Understanding Geometric Pattern and its Geometry Part 13 – Tetradecagonal geometric patterns(1)

Mirosław Majewski

mirek.majewski@gmail.com
New York Institute of Technology, School of Arts & Sciences,
Abu Dhabi campus, UAE

Abstract: *Depending on the region, we may deal with geometric patterns constructed in geometries obtained from different regular polygons. Thus, we have hexagonal patterns created using the geometry of a regular hexagon, octagonal patterns created using the geometry of a regular octagon, decagonal patterns, and patterns in mixed symmetries – dodecagonal and hexagonal, or nonagonal. For example, in Istanbul and Edirne, we find a large group of Ottoman decagonal designs. In Mughal India, we deal with simple octagonal or hexagonal designs. At the same time, in Cairo, we will find patterns created on regular dodecagon as a main geometric shape.*

Surprisingly, we find very few patterns based on the geometry of the regular tetradecagon (14 edges) and regular heptagon (7 edges). These patterns are found in several locations, ranging from Damascus to the Maghreb.

This paper aims to investigate how existing tetradecagonal patterns were created, construct tessellations used to develop these patterns and explore how traditional methods for octagonal and decagonal patterns can be extended to tetradecagonal designs.

Introduction

When working with geometric patterns, we utilize geometries based on various regular polygons. Thus, we often say that one pattern is decagonal, the other one is octagonal, and another one was created in a hexagonal geometry. At the time of writing this text, we are aware of two important statistics regarding patterns formed using different geometries. One of them is a statistic provided by Brian Wichmann (see [8]). This statistic uses graphs but not concrete numbers. However, we can find a clear statement. Some types of geometric patterns were very popular in specific regions, e.g., octagonal in Morocco and Spain, hexagonal in Syria, decagonal in Turkey, and dodecagonal in Egypt etc. In his book, a diagram is presented showing the distribution of patterns by country and region but not the actual numbers or percentages.

Another statistic related to pattern frequency can be found in Bulut's book (see [1]). Bulut's task was simpler as he concentrated on the Anatolia region, where he explored practically all existing mosques and madrasahs. Thus, we get proper numbers from his book. Therefore, we get the following table (we show its modified version here).

[1] All facts described in this paper were invented/discovered in spring 2025. None of them was published anywhere before. Thus this is first publication on geometric patterns in tetradecagonal geometry.

Polygon symmetry	10	6	7	8	9	12 ≤
Pattern description	decagonal	hexagonal	heptagonal	octagonal	nonagonal	dodecagonal and more
Count	109	233	4	241	2	14
Percentage	14%	30%	≤ 1%	31.6%	≤ 1%	2.32%

From this table, we can get quite a good idea of what type of patterns we can find in Anatolia. However, this statistic could look completely different if we examine patterns in Ottoman-ruled regions, such as Istanbul and Edirne.

One can easily notice that we do not have any information about tetradecagonal patterns in both statistics. It appears that tetradecagonal patterns were not particularly popular. In the real world, we can find very few examples of them.

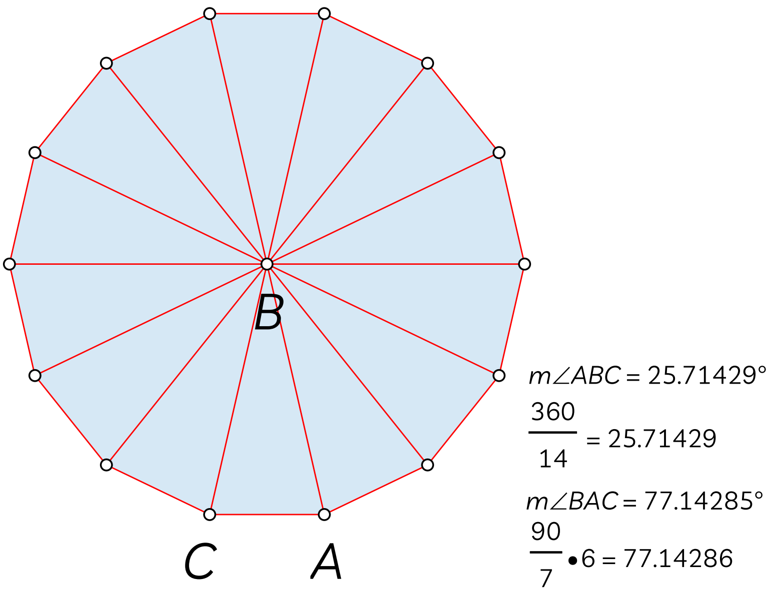
In this paper, we will look at tetradecagonal geometry – the geometry of polygons and tessellations derived from the regular tetradecagon, i.e., a regular polygon with fourteen equal edges and angles. It is worth noticing that we have a situation similar to the one with the regular decagon. In both cases, we have two even numbers: 10 and 14, and two pairs of divisors, 5 and 2, and 7 and 2. No more divisors exist for 10 and 14. All divisors of these two numbers are prime numbers. Therefore, one could expect it to be relatively easy to construct tetradecagonal tessellations and patterns using methods derived from decagonal and dodecagonal geometry.

But there is something that makes both types of regular polygons completely different. Regular decagons and regular pentagons are constructable with straightedge and compasses. These two polygons were well-known to ancient mathematicians. At the same time, a regular heptagon and a regular tetradecagon are not constructible using the tools available to ancient mathematicians. Some authors mention the lost manuscript by Archimedes, *On the Heptagon in a Circle.* There were some attempts to construct a regular heptagon with a marked straightedge. On Wikipedia (see [9]), we can find a modern construction of a regular heptagon using neusis construction. Therefore, ancient mathematicians had no good way to deal with regular tetradecagons. This can be the main reason why we do not have many tetradecagonal and heptagonal geometric patterns.

In this paper, we use Geometer's Sketchpad to develop all geometric constructions. All angles are calculated by Sketchpad, and their values are reduced to five decimal places. We could use another computer tool to produce more accurate values, but this level of accuracy is sufficient for our purposes.

Properties of the regular tetradecagon

A regular tetradecagon is a regular polygon with fourteen equal edges and fourteen equal angles. If we connect its center with each vertex, then we will get fourteen identical triangles.



Angles in regular tetradecagon

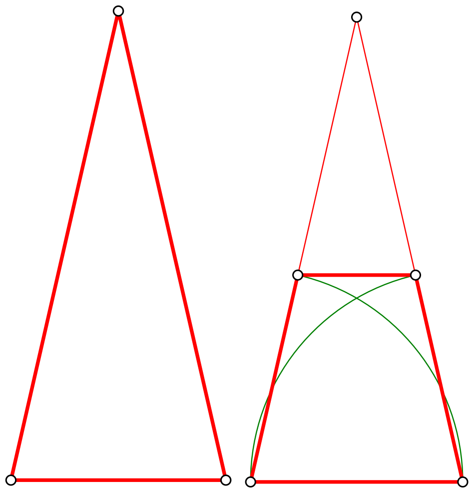
The angles shown here were obtained by using Geometer's Sketchpad. Their values are irrational numbers; we can obtain them using simple trigonometry calculations.

In all our examples, the essential angle unit will be

$$\theta = \frac{90}{7} = 12.85714$$

All other angles will be equal to

$n \cdot \theta$, where n is a natural number.



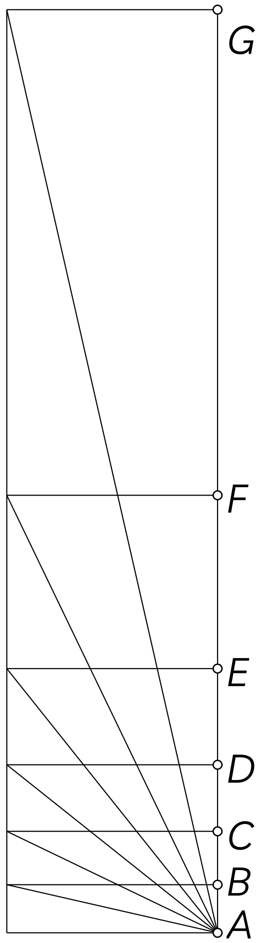
Polygons

Here, we identify the first two polygons in tetradecagonal geometry:

1. Long triangle 1/14 of the regular tetradecagon
2. A tall trapezium obtained from the long triangle by cutting out its top part. It has three equal edges.

Contours and networks

In most examples discussed in this paper, we will deal with patterns filling rectangular contours. These contours will be obtained by using multiple values of the angle θ. The drawing below shows some of them.



Selected rectangular contours in tetradecagonal geometry

Horizontal segments labeled from *A* to *G* show simple contours with the right edge: *AB*, *AC*, *AD*, *AE*, *AF*, and *AG*. Thus, we have six simple contours in tetradecagonal geometry.

Contour *AB* is very flat, and *AG* is very narrow. Both contours have the same ratio $L/s = 4.38129 \ldots$, where L is the length of the long edge, and s is the length of the short edge.

Similarly, contours *AC* and *AF* have the same ratio $L/s = 2.07652$.

