

New Results About Semi-Regular Polygons Circumscribed Around A Given Ellipse

Jean-Jacques Dahan
jjdahan@wanadoo.fr
 IRES of Toulouse
 Toulouse University
 Toulouse FRANCE.

Abstract: In this article, we focus on the class of polygons we previously referred to as semi-regular, which are more commonly known as equiangular polygons circumscribed about a given ellipse. In our earlier work, we formulated a large number of conjectures concerning the variations in the areas of these polygons for all values of n (where n denotes the number of sides), in connection with the behavior of certain relatively complex functions ([8], [3']). This article presents the current state of my research aimed at establishing those conjectures in the general case. I provide complete proofs for the conjectured results when $n=3, 4$ and 6 while the case $n=5$ has so far resisted various lines of attack. The investigation, still largely conducted through experimental exploration using dynamic geometry software, has also led to the discovery of several surprising additional results, some of which have been proven, while others remain conjectural and await proof. This research is far from complete, as it explores a domain that appears not to have been previously studied. This article highlights, among other things, the crucial role of technology-mediated experimentation in the process of discovery ([1], [3]).

1. Reminder

1.1. Construction of semi-regular polygons circumscribed around a given ellipse (Figure 1)

(E) is the ellipse whose foci are f_1 and f_2 ($f_1f_2 = 2c$ with $c < 1$) and where (C) is the principal circle of (E) centered at o (radius $oa=1$). The eccentricity of (E) is c . Figure 1 shows the constructions of all the equiangular pentagons circumscribed around (E) ($m_1m_2m_3m_4m_5$), each pentagon varies depending on the position of point p_1 on (C) (p_2, p_3, p_4 and p_5 are also located on (C)). Note that $\angle p_1f_2p_2 = \angle p_2f_2p_3 = \angle p_3f_2p_4 = \angle p_4f_2p_5 = \angle p_5f_2p_1 = 72^\circ$ ($\frac{360^\circ}{5}$). Note also that the sides of these pentagons are supported by the lines perpendicular to $[f_2p_1], [f_2p_2], [f_2p_3], [f_2p_4]$ and $[f_2p_5]$, respectively at p_1, p_2, p_3, p_4 and p_5 .

The construction, in the general case is obtained in changing the value of n .

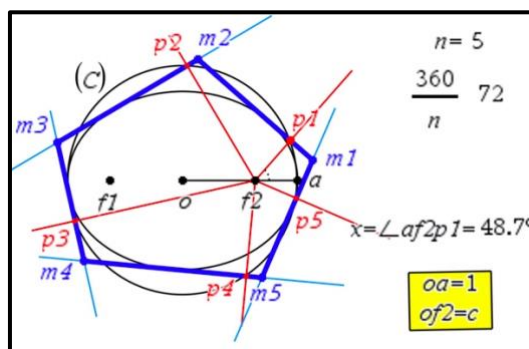


Figure 1: Construction algorithm of semi-regular n -gons circumscribed around a given ellipse

1.2. Conjectures concerning semi-regular n -gons circumscribed around a given ellipse

In an effort to prove a property of the Steiner's ellipse with the aim of simplifying the first complete proof by Minda and Phelps ([5]), inspired by investigations in dynamic geometry, I was led to study the variations in the area of equilateral triangles circumscribed about a given ellipse ([6], ([2'], ([7])). I was able to precisely conjecture how this area varies, and even observed that these variations coincide with those of the sum of the distances from one focus to the three sides of the triangle. As a result, the area variations must match those of the function defined by:

$$\sqrt{1 - c^2 \cdot \sin^2(x)} + \sqrt{1 - c^2 \cdot \sin^2(x + \frac{2\pi}{3})} + \sqrt{1 - c^2 \cdot \sin^2(x + \frac{4\pi}{3})}$$

Unable to prove this result, I turned to the general case, knowing that the equilateral triangle was just a particular instance of an equiangular polygon. Continuing with an experimental approach, I was thus led to precisely conjecture the variations in the areas of equiangular n -gons circumscribed about an ellipse, and also that these variations matched those of the function generalizing the previous one, now defined by:

$$B(x, n) = \sum_{p=0}^{p=n-1} \sqrt{1 - c^2 \cdot \sin^2(x + \frac{2p\pi}{n})}$$

What follows will show that this research has proven to be quite challenging, especially as I have attempted to approach the problem geometrically. I will therefore present the proofs of the conjectures in the cases $n = 3$, $n = 4$ and $n = 6$.

2. Properties of function $B(x, n)$

Here are the parity and periodicity properties of the functions B , established in the general case:

2.1. Case when n is even

Function $B(x, n)$ is always even and its period is $\frac{2\pi}{n}$ because:

$$\begin{aligned} B\left(x + \frac{2\pi}{n}, n\right) &= \sum_{p=0}^{n-1} \sqrt{1 - c^2 \cdot \sin^2\left(x + \frac{2p\pi}{n} + \frac{2\pi}{n}\right)} = \sum_{p=0}^{n-1} \sqrt{1 - c^2 \cdot \sin^2\left(x + \frac{2(p+1)\pi}{n}\right)} \\ &= \sum_{p=1}^n \sqrt{1 - c^2 \cdot \sin^2\left(x + \frac{2p\pi}{n}\right)} \\ &= B(x, n) \end{aligned}$$

Illustration of the $\frac{2\pi}{n}$ -values in Figure 2 below

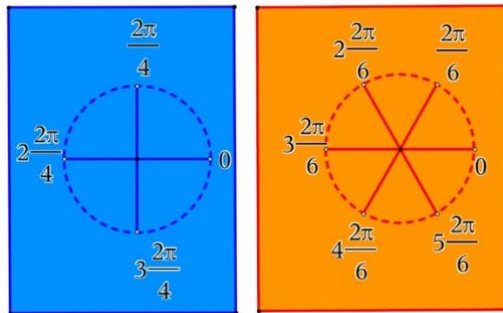


Figure 2: Distribution of the $\frac{2\pi}{n}$ -values on the unit circle (n even and here $n=4$ and $n=6$)

