# Loci of Equal Viewpoints for Two Ellipses

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**Abstract**

*We are given two ellipses, denoted by $E_1$ and $E_2$, respectively. Point $A$ is outside these two ellipses. We draw tangents from $A$ to the two curves, $E_1$ and $E_2$,* respectively. *The angle between these tangents is the angle we see these two ellipses at from point $A$. We investigate the locus of the set of all points where this angle is the same for both ellipses. The original locus problem discussed in this paper is simple, and the algebraic partial solution provided by most students in Section 1.2 is accessible but incomplete. It is indeed virtually impossible to study the locus problem without the help of a computer algebra system. The locus is invariant under the rigid transformations. We shall see how a uniform shear transformation will affect the locus discussed in this paper.*

## 1 Background

Some graduate students raised the original question while the author spent a professional leave at Guangzhou University in China. We are given two disconnected circles, $C_1$, and $C_2$ centered at $O_1$ and $O_2$, respectively. Point $A = (x, y) \in \mathbb{R}^2$ is outside both circles, and we construct two pairs of tangent lines from $A$ toward circles $C_1$ and $C_2$, respectively. We label those points of tangency as $B, C, D$, and $E$, respectively; (see Figure 1). If $\measuredangle BAC = \measuredangle DAE$, we call point $A$ an equal viewpoint.

**Question:** Find the locus of equal viewpoint $A$.

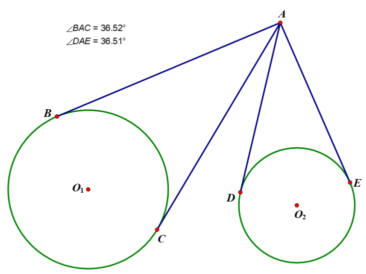


Figure 1. Equal viewpoint $A$.

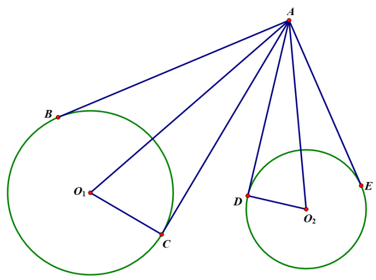


Figure 2. Students' approach

**Remarks:**

1. After this article is completed, we thank one reviewer for pointing out the terminology and concept of *isoptic curve* (see [5] and [4]), which led the author to discover the concept of *equioptic curve*. A equioptic curve is exactly the *true locus* of equal viewpoints for two given smooth (continuously differentiable) curves, which we use in this paper. The author also discovered an interesting paper on equioptic curve (see [2]) after this paper was completed. The methods described in this paper are computational, compared to more theoretical approaches mentioned in [2].

2. The methods discussed in Sections 1.1 and 1.2 were suggested by students.

3. It is trivial, by its definition, that the equal viewpoint $A$ has to be outside two respective circles.

## 1.1 Students' approaches to two circles

We note that Question (1) can also be described as follows: In view of Figure 2, it follows from the assumptions that $\measuredangle O_1AC = \frac{1}{2}\measuredangle BAC = \frac{1}{2}\measuredangle DAE = \measuredangle O_2AD$, and $\measuredangle O_1CA = \measuredangle O_2DA$. Thus, $\triangle O_1AC$ is similar to $\triangle O_2AD$. Hence,

$$\frac{AO_1}{AO_2} = \frac{CO_1}{DO_2} = \frac{R_1}{R_2}, \tag{1}$$

where $R_1$ and $R_2$ are the radii of the circles $C_1$ and $C_2$ respectively.

The preceding Question (1) is similar to the so-called Circle of Apollonius (See [1]).

*Find the locus of those points of equal viewpoint where the ratio of the distances to the centers of two respective circles is a constant (i.e. is a constant, see Figure 2).*

## 1.2 Derivations from algebraic manipulations alone without a CAS are misleading

We demonstrate here that some students may rely on their strong algebraic manipulation skills and come up with a solution. We label the moving locus for the equal viewpoint $A =$

$(x, y)$ , and label the centers of two non-degenerate circles $O_1(x_1, y_1)$ and $O_2(x_2, y_2)$ respectively. Furthermore, because of Eq. (1), if we assume $\frac{R_1}{R_2} = m > 0$, then $\frac{AO_1}{AO_2} = m$. Consequently, we have

$$(x - x_1)^2 + (y - y_1)^2 = m^2\left((x - x_2)^2 + (y - y_2)^2\right)$$

After reordering, we have

$$\begin{aligned}&\left(1 - m^2\right)x^2 + \left(1 - m^2\right)y^2 + 2\left(m^2 x_2 - x_1\right)x + 2\left(m^2 y_2 - y_1\right)y \\ &= m^2\left(x_2^2 + y_2^2\right) - \left(x_1^2 + y_1^2\right).\end{aligned} \tag{2}$$

1. Case 1. If $m \neq 1$, then

   $$x^2 + y^2 + \frac{2\left(m^2 x_2 - x_1\right)}{1 - m^2}x + \frac{2\left(m^2 y_2 - y_1\right)}{1 - m^2}y = \frac{m^2\left(x_2^2 + y_2^2\right) - \left(x_1^2 + y_1^2\right)}{1 - m^2}$$

   By completing square, we obtain

   $$\left(x + \frac{m^2 x_2 - x_1}{1 - m^2}\right)^2 + \left(y + \frac{m^2 y_2 - y_1}{1 - m^2}\right)^2 = \frac{m^2\left(x_1^2 - 2x_2 x_1 + x_2^2 + y_1^2 - 2y_2 y_1 + y_2^2\right)}{\left(1 - m^2\right)^2}. \tag{3}$$

   We see that Eq. (3) represents a circle with center $\left(\frac{m^2 x_2 - x_1}{m^2 - 1}, \frac{m^2 y_2 - y_1}{m^2 - 1}\right)$, and radius $\left|\frac{m}{m^2 - 1}\right| \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ provided that $x_2^2 - x_2 x_1 + y_2^2 - y_2 y_1 > 0$.