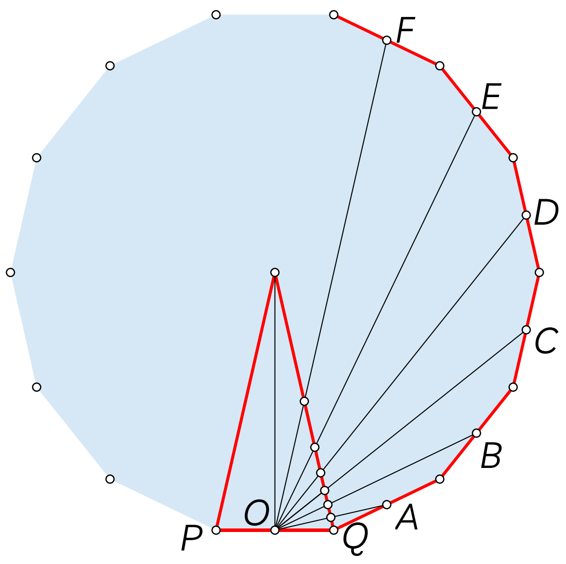
Finally, the ratio L/s for contours *AD* and *AE* equals 1.25396.

These calculations demonstrate that contour *AB* is a 90-degree rotated version of *AG*, *AC* is a rotated version of *AF*, and *AD* is a rotated version of *AE*. Therefore, we only have three simple contours in the tetradecagonal geometry. Remember that in decagonal geometry, we only had two simple contours. Thus, our options for creating tessellations and geometric patterns in tetradecagonal geometry are slightly larger. In future constructions, we will use notation *C1* for the contour with right edge *AB*, *C2* for the contour with edge *AC*, and *C3* for the contour with right edge *AD*. Although the other contours are rotated copies of *C1*, *C2,* and *C3*, we can use the notation *C4*, *C5,* and *C6* if this is convenient.

Discussion of contours in tetradecagonal geometry cannot be limited to simple contours. There are infinite possibilities for creating complex contours obtained by combining two or more simple contours and sometimes subtracting them. From the pattern design point of view, the most useful will be contours *C2* and *C3*.

Tetradecagonal stars and rosettes

Fundamental elements of geometric patterns are stars and rosettes. We will examine ways to create them using a regular tetradecagon. This discussion can be limited to a single long triangle from a regular tetradecagon. Any tetradecagonal star or rosette is determined by one single line or segment passing through the center ***O*** of the bottom edge of the triangle.



Chords in a tetradecagon

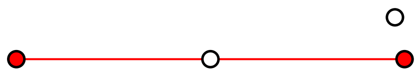
The drawing shows segments *OA*, *OB*, *OC*, *OD*, *OE*, and *OF*. Segments *OA* and *OB* can be successfully used to create rosettes. The remaining segments may not be convenient to produce rosettes. However, we can use them to construct tetradecagonal stars or more complex objects. In the next drawings, we show what can be done for a single long triangle. A complete rosette or star will be constructed by inserting a pattern into each long triangle of the given regular tetradecagon.

Case 1: segment OA

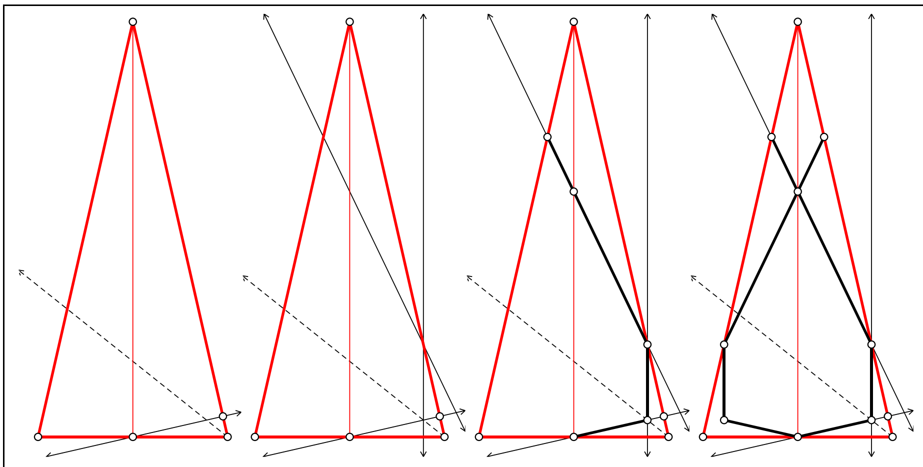
Segment OA determines the smallest angle used in tetradecagonal geometry. Its value is equal to

$$\theta = 90/7 = 12.85714°$$

This angle produces one of the nicest rosette petals seen in geometric patterns. To simplify our constructions, we can create a GSP tool with the two bottom points and the top point of this angle. Something like this:

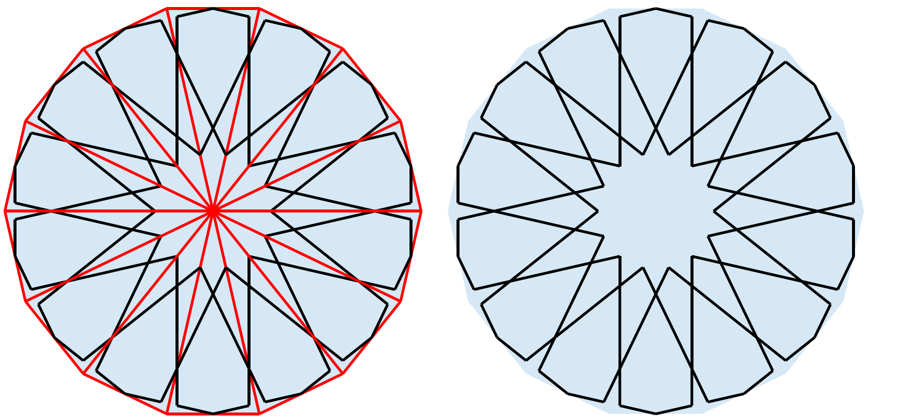


Here, red points are the initial points for the tool. We can skip the red segment, and the red color for points can be ignored. I use it here for reference only. In further cases, we will use similar custom tools for other angles.



Rosette RA

This construction is similar to all rosette constructions presented in my previous papers. The dashed line is the reflection line. The right edge of the triangle and the vertical segment were also used as mirrors.



RA

Left – rosette created by inserting the petal into each triangle of the regular tetradecagon.

Right – the same rosette without outline segments.

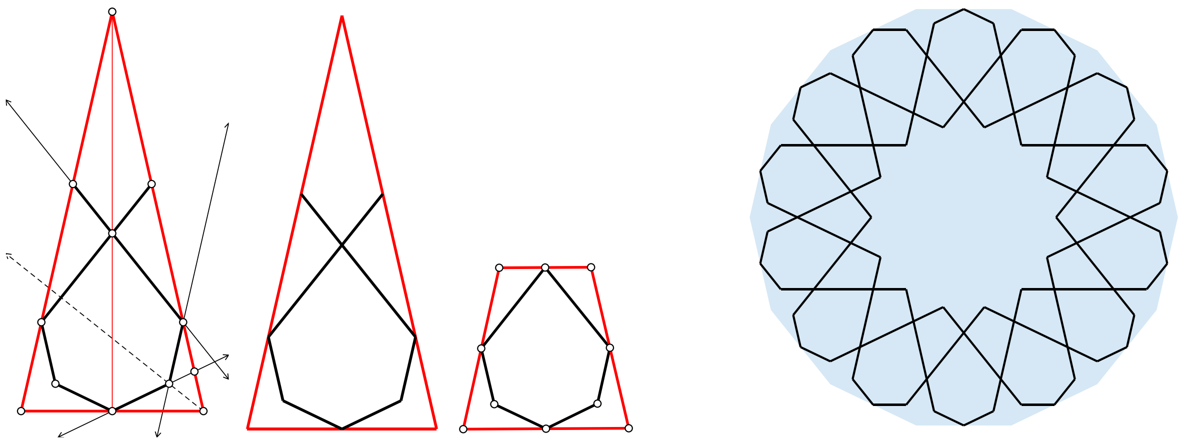
Such a large rosette can be a focal element of the pattern, but usually, the most interesting parts are in the space between rosettes.

Case 2: segment OB

Segment OB determines the second smallest angle used in tetradecagonal geometry. Its value is equal to

$$2\theta = 2 \cdot 90/7 = 25.71429°$$

In the same way, as in Case 1, we can create a custom tool in GSP for this angle and proceed with petal construction without any changes. This way, we will get the petal shown in the next illustration.

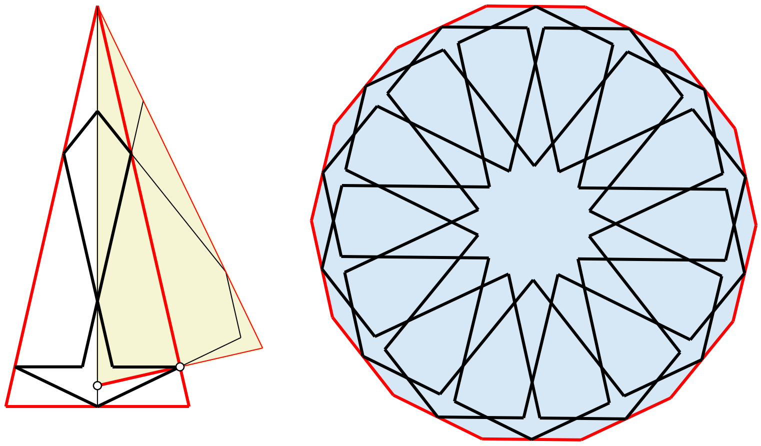


RB

Construction of the rosette fragment for the angle 2θ.