2.2. Case when n is odd

Function $B(x, n)$ is always even and its period is $\frac{\pi}{n}$ because:

As $B(x + \pi, n) = B(x, n)$, $B\left(x + \frac{\pi}{n}, n\right) = B\left(x + \frac{\pi}{n} + \pi, n\right) = B\left(x + \frac{(n+1)\pi}{n}, n\right)$, we have

$$\begin{aligned} B\left(x + \frac{\pi}{n}, n\right) &= \sum_{p=0}^{n-1} \sqrt{1 - c^2 \cdot \sin^2\left(x + \frac{2p\pi}{n} + \frac{\pi}{n}\right)} \\ &= \sum_{p=0}^{n-1} \sqrt{1 - c^2 \cdot \sin^2\left(x + \frac{2p\pi}{n} + \frac{(n+1)2\pi}{2n}\right)} \\ &= \sum_{p=0}^{n-1} \sqrt{1 - c^2 \cdot \sin^2\left(x + \left(p + \frac{(n+1)}{2}\right) \frac{2\pi}{n}\right)} \\ &= \sum_{p=\frac{3n-1}{2}}^{\frac{n+1}{2}} \sqrt{1 - c^2 \left(\sin\left(x + p \frac{2\pi}{n}\right)\right)^2} \end{aligned}$$

With n consecutive integer values of p for $x + p \frac{2\pi}{n}$ and therefore:

$$= \sum_{p=0}^{p=n-1} \sqrt{1 - c^2 \cdot \sin^2\left(x + p \frac{2\pi}{n}\right)} = B(x, n)$$

Illustration of the $\frac{2\pi}{n}$ -values in Figure 3 below

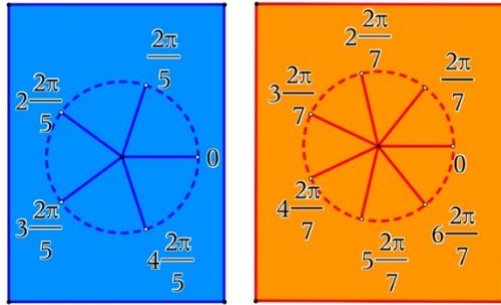


Figure 3: Distribution of the $\frac{2\pi}{n}$ -values on the unit circle (n odd and here $n=5$ and $n=7$)

2.3. Conclusions about functions $B(x, n)$

Theorem 1: Functions $B(x, n)$ are always even functions and their periods are $\frac{2\pi}{n}$ when n is even and $\frac{\pi}{n}$ when n is odd.

3. Final proofs of my conjectures for $n = 3$

We already know that the area of an equilateral triangle circumscribed about a given ellipse (such as the one introduced at the beginning of this article) varies in the same way as the sum of the distances from any interior point of the ellipse to the sides of the triangle (This sum actually represents the height of such a triangle, according to Viviani's Theorem). We conjectured that this area, like the function $B(x, 3)$, is an even function and periodic with period $\frac{\pi}{3}$; the proof of these two properties has just been given above. To fully establish the conjectured variations of both the area and $B(x, 3)$, it is

enough to prove, for example, that $B(x,3)$ is a decreasing function on the interval $[0, \frac{\pi}{6}]$. Since all of my own attempts to prove this result were unsuccessful, a colleague from Singapore proposed the problem to his nephew, Kia Keng GIAM, who was then a final-year high school student in the science track (Temasek Junior College, Singapore). To our great surprise, he succeeded in resolving the question of the sign of the derivative of $B(x,3)$ by using a clever preliminary trick that allowed him to obtain a key factorization with the help of a computer algebra system (CoCalc platform of SAGE). Here is a summary of the proof that ultimately settled the conjecture ([9]):

After defining $f_1(x), f_2(x)$ and $f_3(x)$ as the component functions of $B(x,3)$, he expresses their derivatives in terms of $p=\cos(x)$ and $q=\sin(x)$. He then sets $A=-f_1'(x)$, $B=f_2'(x)$ and $C=-f_3'(x)$. He observes the following equivalence: $f_1'(x)+f_2'(x)+f_3'(x) \leq 0 \Leftrightarrow A+C \geq B$. He then proves that the negativity of the derivative is equivalent to the positivity of $4A^2C^2 - (B^2-A^2-C^2)^2$ over the interval $[0, \frac{\pi}{6}]$. To do this, he has this expression computed by a computer algebra system, which produces an extremely surprising result, one that allows the conclusion to follow easily:

$$\frac{9(1-c^2)(3p^2-q^2)^2(p^2-3q^2)^2c^{12}p^2q^2}{256(1-c^2q^2)^2 \left[1-c^2 \left(\frac{3}{4}p^2 - \frac{\sqrt{3}}{2}pq + \frac{1}{4}q^2 \right) \right]^2 \left[1-c^2 \left(\frac{1}{4}q^2 + \frac{\sqrt{3}}{2}pq + \frac{3}{4}p^2 \right) \right]^2}$$

Theorem 2: The areas of equilateral triangles (or semi-regular polygons with 3 sides) circumscribed about a given ellipse vary like the function $B(x,3)$. These are even, periodic functions with period $\frac{\pi}{3}$, decreasing on the interval $[0, \frac{\pi}{6}]$ and increasing on the interval $[\frac{\pi}{6}, \frac{\pi}{3}]$.

4. Final proofs of my conjectures for $n = 4$

4.1. Variations of the area $A(x,4)$:

$A(x,4)$ represents the area of a semi regular 4-gon circumscribed around a given ellipse. For the study of its variations, it is sufficient to know the variations of this area over the interval $[0, \frac{\pi}{4}]$, since geometrically we see that this function is indeed even with a period of $\frac{\pi}{2}$ due to the double symmetry of the ellipse (see Figure 4 for the extreme cases). Note that when $x = \frac{\pi}{4}$, the rectangle is necessarily a square, again due to the double symmetry of the ellipse.