2. Case 2. If $m = 1$, then the locus, according to Eq. (2), becomes a line of

   $$2\left(x_2 - x_1\right)x + 2\left(y_2 - y_1\right)y - \left(x_2^2 + y_2^2\right) + \left(x_1^2 + y_1^2\right) = 0. \tag{4}$$

**Caution:**

We shall see in Section 4.1 Exercises that Eq. (2) may not represent the true locus for two circles, see Exercises 1 and 2. Furthermore, the locus of equal viewpoints is valid for Case 2, but only when two circles are of the same radius and disconnected with no intersection, see Exercise 2 in Section 4.1.

# 2 Obtaining the first stage quasi-locus equation

Note that the preceding algebraic method in Section 1.2 works only for special circle cases; it cannot be extended to ellipses, for example. We thus discuss an alternative method, which uses the angles of the slopes of tangent lines at respective figures, and see how the problem can be extended to two ellipses. The methods described here are accessible to high school students; however, it shall become apparent that the students would need to have access a computer algebra system for in-depth investigations. We shall see that first stage quasi-locus (13) is a degree 6 polynomial equation of $x$ and $y$. Next, we shall express $y$ in terms of $x$. Consequently, we obtain at most six branches of functions, and not all branches of functions are suitable answers. We need to extract branches of functions to form the true locus, as we shall see in Section 3.

Unless otherwise specified, the ellipses we discuss in this paper are non-degenerate cases.

1. We start with a simple case with the following two respective equations of ellipses, $E_1$ (5) and $E_2$ (6) respectively:

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \tag{5}$$

and

$$\frac{(x-c)^2}{a_2^2} + \frac{(y-d)^2}{b_2^2} = 1. \tag{6}$$

We assume $a_1, b_1, a_2$, and $b_2 > 0$. We set up the tangent line equation with slope $k_1$ for $E_1$, which passes through the equal viewpoint $A(x_0, y_0)$ as follows

$$y - y_0 - k_1(x - x_0) = 0. \tag{7}$$

Similarly, We set up the tangent line equation with slope $k_2$ for $E_2$, which passes through the equal viewpoint $A(x_0, y_0)$ as follow

$$y - y_0 - k_2(x - x_0) = 0. \tag{8}$$

2. We substitute first tangent line equation (7) into the first ellipse (5). After simplifications, we obtain the following:

$$a_1^2k_1^2x^2 - 2a_1^2k_1^2xx_0 + a_1^2k_1^2x_0^2 + 2a_1^2k_1xy_0 - 2a_1^2k_1x_0y_0 - a_1^2b_1^2 + a_1^2y_0^2 + b_1^2x^2 = 0,$$

and we collect the coefficients for $x^2, x$ and $x^0$ as follows respectively:

$$\begin{aligned}
A &= a_1^2k_1^2 + b_1^2, \\
B &= -2a_1^2k_1^2x_0 + 2a_1^2k_1y_0, \\
C &= a_1^2k_1^2x_0^2 - 2a_1^2k_1x_0y_0 - a_1^2b_1^2 + a_1^2y_0^2.
\end{aligned}$$

3. Because of the tangent line equations, we set the discriminant $B^2 - 4AC = 0$, and consider the left-hand side of (9) as a polynomial in $k_1$

$$4a_1^4b_1^2k_1^2 - 4a_1^2b_1^2k_1^2x_0^2 + 8a_1^2b_1^2k_1x_0y_0 + 4a_1^2b_1^4 - 4a_1^2b_1^2y_0^2 = 0. \tag{9}$$

4. We collect the coefficients for $k_1^2, k_1$, and the constant term, $k^0$ respectively as follows:

$$\begin{aligned}
d &= 4a_1^4b_1^2 - 4a_1^2b_1^2x_0^2, \\
e &= 8a_1^2b_1^2x_0y_0, \\
f &= 4a_1^2b_1^4 - 4a_1^2b_1^2y_0^2.
\end{aligned}$$

5. We recall that if $m_1$ and $m_2$ are two slopes for two tangent lines, then we have

$$\begin{aligned}
m_1 + m_2 &= -\frac{e}{d}, \\
m_1m_2 &= \frac{f}{d}.
\end{aligned}$$

Furthermore, the angle between these two lines, which we denote it as $\alpha_1$, is

$$\tan\alpha_1 = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2} = \frac{\sqrt{e^2 - 4df}}{d + f},$$

and hence

$$\alpha_1 = \tan^{-1}\left(\frac{\sqrt{e^2 - 4df}}{d + f}\right).$$

6. Analogously, we proceed to find the angle $\alpha_2$ between two tangent lines for the second ellipse $E_2$ as follows

$$\alpha_2 = \tan^{-1}\left(\frac{\sqrt{e_1^2 - 4d_1 f_1}}{d_1 + f_1}\right)$$

7. We solve for the equal viewpoint $\{x_0, y_0\}$ from the following equation (10)

$$\alpha_1 - \alpha_2 = 0. \tag{10}$$

It can be shown, after substituting $x = x_0$, and $y = y_0$, that Eq. (10) is equivalent to

$$\begin{aligned} &\left[a_1^2 b_1^2\left(\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1\right)\right]\left[\left(a_2^2 + b_2^2 - \left((x-c)^2 + (y-d)^2\right)\right)^2\right] \\ &- \left[\left(a_1^2 + b_1^2 - (x^2 + y^2)\right)^2\right]\left[a_2^2 b_2^2\left(\frac{(x-c)^2}{a_2^2} + \frac{(y-d)^2}{b_2^2} - 1\right)\right] = 0. \end{aligned} \tag{11}$$

If we consider the following 2 by 2 matrix

$$M_d = \begin{bmatrix} a_1^2 b_1^2\left(\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1\right) & a_2^2 b_2^2\left(\frac{(x-c)^2}{a_2^2} + \frac{(y-d)^2}{b_2^2} - 1\right) \\ \left(a_1^2 + b_1^2 - (x^2 + y^2)\right)^2 & \left(a_2^2 + b_2^2 - \left((x-c)^2 + (y-d)^2\right)\right)^2 \end{bmatrix}, \tag{12}$$

then $\det(M_d) = 0$ is equivalent to Eq. (10).

