Pedal curves of conics:
a tale of cubics, sextics, octics and more

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Abstract

Constructions and exploration of plane algebraic curves has received a new push with the development of automated methods, whose algorithms are continuously improved and implemented in various software packages. This push revived the study of classical plane curves, together with the discovery of new interesting curves. Here, we use automated methods to explore pedal curves of conics. This exploration provides constructions of interesting geometric loci, given at first by parametric representations. Implicitization is then an important process which uses a strong algebraic machinery. Its output is the discovery that the pedal curves under study are sextics, octics and other curves of higher degree. We explore their irreducibility and their singular points (crunodes, cusps, etc.).

1 Introduction

Plane algebraic curves are a traditional topic, which drew continuous interest, from ancient times to nowadays. A vast knowledge is offered in books such as [33, 36] and websites [22]. Algorithms to work in polynomial rings (computation of Gröbner bases, elimination, etc.) led to both the revival of traditional topics and to the exploration of varieties in 2D and 3D spaces (and beyond) which have been somehow forgotten or at least left aside for a long time. Among these are envelopes of families of curves dependent on a parameter [**?**, 6], isoptic and bisoptic curves [16, 13, 7, 14, 15] and also constructions and exploration of curves of higher degree. Some of them appear as an outcome of a "pure geometric" question, we mean geometric loci [18, 19]. For example, we can mention hyperbolisms of curves with respect to a point and a line [8], which have been rarely studied. Other interesting non trivial constructions leading to high order curves are also shown in [35].

The exploration is based on the usage of software, originally called either Computer Algebra Systems (CAS) or Dynamic Geometry Software (DGS). Different software may have similar features, and the distinction between CAS and DGS became quite fuzzy, as they evolved to multi-purpose software. For example, a Computer Algebra System called Giac [26] is embedded into GeoGebra. Therefore we may not call GeoGebra a DGS anymore, but rather a Dynamic Mathematical Software (DMS). Anyway, the different possibilities offered by the different packages make a dialog between them very useful. Networking between them is of the utmost importance [17]. A pitfall is that transferring data from one package to another one must often be made "by hand", i.e. by copy-paste using the mouse. An automatic dialog between them is a goal mentioned a long time ago [31], but has not been achieved yet.

In the present work, we explore *pedal curves* of conics, namely parabolas and ellipses. As the reader will see later, these curves can be realized as geometric loci, as in Definition 1, but also as envelopes of some families of circles (see Section 2). As conics can themselves been defined as geometric loci, we have here a double such construction, a situation which can induce some problems with the DMS.

Definition 1 *Let be given a plane curve $\mathcal{C}$ and a point D. The* pedal *of $\mathcal{C}$ with respect to D (called the* pole*) is the geometric locus of the feet of the lines passing by D and perpendicular to the tangents to $\mathcal{C}$.*

We will generally denote that curve by $\mathcal{P}$.

In this work, we observe and explore curves of degree 3 (cubics), 6 (sextics), 8 (octics) and in some cases curves of degree 20. These will be reducible. A central issue is to factor polynomials in order to check whether the obtained pedal curves are irreducible or not. If the polynomial P is irreducible, the variety $V(P)$ may have components impossible to distinguish by algebraic means. Such an issue has already been discussed in [**?**]. It has been shown in [13, 14] how to make such a distinction using the DMS.

We look at the curve under study as an intriguing curve, in the spirit of [21]. The construction process provides a parametric representation, which is implicitized to obtain a polynomial equation. Different methods exist, based on resultants or on Gröbner bases [5], or more recently on the Gröbner cover [29]. In particular, this book displays numerous examples of determination of geometric loci, which is our concern here. Elimination is almost ubiquitous in the papers mentioned above, and algorithms for elimination are implemented in both kinds of software that we use:

- The Computer Algebra System Maple 2024, in particular for its strong *PolynomialIdeals* package. Maple has also an *algcurves* package, with a command for automatic implicitization, but we preferred not to use it,as it is valid in specific cases.
- GeoGebra Discovery (GD).[1], a package for automated commands, which works on the basis of the regular GeoGebra.[2]

GD provides accurate answers for the determination of a geometric locus (in several settings), based on symbolic algorithms. It may provide non only a plot, but under certain conditions, also a polynomial equation for the geometric locus. It is helpful to decide whether the determined locus is reducible or not, but sometimes a stronger algebraic tool is needed, therefore we use Maple. This will be illustrated in Section 3.

[1]Freely downloadable from `https://github.com/kovzol/geogebra-discovery` We recommend to check frequently for the newest release.

[2]Freely downloadable from `http://www.geogebra.org`.

2 Pedal curves of a parabola

In what follows, we work out specific examples, which are good representatives of a general situation. We could have worked with more general equations involving some parameters. The CAS can handle them, but then the output is heavier. For the reader's sake, we chose the examples so that the output is not too complicated, and the topology of the curves is clear enough on the plots. Working with parameters enable to construct animations with the DMS; the reader is invited to do it. Of course, what you see may not be what really exists and conversely, a situation which has been analyzed in [11]; the visualization is an incitement to develop a formal proof.

