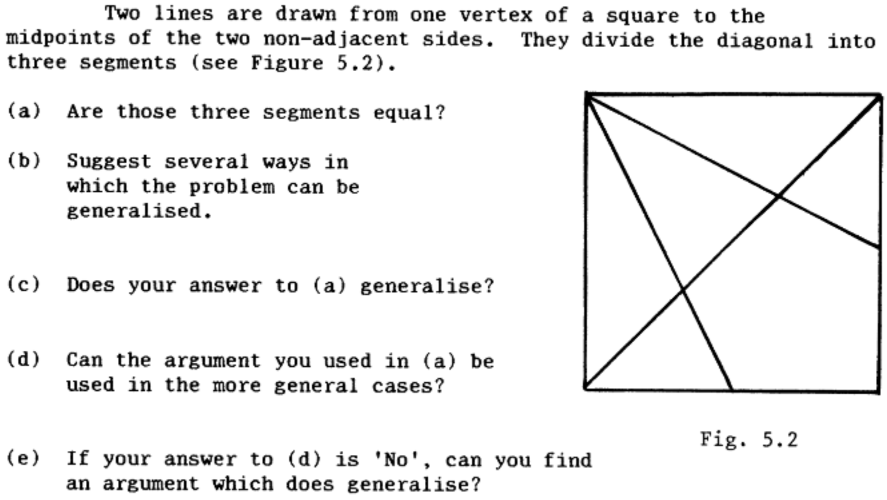
These systems employ a variety of methods and algorithms to derive theorems, making the learning process more dynamic. They facilitate geometry teaching by providing students with a practical way to understand and apply concepts. Furthermore, the interactivity offered by these tools allows students to actively explore conjectures, promoting more effective learning [13, 14, 15].

The integration of DGS and automated deduction systems in education offers several benefits:

- **Conjecture Exploration**: Allows for interactive and dynamic exploration and validation of geometric conjectures.
- **Technological Integration**: Provides a richer and more engaging learning experience.
- **Immediate Feedback**: Students receive instant feedback on their conjectures, allowing for real-time adjustments.
- **Improved Understanding**: These tools deepen understanding of geometric concepts, facilitating more meaningful learning.
- **Active Learning**: Automated testers promote active learning, where students become protagonists in the construction of their knowledge.

4 Proposed Classroom Activity

This (adapted) activity was designed for secondary education students, in which the generalization of a paradigmatic problem proposed in the ICMI study "School Mathematics in the 90's" [8] related to the partition of the diagonal AD of a square $ABCD$ into different segments, described by the intersection of the diagonal with straight lines from D to points A_i on side AB, after dividing this side into n equal parts (Figure 1).



Two lines are drawn from one vertex of a square to the midpoints of the two non-adjacent sides. They divide the diagonal into three segments (see Figure 5.2).

(a) Are those three segments equal?

(b) Suggest several ways in which the problem can be generalised.

(c) Does your answer to (a) generalise?

(d) Can the argument you used in (a) be used in the more general cases?

(e) If your answer to (d) is 'No', can you find an argument which does generalise?

Fig. 5.2

Figure 1: Activity proposal [8, p. 59]