If we expand Eq. (11), it becomes an equation of a degree 6 polynomial in two real variables $x$ and $y$, which can be expressed as

$$P(x, y) = \sum_{i=0}^{6}\sum_{j=0}^{6-i} a_{ij} x^i y^j = 0, \tag{13}$$

where $a_{ij}$ are real constants. The plot of (11), an algebraic curve, could be complicated involving sharp corners (or cusps). We call Eq. (11) to be a *quasi-locus* equation, and we denote the implicit plot of Eq. (11), the quasi-locus by $L^Q$. Since $\det(M_d) = 0$ is the equation for the quasi-locus for (5) and (6). and if we fix the ellipse (5), but vary the values of $c$ or $d$ for the ellipse (6), we invite readers to explore how a vertical or horizontal shifting on the second ellipse (6) affects their corresponding plots of quasi-locus respectively.

# 3 The need to extract true locus from a quasi-locus

Algebraically, we shall show that the final *true locus* for the problem is extracted from these possible 6 branches of functions. Due to the complexity of Eq. (11), we will not display these six possible branches of functions symbolically. Instead, we make use of the plots of these possible branches of functions for further investigations. We recall that, geometrically, if a point $A = (x, y) \in \mathbb{R}^2$ belongs to a true locus, then $A$ satisfies $\measuredangle BAC = \measuredangle DAE$ in Figure 1. Our approach is to use an algebraic approach to extract the true locus from the quasi-locus Eq. (11), so that points in the true locus will be equal viewpoints in $\mathbb{R}^2$ from geometric point of view. However, obtaining the quasi-locus Eq. (11) is only the first step, it is necessary to use a technological tool to extract proper equal viewpoints in $\mathbb{R}^2$ from those possible 6 branches of functions, which we describe in the following steps.

1. We consider Figure 3 below, where the black curve shows the plot of a branch of $L^Q$.

2. Suppose we pick a point $A \in L^Q$, and draw two respective tangent lines toward the two respective ellipses, (5) and (6), which are shown in blue and green, respectively. The point $A$ is chosen in such a way that Eq. (11) is satisfied and $\angle BAC = \angle DAF$. However, point $F$ does not lie on the second ellipse, (6). Therefore, the pair of equal angles, $\angle BAC = \angle DAF$, is not what we need.

   (a) We recall that our objective is to find points $B$ and $C$ on (5), lying on the respective tangent lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, and then find points $D$ and $E$ on (6), lying on the respective tangent lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{AE}$ so that

   $$\angle BAC = \angle DAE. \tag{14}$$

   (b) However, simply by visual inspection, we see $\angle BAC \neq \angle DAE$ in Figure 3. In fact, the quasi-locus from (11) collects all points $A$ satisfying either (14) or

   $$\angle BAC = \pi - \angle DAE. \tag{15}$$

   Therefore, point $A \in L^Q$, in this case, cannot be an equal viewpoint. In other words, we need to extract the quasi-locus $L^Q$ further to form the true locus of equal viewpoints.

   (c) Simply put, if $\theta_1$ is the angle between two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and if $\theta_2$ is the angle two vectors $\overrightarrow{AD}$ and $\overrightarrow{AE}$, we need

   $$\theta_1 - \theta_2 = 0, \tag{16}$$

   or close to 0 after ignoring insignificant decimal, before we say point $A$ belongs to the true locus.

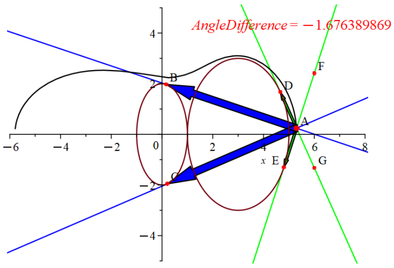


Figure 3. Plot of point A that
is not an equal viewpoint

We see that if $A \in L^Q$, then either $A$ is a true locus or $A$ is not. We, therefore, use $L^T$ to denote the true locus points and $L^N$ to denote those that are not true locus points. Subsequently, we see

$$L^Q = L^T \cup L^N.$$

The following is a simple observation from factorization using a computer algebra system.

**Theorem 1** *We consider Question (1). If the ellipses (5) and (6), are two equal-sized ellipses with* $c > 0$, *and* $d = 0$. *Then the quasi-locus equation can be expressed as* $c\,(c - 2x) \cdot p\,(x, y) = 0$, *where* $p(x, y)$ *is a degree* 4 *polynomial of* $x$ *and* $y$.

**Proof:** By assumption, we set $a_1 = a_2, b_1 = b_2$ , and $d = 0$. Then the quasi-locus (11) can be factorized as

$$c\,(c - 2x) \cdot p\,(x, y) = 0,$$

where

$$p(x,y) = \left( \begin{array}{c} a_2^4 b_2^2 - 2a_2^4 y^2 - a_2^2 b_2^2 c^2 + 2a_2^2 b_2^2 cx - 2a_2^2 b_2^2 x^2 - 2a_2^2 b_2^2 y^2 + \\ a_2^2 c^2 y^2 - 2a_2^2 cx\, y^2 + 2a_2^2 x^2 y^2 + 2a_2^2 y^4 - b_2^6 + 2b_2^4 y^2 + b_2^2 c^2 x^2 - 2b_2^2 c\, x^3 + b_2^2 x^4 - b_2^2 y^4 \end{array} \right). \blacksquare$$

In view of Figure 3, we see that the quasi-locus $A$ is the set of points that satisfies either $\angle BAC = \angle DAF$ or $\angle BAC = \angle DAE$, and we note that $\angle DAF$ and $\angle DAE$ are supplementary angles of each other. They will not be the same unless they are $\frac{\pi}{2}$, or one of them is 0, in which case, the quasi-locus is a straight line.