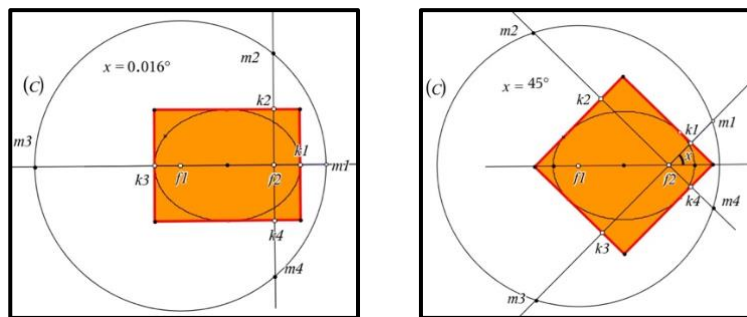


Figure 4: Extreme cases for rectangles circumscribed around a given ellipse

Figure 5 illustrates, in the general case, the calculation of the dimensions of the rectangle and consequently the evaluation of its area as a function of x .

We know that $f_2k_1 = -c \cdot \cos(x) + \sqrt{1-c^2 \cdot \sin^2(x)} = u(x)$. Therefore:

$$f_2k_2 = u \left(x + \frac{\pi}{2} \right) = c \cdot \sin(x) + \sqrt{1-c^2 \cdot \cos^2(x)}$$

$$f_2 k_3 = u\left(x + \frac{2\pi}{2}\right) = c \cdot \cos(x) + \sqrt{1 - c^2 \cdot \sin^2(x)}$$

$$f_2 k_4 = u\left(x + \frac{3\pi}{2}\right) = -c \cdot \sin(x) + \sqrt{1 - c^2 \cdot \cos^2(x)}$$

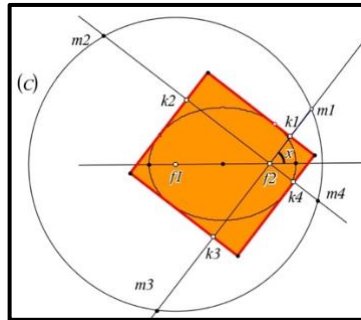


Figure 5: Rectangle circumscribed around a given ellipse

Therefore, the lengths and widths of the rectangle circumscribed around the ellipse are equal to:

$$l_1 = f_2 k_1 + f_2 k_3 = 2\sqrt{1 - c^2 \cdot \sin^2(x)} \text{ and}$$

$$l_2 = f_2 k_2 + f_2 k_4 = 2\sqrt{1 - c^2 \cdot \cos^2(x)}$$

The area of the rectangle is equal to $4\sqrt{1 - c^2 \cdot \sin^2(x)} \cdot \sqrt{1 - c^2 \cdot \cos^2(x)}$ when the inclination of the rectangle is x with respect to the direction of the major axis of the ellipse.

Let us therefore study the variations of $A(x, 4)$ by determining the sign of its derivative $A'(x, 4)$ over the interval $[0, \frac{\pi}{4}]$.

By using TI-Nspire, where we have denoted $a(x)$ function $A(x, 4)$, we obtain:

$$a(x) := 4 \cdot \sqrt{1 - (c \cdot \sin(x))^2} \cdot \sqrt{1 - (c \cdot \cos(x))^2}$$

$$da(x) := \frac{d}{dx}(a(x)) \quad \blacktriangleright \text{Done}$$

$$da(x) \blacktriangleright \frac{4 \cdot c^2 \cdot \sin(x) \cdot \sqrt{1 - c^2 \cdot (\sin(x))^2} \cdot \cos(x)}{\sqrt{1 - c^2 \cdot (\cos(x))^2}}$$



$$\frac{4 \cdot c^2 \cdot \sin(x) \cdot \cos(x) \cdot \sqrt{1 - c^2 \cdot (\cos(x))^2}}{\sqrt{1 - c^2 \cdot (\sin(x))^2}}$$

$$\text{comDenom}(da(x)) \blacktriangleright \frac{8 \cdot c^4 \cdot \sin(x) \cdot (\cos(x))^3 - 4 \cdot c^4 \cdot \sin(x) \cdot \cos(x)}{\sqrt{1 - c^2 \cdot (\sin(x))^2} \cdot \sqrt{1 - c^2 \cdot (\cos(x))^2}}$$

The numerator of the derivative, which gives the sign of the derivative, can be factored as follows:

$$4c^4 \sin(x) \cdot \cos(x) \cdot (2(\cos(x))^2 - 1) = 4c^4 \sin(x) \cdot \cos(x) \cdot (\cos^2(x) - \sin^2(x))$$

The sign of the derivative is therefore given by the sign of $(\cos^2(x) - \sin^2(x))$, that is to say the sign of $((\cos(x) + \sin(x)) \cdot ((\cos(x) - \sin(x)))$ or ultimately of $\cos(x) - \sin(x)$ over $[0, \frac{\pi}{4}]$ because $\cos(x) + \sin(x)$ is positive over this interval. We easily obtain the following result:
 $A'(x, 4) > 0$ over $[0, \frac{\pi}{4}]$, from which the variations of $A(x, 4)$ over a period:

x	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$
$A'(x,4)$		-	+
$A(x,4)$			

4.2. Variations of the function $B(x, 4) = 2\sqrt{1 - c^2 \cdot \sin^2(x)} + 2\sqrt{1 - c^2 \cdot \cos^2(x)}$

From paragraph 1 we already know that this function is even and its period is $\frac{\pi}{2}$. Hence the study of variations is sufficient here as well, over $[0, \frac{\pi}{4}]$.