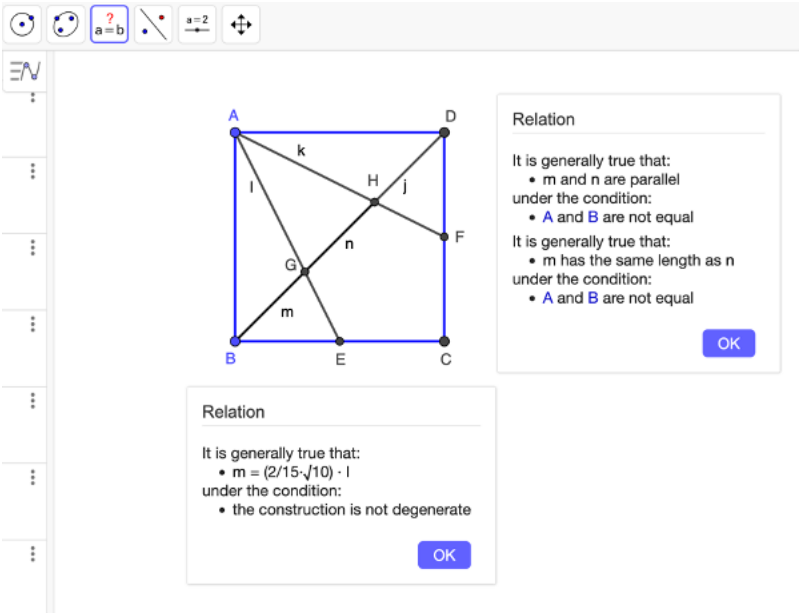
First approach

In this activity, the 'Relation' tool in GeoGebra (Figure 2) allows you to determine the relationships that exist between objects.

GeoGebra's 'Relation' tool, the student must first select the objects they wish to analyze, and then click on the 'Relation' icon on the toolbar. GeoGebra will display a dialog box providing information about the relationship between the selected objects. This button appears when symbolic support is available. This means that the statement can be rigorously verified for the general case with arbitrary coordinates, not just for the displayed geometric construction. Clicking this button will display another dialog box with more details about the relationship, such as the validity of the relationship under certain conditions.

This command allows the student to determine numerically (i.e., based on the construction drawn with coordinates precisely assigned to each point) whether [13]:

- two lines are perpendicular;
- two lines are parallel;
- two (or more) objects (points, segment lengths, polygon areas) are equal;
- a point is on a line or conic;
- a line is tangent or a line that passes through a conic;



Relation

It is generally true that:
- m and n are parallel

under the condition:
- A and B are not equal

It is generally true that:
- m has the same length as n

under the condition:
- A and B are not equal

Relation

It is generally true that:
- m = (2/15·√10) · l

under the condition:
- the construction is not degenerate

Figure 2: Use of the 'Relation' tool

- three points are collinear;
- three lines are concurrent (or parallel);
- four points are concyclic (or collinear)

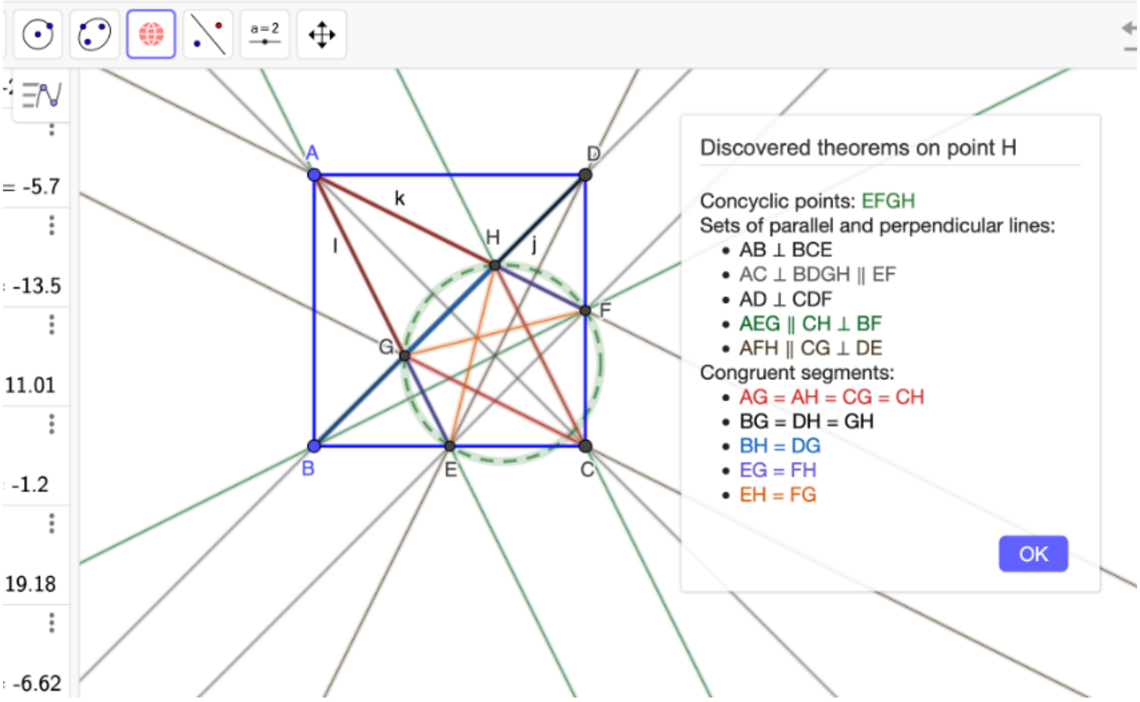
Second approach

'Discover' tool in GeoGebra (`https://autgeo.online/geogebra-discovery/`) allows you to discover all possible relationships of a certain type between the elements of a construction that involves a chosen element (Figure 3).