**Theorem 2** *We consider the locus in Question (1) for the ellipses (5) and (6), see Figure 3. If* $\angle BAC = \angle DAE = \angle DAF = \frac{\pi}{2}$, *then the real solutions of the intersections*

$$\left\{ \begin{array}{c} \left(a_1^2 + b_1^2 - (x^2 + y^2)\right) = 0, \\ \left(a_2^2 + b_2^2 - \left((x - c)^2 + (y - d)^2\right)\right) = 0 \end{array} \right. \tag{17}$$

*are the intersecting points between the quasi-locus curve that contains the equal viewpoints and the quasi-curve that does not contain the equal view points.*

**Proof:** We refer to Figure 3 and assume $\angle BAC = \angle DAE = \angle DAF = \frac{\pi}{2}$, then $A$ can be either an equal viewpoint or not an equal view point. We proceed and set $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, and since $\tan\frac{\pi}{2}$ is undefined, we obtain $(a_1^2 + b_1^2 - (x^2+y^2)) = \left(a_2^2 + b_2^2 - \left((x-c)^2 + (y-d)^2\right)\right) = 0$, and it follows from Eq. (11) that when $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, point $A$ lies on the intersections of Eq. (17). ■

**Remarks:** We remark that for simplicity and with no confusion, we denote the intersection points in Theorem (2) by $L^T \cap L^N$. In addition, since the implicit plot of Eq. (11) is continuous throughout an interval, we have the following observations:

1. Suppose the quasi-locus (11) is expressed in branches of functions, $\{A_i(x)\}_{i=1}^n$, and if $A_i(x)$ is a real branch, non-differentiable at $x = x^*$, and $x^*$ is an equal viewpoint, then $x^*$ is either a start or an endpoint for $A_i(x)$.

2. If $x = x^*$ is a start point and an equal viewpoint for $A_i(x)$, then $A_i(x)$ has equal viewpoints in some interval $[x^*, x^*+\alpha)$ for some $\alpha > 0$. If $x = x^*$ is an endpoint and an equal viewpoint for $A_i(x)$, then $A_i(x)$ has equal viewpoints in some interval $(x^* - \alpha, x^*]$ for some $\alpha > 0$. On the other hand, if $x = x^*$ is not an equal viewpoint for $A_i(x)$, then $A_i(x)$ does not have equal viewpoints in some interval containing $x^*$. To summarize, we have

**Theorem 3** *We consider the locus in Question (1) for the ellipses (5) and (6), when A is outside both ellipses. Then the plot of the true locus $L^T$ is a continuous curve in a certain interval.*

## 3.1 An example

Suppose we use $a_1 = 1, b_1 = 2, a_2 = 2, b_2 = 3$, $c = 3$ and $d = 0$ for (5) and (6), and we call these ellipses $E_1$ and $E_2$, respectively. We include all detailed computations in [S1] but we summarize as follows:

1. We set $\alpha_1 - \alpha_2 = 0$ and obtain the equation:

$$\arctan\left(\frac{2\sqrt{\dfrac{9x^2+4y^2-54x+45}{(x^2+y^2-6x-4)^2}} - 2\sqrt{\dfrac{4x^2+y^2-4}{(x^2+y^2-5)^2}}}{4\sqrt{\dfrac{4x^2+y^2-4}{(x^2+y^2-5)^2}}\sqrt{\dfrac{9x^2+4y^2-54x+45}{(x^2+y^2-6x-4)^2}}+1}\right) = 0. \tag{18}$$

   After simplifying with (11), (18) becomes a polynomial degree 6 of $x$ and $y$ :

$$\begin{aligned} &5x^6 + 13x^4y^2 + 11x^2y^4 + 3y^6 - 6x^5 - 48x^3y^2 - 42xy^4 - 153x^4 \\ &-28x^2y^2 + 17y^4 + 300x^3 + 444xy^2 - 177x^2 - 398y^2 - 1158x + 1189 = 0, \end{aligned} \tag{19}$$

   which we recall that the implicit plot of (19) is called $L^Q$.

2. In this paper, we express $y$ in terms of $x$ in the degree 6 polynomial equation (19) to obtain at most 6 functions by using the following commands using *[3]*. First, we use

‘prelocus’ for quasi-locus to represent the Eq. (11) below:

```
A:=solve(prelocus,y):
A1:=unapply(solve(prelocus,y)[1],x):
A2:=unapply(solve(prelocus,y)[2],x):
:
A6 :=unapply(solve(locus, y)[6],x):
```

(a) It follows from the nature of constructing the quasi-locus from Eq. (11), points in Eq. (11) are in $\mathbb{R}^2$. Therefore, we ignore some of the branch(es) that may yield complex outputs.

3. We save the plot for each branch as follows:

```
branch1:=plot(A1(x),x=-6..6,y=-5..5);
branch2:=plot(A2(x),x=-6..6,y=-5..5);
:
branch6:=plot(A6(x),x=-6..6,y=-5..5);
```

4. In [S1], readers see that branches 3 and 4 are empty plots since complex values were detected.

5. We sketch the ellipses $E_1$ and $E_2$, in red together with branch 1, branch 2, branch 5 and branch 6 in black, green, cyan, and pink, respectively in Figure 4. We notice that $A3(x)$ and $A4(x)$ produce complex outputs, thus we do not see branch 3 and branch 4.

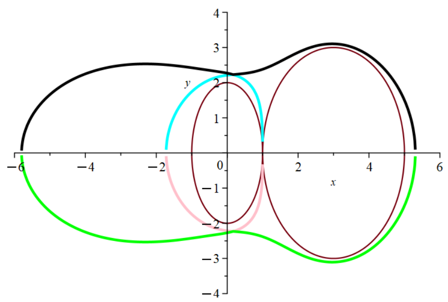

Figure 4. Plots of different branches

6. Since the ellipses $E_1$ and $E_2$ are symmetric with respect to the $x - axis$, it is easy to see that branch 1 (black) and branch 2 (green) are symmetric with respective to the $x - axis$, as is branch 5 (cyan) and branch 6 (pink).