A larger angle significantly alters the shape of the petal, and the rosette appears quite different. Note also that this petal can fit into the tetradecagonal trapezium with three equal edges.

We may not like the shape of the petals in this rosette. Thus, there is a way to push the rosette towards the center of the tetradecagon. Here, we show how it can be done.



RA1

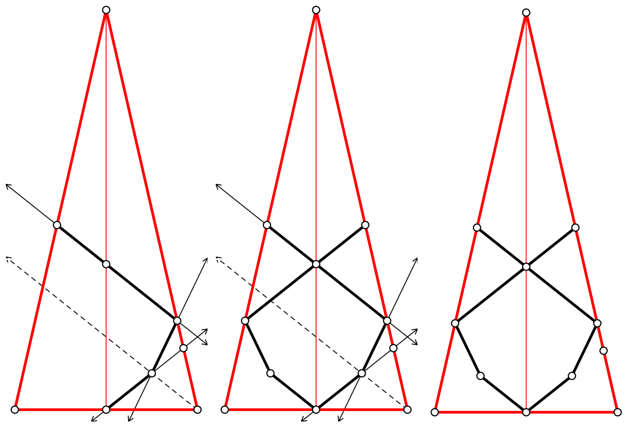
Rosette moved toward the center

The left drawing shows a new long triangle created on the side of the previous triangle. This way, we could create a new rosette, smaller than the previous one and with different shapes of petals.

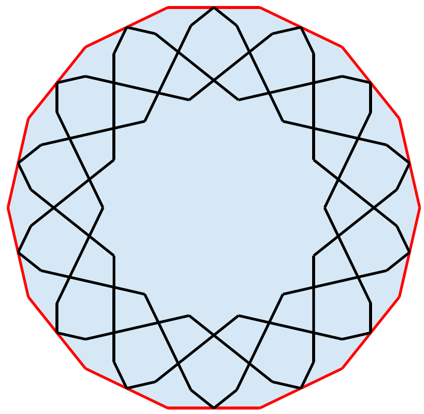
Case 3: segment OC

Now, we can proceed to the next angle

$$3\theta = 3 \cdot 90/7 = 38.57143°.$$



Three stages of creating part of the rosette inside the long triangle

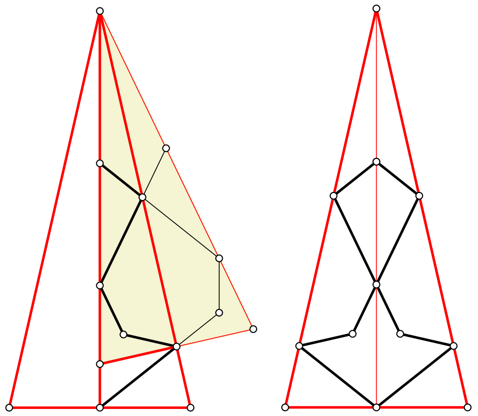


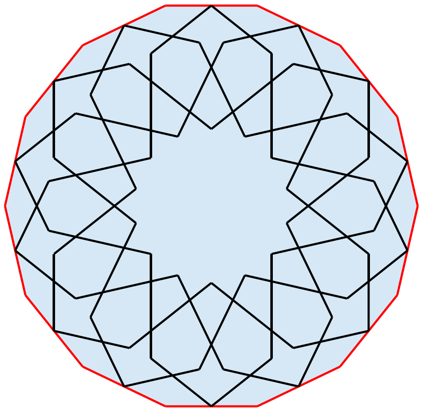
RC

A rosette obtained using the angle

$$3\theta = 3 \cdot 90/7 = 38.57143°$$

The petals in the obtained rosette resemble star elements, and the overall construction leaves a large empty area in the middle. This is the place where we can add some extra elements, say another rosette.





Solution for the large space inside the rosette in case 3

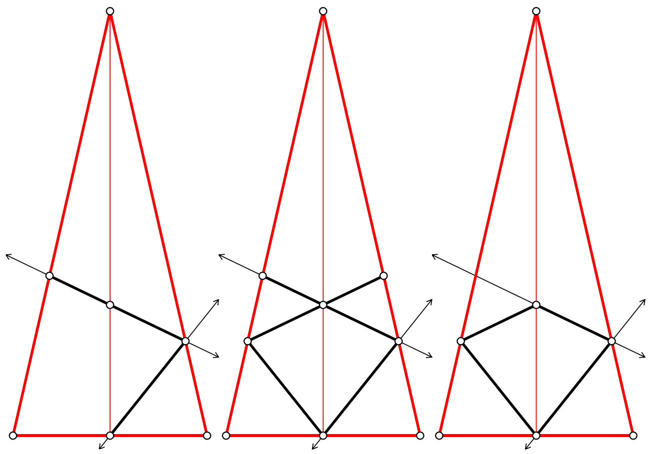
This way, we get a more natural-looking rosette inside the empty area of the regular tetradecagon. It is the same rosette petal that we created in case 2. But we also get some extra shapes below it.

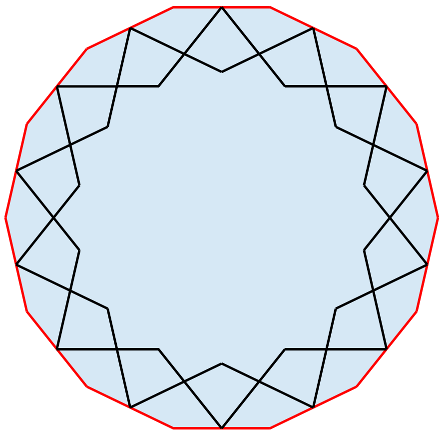
A sample tetradecagonal rosette using the extension to the angle in case 3. In many cases, this rosette will fit much better in a design than the previous one.

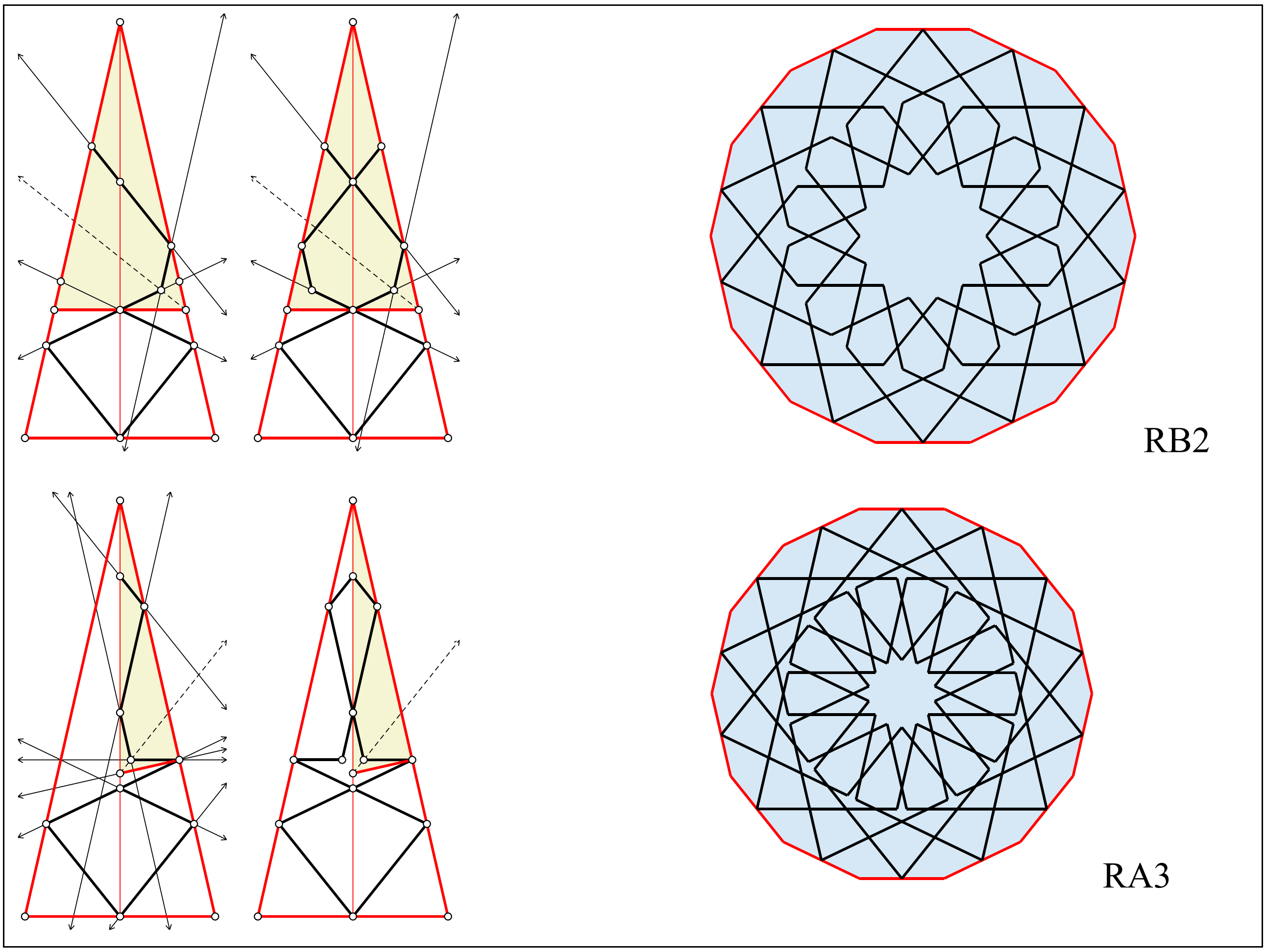
Case 4: segment OD

This time, the angle is even larger than the one used for the previous case:

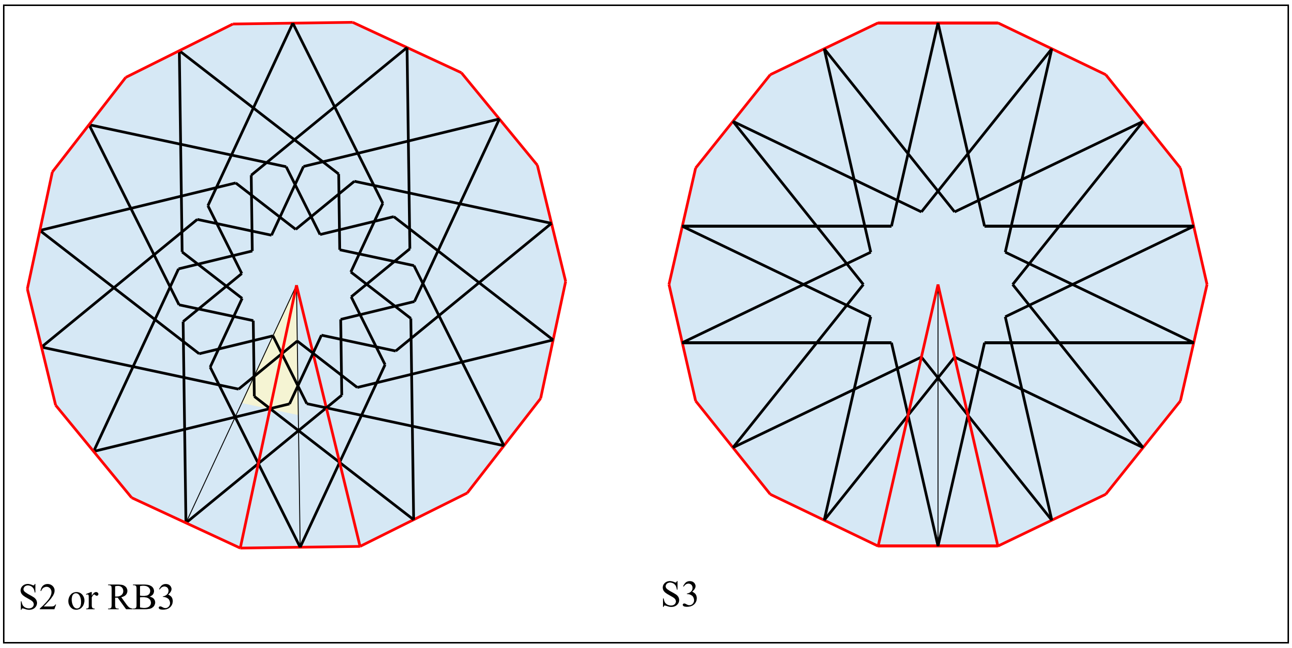
$$4\theta = 4 \cdot 90/7 = 51.42857°$$







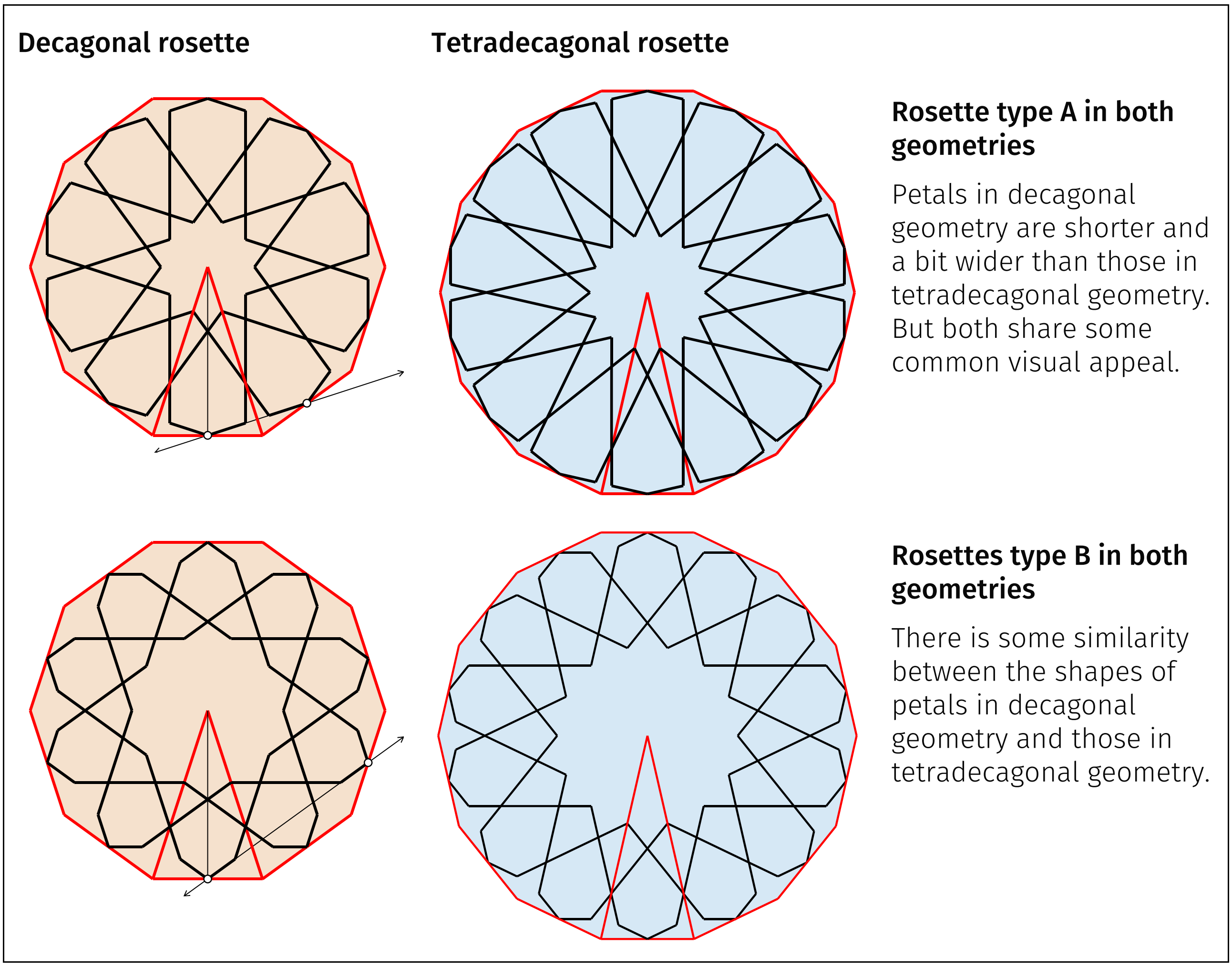
Cases 5 & 6: segment OE and OF



In both cases, we obtain stars with very sharp corners. Their applications may be very limited or, due to sharp angles, sometimes even impossible. We show them to have a reasonably complete description of what happens in each of these cases. The center of each of these stars can be filled with a rosette type A or B. But we will get a combination of very large and very small shapes. Thus, the image may not be acceptable for aesthetical reasons.

Summary of rosettes and stars in regular tetradecagon

The analysis performed in the previous sections yielded several different results. We have three different types of rosettes, sometimes touching the edges of the tetradecagon (**RA**, **RB**) and sometimes pushed towards the center of the tetradecagon, one level (**RA1**, **RB1**), or two levels (**RA2**, **RB2**). We also obtained a rather unusual rosette **RC**. Each of the geometric constructions is correct and follows precisely rules of gereh. But some of them may be more appealing to our sense of beauty than others. Constructions with rosette **A** are very similar to those in decagonal patterns. Constructions with rosette **B** are less favorite, and in decagonal patterns, we have designs using similar shapes.