'Discover' tool allows [9]:

- check the veracity/failure of a conjectured relationship (by the user);
- find missing hypotheses for a given relationship to be true;
- formulate a thesis (for example, assert that a given point is on a given line, the perpendicularity of two lines, find the ratio between two lengths or an inequality involving different lengths) maintained under the set of hypotheses that describe the geometric construction;
- automatically discover all possible relationships of a certain type between the elements of a construction involving a chosen element.

When selecting the command in GeoGebra it will 'discover' theses that involve (for example) point H. The answer is quite complete. On the one hand, we find the desired properties: line AC is perpendicular to line BD and BD is parallel to EF (in gray), but also that $EH = FG$ (in orange) and that $EG = FH$ (in purple), that is, $EFHG$ is an isosceles trapezoid!



Discovered theorems on point H

Concyclic points: EFGH
Sets of parallel and perpendicular lines:
- AB ⊥ BCE
- AC ⊥ BDGH || EF
- AD ⊥ CDF
- AEG || CH ⊥ BF
- AFH || CG ⊥ DE

Congruent segments:
- AG = AH = CG = CH
- BG = DH = GH
- BH = DG
- EG = FH
- EH = FG

Figure 3: Use of the 'Discover ' tool presents several theses involving point H

Third approach

JGEx[1] is a software that combines a DGS and a bit of automated proving systems to create and manipulate formal visual proofs in mathematics. Its main features include [13, 21]:

- **User-friendly interface**: *JGEx* has a user-friendly interface that allows users to easily create and manipulate geometric constructions, draw diagrams, and annotate their work with text labels and other visual elements.
- **Multiple provers**: includes algebraic, synthetic, and semi-synthetic provers.
- **Visually dynamic demonstration presentations**: *JGEx* allows users to animate their constructions and demonstrations, making it easier for audiences to follow the reasoning in the formal demonstration. This can be particularly useful in classroom settings, where teachers can use *JGEx* to create interactive presentations that engage their students.

In this activity, synthetic testers, also called GDD (Geometry Deductive Database), or the deductive bases method. These tools are based on a progressive deductive reasoning model, in which one starts from the initial geometric construction and iteratively applies rules of logical inference with the aim of reaching the desired conclusion [15].

Throughout this process, geometric proofs are generated that maintain a strong connection to the original construction, making reasoning more accessible and meaningful for students. Continuous visualization of the proof steps, in conjunction with the geometric figure, contributes