2.1 Exploration with GeoGebra Discovery

WLOG, we consider a parabola $\mathcal{C}$, with focus $F(0,1)$ and whose directrix has equation $y=-1$. This parabola has equation $x^2-4y=0$. Many commands require the construction to be purely geometric. Therefore, we make the following construction (the notations appear in the figures):

(i) Plot two points A and B to determine a line, which will be the directrix of the parabola, denoted by c.

(ii) Plot a point F not on the line AB, the focus of the parabola.

(iii) The command **Locus(<Point defining the locus>, <Moving Point>)** plots the parabola.

(iv) Plot a point E on the parabola, with the button *Point on Object* (a written command is also available).

(v) Determine the tangent to the parabola at E.

(vi) Plot any point D in the plane.

(vii) Determine the normal h through D to the tangent g to the parabola at E.

(viii) Determine the point of intersection H of this normal with g.

(ix) The command **Locus(*H*,*E*)** provides a plot of the pedal curve of the parabola with respect to the point E.

(x) The command **LocusEquation(*H*,*E*)** provides a symbolic equation for this curve.

Figure 1 shows screenshots of the exploration. The differential properties of the obtained curves can be proven using symbolic equations, which are derived from the algebraic work in next subsection. The obtained curves show 3 different topologies:

(i) If D is a point inside the parabola, the pedal curve is a smooth cubic, as shown in Figure 1(a).

(ii) Id D is a point on the parabola, then the pedal curve has a cusp at D; see Figure 1(b)

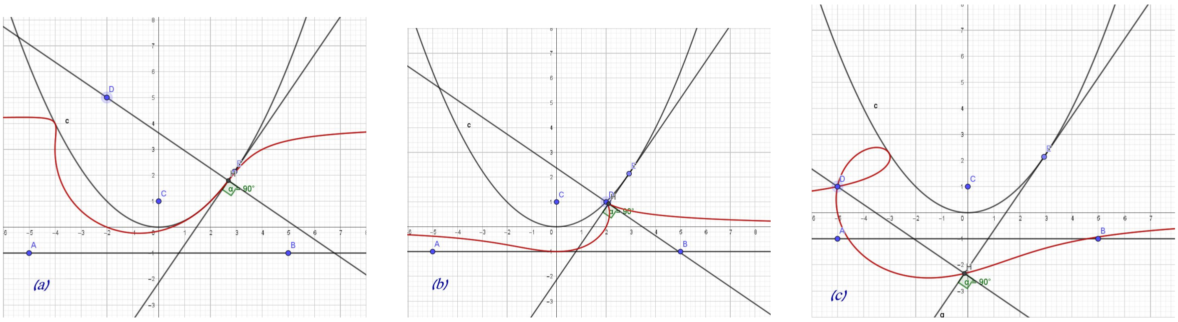

Figure 1: Pedal curve of a canonical parabola with respect to a point – the 3 cases

(iii) If D is a point out of the parabola, the pedal curve is a *crunodal cubic*, i.e. it has a crunode (a double point, where the curve intersects itself with two different tangents; see [33]), as shown in Figure 1(c).

These properties can be proven in general. We study examples in the next section. The polynomial equations in these examples are obtained via algebraic computations with Maple, and also been obtained with GeoGebra's **LocusEquation** command.

2.2 Algebraic work with Maple

WLOG, we consider a parabola $\mathcal{C}$, whose focus is the point $F(0,1)$ and the directrix has equation $y=-1$. The parabola $\mathcal{C}$ has equation $x^2-4y=0$. At a point $A_0=(x_0,y_0)$ on $\mathcal{C}$, the tangent T_{A_0} has equation $y=\frac{1}{2}xx_0-\frac{1}{4}x_0^2$. Denote $D=(x_D,y_D)$; then the perpendicular to T_{A_0} through D has equation $y-y_D=-\frac{2}{x_0}(x-x_D)$. The foot H of this perpendicular on the tangent T_{A_0} is thus the point verifying the following system of equations:

$$\begin{cases} y = \frac{1}{2}xx_0 - \frac{1}{4}x_0^2 \\ y - y_D = -\frac{2}{x_0}(x - x_D) \end{cases} \tag{1}$$

The solution is given by:

$$\begin{cases} x_H = \frac{x_0^3+4x_0y_D+8x_D}{2(x_0^2+4)} \\ y_H = \frac{x_0(x_0y_D-x_0+2x_D)}{x_0^2+4} \end{cases} \tag{2}$$

Maybe the general formula is not clear enough to be an incitement to distinguish three cases, but the dynamic experiment with the DMS provides it.

There exist infinitely many parametric representations for the given parabola $\mathcal{C}$; we take $(x,y)=(2t,t^2),\ t\in\mathbb{R}$. The following holds:

$$\begin{cases} x_H = \frac{8t^3+8ty_D+8x_D}{2(4t^2+4)} \\ y_H = \frac{2t(2ty_D-2t+2x_D)}{4t^2+4} \end{cases} \tag{3}$$

From now on, we work with the CAS (here Maple 2024). Define two polynomials and the ideal that they generate, using the commands:

```
parab:= implicitplot(4y=x^2,x=-8..8,y=-3..8,color=blue);
tgt := y = 1/2*x*x0 - 1/4*x0^2;
nrml := y - yD = -2*(x - xD)/x0;
H := solve({nrml, tgt}, {x, y})
H := subs(x0 = 2*t, subs(y0 = t^2, H));
P1 := x*denom(rhs(H[1])) - numer(rhs(H[1])):
P2 := y*denom(rhs(H[2])) - numer(rhs(H[2]));
J:=<P1,P2>;
```