## 3.2 Procedure for forming a true locus from a quasi-locus

In view of Eq. (16), to say $A \in L^T$, we need to have $\theta_1 - \theta_2 = 0$ or close to 0 after ignoring insignificant decimal. Since Eq. (11) is a continuous curve, it follows from Theorem (3) if a point $A \in L^T$, then $A$ should travel along a continuous curve; same thing can be said if $A \in L^N$. Therefore, if we start with $A \in L^Q$, we need to check computationally and verify if $A \in L^T$ or $A \in L^N$. However, when (13) is expressed as branches of functions, we shall see that individual branches of functions may not be a smooth curve. As a result, we call a sharp corner $A$ or cusp of a non-smooth curve a *turning point*, since $A$ may be a point turning from $L^T$ to $L^N$ or vice versa. Furthermore, if $A \in L^T \cap L^N$, the Theorem (2) helps to determine if a point $A$ is a starting, turning point or an ending point for $L^T$ or $L^N$. Therefore, we call $A$ to be a *critical point* for a branch if it is a beginning, ending, or a turning point. We remark that the computational process of finding the critical points involve serious numerical computation; we thus set the `Digits:=15` when we use the computer algebra system [3]. Secondly, we need to take special care to the point $(1, 0)$, where both ellipses share a vertical tangent line, and thus the slope is undefined. However, it is an equal viewpoint obviously. We describe how we proceed in the following steps:

1. The positions of two ellipses are determined. We use Theorem (2) to find intersection points $A \in L^T \cap L^N$.

2. We identify the critical points for a branch, and proceed with a point $A = (x_0, y_0)$ in a branch of the quasi-locus plot $L$.

3. We solve Eq. (9) for the slopes of the tangent lines for the ellipse (5) and call them $k_{11}$ and $k_{12}$ respectively.

4. We find the respective tangent line equations for $k_{11}$ and $k_{12}$, respectively, and call them $l_{11}$ and $l_{12}$ respectively.

5. We find the tangent lines $l_{21}$ and $l_{22}$ respectively for the ellipse (6).

6. We next find the tangencies for these two respective ellipses, (5) and (6).

   (a) We solve the intersections between $l_{1j}, j = 1, 2$, and ellipse (5) respectively, and label them $(x_{11}, y_{11})$ and $(x_{12}, y_{12})$ respectively.

   (b) We do the same and find the intersections between $l_{2j}, j = 1, 2$, and ellipse (6) respectively, and label them $(x_{21}, y_{21})$ and $(x_{22}, y_{22})$ respectively.

   (c) We form the vector
   $$v_{ij} = (x_{ij} - x_0, y_{ij} - y_0),$$
   where $i, j = 1, 2$.

   (d) We verify if the angle between $v_{11}$ and $v_{12}$ is the same as the angle between $v_{21}$ and $v_{22}$ up to some significant decimal places. If the pairs of angles are the same, then $A \in L^T$; otherwise $A \in L^N$. We write a program to speed up checking all points $x$ in an interval $[a, b]$ to be an equal viewpoint or not.

7. When checking if a point $A$ is an equal viewpoint. It is possible to run into $A$ possesses either a horizontal or vertical tangent. In such case, we shall manually select $l_{ij}$, for $i = 1, 2$, before verifying if $A$ is an equal viewpoint.

8. In view of Figure 4, some branches are symmetric with respect to the $x - axis$, it is sufficient to work on branches of functions 1 and 5 respectively.

**Branch 1:** If we zoom in the plot for branch 1, as shown on in Figure 5(a), we find a point of non-differentiability for branch 1. This suggests that such point is in $L^T \cap L^N$.

1. In view of Theorem (2), we solve Eq. (17) when $a_1 = 1, b_1 = 2, a_2 = 2, b_2 = 3$, $c = 3$ and $d = 0$ for (5) and (6). We obtain two intersections at $P_1 = \left[\frac{1}{6}, \frac{\sqrt{179}}{6}\right]$ and $P_2 = \left[\frac{1}{6}, \frac{-\sqrt{179}}{6}\right]$, respectively. The turning point $P_1 = \left[\frac{1}{6}, \frac{\sqrt{179}}{6}\right]$ for branch 1 can be also seen from the discontinuous plot of the derivative of $A1(x)$ , where there is an extraneous vertical line for the derivative of $A1(x)$ between $x = 0$ and 1 (see Figure 5(b) and [S1]). If we solve the denominator of $\frac{d}{dx}(A1(x)) = 0$, we obtain $x = \frac{1}{6}$.

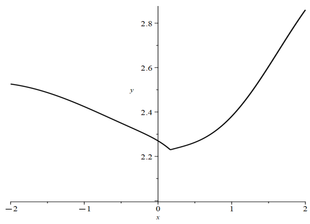

Figure 5(a) Plot of $A1(x)$with a sharp corner

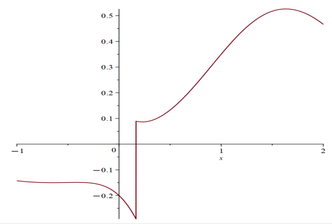

Figure 5(b). Plot of $\frac{d}{dx}(A1(x))$with a vertical tangent

2. We refer to reader to [S1], and see computationally that $P_1 = \left[\frac{1}{6}, \frac{\sqrt{179}}{6}\right]$ is a equal view-point from branch 1, which we demonstrate in Figure 6 below. We see if $A = P_1$, and angle $\theta_1$ is between two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ , and angle $\theta_2$ is between two vectors $\overrightarrow{AD}$ and $\overrightarrow{AE}$. Then

$$\theta_1 - \theta_2 = -4 \cdot 10^{-14}. \tag{20}$$

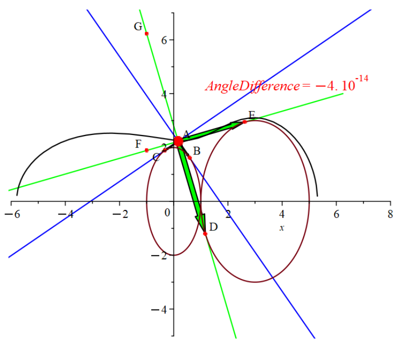


Figure 6. Plot of an equal viewpoint at $A$

3. Next, we find the start and end points for $A1(x)$. First, we obtain two $x - intercepts$ for branch 1, which are $x_1 = -5.8$ and $x_2 = 5.30758887812584$, and are start and end points for $A1(x)$ respectively.