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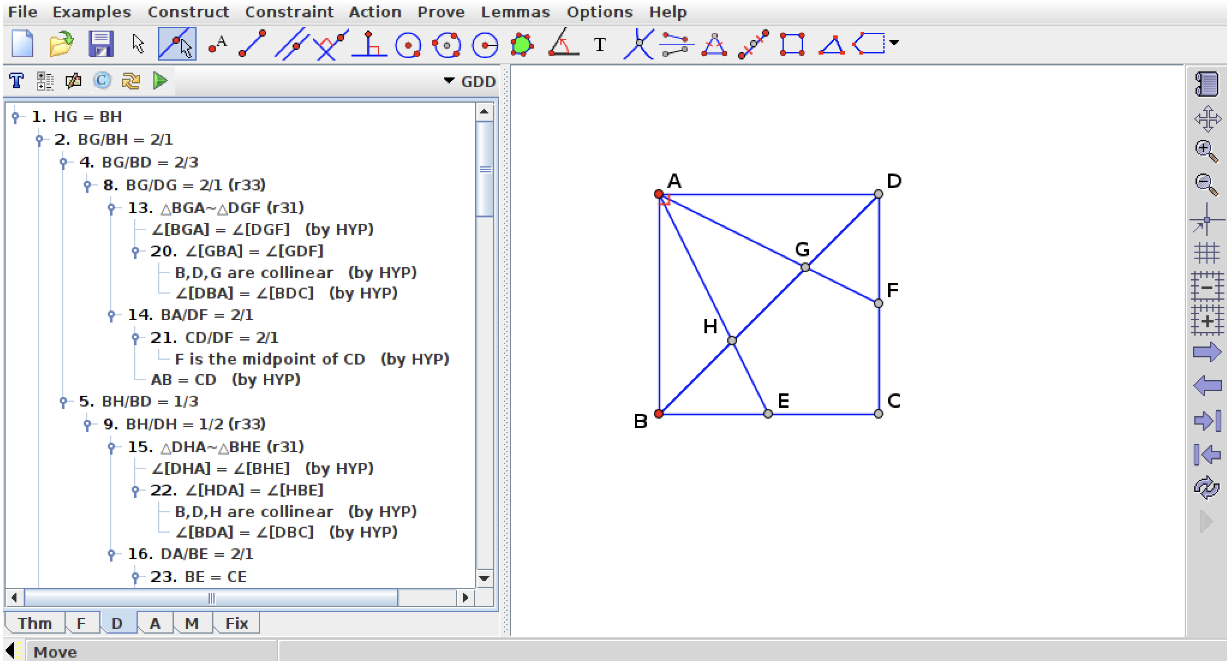
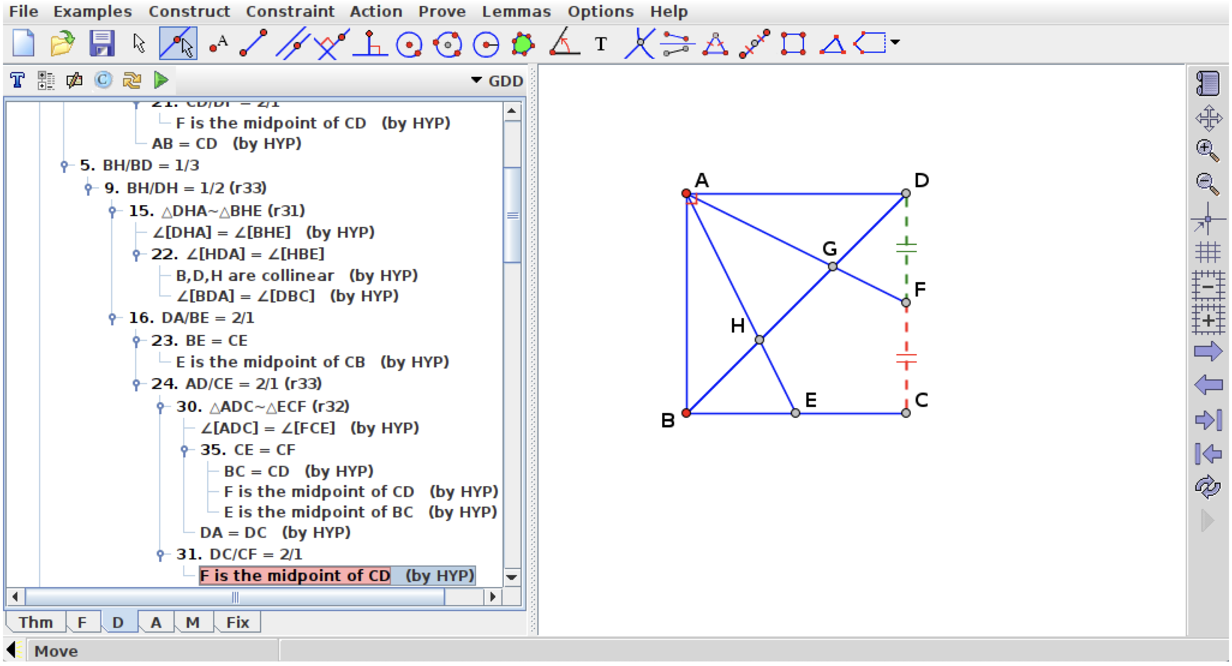
[1] `https://github.com/kovzol/Java-Geometry-Expert` and `https://github.com/kovzol/Java-Geometry-Expert/releases`

to a clearer understanding of the arguments used, promoting a more intuitive and structured approach to deductive thinking (Figures 4–8).