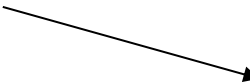

$$B'(x, 4) = 2 \cdot \left(\frac{-c^2 \sin(x) \cdot \cos(x)}{\sqrt{1 - c^2 \cdot \sin^2(x)}} + \frac{+c^2 \sin(x) \cdot \cos(x)}{\sqrt{1 - c^2 \cdot \cos^2(x)}} \right)$$

$$B'(x, 4) = 2 \cdot c^2 \sin(x) \cdot \cos(x) \cdot \left(\frac{-\sqrt{1 - c^2 \cdot \cos^2(x)} + \sqrt{1 - c^2 \cdot \sin^2(x)}}{\sqrt{1 - c^2 \cdot \sin^2(x)} \cdot \sqrt{1 - c^2 \cdot \cos^2(x)}} \right), \text{ that is to say:}$$

$$2 \cdot c^2 \sin(x) \cdot \cos(x) \cdot \left(\frac{c^2 \cdot (\cos^2(x) - \sin^2(x))}{\sqrt{1 - c^2 \cdot \sin^2(x)} \cdot \sqrt{1 - c^2 \cdot \cos^2(x)} \cdot (\sqrt{1 - c^2 \cdot \sin^2(x)} + \sqrt{1 - c^2 \cdot \cos^2(x)})} \right)$$

$B'(x, 4)$ is therefore of the sign of $(\cos^2(x) - \sin^2(x))$, that is to say of $((\cos(x) + \sin(x)) \cdot ((\cos(x) - \sin(x)))$ or ultimately of $\cos(x) - \sin(x)$ over $[0, \frac{\pi}{4}]$ because $\cos(x) + \sin(x)$ is positive over this interval. We easily obtain the following result:

$B'(x, 4) > 0$ over $[0, \frac{\pi}{4}]$, from which the variations of $B(x, 4)$ over a period:

X	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$
$B'(x, 4)$		-	+
$B(x, 4)$			

We thus obtain the same variations as those of the area.

4.3. Conclusion

Theorem 3: The areas of rectangles (or semi-regular polygons with 4 sides) circumscribed about a given ellipse vary like the function $B(x, 4)$. These are even, periodic functions with period $\frac{\pi}{2}$, decreasing on the interval $[-\frac{\pi}{4}, 0]$ and increasing on the interval $[0, \frac{\pi}{4}]$

5. The case when $n = 5$. Refocusing on the research

Naturally, my research then focused on the case $n=5$; this investigation proved unsuccessful despite all my geometric and even algebraic attempts. However, as is often the case in dynamic explorations, unexpected conjectures emerge, some of which may seem useless, at least initially. That is why I believe it is worthwhile to present those that appear to me as relevant.

5.1. The path of the semi-regular star-shaped pentagons

At that point in my research, I had the idea to focus on star-shaped pentagons based on the semi-regular pentagons circumscribed around the given ellipse (see Figure 6 left, where the vertices of this polygon are the points s_1, s_2, s_3, s_4 and s_5). I was interested in the variations in the area of these pentagons and to my great surprise, I was able to observe experimentally that these variations seemed the same as those of the semi-regular pentagons from which we started.

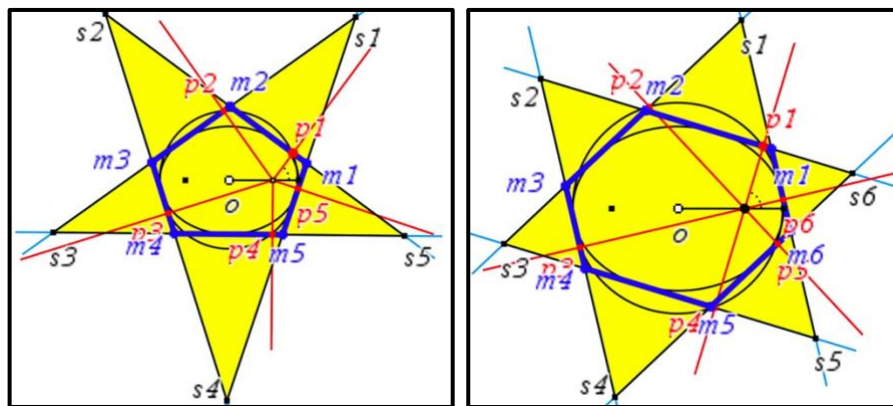


Figure 6: Semi-regular star-shaped pentagons and hexagons

At first, I thought that this property might help me move towards proving my conjectures for $n = 5$. This was not the case. I decided, however, to investigate the variations of the areas of semi-regular star-shaped polygons for other values of n , hoping to generalize the previous conjecture.

5.2. The case of the semi-regular star-shaped hexagons

So, I constructed the semi-regular star-shaped hexagon based on the semi-regular hexagon circumscribed to the given ellipse (Figure 6 right). And there a bigger surprise awaited me: the area of this star-shaped hexagon seemed constant: in reality, it is, and the proof will be given later. We will see that this result will allow me to overcome my conjectures in the case $n = 6$.

5.3. Variations of the areas of the other semi-regular star-shaped polygons (Figure 7)

I observe experimentally that, for $n = 7$, the variations are exactly opposite to those of the corresponding semi-regular heptagon. The same observations are made in the cases $n = 8$ and $n = 9$.

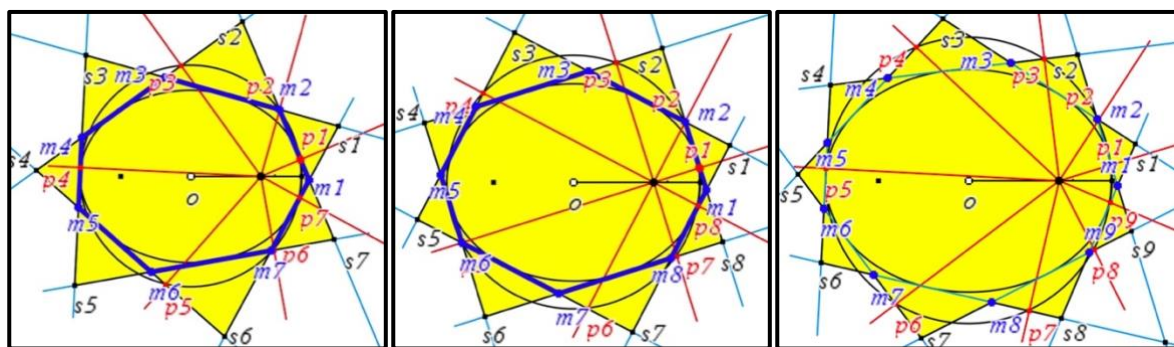


Figure 7: Three cases of semi regular star-shaped polygons

My conjecture on the variations of the areas of semi-regular star-shaped polygons circumscribed to a given ellipse from $n = 5$ is therefore the following

- For $n = 5$, same variations
- For $n = 6$, the area is constant
- For $n \geq 7$, opposite variations

6. Final proofs of my conjectures for $n = 6$

6.1. Preliminary properties of certain convex or star-shaped equiangular hexagons

Shown below are two overlapping equilateral triangles with corresponding sides parallel two by two.