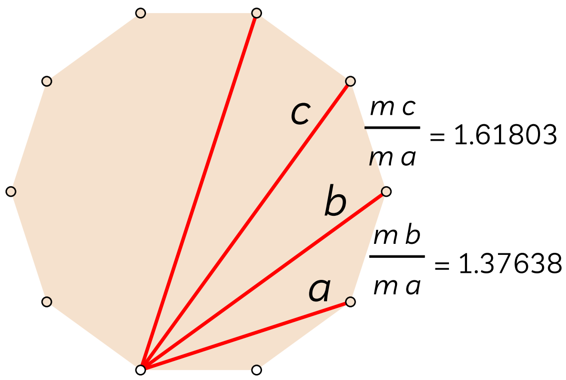


Due to the similarity of shapes and techniques used, we may expect tetradecagonal patterns will have more common features as those in decagonal patterns.

In decagonal patterns, we can follow the same systematic analysis as the one presented in this paper. However, most designers focused primarily on patterns using types **A** and **B** of rosettes. For the sake of space, the following examples will primarily use rosettes **A** and **B**. However, continuing research on using other types of rosettes and stars may lead to some interesting designs.

The golden ratio and ratios in tetradecagonal geometry

While examining decagonal patterns, in the past, we observed how the golden ratio is present in decagonal rosettes. First, let's examine where we can find the golden ratio in a regular decagon.

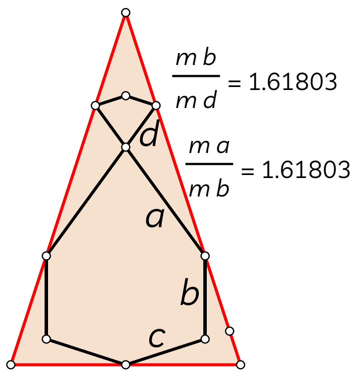


Golden ratio in a regular decagon

We can notice that the ratio mc/ma calculated in GSP is a rounded decimal representation of the golden ratio:

$$1.618033988749....$$

We obtained another specific number, 1.37638..., which is also a rounded decimal representation of another irrational number occurring in the regular decagon.



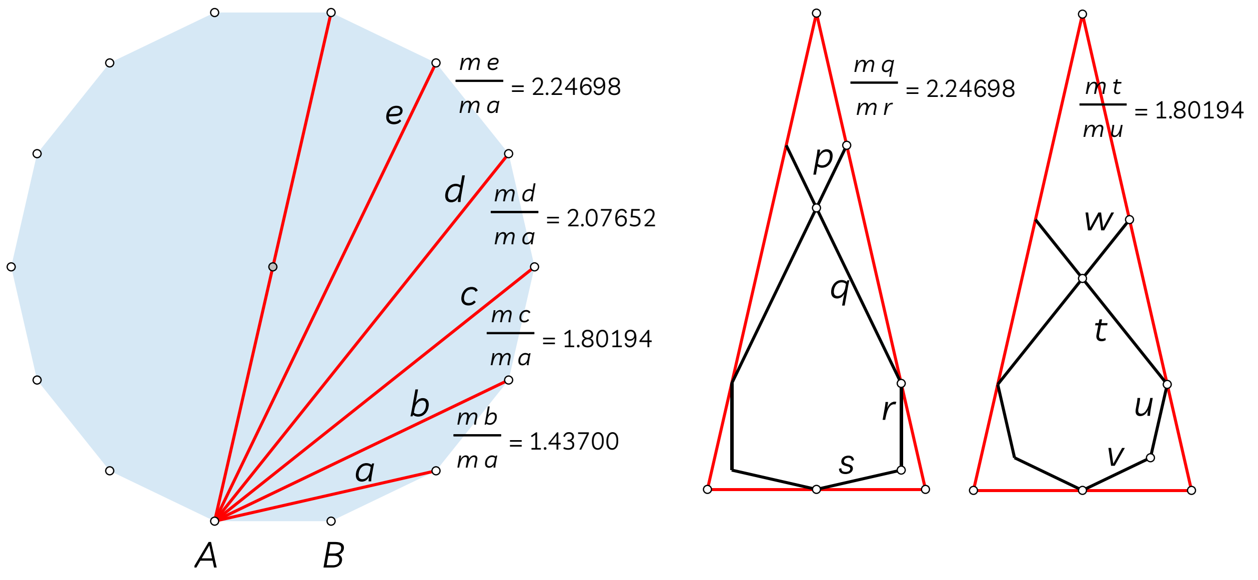
Ratios in a decagonal rosette

Calculations obtained in GSP confirm that the decagonal rosette type A is created from segments that fulfill the golden ratio equation:

$$mb/md = ma/mb = 1.61803\ ...$$

Formal proof can confirm this result.

We can investigate ratios for the regular tetradecagon and see if we get similar conclusions.



Ratios of diagonals in a tetradecagon and tetradecagonal rosettes

In the regular tetradecagon, we find four different ratios. Two of them also occur in rosette types *A* and *B*. One of them we denote as $\rho = 1.80194...$ and it occurs in tetradecagonal rosette type *B*. Another one is denoted as $\sigma = 2.24698\ ...$ and it occurs in rosette type *A*.

Numbers ρ and σ were investigated by Peter Steinbach in [7] as ratios for the regular heptagon. His approach was slightly different than the one presented here. He compared the edge of the heptagon with its diagonals. We compare the diagonals of the tetradecagon here.

Steinbach proved several interesting properties of both numbers. Here are some of them.

$$\rho^2 = 1 + \sigma \qquad \rho\sigma = \rho + \sigma \qquad \frac{\sigma}{\rho} = \sigma - 1 \qquad \frac{\rho}{\sigma} = \rho - 1$$

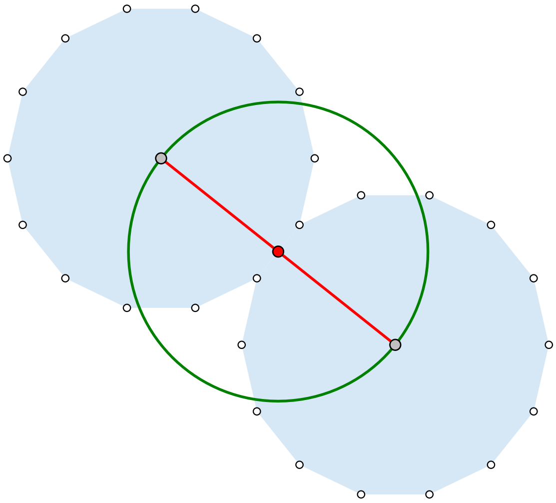
$$\sigma^2 = 1 + \rho + \sigma \qquad \frac{1}{\rho} = 1 + \rho - \sigma \qquad \frac{1}{\sigma} = \sigma - \rho \qquad \frac{1}{\rho} + \frac{1}{\sigma} = 1$$

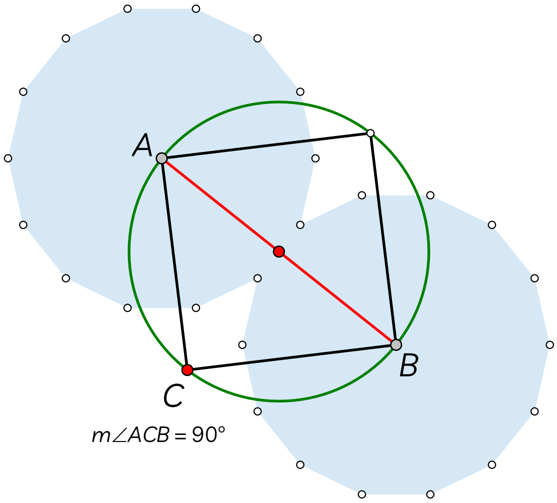
The analysis of the ratios in the regular tetradecagon produced four different ratios—two of them we found in the petals of the two rosettes we created. But there are four different ratios. Thus, it could be interesting to investigate these four rations and their properties. Can we find all of them in geometric patterns created with the regular tetradecagon? It could be interesting to investigate ratios of diagonals in other regular polygons.

Investigating the geometry of the two tangent regular tetradecagons

If we want to create geometric patterns in tetradecagonal geometry, then the simplest case would be to see what we can obtain with two tangent regular tetradecagons. In decagonal patterns, we were able to produce a large collection of geometric patterns. What we may get here?

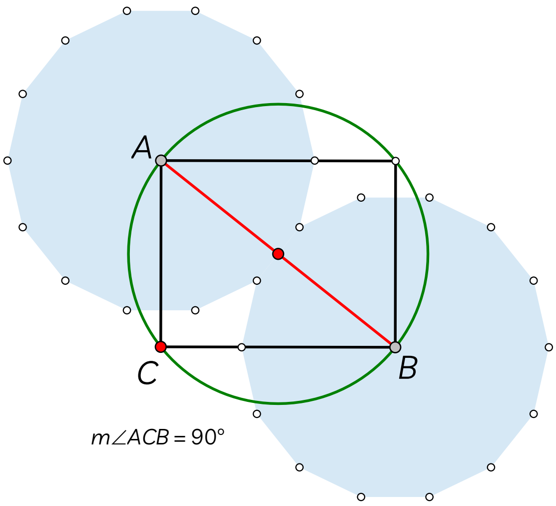
To examine this case, we need to take two tangent regular tetradecagons and see how many tessellations we can construct. For this reason, we will produce a very simple animated model in GSP.