4. We refer readers to [S1] to confirm that indeed $x_1 = -5.8$ is an equal viewpoint from computational point of view. We depict the plot in Figure 7:

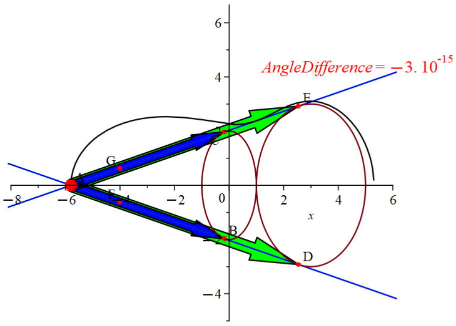


Figure 7. Plot of an equal viewpoint at $x_1 = -5.8$

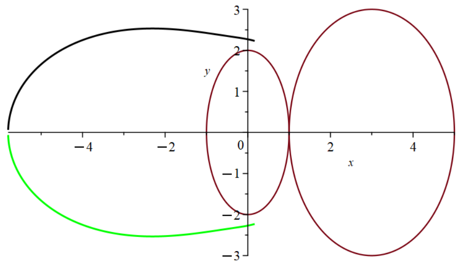


Figure 8. True locus of branch 1 and branch 2

5. It follows from Theorem (3) and its Corollary that points throughout the interval $(-5.8, \frac{1}{6}]$ are equal viewpoints. We make use of an animation program (See [S1]) to check if points in $(-5.8, \frac{1}{6}]$ are all equal viewpoints. It is worth noting the following:

```
an:=i->TestprocAnim(aa+bb*i/30,Re(evalf(AA1(aa+bb*i/30))));

plots-display(seq(an(i),i=1..30),insequence=true);
```

   (a) We remark that `aa:=-5.8;bb:=(1/6+5.8);` `where aa is the start point, bb is the length between aa and 1/6.`

   (b) In addition, we write a program and demonstrate computationally that the 30 sequence points $a_i$ in the interval $[-5.8, \frac{1}{6}]$ , where

$$a_i = \left(aa + \frac{bb \cdot i}{30}, A1\left(aa + \frac{bb \cdot i}{30}\right)\right),$$
$$i = 1, 2, ...30,$$

   are equal viewpoints. Since $x_1 = -5.8$ has been verified to be an equal viewpoint, we start with $i = 1$ (instead of $i = 0$) for $a_i$.

6. Similarly, we run the animation program in [S1] to verify that $A1$ does not have equal viewpoints in the interval $\left(\frac{1}{6}, 1\right)$. Furthermore, as we see in [S1] that the second $x - intercept$, $x_2 = 5.30758887812584$, is not an equal viewpoint. We use the animation program and see from Theorem (3) and its Corollary that all points throughout $(1, x_2]$ are not equal viewpoints either. On the one hand, $x \in L^T$ if $x \in [-5.8, \frac{1}{6}]$; on the other hand, $x \in L^N$, if $x \in \left(\frac{1}{6}, 1\right) \cup (1, x_2]$, which reinforces the idea of $P_1 = \left[\frac{1}{6}, \frac{\sqrt{179}}{6}\right]$ being a turning point (from $L^T$ to $L^N$).

7. We recall that branch 2 is symmetric to branch 1 with respect to the $x-$axis, and we will obtain similar results as those from branch 1. Therefore, we plot branch 1 and branch 2 in the interval $[-5.8, 1]$ together with $E_1$ and $E_2$ in Figure 8.

**Branch 5**

1. We first find the $x - intercepts$ for branch 5 by solving $A5(x) = 0$, and yield $x = -1.72249803155844$ and $x = 1$ respectively.

2. Analogous to $P_1 = \left[\frac{1}{6}, \frac{\sqrt{179}}{6}\right]$ being the turning point for branch 1, readers can explore and conclude that $P_1$ is a turning point for branch 5. We also refer readers to [S1] that if $x \in [-1.72249803155844, \frac{1}{6})$, $x$ is not an equal viewpoint for branch 5. We only depict that the $x = -1.72249803155844$ is not an equal viewpoint, and the angle difference between $\theta_1$ and $\theta_2$ is not close to 0 in Figure 9:

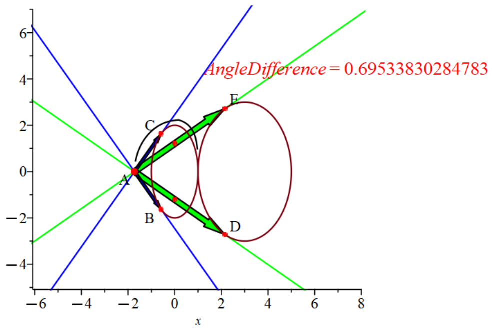


Figure 9. Plot of a non-equal viewpoint

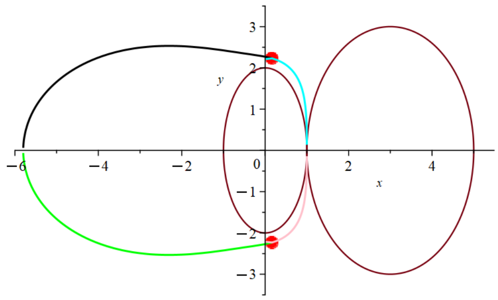


Figure 10(a). Turning points in red, $P_1$ and $P_2$

3. We remark that the intersecting point for the branch 1 and branch 5 is at $x = \frac{1}{6}$. It is worth noting that both $A5(x)$ and $A1(x)$ have a point of non-differentiability or turning point at $x = \frac{1}{6}$, and we verify that $x = \frac{1}{6}$ is indeed an equal viewpoint for $A5(x)$.

4. We remark that there is no contradiction that both $A1(x)$ and $A5(x)$ has equal viewpoint at $P_1 = \left[\frac{1}{6}, \frac{\sqrt{179}}{6}\right]$. As we have mentioned that the final locus should be continuous, it is expected that $\left(\frac{1}{6}, A1\left(\frac{1}{6}\right)\right) = \left(\frac{1}{6}, A5\left(\frac{1}{6}\right)\right)$. Similar conclusions can be drawn for $A2(x)$ and $A6(x)$.