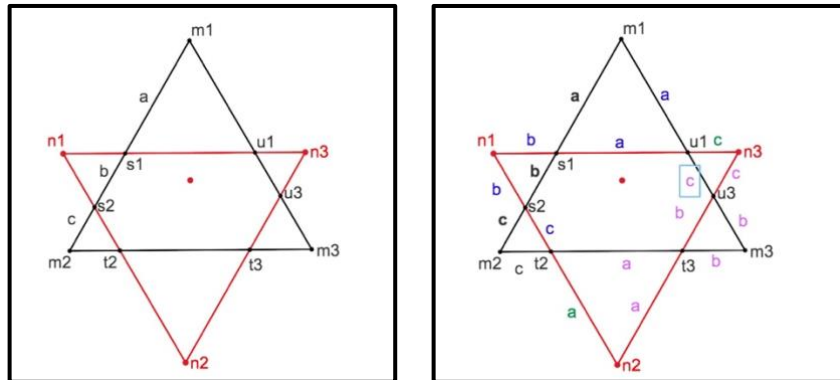


Figure 8: Overlapping equilateral triangles

The two triangles $m_1m_2m_3$ and $n_1n_2n_3$ are isometric, with their sides parallel. All the other triangles are equilateral (with 60° angles). We denote: $m_1s_1 = a$, $s_1s_2 = b$, and $s_2m_2 = c$ (Figure 8 left), from which we deduce what is shown in Figure 8 on the right: first the blue values, then the green ones, and finally the purple ones. We have thus proven that the small triangles are pairwise isometric.

Properties of the equiangular hexagon defined by the previous configuration (Figure 9):

The segments $[s_1t_3]$, $[s_2u_3]$, and $[t_2u_1]$ therefore intersect at their common midpoint o , since the quadrilaterals $s_1t_2t_3u_1$ and $s_1s_2t_3u_3$ are parallelograms. Hence, the point o is the center of symmetry of the equiangular hexagon $s_1s_2t_2t_3u_3u_1s_1$ (Figure 9 left). It follows that $[m_1n_2]$, $[n_1m_3]$, and $[m_2n_3]$ also intersect at their common midpoint o . Thus, the point o is also the center of symmetry of the star-shaped equiangular hexagon (Figure 9 right).

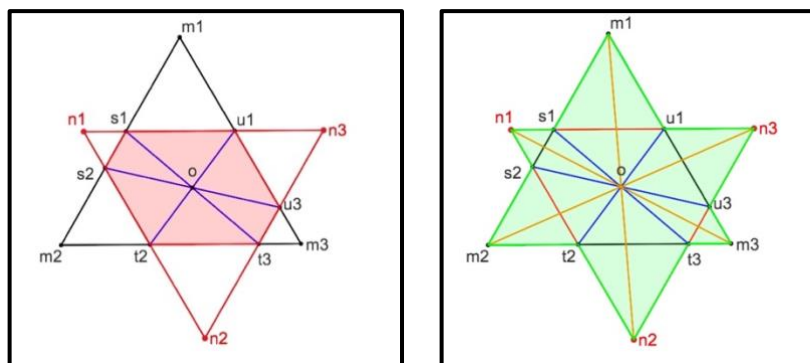


Figure 9: A property of some overlapping equilateral triangles

Conclusion: Our equiangular hexagon thus has its opposite sides equal in pairs. In addition, since the diagonals meet at a common point, it follows from Brianchon's theorem that the hexagon is circumscribed about an ellipse.

6.2. Equiangular hexagons circumscribed around an ellipse

6.2.1. Summary of the constructions with revised notations (Figure 10)

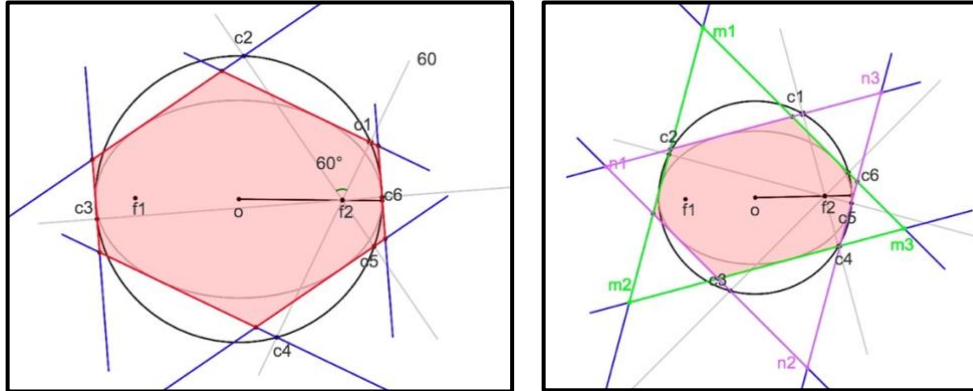


Figure 10: Construction algorithm of semi-regular and star shaped hexagons

6.2.2. Behavior of the sum of a point's distances to the sides of such a hexagon (Figure 11)

It is worth noting that, by the generalized Viviani property for convex equiangular polygons ([1']), this sum remains constant regardless of the point chosen within the ellipse.