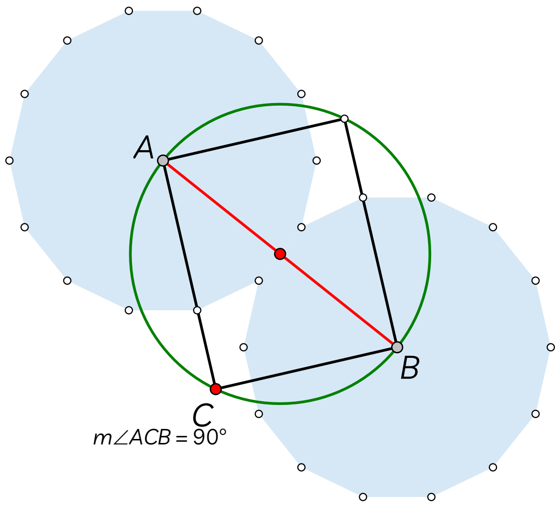


Let us assume that we have two tangent regular tetradecagons, a segment connecting their centers and a circle with its center at the midpoint of the segment and passing through centers. Every point **C** selected on this circle will determine a right angle **ACB**. Therefore, we can produce a rectangle shown above.

The edges of the rectangle should pass along the symmetry lines of both tetradecagons. How many such situations are possible? By moving point **C** along the circle, we can obtain only two possible solutions (shown below). We can also notice that solution (2) is the same as solution (1) rotated. So, in reality, we have only one possible tessellation with two tangent regular tetradecagons.

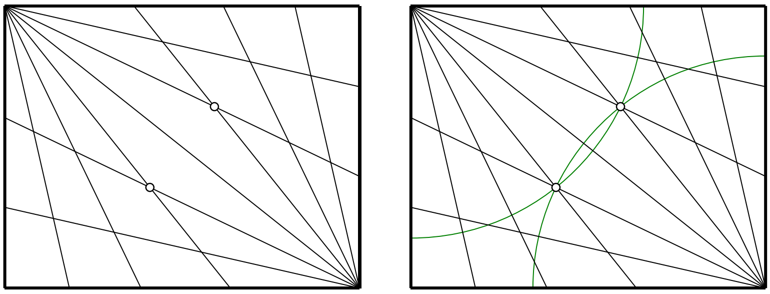


(1)



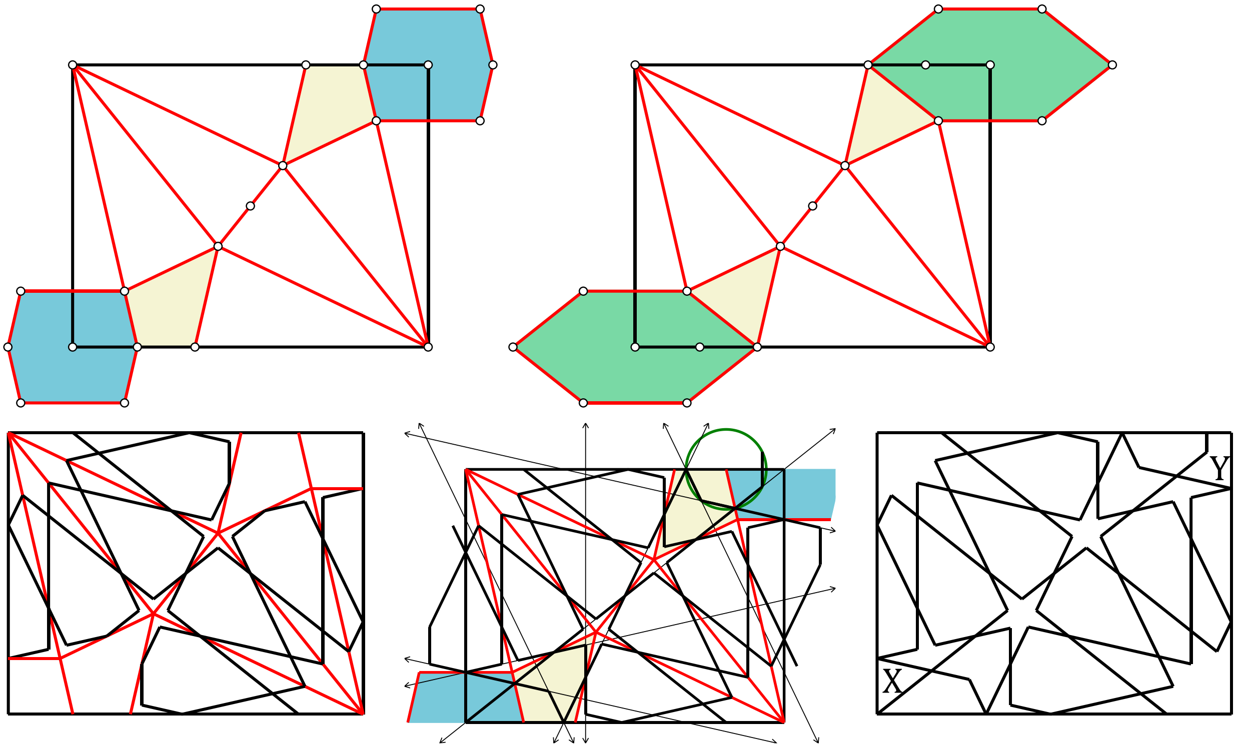
(2)

From here, we can develop complete tessellation and start developing patterns using fills for the regular tetradecagon. We know what to put inside tetradecagons, and we will have to invent pattern elements for the spaces between them.



Tessellations for two tangent tetradecagons

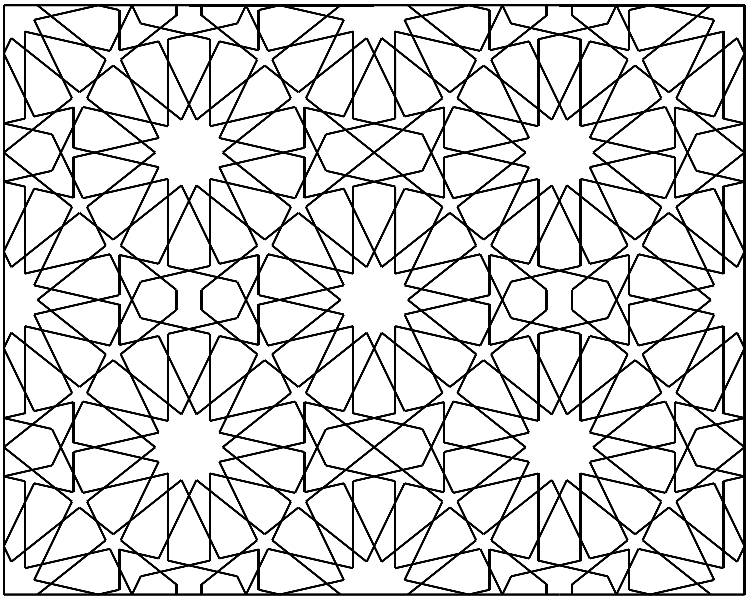
The drawing illustrates the process of creating a tessellation. Below are shown two possible tessellations for this example.



Construction of the tessellation and pattern using rosette RA

With the ready elements, filling tetradecagons is an easy task. The only thing that can be added is the pattern between tetradecagons. Here, in row two, we show how it can be done for the two new polygons (green and yellow). Note the pattern for the bottom-left corner tiles was constructed differently than the pattern for the right-up corner. Both solutions, X and Y, are correct.

In this example, we created four new polygons in tetradecagonal geometry. Each of them can be constructed as a part of the regular tetradecagon.



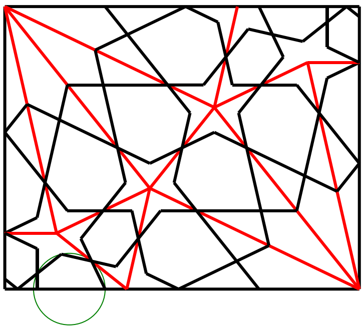
P1

Left – design from Wakala al-Ghuri caravanserai in Cairo

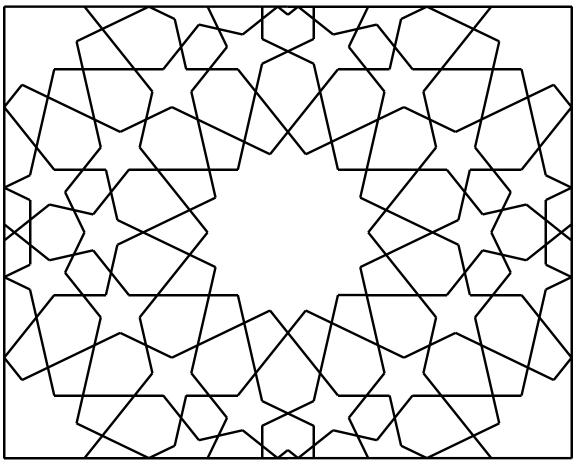
Right – pattern created from the template created on the previous page. It uses two different constructions for the corners.

The same tessellation can be used with rosette **RB** and other rosettes we created in the previous sections.