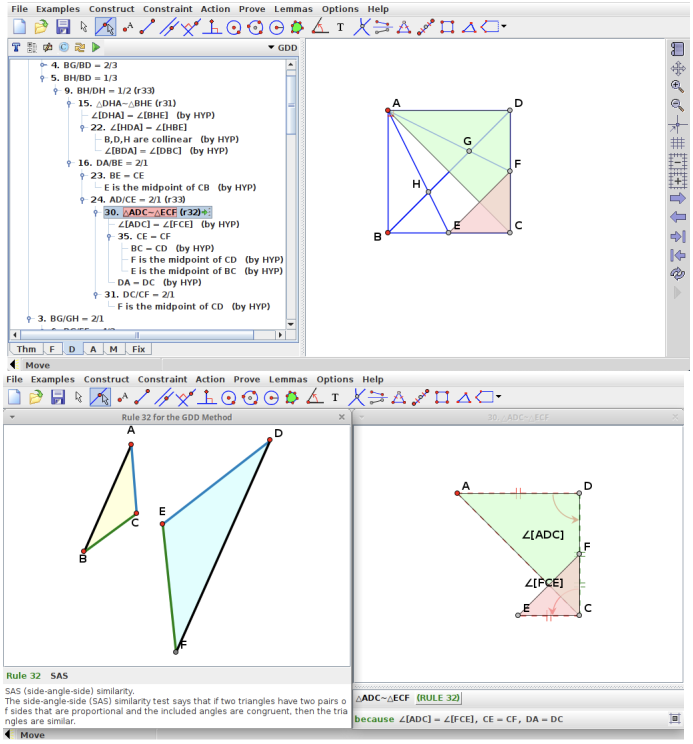
```
1. HG = BH
  2. BG/BH = 2/1
    4. BG/BD = 2/3
      8. BG/DG = 2/1 (r33)
        13. △BGA~△DGF (r31)
          ∠[BGA] = ∠[DGF]   (by HYP)
          20. ∠[GBA] = ∠[GDF]
            B,D,G are collinear   (by HYP)
            ∠[DBA] = ∠[BDC]   (by HYP)
        14. BA/DF = 2/1
          21. CD/DF = 2/1
            F is the midpoint of CD   (by HYP)
          AB = CD   (by HYP)
    5. BH/BD = 1/3
      9. BH/DH = 1/2 (r33)
        15. △DHA~△BHE (r31)
          ∠[DHA] = ∠[BHE]   (by HYP)
          22. ∠[HDA] = ∠[HBE]
            B,D,H are collinear   (by HYP)
            ∠[BDA] = ∠[DBC]   (by HYP)
        16. DA/BE = 2/1
          23. BE = CE
```

Figure 4: Construction in *JGEx*



```
            F is the midpoint of CD   (by HYP)
          AB = CD   (by HYP)
    5. BH/BD = 1/3
      9. BH/DH = 1/2 (r33)
        15. △DHA~△BHE (r31)
          ∠[DHA] = ∠[BHE]   (by HYP)
          22. ∠[HDA] = ∠[HBE]
            B,D,H are collinear   (by HYP)
            ∠[BDA] = ∠[DBC]   (by HYP)
        16. DA/BE = 2/1
          23. BE = CE
            E is the midpoint of CB   (by HYP)
          24. AD/CE = 2/1 (r33)
            30. △ADC~△ECF (r32)
              ∠[ADC] = ∠[FCE]   (by HYP)
              35. CE = CF
                BC = CD   (by HYP)
                F is the midpoint of CD   (by HYP)
                E is the midpoint of BC   (by HYP)
              DA = DC   (by HYP)
            31. DC/CF = 2/1
              F is the midpoint of CD   (by HYP)
```