We eliminate the parameter t using the *PolynomialIdeals* package and its **EliminationIdeal** command, and discover that the obtained ideal is generated by a unique polynomial of degree 3; then we show that the generator is irreducible. The code is as follows:

```
JE := EliminationIdeal(J, {x, xD, y, yD});
ped := Generators(JE)[1];
factors(ped);
```

Note that we used here the factor command. In order to be more confident that the answer is true, we could have performed absolute factorization, as we do later in this subsection (this is one of the reasons to factorize with the CAS, not with the CAS embedded in GD); see [18, 19]. The generator of the ideal (denoted by *ped* in the code) provides a symbolic equation for the desired pedal curve is:

$$\begin{aligned} G(x,y) = {} & (x_D^2 + y_D^2 - 2y_D + 1)\, y^3 + (x_D^2 + y_D^2 - 2y_D + 1)x^2 y \\ & - (x_D^2 y_D + y_D^3 + x_D^2 - 3y_D^2 + 3y_D + 1)\, x^2 - (x_D^3 + x_D y_D^2 - 2x_D y_D + x_D)\, xy \\ & - (2x_D^2 y_D + 2y_D^3 - 4y_D^2 + 2y_D)\, y^2 + (x_D^2 y_D^2 + y_D^4 - 2y_D^3 + y_D^2)\, y \\ & + (x_D^3 y_D + x_D y_D^3 - 2x_D^3 - 4x_D y_D^2 + 5x_D y_D - 2x_D)\, x \\ & + (x_D^4 + x_D^2 y_D^2 - 2x_D^2 y_D + x_D^2). \end{aligned} \tag{4}$$

The **factors** command (or the command or **evala(AFactor)** (for absolute factorization, as we will see in next Section), proves that this generator is irreducible. Actually, we could have conjectured this, as a reducible cubic polynomial can be factorized only with at least one linear factor, and the experimental work with GeoGebra did not reveal any line.

It is now easy to substitute specific coordinates for the point D; the resulting cubic curves are identical to those which are obtained by interactive work with GeoGebra.

1. Take $D(-2, 6)$; this point lies inside the parabola. The equation of the corresponding pedal curve is
$$x^2y - 5x^2 + 2xy - 8x + y^3 - 12y^2 + 36y + 4 = 0 \tag{5}$$
Denote by $F(x, y)$ the left-hand side of Equation (5). The systems of equations $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ has one real solution, namely the point D, and non real solutions. All of these are irrelevant, thus the curve has no singular point.

2. Take now $D(4, 4)$, a point on the parabola. The equation of the pedal curve is
$$x^2y - 3x^2 - 4xy + 8x + y^3 - 8y^2 + 16y + 16 = 0 \tag{6}$$
Denote by $F(x, y)$ the left-hand side of Equation (5). The systems of equations $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ has 3 real solutions. Two of them correspond to points which do not lie on the

curve. The 3rd one is the point D itself, which is a singular point. In order to show that it is a cusp, it is more convenient to use a parametric representation of the curve. We derive such a parametrization by a classical way: as D is a singular point, we consider a line l_D through D with slope t. Intersecting this line with the curve $\mathcal{P}$, we find

$$\begin{cases} x_1(t) = \frac{4t^3-4t^2-1}{t(t^2+1)} \\ y_1(t) = -\frac{4t-3}{t^2+1} \end{cases} \tag{7}$$

which is a rational parametrization of $\mathcal{P}$.

Solving the system of equations $(x_1(t) = y_1(t) = 4$, we find that the point D corresponds to $t = -1/2$. Now we differentiate $x_1(t)$ and $y_1(t)$ with respect to t, and obtain

$$\begin{cases} \frac{dx_1}{dt} = \frac{4t^4+8t^3-t^2+1}{t^2(t^2+1)^2} \\ \frac{dy_1}{dt} = \frac{4t^2-6t-4}{(t^2+1)^2} \end{cases} \tag{8}$$

Denote $\overrightarrow{V^{[1]}}(t) = \left(\frac{dx_1}{dt}, \frac{dy_1}{dt}\right)$. We have $\overrightarrow{V_1}(4) = \vec{0}$, which confirms that D is a singular point of the curve.

Now, denote $\overrightarrow{V^{[2]}}(t) = \frac{d}{dt}\overrightarrow{V^{[1]}}(t)$. We have:

$$\overrightarrow{V^{[2]}}(t) = \left(\frac{-8t^6-24t^5+12t^4+8t^3-6t^2-2}{t^3\left(t^2+1\right)^3}, \frac{-8t^3+18t^2+24t-6}{\left(t^2+1\right)^3}\right)$$

and by substitution obtain that $\overrightarrow{V^{[2]}}\left(-\frac{1}{2}\right) = \frac{32}{5}(2,-1)$. We differentiate once again:

$$\begin{aligned} \overrightarrow{V^{[3]}}(t) &= \frac{d}{dt}\overrightarrow{V^{[2]}}(t) \\ &= \left(\frac{24t^8+96t^7-84t^6-96t^5+54t^4+24t^2+6}{t^4\left(t^2+1\right)^4}, \frac{24\left(t^4-3t^3-6t^2+3t+1\right)}{\left(t^2+1\right)^4}\right) \end{aligned}$$

and obtain $\overrightarrow{V^{[3]}}\left(-\frac{1}{2}\right) = \frac{384}{25}(7,-1)$. The fact that the vectors $\overrightarrow{V^{[2]}}\left(-\frac{1}{2}\right)$ and $\overrightarrow{V^{[3]}}\left(-\frac{1}{2}\right)$ are linearly independent proves that D is a cusp of the pedal curve under study; see [1] for a general classification of singular points.