If we extend the supports of the sides of the hexagon, we can see two equilateral triangles $m_1m_2m_3$ and $n_1n_2n_3$ entangled as in 6.1. which are circumscribed to the given ellipse. Indeed, these triangles are equilateral by construction; they are also isometric because the area of such a triangle is periodic with a period of 60° (see properties obtained for $n = 3$). The area function of such a triangle is even and we know that it is decreasing over $[0, \frac{\pi}{6}]$. We also know that the variations of the sum of the distances from a point inside the triangle to the three sides are the same as those of the area of the triangle.

We want to evaluate the variations of the sum of the distances from f_2 to the sides of the equiangular hexagon circumscribed to the given ellipse. Knowing that this sum is independent of the point chosen inside an equiangular polygon, we can evaluate this sum from the center of the ellipse and we obtain (see Figure 11): $(h_1+h_2+h_3) + (k_1+k_2+k_3)$ which because of the symmetry of the figure is equal to: $2 \cdot (h_1+h_2+h_3)$. Or $(h_1+h_2+h_3)$ being the sum of the distances to the sides of a circumscribed equilateral triangle these variations are known.

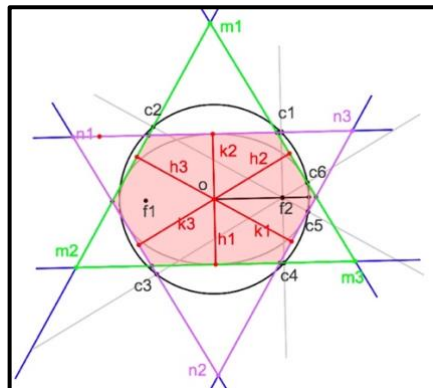


Figure 11: Sum of the squares of the distances from a point to the sides of a semi-regular hexagon

Theorem 4: the variations of the distances from an interior point to an equiangular hexagon circumscribed to an ellipse are the same as those concerning equilateral triangles with respect to the same variable x .

6.2.3. Variations in the area of such a hexagon

6.2.3.1. Area of the equiangular star-shaped hexagon

To begin, let us prove the conjecture made in 5.2. which states that the area of the equiangular star-shaped hexagon resting on the semi-regular hexagon circumscribed around a given ellipse is constant. The area of such a star-shaped hexagon is equal to the area of an equilateral triangle like $m_1m_2m_3$ whose height is equal to $h_1+h_2+h_3$ to which we must add the areas of three small equilateral triangles with vertices n_1, n_2 and n_3 respectively whose respective heights are $h_1+h_2-h_3, h_2+h_3-h_1$ et $h_1+h_3-h_2$ (the first being for example obtained by $h_1+h_2+h_3-2h_3$). This area is therefore obtained as follows:

$$\frac{(h_1+h_2+h_3)^2}{\sqrt{3}} + \frac{(h_1+h_2-h_3)^2}{\sqrt{3}} + \frac{(h_2+h_3-h_1)^2}{\sqrt{3}} + \frac{(h_1+h_3-h_2)^2}{\sqrt{3}}, \text{ that is to say:}$$

$$\frac{4 \cdot (h_1^2 + h_2^2 + h_3^2)}{\sqrt{3}}$$

We will now prove the constancy of this result with respect to x by proving the constancy of

$$h_1^2 + h_2^2 + h_3^2$$

Let us use Figure 12

We know that $f_2c_1 = -c \cdot \cos(x) + \sqrt{1 - c^2(\sin(x))^2}$. By analogous calculation (or for reasons of symmetry), we get: $f_1c_1' = c \cdot \cos(x) + \sqrt{1 - c^2(\sin(x))^2}$.

As $(f_1c_1') \parallel (ot) \parallel (f_2c_1)$ and since o is the midpoint of $[f_1f_2]$, we have:

$$ot = h_1 = \frac{1}{2} \cdot (f_1c_1' + f_2c_1) = \sqrt{1 - c^2(\sin(x))^2}.$$

Eventually

$$h_1^2 + h_2^2 + h_3^2 = (1 - c^2 \cdot \sin^2(x)) + (1 - c^2 \cdot \sin^2(x + \frac{2\pi}{3})) + (1 - c^2 \cdot \sin^2(x + \frac{4\pi}{3}))$$

$$= 3 - c^2 \cdot \sin^2(x) - c^2 \cdot \sin^2(x + \frac{2\pi}{3}) - c^2 \cdot \sin^2(x + \frac{4\pi}{3})$$

As $\sin^2(x) + \sin^2(x + \frac{2\pi}{3}) + \sin^2(x + \frac{4\pi}{3})$

$$= \sin^2(x) + (-\frac{1}{2} \cdot \sin(x) + \frac{\sqrt{3}}{2} \cdot \cos(x))^2 + (\frac{1}{2} \cdot \sin(x) + \frac{\sqrt{3}}{2} \cdot \cos(x))^2$$

$$= \frac{3}{2} \sin^2(x) + \frac{3}{2} \cos^2(x)$$

$$= \frac{3}{2}$$

Eventually, $h_1^2 + h_2^2 + h_3^2 = 3 - \frac{3}{2}c^2$

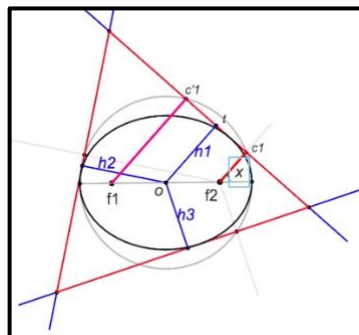


Figure 12: Evaluating $h_1^2 + h_2^2 + h_3^2$

Eventually, the area of the star-shaped hexagon is always equal to $\frac{4}{\sqrt{3}}(3 - \frac{3}{2}c^2)$

Note: The previous result, $\sin^2(x) + \sin^2(x + \frac{2\pi}{3}) + \sin^2(x + \frac{4\pi}{3}) = \frac{3}{2}$, made me want to consider a generalization of the type:

$$\text{Conjecture: } \sum_{k=0}^{n-1} \sin^2\left(x + k \cdot \frac{2\pi}{n}\right) = \frac{n}{2}$$

After verifying this identity for the values 3, 4, 5 and 6 of n , I was able to conjecture this result which I corroborated experimentally by displaying the curve of this function for different values of n and by displaying the ordinate of any point of this curve with a precision of 7 decimal places ([2]).