Figure 5: *F* is the midpoint of the line segment *CD*, in *JGEx*, using the deductive bases method



```
4. BG/BD = 2/3
5. BH/BD = 1/3
  9. BH/DH = 1/2 (r33)
    15. △DHA~△BHE (r31)
      ∠[DHA] = ∠[BHE]   (by HYP)
      22. ∠[HDA] = ∠[HBE]
        B,D,H are collinear   (by HYP)
        ∠[BDA] = ∠[DBC]   (by HYP)
    16. DA/BE = 2/1
      23. BE = CE
        E is the midpoint of CB   (by HYP)
      24. AD/CE = 2/1 (r33)
        30. △ADC~△ECF (r32)
          ∠[ADC] = ∠[FCE]   (by HYP)
          35. CE = CF
            BC = CD   (by HYP)
            F is the midpoint of CD   (by HYP)
            E is the midpoint of BC   (by HYP)
          DA = DC   (by HYP)
        31. DC/CF = 2/1
          F is the midpoint of CD   (by HYP)
3. BG/GH = 2/1
```

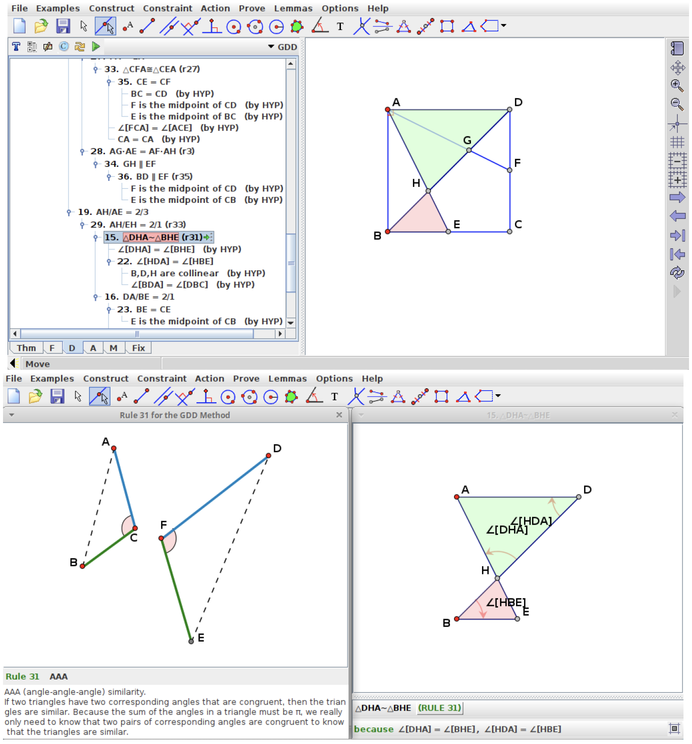
Rule 32 **SAS**

SAS (side-angle-side) similarity.
The side-angle-side (SAS) similarity test says that if two triangles have two pairs of sides that are proportional and the included angles are congruent, then the triangles are similar.

△ADC~△ECF (RULE 32)

because ∠[ADC] = ∠[FCE], CE = CF, DA = DC

Figure 6: Triangles ADC and ECF are similar by the side-angle-side rule, in *JGEx*, using the deductive bases method



```
33. △CFA≅△CEA (r27)
  35. CE = CF
    BC = CD   (by HYP)
    F is the midpoint of CD   (by HYP)
    E is the midpoint of BC   (by HYP)
  ∠[FCA] = ∠[ACE]   (by HYP)
  CA = CA   (by HYP)
28. AG·AE = AF·AH (r3)
  34. GH ∥ EF
    36. BD ∥ EF (r35)
      F is the midpoint of CD   (by HYP)
      E is the midpoint of CB   (by HYP)
19. AH/AE = 2/3
  29. AH/EH = 2/1 (r33)
    15. △DHA~△BHE (r31)
      ∠[DHA] = ∠[BHE]   (by HYP)
      22. ∠[HDA] = ∠[HBE]
        B,D,H are collinear   (by HYP)
        ∠[BDA] = ∠[DBC]   (by HYP)
    16. DA/BE = 2/1
      23. BE = CE
        E is the midpoint of CB   (by HYP)
```