3. We take now $D(-6,2)$, a point out of the parabola. A proof that the pedal curve has a crunode (a point of self-intersection) may be not easy, but an efficient method is explained in [30] and has been used in [20]. The equation of the pedal curve is

$$x^2y + y^3 - x^2 + 6xy - 4y^2 + 4y + 36 = 0. \tag{9}$$

Using the intersection of a general line through D (whose equation is $y - 2 = t(x+6)$ with the pedal curve $\mathcal{P}$, we obtain the following parametrization for $\mathcal{P}$:

$$\begin{cases} x = -\frac{6t^3+2t^2+1}{t(t^2+1)} \\ y = \frac{6t+1}{t^2+1} \end{cases} \tag{10}$$

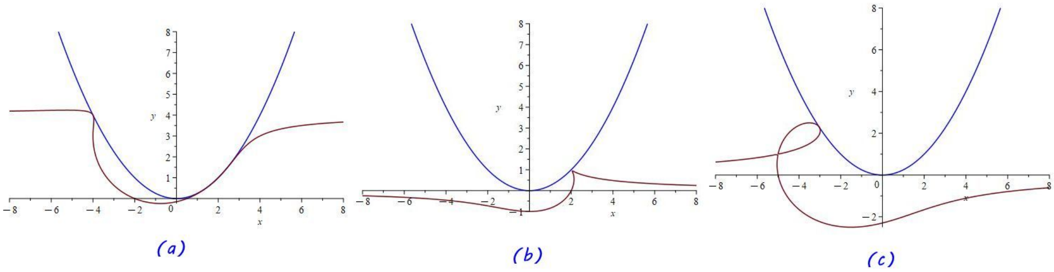

Figure 2: Pedal curve of a canonical parabola w.r.t a point, plotted with Maple

Compare the plots in Figure 2 with Figure 1. In order to prove that D is a point of self-intersection, we have to find two different values of the parameter corresponding to the same point, i.e., we need to solve the following system of equations:

$$\begin{cases} \frac{6t_1^3+2t_1^2+1}{t_1\left(t_1^2+1\right)} = \frac{6t_2^3+2t_2^2+1}{t_2\left(t_2^2+1\right)} \\ \frac{6t_1+1}{t_1^2+1} = \frac{6t_2+1}{t_2^2+1} \end{cases} \tag{11}$$

Maple's **solve** command yields the following output:

$$(t_1, t_2) = \left(\frac{3}{2} + \frac{\sqrt{7}}{2}, \frac{3}{2} - \frac{\sqrt{7}}{2}\right). \tag{12}$$

An easy substitution confirms that these values of the parameter correspond both to the point D.

The examples above lead to prove the following theorem:

Theorem 2 *Let $\mathcal{C}$ be a parabola and D a point in the plane. We denote by $\mathcal{P}$ the pedal curve of $\mathcal{C}$ with respect to D.*

1. *If D is a point internal to $\mathcal{C}$, then $\mathcal{P}$ has no singular point.*
2. *If D is on $\mathcal{C}$, then it belongs also to $\mathcal{P}$ and is a cusp of $\mathcal{P}$.*
3. *If D is a point external to $\mathcal{C}$, then $\mathcal{P}$ has a point of self-intersection at D, and it is the only singular point of $\mathcal{P}$.*

The proof follows exactly what has been shown in the examples, starting from Equation (4). The CAS performs the requested computations, but as the output is really heavy, we do not wish to present it here; the interested reader can run the Maple code.

3 Pedal curves of a canonical ellipse with respect to an external point

We illustrate a general question by means of an example, and display a software-based construction in two different ways.

3.1 The ellipse is given by a canonical equation

Let $\mathcal{C}$ be the ellipse whose equation is $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Now we follow the following steps:

(i) Plot a point E in the plane.

(ii) Plot the two foci $F1$ and $F2$, and a point A in the plane; for 4, we chose $F1(3,0)$, $F2(-3,0)$ and $A(5,0)$.

(iii) With GeoGebra command for the building of an ellipse, when the foci and a point are given, construct the ellipse.

(iv) Plot a point B on $\mathcal{C}$.

(v) Plot the tangent to $\mathcal{C}$ at B.

(vi) Plot the perpendicular to this tangent via E. The foot of this perpendicular is denoted by B.

(vii) Use the **Locus(C,B)** command to determine the pedal curve of $\mathcal{C}$ with respect to E.

(viii) Use the **LocusEquation(C,B)** command to obtain an equation for the pedal curve of $\mathcal{C}$ with respect to E.

The graphical output of the last commands are not identical; note that they have been obtained by two different algorithms. In order to have more insight, we switch to the CAS window in order to factorize the polynomial of order 10 which has been displayed. The output of a GeoGebra session is displayed in 3; the Construction Protocol displays the session according to the actual order of commands.