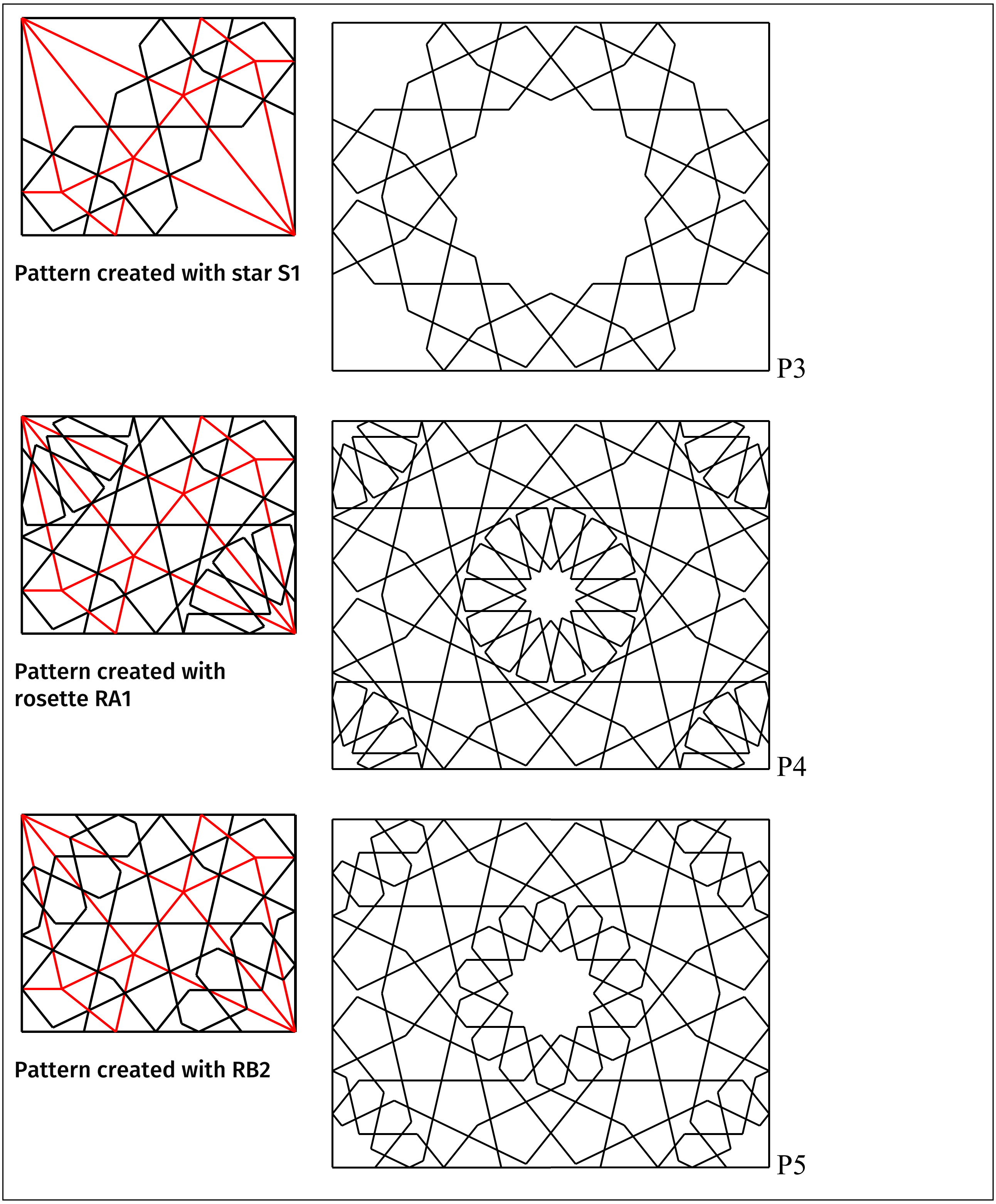
We can continue to fill the obtained tessellations with other rosettes. Our constructions may be successful in some cases and fail in others.



Geometric pattern using rosette RB



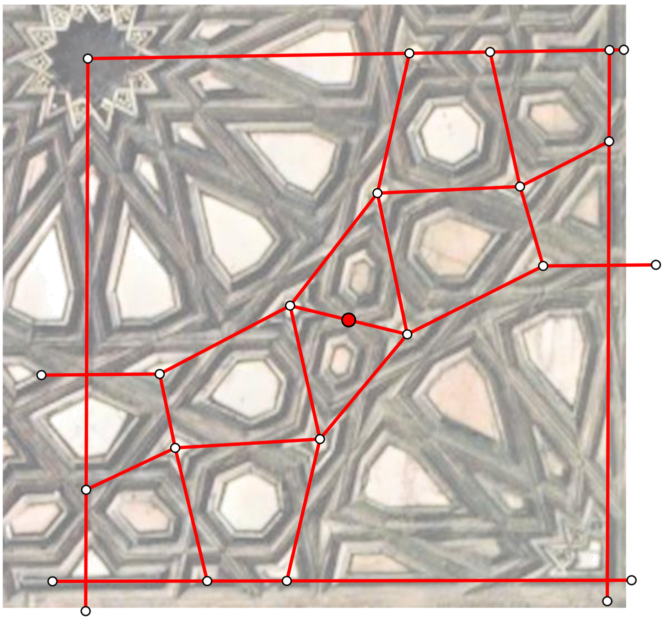
P2



The remaining fills for the regular tetradecagon do not produce anything interesting. The sharp angles make it very difficult to fill the corner spaces between tetradecagons. As we can see, some of the obtained patterns were quite acceptable from an aesthetical point of view. This means patterns **P1**, **P2**, and **P3**.

Neusis in tetradecagonal designs

We know that regular tetradecagon and tetradecagonal patterns cannot be constructed using compasses and rulers – the traditional tools of medieval pattern designers. Thus, the only good solution was the neusis construction. In the next example, we will demonstrate how the neusis construction can be applied. Of course, we will never be certain if the solution shown here was applied, but we do not have much of a choice. This was the only tool available at the time, probably to some.

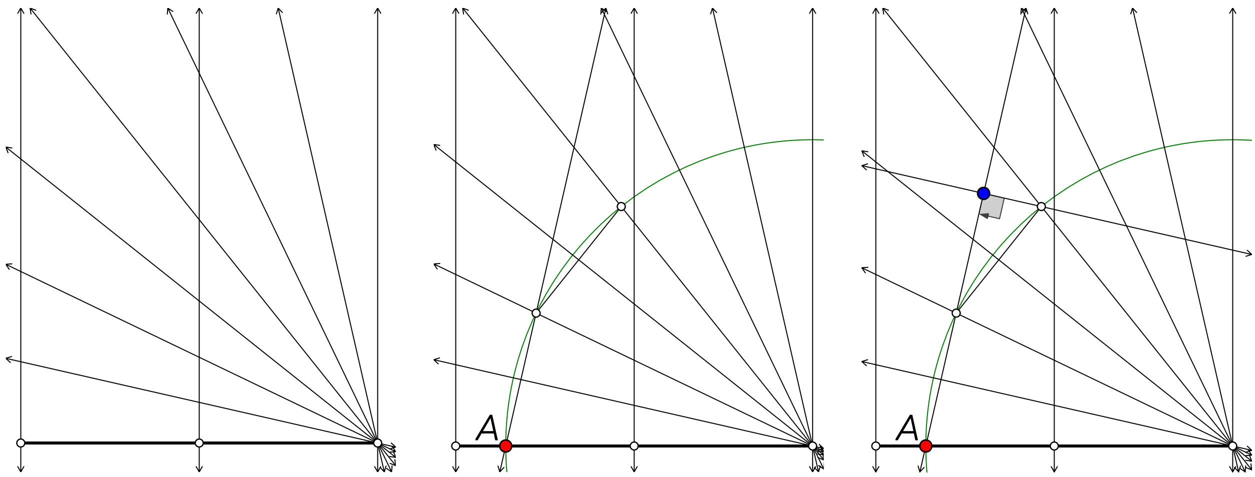


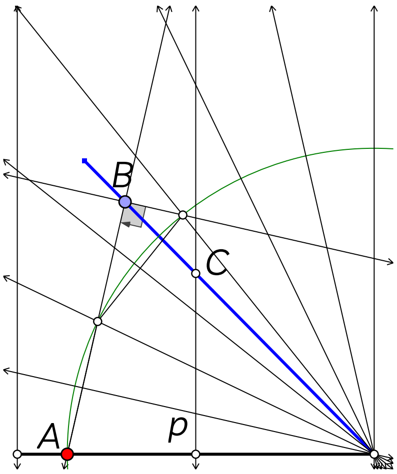
Fragment of an old photo of the Sultan al-Muayyad Sheikh Mosque in Cairo

The photo was restored and enhanced. The red lines were drawn by hand by the author. It displays a tetradecagonal pattern featuring rosettes formed with the RA1 motif. The remaining shapes are triangles and tall trapeziums with three equal edges. The contour for this pattern does not fit into the list of contours discussed earlier in this paper, and it is not a square either.

The red point is the center of this rectangular shape. If we can find this point, then we can draw the entire tessellation and, subsequently, the pattern.