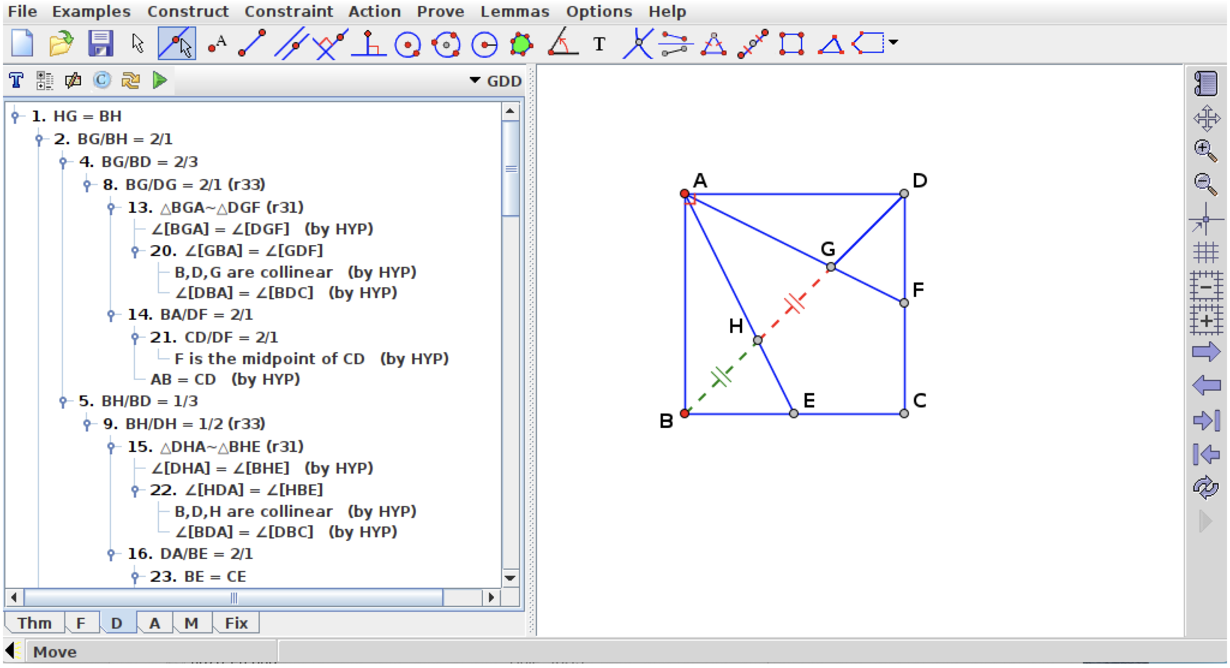
Rule 31 **AAA**

AAA (angle-angle-angle) similarity.
If two triangles have two corresponding angles that are congruent, then the triangles are similar. Because the sum of the angles in a triangle must be π, we really only need to know that two pairs of corresponding angles are congruent to know that the triangles are similar.

△DHA~△BHE (RULE 31)

because ∠[DHA] = ∠[BHE], ∠[HDA] = ∠[HBE]

Figure 7: Triangles DHA and BHE are similar by the angle-angle-angle rule, in *JGEx*, using the deductive bases method



```
1. HG = BH
  2. BG/BH = 2/1
    4. BG/BD = 2/3
      8. BG/DG = 2/1 (r33)
        13. △BGA~△DGF (r31)
          ∠[BGA] = ∠[DGF]   (by HYP)
          20. ∠[GBA] = ∠[GDF]
            B,D,G are collinear   (by HYP)
            ∠[DBA] = ∠[BDC]   (by HYP)
        14. BA/DF = 2/1
          21. CD/DF = 2/1
            F is the midpoint of CD   (by HYP)
          AB = CD   (by HYP)
    5. BH/BD = 1/3
      9. BH/DH = 1/2 (r33)
        15. △DHA~△BHE (r31)
          ∠[DHA] = ∠[BHE]   (by HYP)
          22. ∠[HDA] = ∠[HBE]
            B,D,H are collinear   (by HYP)
            ∠[BDA] = ∠[DBC]   (by HYP)
        16. DA/BE = 2/1
          23. BE = CE
```

Figure 8: The length of the line segments HG and BH as requested to prove, in *JGEx*, using the deductive bases method

This integration of visual and logical elements proves to be particularly effective in the context of teaching and learning Geometry, encouraging the development of mathematical reasoning in an active and well-founded way.

5 Discussion

Tools such as GeoGebra's 'Relation' and 'Discover' have proven particularly useful in the context of geometry education, offering students new ways to explore and understand geometric relationships.