The **LocusEquation(C,B)** command yields a much more complicated plot and a polynomial equation $P(x,y) = 0$. Working in the CAS window, we show that $P(x,y)$ is reducible and has two non constant factors $P_1(x,y)$ and $P_2(x,y)$ given as follows:

$$\begin{aligned}
P_1(x,y) &= x^4 - 2x^3 + 2x^2y^2 - 14x^2y - 24x^2 - 2xy^2 + 14xy + 50x + y^4 \\
&\quad - 14y^3 + 33y^2 + 224y - 809 \\
P_2(x,y) &= 16x^6 - 64x^6 + 57x^6y^6 - 574x^4y + 1321x^4 - 146x^3y^2 + 1372x^3y \\
&\quad - 2514x^3 + 66x^2y^4 - 1274x^2y^3 + 8174x^2y^2 - 17038x^2y - 2728x^2 - 82xy^4 \\
&\quad + 1498xy^3 - 8788xy^2 + 15106xy + 7938x + 25y^6 - 700y^5 + 7366y^4 \\
&\quad - 34524y^3 + 60728y^2 + 1134y - 3969.
\end{aligned} \tag{13}$$

The equation $P_1(x,y) = 0$ determines a quartic, a Limaçon like curve $\mathcal{P}_1$ (the green curve in Figure 4(b)) and the equation $P_2(x,y) = 0$ determines a sextic $\mathcal{P}_2$ (in red). Actually, the first output is as displayed in Figure 4(a)), it has been corrected with the **Plot2D** command, available in GD only[3] .

What is left to do is to identify that only the quartic $\mathcal{P}_1$ answers the question. It seems that this can be done only by dragging the point B along the given ellipse, not by algebraic means.

[3]The **Plot2D** command uses symbolic algorithms, different from those used by the regular Plot command. It overcomes the impossibility for the regular command to plot a curve in a neighborhood of a singular point; see [4].

No.	Name	Description	Value
1	Point F1		F1 = (3, 0)
2	Point F2	Point on xAxis	F2 = (-3, 0)
3	Point A	Point on xAxis	A = (5, 0)
4	Ellipse c	Ellipse with foci F2, F1 passing through A	c: 16x² + 25y² = 400
5	Point B	Point on c	B = (1.99, 3.67)
6	Line f	Tangent to c through B	f: 509.31x + 1467.89y = 6400
7	Point E		E = (1, 7)
8	Line g	Line through E perpendicular to f	g: -1467.89x + 509.31y = 20...
9	Point C	Intersection of f and g	C = (0.07, 4.33)
10	Angle α	Angle between f, g	α = 90°
11	Locus loc1	Locus(C, B)	loc1 = Locus(C, B)
12	Implicit Curve eq1	LocusEquation(C, B)	eq1: 16x¹⁰ - 96x⁹ + 89x⁸ y² - ...
13	CAS Cell $1	LeftSide(eq1) - RightSide(eq1)	16x¹⁰ + 25y¹⁰ + 116x² y⁸ + 2...
14	CAS Cell $2	Factor($1)	(x⁴ - 2x³ + 2x² y² - 14x² y - 24...
15	CAS Cell $3	Element($2,1)	x⁴ - 2x³ + 2x² y² - 14x² y - 24x...
16	CAS Cell $4	Element($2,2)	16x⁶ - 64x⁵ + 57x⁴ y² - 574x⁴...
17	Implicit Curve eq2	eq2	eq2: x⁴ + y⁴ + 2x² y² - 2x³ - 1...
18	Implicit Curve eq3	eq3	eq3: 16x⁶ + 25y⁶ + 66x² y⁴ +...

Figure 3: Construction Protocol of the pedal curve

3.2 Algebraic work with Maple

Maple's **evala(AFactor)** command provides the same factorization as GD. What is left to do is to identify whether the curve $\mathcal{P}_1$ is a Pascal Limaçn as conjectured, or not. In the previous subsection, we worked on purpose with a point E not on one of the symmetry axes of the ellipse. After all, we worked already with a canonical ellipse. If we had chosen a point E on the $y-$axis, the pedal curve would have been symmetric about the $y-$axis. For example, if $E(0,6)$, the pedal curve has the following quartic equation:

$$P_1(x,y) = x^4 + 2x^2y^2 - 12x^2y - 25x^2 + y^4 - 12y^3 + 20y^2 + 192y - 576 \tag{14}$$

A canonical equation of a Pascal Limaçn is given by

$$L(x,y) = (x^2 + y^2 + a\ e\ y)^2 - a^2(x^2+y^2). \tag{15}$$

Let us try to find a change of coordinates of the form

$$\begin{cases} x = p_1\ X + q_1\ Y + r_1 \\ y = p_2\ X + q_2\ Y + r_2 \end{cases} \tag{16}$$

The real parameters in (16) must be determined; we do not need terms in xy as we conjecture that no rotation has to be involved. By substitution into (15), we obtain:

$$\begin{aligned} L(x,y) = &\left[(p_1\ X + q_1\ Y + r_1)^2 + (p_2\ X + q_2\ Y + r_2)^2 + a\ e\ (p_2\ X + q_2\ Y + r_2)\right]^2 \\ &- a^2((p_1\ X + q_1\ Y + r_1)^2 + (p_2\ X + q_2\ Y + r_2)^2). \end{aligned} \tag{17}$$