5. We use an animation program and verify that $A5(x)$ does have equal viewpoints in the interval $[\frac{1}{6}, 1]$. Thus, point $P_1 = \left[\frac{1}{6}, \frac{\sqrt{179}}{6}\right]$ is indeed a turning point for branch 5 since $x \in L^N$ if $x \in [-1.72249803155844, \frac{1}{6})$, and $x \in L^T$ if $x \in \left[\frac{1}{6}, 1\right]$.

6. We recall that branch 6 is symmetric to branch 5 respective to the $x - axis$. The true locus plot for this problem is to combine all the equal viewpoints from branch 1 (in black), branch 2 (in green), branch 5 (in cyan), branch 6 (in pink), and the turning points $P_1$

and $P_2$ (in red), which we show in Figure 10(a). We also depict the plot of those points that are not equal viewpoints in Figure 10(b):

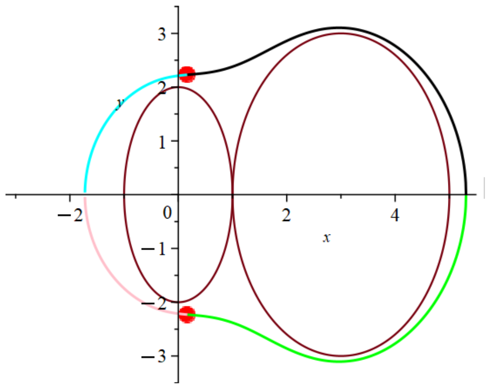


Figure 10(b). Points in $L^N$.

# 4 Uniform scaling on a locus

We recall that rigid transformations are geometric transformations that preserve distances and angles. For example, translations, rotations, and reflections are examples of rigid transformations. It is easy to see that the locus is invariant under the rigid transformations or compositions of any rigid transformations. In addition, uniform scaling is a linear transformation where all dimensions of a shape are scaled by the same factor. We prove that if the compressions or expansions is applied on both (5) and (6) with the same rate in both $x$ and $y$ directions, then we can adjust the locus Eq. (11) (with the same rate in both $x$ and $y$ directions) accordingly without calculating the new locus for two respective new equations. First, the quasi-locus equation (11) is valid under shifting and rotations. Next, we prove the following about uniform scaling, where we show that the locus Eq. (11) can be adjusted accordingly if the compressions or expansions is applied on both (5) and (6) with the same rate in both $x$ and $y$ directions.

**Theorem 4** *Suppose $a, b \neq 0$, and consider two ellipses after we substitute $x$ by $a \cdot x$ and $y$ by $b \cdot y$ in both (5) and (6). We then compute its quasi-locus equation directly, and call it $L^1_{ab}$. On the other hand, we substitute $x = a \cdot x$ and $y = b \cdot y$ directly in the quasi-locus Eq. (11), and call it $L^2_{ab}$. Then we see $L^1_{ab} \neq L^2_{ab}$. However, if we choose $a = b \neq 0$, then we have*

$$\frac{L^1_{aa}}{a^8} = L^2_{aa}.$$

**Proof:** We first substitute $x$ by $a \cdot x$ and $y$ by $b \cdot y$ in both (5) and (6) to obtain two new ellipses, and follow the processes of constructing the quasi-locus equation for these two new ellipses. It can be shown that $L^1_{ab}$ is the same as when setting the determinant of the following matrix equal to 0. In other words, if

$$M_{ab} = \begin{bmatrix} a^2b^2a_1^2b_1^2\left(\frac{(ax)^2}{a_1^2}+\frac{(by)^2}{b_1^2}-1\right) & a^2b^2a_2^2b_2^2\left(\frac{(ax-c)^2}{a_2^2}+\frac{(by-d)^2}{b_2^2}-1\right) \\ \left(b^2a_1^2+a^2b_1^2-a^2b^2\left(x^2+y^2\right)\right)^2 & \left(b^2a_2^2+a^2b_2^2-\left(b^2\left(ax-c\right)^2+a^2\left(by-d\right)^2\right)\right)^2 \end{bmatrix}.$$

Then

$$L^1_{ab} = \det M_{ab} = 0.$$

On the other hand, if we replace $x$ by $ax$, and $y$ by $by$ in (12) and denote it by (21).

$$M_{dab} = \begin{bmatrix} a_1^2 b_1^2 \left( \frac{(ax)^2}{a_1^2} + \frac{(by)^2}{b_1^2} - 1 \right) & a_2^2 b_2^2 \left( \frac{(ax-c)^2}{a_2^2} + \frac{(by-d)^2}{b_2^2} - 1 \right) \\ \left( a_1^2 + b_1^2 - \left( (ax)^2 + (by)^2 \right) \right)^2 & \left( a_2^2 + b_2^2 - \left( (ax-c)^2 + (by-d)^2 \right) \right)^2 \end{bmatrix}. \tag{21}$$

Then we see $L^2_{ab} = \det M_{dab} = 0$. Finally, if we define two functions $f(a, b, a_1, b_1, a_2, b_2, c, d, x, y) = \det\ M_{ab}$ and $g(a, b, a_1, b_1, a_2, b_2, c, d, x, y) = \det\ M_{dab}$. Then it can be shown, using a computer algebra system such as (see [S2]), that only when $a = b$, will we see

$$\frac{f(a, a, a_1, b_1, a_2, b_2, c, d, x, y)}{a^8} = g(a, a, a_1, b_1, a_2, b_2, c, d, x, y),$$

see ([S2]). Since $a \neq 0$, the proof is complete. ■

## 4.1 Exercises

1. We consider the case of two non-intersecting and different sizes of circles. For example, we consider the unit circle $C_1$ of $x^2 + y^2 - 1 = 0$, and $C_2$ of $(x-4)^2 + y^2 = 4$. We follow the procedure in this paper to find that the true locus is a circle of center $\left(-\frac{4}{3}, 0\right)$ and radius of $\frac{8}{3}$, which is identical to Eq. (3) where $m = \frac{1}{2}$, center is $\left(\frac{m^2 x_2 - x_1}{m^2 - 1}, \frac{m^2 y_2 - y_1}{m^2 - 1}\right)$, and radius $\left|\frac{m}{m^2-1}\right| \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