6.2.3.2. Consequence of the previous proved result on the variations of the area of equiangular hexagons circumscribed around a given ellipse

We denote by A the area of an equilateral triangle circumscribed to the ellipse (for example $m_1m_2m_3$). If we denote by B the area of the equiangular hexagon circumscribed to the ellipse, we can write:

$$2A - B = \frac{4}{\sqrt{3}}(3 - \frac{3}{2}c^2) \text{ or: } B = 2A - \frac{4}{\sqrt{3}}(3 - \frac{3}{2}c^2).$$

This last relation shows that the variations of B are the same as those of A which are known.

Theorem 5: The areas of semi-regular hexagons (or equiangular polygons with 6 sides) circumscribed about a given ellipse vary like the function $B(x,6)$. These are even, periodic functions with period $\frac{\pi}{3}$, decreasing on the interval $[0, \frac{\pi}{6}]$ and increasing on the interval $[\frac{\pi}{6}, \frac{\pi}{3}]$.

Remark: These properties are exactly the same as those obtained for equilateral triangles circumscribed to a given ellipse.

7. Other results and other conjectures

7.1. Sum of the squares of the distances from a point to the sides of an equilateral triangle

When I attempted to assess the constancy of $h_1^2 + h_2^2 + h_3^2$ in the previous paragraph, I thought it might be relevant to consider the sum of the squares of the distances from a point to the sides of an equilateral triangle, in hopes of possibly discovering a generalization of the well-known Viviani property, which concerns the sum of distances, not the squares of the distances ([1]). Naturally, my investigations showed that this sum of squares is not constant and clearly depends on the position of the point inside the triangle. However, I did not abandon this line of inquiry. Inspired by the technique used in the study of Lhuillier's problem ([4]), I placed my variable point on a circle centered at the centroid of the equilateral triangle (which is also the center of its circumscribed circle). To my pleasant surprise, I observed that the sum of the squared distances remained constant for any such circle centered at that special point. Below is the formal proof of this result, which only became apparent to me through dynamic exploration.

We use the notations from Figure 13: ABC is a given equilateral triangle with height h . Let (C) be a circle of radius r , centered at O , the center of the circumscribed circle of triangle ABC , which we take as the origin of the orthonormal coordinate system in which we will carry out our calculations. Let M be an arbitrary point on this circle, with coordinates expressed as a function of the angle a .

In this coordinate system, we will evaluate the sum of the squares of the distances from point M to the three sides of triangle ABC .

We begin by determining the cartesian equations of the three lines (CB) , (BA) and (AC) :

$$(CB): 3y + h = 0$$

$$(BA): 3\sqrt{3}.x - 3y + 2h = 0$$

$$(AC): 3\sqrt{3}.x + 3y - 2h = 0$$

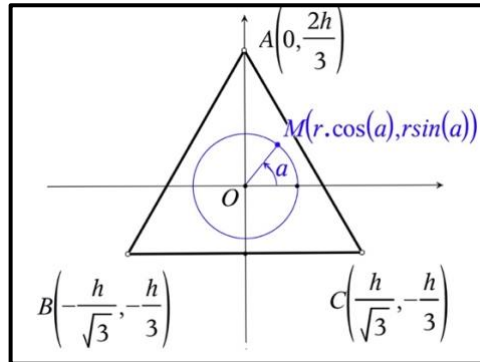


Figure 13: Sum of squares of distances

We then obtain the distances from M to each of the three lines:

$$\text{Distance from } M \text{ to } (CB) = d_1 = \frac{|3r\sin(a)+h|}{3}$$

$$\text{Distance from } M \text{ to } (BA) = d_2 = \frac{|3\sqrt{3}.r.\cos(a)-3r.\sin(a)+2h|}{6}$$

$$\text{Distance from } M \text{ to } (AC) = d_3 = \frac{|3\sqrt{3}.r.\cos(a)+3r.\sin(a)-2h|}{6}, \text{ therefore:}$$

$d_1^2 + d_2^2 + d_3^2 = \frac{9.r^2+2.h^2}{6}$ (These calculations were done by hand and confirmed by symbolic computation using TI-Nspire). Since the result is independent of a , it remains constant over the circle (C) .

Theorem 6: The sum of the squares of the distances from a point in the plane to the sides of an equilateral triangle remains constant when the point lies on a circle centered at the center of the triangle's circumcircle. Moreover, this sum is given by the formula $\frac{9.r^2+2.h^2}{6}$ where h is the height of the equilateral triangle and r is the radius of the circle considered.

7.2. Geometric properties of semi-regular polygons (convex or star-shaped)

7.2.1. Case when n is even (Figure 14): By construction, the sides are pairwise parallel and therefore symmetric with respect to the center o of the ellipse. Consequently, the vertices of such a semi-regular polygon are also symmetric with respect to o , since they are formed by the intersections of pairs of parallel tangent sides. This justifies the equality of the pairwise parallel sides for both types of semi-regular polygons, whether convex or star-shaped.

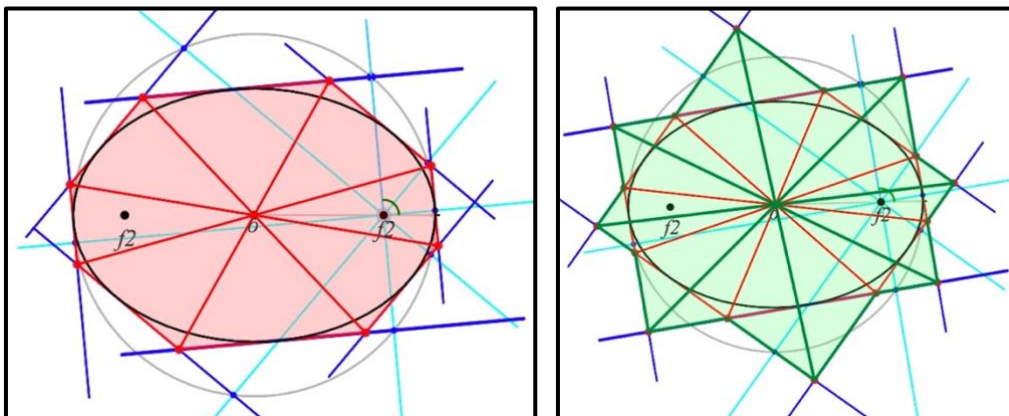


Figure 14: Some geometric properties of semi-regular polygons (convex or star-shaped)