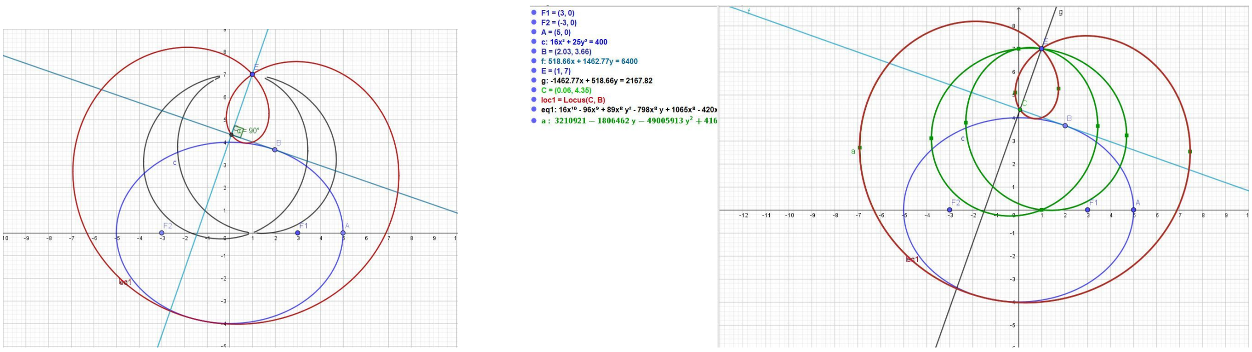

Figure 4: Pedal curve of a canonical ellipse with respect to an external point – total geometric construction

We expand the polynomial $L(x, y)$ and

after identification of its coefficients with the coefficients in between Equations (14), we have to solve the following system of equations:

$$\left\{\begin{aligned} p_1^4 + 2p_1^2p_2^2 + p_2^4 &= 1\\ 6p_1^2q_1^2 + 2p_1^2q_2^2 + 8p_1p_2q_1q_2 + 2p_2^2q_1^2 + 6p_2^2q_2^2 &= 2\\ 2aep_1^2q_2 + 4aep_1p_2q_1 + 6aep_2^2q_2 + 12p_1^2q_1r_1 + 4p_1^2q_2r_2 + 8p_1p_2q_1r_2 &\\ +8p_1p_2q_2r_1 + 4p_2^2q_1r_1 + 12p_2^2q_2r_2 &= -12\\ a^2e^2p_2^2 + 2aep_1^2r_2 + 4aep_1p_2r_1 + 6aep_2^2r_2 - a^2p_1^2 - a^2p_2^2 + 6p_1^2r_1^2 + 2p_1^2r_2^2 &\\ +8p_1p_2r_1r_2 + 2p_2^2r_1^2 + 6p_2^2r_2^2 &= -25\\ q_1^4 + 2q_1^2q_2^2 + q_2^4 &= 0\\ 6aeq_2^2r_2 + 8q_1q_2r_1r_2 + 4aeq_1q_2r_1 + 2aeq_1^2r_2 + 6q_1^2r_1^2 + 2q_1^2r_2^2 &\\ +a^2e^2q_2^2 - a^2q_2^2 + 2q_2^2r_1^2 + 6q_2^2r_2^2 &= 20\\ -2a^2q_1r_1 + 2a^2e^2q_2r_2 + 2aeq_2r_1^2 + 6aeq_2r_2^2 + 4q_1r_1r_2^2 + 4aeq_1r_1r_2 + 4q_1r_1^3 &\\ -2a^2q_2r_2 + 4q_2r_1^2r_2 + 4q_2r_2^3 &= 192\\ -a^2r_1^2 - a^2r_2^2 + r_2^4 + 2aer_2^3 + 2aer_1^2r_2 + 2r_1^2r_2^2 + r_1^4 + a^2e^2r_2^2 &= 576 \end{aligned}\right. . \qquad (18)$$

Note that an expanded form of $L(x, y)$ is a huge expression, but to determine the coefficients we do not need the entire expression.

As the algebraic work is heavy, the explorer may try to have a geometric experimentation, illustrated in 5, performed interactively using GD. For this exploration, we chose a point E on the $y-$axis, which ensures that the entire situation is symmetric about the $y-$axis, and enables to use a “canonical” equation for the limaqn available in the literature. Plot the limaçon given by the equation[4]

$$(x^2 + y^2 + e\ a\ y)^2 - a^2\ (x^2 + y^2) = 0. \qquad (19)$$

As the letter e has a specific meaning in GD, we used b instead of it in GD commands . Now denote $\vec{u} = \overrightarrow{OE}$ and translate the plotted limaçon by this vector. Now change the values of the parameters with the sliders in order to try to have the translated limaqn (in black) coalesce with the pedal curve obtained previously (in red). Of course, not obtaining a true coalescence is not a complete proof, but may be enough for the explorer. Such a situation has been already

[4]This equation has been adapted from the one given in `https://mathcurve.com/courbes2d.gb/limacon/limacon.shtml`.