Let us begin with the concept of neusis construction for the pattern from the Muayyad Sheikh Mosque.

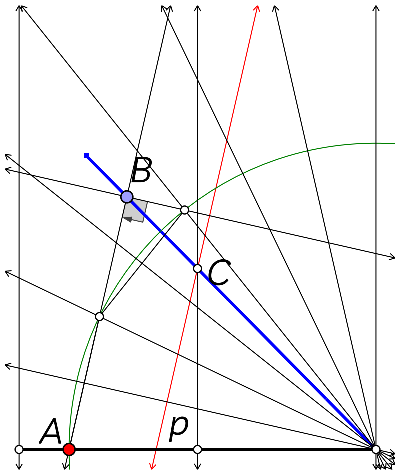


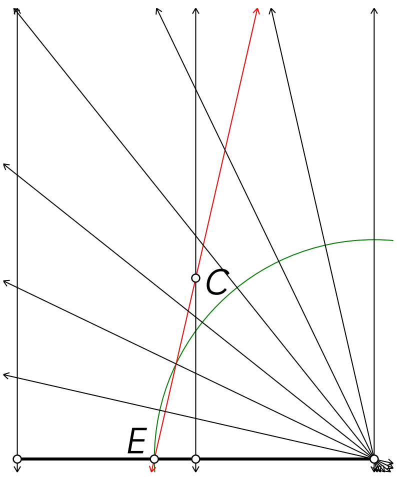


Above –point **A** is an arbitrary point selected on the horizontal segment. It can be moved along this segment. We create a part of a regular tetradecagon starting from **A**.

Left – in GSP, we select point **B** and point **A,** and from the menu **Construct**, we select **Locus**. This way, we get the blue segment. This is our neusis result.

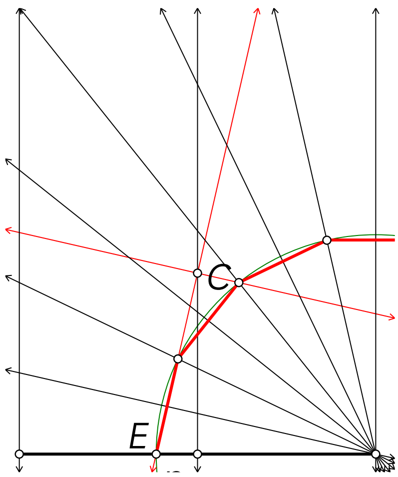
Right – we use point **C** to draw a line parallel to **AB**.

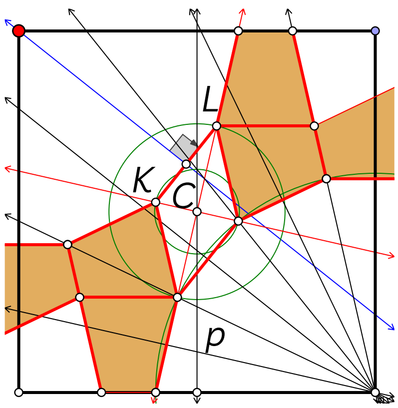




Left – we hide all elements of the neusis construction and leave only line **EC**.

Right – we construct here a quarter of the first tetradecagon.

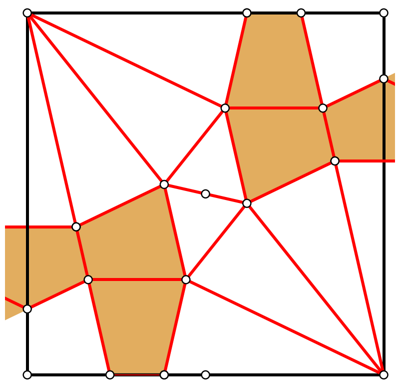


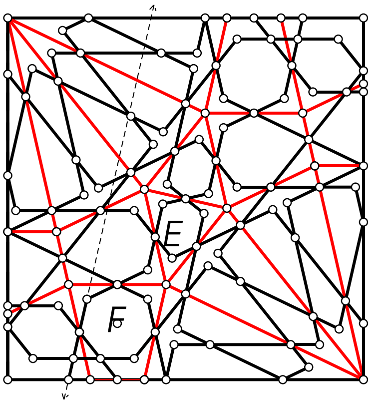


Left – construction of the rhombus in the middle of the drawing and construction of the left-top vertex of the contour.

Right - complete tessellation

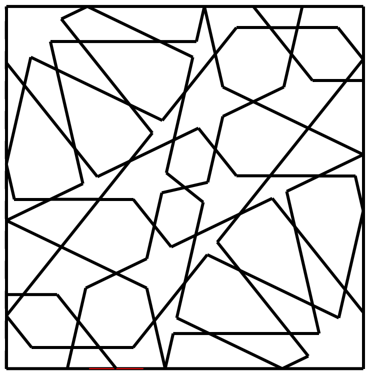
The edges of the contour cut some of the tessellation tiles. This way, we obtained two new tessellation polygons.

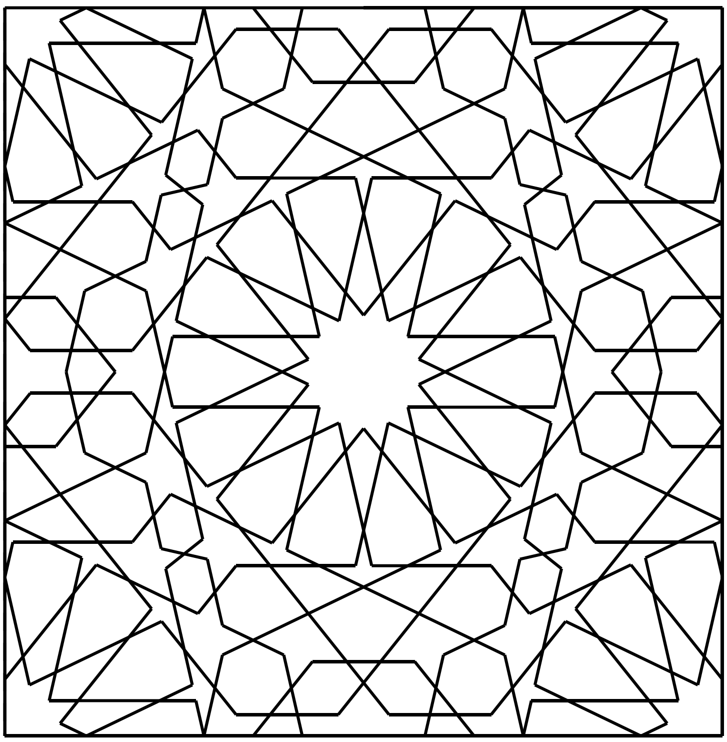




Left – tessellation polygons filled with appropriate motifs. Some of them were modified to get a consistent pattern. We could find a few slightly different motifs than those used here.

Right – a complete contour for the pattern from Muayyad Sheikh Mosque.





A large replica of the pattern from Muayyad Sheikh Mosque

This pattern is one of the most elaborated tetradecagonal patterns known. It can be extended in any direction by using multiple copies of it (translations or reflections about an edge).

Final comments

In this paper, we described some foundations of the tetradecagonal geometry and a few tetradecagonal patterns. There are many other topics related to tetradecagonal geometry that we could discuss. This discussion will be continued in the next paper or papers where we will provide a catalog of existing tetradecagonal patterns, design of tetradecagonal panjara patterns, structural design in tetradecagonal geometry, alternative tessellations, patterns without stars and rosettes, and a partial summary of tessellations used in tetradecagonal designs.

References(2)

[1] Bulut M. (2019). *Selçuklu Çizgileri-Anadolu Selçuklu Geometrik Kompozisyonları*. İnkılab Yayınları. Turkey.

[2] Coxeter. H. S. *Mathematical Recreations and Essays*, Thirteenth edition, p.141

[3] Majewski. M. (2020). *Practical Geometric Pattern Design: Geometric Patterns from Islamic Art*. Kindle Direct, Independently published (February 10, 2020).

[4] Majewski. M. (2020). *Understanding Geometric Pattern and its Geometry (Part 1)*, eJMT, vol. 14, Nr 2, pages 87-106.

[5] Majewski. M. (2020). *Understanding Geometric Pattern and its Geometry (Part 2) – Decagonal Diversity*, eJMT, vol. 14, Nr 3, pages 147-161.

[6] Majewski. M. (2021). *Practical Geometric Pattern Design: Decagonal Patterns in Persian Traditional Art*. Kindle Direct, Independently published (March 29, 2021)

[7] Steinbach. P. (2000). *Sections Beyond Golden*. Bridges Mathematical Connections in Art, Music, and Science.

[8] Wichmann B., Wade D.(2017). *Islamic Design: A Mathematical Approach*, Birkhäuser.

[9] Wikipedia Tetradecagon: https://en.wikipedia.org/wiki/Tetradecagon

[10] Wikipedia Regular Polygons: https://en.wikipedia.org/wiki/Category:Polygons_by_the_number_of_sides

[2] All papers by M. Majewski can be downloaded from: https://www.researchgate.net/profile/Miroslaw-Majewski