The 'Relation' tool is especially valuable as it allows students to automatically identify the geometric relationships between the different elements constructed within the GeoGebra environment. This not only facilitates the visualisation of geometric properties but also supports the understanding of how different objects interact, for example. By automating this process, the software enables students to focus on interpreting the underlying properties and their implications. The real-time visualisation of how changes in one element affect others creates an interactive and effective learning experience that encourages deeper exploration of geometric concepts.

On the other hand, GeoGebra's 'Discover' tool offers a more exploratory approach, allowing students to formulate conjectures and test their validity through dynamic proofs. This feature does more than simply present final results, it actively involves students in the logical deduction process, enabling them to validate hypotheses interactively while developing deductive reasoning and mathematical argumentation. By constructing and testing their own conjectures, students are encouraged to question, investigate, and reflect on their ideas, which enhances their understanding and application of geometric concepts. In this way, 'Discover' tool not only facilitates active exploration of geometry but also reinforces the importance of proof and justification in mathematics.

By integrating these two tools, GeoGebra transforms the learning of geometry, making it more accessible, intuitive, and engaging. The use of these features enables students to explore geometric properties dynamically, fostering more meaningful and collaborative learning experiences. The software not only supports the visualisation of complex concepts but also helps students develop logical reasoning and the ability to construct and justify arguments, key skills in school mathematics.

In a third approach, another dynamic geometry software, *JGEx* (Java Geometry Expert), was used and proved equally valuable in exploring geometric properties. *JGEx* offers a distinct approach based on progressive reasoning. The tool allows students to begin with a geometric construction and then apply inference rules repeatedly to reach specific conclusions. The construction process is guided by logical reasoning, with geometric properties being explored in a gradual and systematic way.

One of the major advantages of *JGEx* is its ability to produce readable and visual geometric proofs. By following the inference rules, the software generates proofs that are not only rigorous but also easily understood by students, as they are directly linked to the visual construction being created. The use of visual resources makes the proof process more accessible, allowing students to follow each step and clearly understand the relationship between concepts and conclusions.

In summary, both GeoGebra and *JGEx* offer innovative and effective approaches to teaching geometry, promoting active, interactive, and visual learning. These tools contribute significantly to the development of students' logical reasoning, deduction, and mathematical argumentation skills, making the study of geometry more engaging and comprehensible.

6 Conclusion

The teaching of geometric demonstrations in the Portuguese education system faces challenges that require reflection and innovation. Although curricula promote the development of deductive thinking, teaching practices do not always foster deep understanding on the part of students. According to Hanna [5], a teaching activity that includes formal or informal reasoning can only be considered valuable to the extent that it promotes greater understanding.

JGEx serves as an excellent tool for visualizing and analyzing problems in the classroom. This software combines a DGS and some automatic geometric provers, allowing the dynamic creation and manipulation of geometric constructions. Hanna [4] focuses on the importance of proof in mathematics teaching, particularly in the context of geometry, and discusses the role of dynamic geometry software in teaching proofs, including exploration and visualization. The author believes that geometric proofs are fundamental in mathematics teaching and investigated the use of dynamic geometry software for teaching and learning proofs.

According to Loureiro [10], "this entire exploration would be arid if we resorted to algebraic solutions, involving various segment ratios and area calculations. By using a DGS, we solved the original problem and made a very powerful generalization" (p. 110). The author emphasizes that using a DGS not only allows for more intuitive and visual problem solving but also paves the way for the formulation of meaningful generalizations. This highlights the potential of DGS to transform geometry learning, making it more accessible, exploratory, and rich in opportunities for the development of mathematical thinking.

It is important to support teachers through ongoing professional development focused on didactic demonstration, as well as to encourage the use of visual and technological resources that facilitate the transition from empirical to deductive reasoning.

Future research could explore in greater depth the impact of teaching sequences based on geometric problem-solving, particularly on the development of students' deductive thinking and argumentative skills. This approach provides more meaningful learning contexts, in which proof emerges as a natural response to the need to justify and validate conjectures.

At the same time, it will be important to analyze the more systematic integration of digital tools in the teaching of mathematical proof, assessing how these technologies can enhance conceptual understanding, the visualization of geometric relationships, and the construction of logical arguments. The combination of problem-solving strategies and the intentional use of digital resources could play a decisive role in promoting more accessible, engaging, and effective pedagogical practices in teaching proof.

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