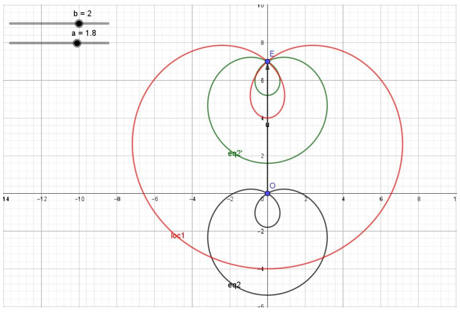

Figure 5: Checking whether we obtained a Limaçon or not

met in [8], where curves “looking like but different from” an Agnesi witch have been obtained. There, observing the equations was an important part of the conviction (not of the proof).

4 Pedal curves of a canonical ellipse with respect to the origin

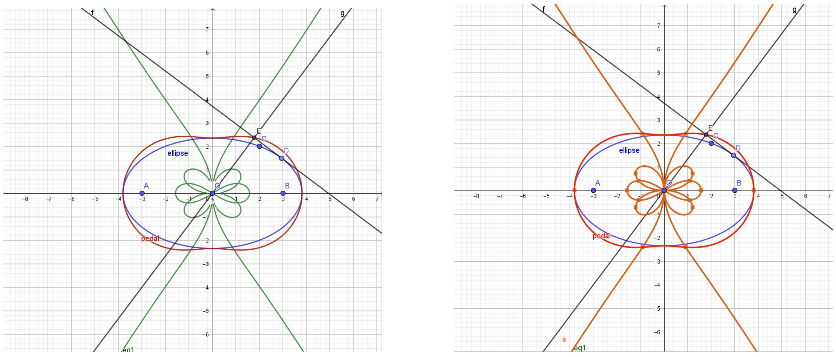

Figure 6: Pedal curve of a canonical ellipse

The **LocusEquation** command of GD provides the following polynomial, defining the curve displayed in Figure 6(a):

$$\begin{aligned} f(x,y) = \quad & 4x^{20} + 41x^{18}y^2 + 180x^{16}y^4 + 444x^{14}y^6 + 672x^{12}y^8 + 630x^{10}y^{10} + 336x^8y^{12} \\ & + 60x^6y^{14} - 36x^4y^{16} - 23x^2y^{18} - 4y^{20} - 68x^{18} - 566x^{16}y^2 - 1738x^{14}y^4 \\ & - 2354x^{12}y^6 - 754x^{10}y^8 + 1762x^8y^{10} + 2354x^6y^{12} + 1162x^4y^{14} + 206x^2y^{16} \\ & - 4y^{18} + 144x^{16}729x^{14}y^2 - 2880x^{12}y^4 - 15777x^{10}y^6 - 24624x^8y^8 - 15777x^6y^{10} \\ & - 2880x^4y^{12} + 729x^2y^{14} + 144y^{16} - 324x^{12}y^2 + 17982x^{10}y^4 + 24138x^8y^6 \\ & - 12474x^6y^8 - 23814x^4y^{10} - 5508x^2y^{12} - 26244x^8y^4 + 59049x^6y^6 + 26244x^4y^8 \end{aligned} \tag{20}$$

Switching to the CAS window of GD, we apply the **factor** command. The output of the **factor** command is a product of two polynomials, one of degree 12 and one of degree 8, namely:

$$\begin{aligned}P_1(x,y) =&4x^{12}+25x^{10}y^2+56x^8y^4+54x^6y^6+16x^4y^8-7x^2y^{10}\\&-4y^{12}-9x^8y^2+135x^6y^4+297x^4y^6+153x^2y^8-729x^4y^4\\P_2(x,y) =&x^8+4x^6y^2+6x^4y^4+4x^2y^6+y^8-17x^6\\&-33x^4y^2-15x^2y^4+y^6+36x^4-81x^2y^2-36y^4\end{aligned}$$

This means already that the obtained curve is reducible, but as we will see this decomposition is not the ultimate one.

We switch now to Maple; its **factor** command provides the same answer as Giac. Separate plots for these polynomials are displayed in 7. But we could go further, with a more advanced

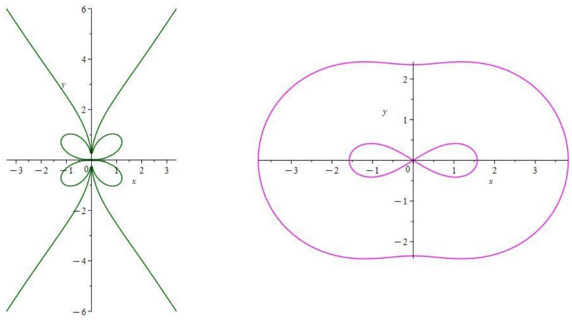

Figure 7: Separate plots with Maple

Maple command for factorization (as in [18]).

```
evala(AFactor(f));
F := allvalues(%);
```

The first command provides an answer involving the place holder *RootOf*, which can be resolved using the **allvalues** command. The output provides the following polynomial, whose components are of multiplicity 2.