2. We consider two intersecting circles with different radii of circles. For example, we consider the unit circle $C_1$ of $x^2 + y^2 - 1 = 0$, and $C_2$ of $(x-2)^2 + y^2 = 4$. The quasi-locus (in cyan) and respective circles (in red and blue respectively) for this problem is sketched in Figure 11(a), which we see that Eq. (3) does not work in this case. Nevertheless, readers should be able to sketch and conjecture the true locus from the quasi-locus after discussions from this paper. [Hints: 1. The portion inside the intersections between two given circles in cyan cannot belong to true locus by definition. 2. Try the circle using Eq. (3) minus the portion inside the intersections between two given circles. See the true locus shown in

dark green in Figure 11(b).]

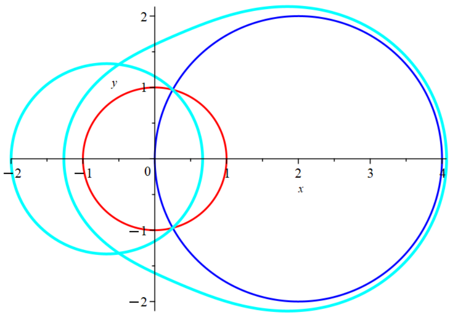

Figure 11(a). The plot of quasi-locus in cyan and two respective circles

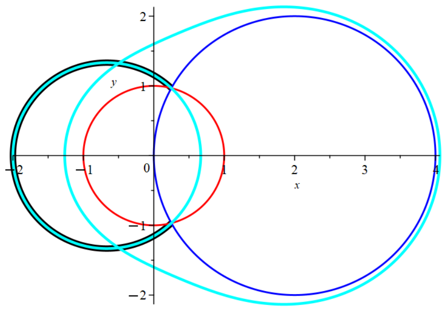

Figure 11(b) True locus shown in dark green

## 5 Discussions and future work

In this paper, the quasi-locus $L^Q = L^T \cup L^N$ is a continuous curve for two ellipses, and we observe the Figures 10(a) and (b) of Section 3.1 that both $L^T$ and $L^N$ are continuous curves. It is natural to ask if we can express $L^T$ or $L^N$ in parametric form, so that a point $A$ is either traveling continuously on curve $L^T$ or continuously on $L^N$.We may adopt the strategies mentioned in this paper to explore finding true locus for three or more closed smooth curves of degree 2. Subsequently, we can explore finding an equal viewpoint that will work for infinitely many circles. We certainly can extend the method discussed in this paper to find the true locus for two conic curves such as combinations of ellipses, parabolas and hyperbolas. We expect the complexity of extracting the true locus from quasi-locus when involving the asymptotes. Moreover, it remains to be seen if we can extend finding the equal viewpoints for two smooth curves with even degrees that are greater than 2.

Finally, for the 3D extensions, we shall explore the 3D equal viewpoints for spheres. In view of the original definition and Figure 2 in 2D, we let $A$ be an equal viewpoint for two spheres $S_1$ and $S_2$. Then $A$ is the vertex for two cones $N_1$ and $N_2$ with angle $\theta$ such that $S_i$ is tangent to $N_i, i = 1, 2$, respectively.

## 6 Conclusions

We see that the original locus problem discussed in this paper is simple and the algebraic partial solution provided by most students in Section 1.2 is accessible but incomplete. Nevertheless, it is indeed difficult task to tackle the quasi-locus, which is a degree 6 polynomial equations, and analyze the corresponding final true locus without the help of a computer algebra system. We recall that a uniform shear can be represented as a combination of rotation, uniform scaling, and another rotation. In this paper, we see that suppose we obtain the quasi-locus plot $L^Q$ for the ellipses, $E_1$ (5) and $E_2$ (6), and if we apply a uniform shear map $f$ on the ellipses, $E_1$ (5) and $E_2$ (6), then the quasi-locus plot for $f(E_1) \cup f(E_2)$ will be $f\left(L^Q\right)$. In other words, we don't need to calculate the quasi-locus for $f(E_1) \cup f(E_2)$ all over again. This locus problem is a strong example demonstrating why students need to be technologically competent before they graduate, so they can handle many real-life challenging problems. In addition, we see how a simple problem can lead to serious investigations after exploring with the help of technological tools.

## 7 Acknowledgements

The author would like to thank Harald Pleym of Norway for giving critical technical support for plotting branches of functions using the computer algebra system (see [3]) and creating animations for checking if $A \in L^T$ or $A \in A^N$ in an interval. In addition, the author would like to thank the following international colleagues for giving invaluable suggestions while solving the problem: Qiuxia Li, Douglas Meade, Sylvester Thompson, and Zoltan Kovacs. Finally, the author thanks reviewers for proofreading and pointing out important terminologies and concepts that are related to this article.

## 8 Supplementary Electronic Materials

[S1] Maple worksheet: https://atcm.mathandtech.org/EP2025/invited/22224/S1.mw

[S2] Proof of Theorem 4: https://atcm.mathandtech.org/EP2025/invited/22224/S2.mw

## References

[1] Circle of Apollonius, see https://en.wikipedia.org/wiki/Circle#Circle_of_Apollonius/.

[2] B. Odehnal, *Equioptic curves of conic sections*, Journal for Geometry and Graphics, January 2010, https://www.geometrie.tuwien.ac.at/odehnal/tr200.pdf.

[3] A product of Maplesoft, see http://Maplesoft.com/. The 2024 version.

[4] Th. Dana-Picard, G. Mann and N. Zehavi (2011): *From conic intersections to toric intersections: the case of the isoptic curves of an ellipse,* The Montana Mathematical Enthusiast 9 (1), pp. 59-76.

[5] Wolfram MathWorld: https://mathworld.wolfram.com/IsopticCurve.html.