7.2.2. Case when n is odd: I must acknowledge that at this stage of my research, I have not been able to identify a geometric invariant as interesting as in the even case, despite all my dynamic investigations. Nevertheless, in this case, my dynamic experiments led me to conjecture that when the area is maximal or minimal, these semi-regular polygons are isosceles. This means that they have an axis of symmetry, which is either the major or the minor axis of the ellipse, and that their sides symmetric with respect to this axis are equal. This conjecture is illustrated in the case of $n = 5$ in Figure 15.

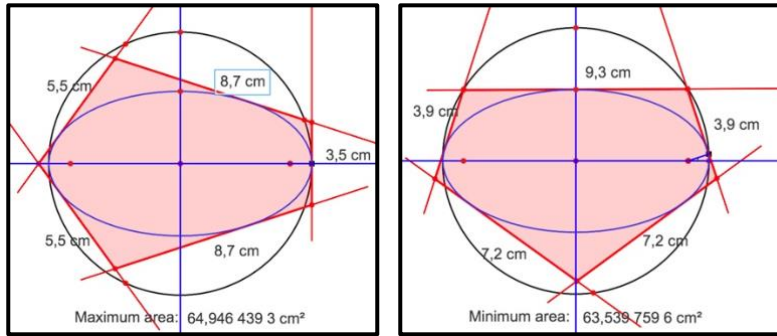


Figure 15: Some isosceles semi-regular pentagons

7.3. Further research directions: semi-irregular polygons

To generalize is, in my view, to take a step back from overly local investigations that lead nowhere or remain incomplete. That is why I came to realize that my constructions of semi-regular polygons circumscribed around a given ellipse could be extended to polygons that are not equiangular but still circumscribed around the same ellipse. These are the polygons I would call semi-irregular polygons.

Illustrated in Figure 16 is the construction algorithm for the case $n = 3$ (i.e., a triangle with given angles). The values of the first two angles, a and b , of the triangle are given and controlled by two sliders. $p1$ is a given point on the main circle of the given ellipse. The line perpendicular to $[f2p1]$ at $p1$ serves as the support for the first side of triangle ABC . $[f2p2]$ is the image of $[f2p1]$ under a rotation centered at $f2$ with angle $180^\circ - a$. The line perpendicular to $[f2p2]$ at $p2$ defines the second side of triangle ABC . $[f2p3]$ is the image of $[f2p2]$ under a rotation centered at $f2$ with angle $180^\circ - b$. The line perpendicular to $[f2p3]$ at $p3$ defines the third side of triangle ABC . Moving $p1$ along the main circle of the ellipse generates all triangles ABC with respective angles a , b , and $180^\circ - a - b$ that are circumscribed around the ellipse.

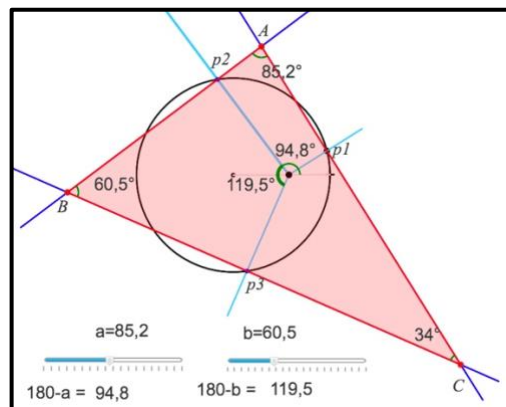


Figure 16: Irregular triangle circumscribed around a given ellipse

The construction algorithm for a convex n -sided polygon with given angles a_1, a_2, \dots, a_n , where $a_n = \pi - a_1 - a_2 - a_3 - \dots - a_{n-1}$, is similar to the one used for triangles, but it requires the specification of the first $n - 1$ angle values. One can already envision a wide range of investigations generalizing those carried out with semi-regular polygons, potentially leading to numerous results that extend our previously proven results or current conjectures awaiting proof.

8. Conclusion

The initial goal of the research presented here was to prove the conjectures put forward in my previous article concerning semi-regular polygons circumscribed around a given ellipse (i.e., equiangular polygons), and certain associated functions of relatively complex form. This research proved to be challenging, as for a long time, despite all my efforts, none of the avenues I explored led to any conclusive results. A breakthrough came with the proof of the case $n = 3$ by a student from Singapore, which enabled me to quickly resolve the case $n = 4$. However, the case $n = 5$ has remained unsolved to this day. I soon realized that our existing knowledge of circumscribed hexagons might offer valuable insight, and indeed, I was able to prove my conjectures for the case $n = 6$ through a clever detour involving star-shaped semi-regular polygons. Dynamic investigations demonstrated their full potential when they revealed the constancy of the areas of these star-shaped semi-regular polygons in the case $n = 6$. This proven result turned out to be the key to solving the $n = 6$ case. It is worth noting that the path to discovery is often paved with unexpected collateral results, such as the one concerning the sum of the squares of the distances from a point to the sides of an equilateral triangle, which can be viewed as a kind of generalization of Viviani's theorem on the sum of distances from a point to the sides of an equiangular polygon ([1]). Much work remains to be done to resolve the conjectures in the general case. While it may seem disappointing to have resolved only three specific cases, the work accomplished, along with the techniques developed, will undoubtedly facilitate progress toward other partial solutions or, perhaps, even a complete solution. The extension proposed at the end of this article to semi-irregular polygons circumscribed around a given ellipse demonstrates the power of dynamic geometry as a powerful experimental tool for research ([3]).

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