$$\begin{aligned}F =4&\left[x^4+\left(2y^2-\frac{\sqrt{145}}{2}-\frac{17}{2}\right)x^2+y^4+\left(-\frac{\sqrt{145}}{2}+\frac{1}{2}\right)y^2\right]^2\\&\left[x^4+\left(2y^2+\frac{\sqrt{145}}{2}-\frac{17}{2}\right)x^2+y^4+\left(\frac{\sqrt{145}}{2}+\frac{1}{2}\right)y^2\right]^2\\&\left[x^6+\left(-\frac{\sqrt{145}}{8}+\frac{25}{8}\right)y^2x^4+\left(\left(-\frac{\sqrt{145}}{4}+\frac{13}{4}\right)y^4+\left(\frac{9\sqrt{145}}{8}-\frac{9}{8}\right)y^2\right)x^2+\left(-\frac{\sqrt{145}}{8}+\frac{9}{8}\right)y^6\right]^2\\&\left[x^6+\left(\frac{\sqrt{145}}{8}+\frac{25}{8}\right)y^2x^4+\left(\left(\frac{\sqrt{145}}{4}+\frac{13}{4}\right)y^4+\left(-\frac{9\sqrt{145}}{8}-\frac{9}{8}\right)y^2\right)x^2+\left(\frac{\sqrt{145}}{8}+\frac{9}{8}\right)y^6\right]^2\end{aligned}$$

An interactive exploration may provide conviction which part of the plot is relevant to the actual question. Once again, irrelevant components appear for topological reasons, as in [12].

5 Discussion

5.1 About the geometry

Pedal curves are a classical topic in plane geometry. Many catalogues are devoted to plane curves [23, 24, 36, 33] and may mention when some specific curves are obtained as pedal curves. We wish to emphasize the fact that for a given curve, the choice of the point has a strong influence on the shape (and the topology) of the pedal curve. In Section 2.2 we considered pedals of a parabola. Using the mouse in the GeoGebra-Discovery session, the user can mimic continuous changes of the pole D and discover that this is not translated into conservation of the curve topology.

Pedal curves are only a particular case in the vast topic of geometric loci; we refer here to a small sample of papers such as to [3, 19]. In this work, we could shed a new light on the topic. One of the main products is the discovery and study of algebraic curves of higher degree than what is traditionally explored. In the recent past, such studies yielded new contributions to the study of quartics (such as Cassini ovals in [13]) and the study of sextics [20] and octics [18]. All the curves of degree 2,3 and 4 are well known. The higher the degree, the larger the set of curves of this degree.

5.2 About the software

The software in use depends strongly on the needed affordances, and also on which software is available, as a consequence of the policy of the institution and its decisions about funding - an institution may not afford a commercial license, in which situation researchers and students use only free software. In our work we use both a commercial CAS and a free DMS, namely GD. We do not know of another software which provides the same kind of automated commands sd GD.

In this work, we present pedal curves of ellipses with respect to an external point and with respect to the origin. Trying to determine a pedal curve of an ellipse with respect to a point on the ellipse, the **Locus** command produced a graphical representation looking like a limaȩn, but the command **LocusEquation** was unable to provide an equation. Several trials have been performed, using different ways to define the ellipse, but none provided an equation. This could have been obtained using the algebraic method presented above.

Now we wish to emphasize the role of GD's **Plot2D** command: it happens often that software has hard time to plot a curve close to a singular point. Generally, a hole appears instead of an arc around the point. Using an algorithm different from the one implemented in the normal **Plot** command, **Plot2D** is able to give an accurate plot of the curve, without holes, and the singular point is emphasized. This command is efficient here, as it is in [20].

5.3 About education

The author had an opportunity to teach a course for in-service mathematics teachers learning towards an advanced degree. The group counted 12 teachers whose CAS and DMS literacy was almost nonexistent. Prior to the formal meetings they learnt the basic GeoGebra features using videos prepared by the author, then they learnt simultaneously new topics in mathematics and the corresponding usage of the software, sometimes a little beyond. About half of the topics were related to plane algebraic curves as in [**?**, 8]. By that way, they internalized the fact that the newly acquired technological skills are an integral part of the new mathematical knowledge, totally in line with Artigue [2]. Some of the topics had also byproducts with art creation (such as in [10] for periodic curves and [9] for string art and loci, presented at a previous ATCM conference), maybe the first encounter of these students with a STEAM approach. There was no exam, but a series of tasks to fulfill with this new knowledge and mathematical thinking; they provide very interesting outcomes. The participants' feedback was enthusiastic all along the course, as they discovered that mathematics is a very dynamic field. A good surprise for the author was the very positive feedback that they sent to their department head, saying that this was the most significant course in the whole degree.

Generally, working in a technology-rich environment enables to follow a non-traditional sequence, namely exploration-conjecture-proof. Slowly, this becomes a main stream way of working in geometry. Sendra and Winkler's book [34] is devoted to parametric curves with a Computer Algebra approach. Automated methods are under constant development [32], as we saw with GeoGebra-Discovery, and other packages are available. Once again here, as in [17, 19] we face a situation where a single package of software does not provides the whole picture that the user is trying to build. in the present state-of-the-art, the data transfer from one software to the other is made "by hand" (copy-paste and then fine tuning to adapt what is written to the specific syntax of the receiving software). Following Roanes-Lozano's wish [31], we hope that this dialog will become more and more automatic.

As a whole, the ecosystem provided by the automated methods implemented in GD (see [32, 27] is a environment enabling to adopt a modern approach to classical topics in geometry. IMHO, for the new generations of students and teachers, we have an opportunity to develop more motivation and engagement among students. This is a large issue beyond the scope of ten present paper.

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