Some Poncelet invariants for bicentric hexagons

Grant Keady and Alasdair McAndrew*
∗Corresponding Author: grantkeady@gmail.com,
Curtin University, Perth, Australia
and amca01@gmail.com
Victoria University, Melbourne, Australia

Abstract

Tangential polygons are (convex) polygons for which every side is tangent to an inscribed circle. Cyclic polygons are those for which every vertex lies on a circle, the circumcircle. Bicentric n-gons are those which are both tangential and cyclic. Every triangle is bicentric. Bicentric quadrilaterals are those for which the sum of the lengths of opposite sides is the semiperimeter and for which opposite angles sum to π. Here we give some results pertaining to invariants of (convex) bicentric hexagons.

A remarkable result of Poncelet is that if one has a pair of circles admitting a bicentric n-gon, then for every point on the circumcircle can be a vertex for a bicentric n-gon. This is illustrated in the animation at
`https://mathworld.wolfram.com/PonceletsPorism.html`

The animation indicates that, along with the incentre and circumcentre, the point of intersection of the principal diagonals of a $2m$-gon is invariant under the motion. Such invariants – here called Poncelet invariants – have been studied for two centuries, in particular for triangles and bicentric quadrilaterals. We present results, for bicentric hexagons, that various combinations of distances between vertices - lengths of diagonals and of sides - are invariant. CAS supplements are available for checking the results.

1 Introduction

Several years ago there was an ATCM Invited Talk on bicentric polygons and Poncelet's Porism [2]. The second author of this paper has presented on them too: [4, 5]. The area - Poncelet invariants, bicentric polgons - has a colourful history. It and adjacent problem areas, e.g. elliptic billiards, remain active areas of research: there is a huge and growing literature. References in this paper post-dating 2020 are [11, 6, 15, 14]. Some of the literature is very technical (e.g. [6, 3], the latter of which has a good historical account but in later parts is technical). One function of this paper is to illustrate, yet again, that classical polynomial algebra continues to yield results which, once found, could easily be explained to undergraduate mathematics students. Theorem 1 in 1.3 is an example.

Poncelet was an officer in Napoleon's Grande Arm'ee and was taken prisoner during the invasion of Russia in 1812. While in prison, he discovered "invariants of bicentric polygons" (and made really major contributions to projective geometry). See [1, 3].

The history, and the MathWorld animations referenced in the abstract, make some of the area appropriate in various undergraduate mathematics courses. For example, it might be an aside in a history of mathematics course when Poncelet is being treated.

1.1 Notation

Label the vertices of the (convex) polygon, n-gon, by numbers from 1 to n, increasing in the counter-clockwise direction. The interior angle at vertex j is denoted α_j: the distance between vertices i and j by d_{ij}. When we use Cartesian coordinates we will take the origin of coordinates at the incentre of the tangential polygon and denote the coordinates of the vertices by z_j. We have $d_{ij} = |z_i - z_j|$. From §1.4 onwards we take the inradius ρ to be 1 (with occasional reminders on this).

1.2 Symmetric polynomials

Given n angles α_j with $0 < \alpha_j < \pi$ there exists a tangential n-gon with these angles. Denote the inradius by ρ. If one fixes the incircle and permutes the 'ears', vertex angles, the n-gon remains tangential. The perimeter L and area A satisfy

$$L = 2\rho \sum_{j=1}^{n} T_j, \qquad A = \frac{1}{2}\rho L.$$

Amongst n-gons with sides $d_{j,j+1}$ there are cyclic n-gons each of which has the maximum area. Different cyclic n-gons, with the same circumradius R and area A are formed as one permutes the sides.

Elementary symmetric polynomials are appropriate in studies of these: polynomials $e_j(\mathbf{T})$ for tangential n-gons, with the obvious first example that e_1 is the semi-perimeter. For an example of the use of symmetric polynomials in connection with cyclic n-gons see the `mathworld.com` page on 'Cyclic Hexagon' and §3.2.

Bicentric n-gons are tangential n-gons which are also cyclic. Some of the invariants, e.g. that of equation (22) (and related results associated with $\cos(\alpha_j)$) are special cases of results first derived for tangential n-gons, e.g. equation (21). The key steps in moving from this result for tangential hexagons to bicentric hexagons is due to Radic, equations (3,4) and implications for the $e_j(\mathbf{T})$ given at equations (9). All invariants are functions of the single variable e_2.

1.3 Sides and the circumradius

As stated in the abstract, much is popularly known for triangles and bicentric quadrilaterals. The wikipedia entry for bicentric quadrilaterals notes that the product of the diagonals $(d_{13}\, d_{24})$ is a Poncelet invariant, specifically,

$$\frac{d_{13}\, d_{24}}{\rho^2} - \frac{4R^2}{d_{13}\, d_{24}} = 1, \qquad \text{for a bicentric quadrilateral ,}$$

where ρ is the inradius and R the circumradius. This re-writes to

$$4\,R^2 = d_{13}\, d_{24} \left(\frac{d_{13}\, d_{24}}{\rho^2} - 1 \right) \qquad \text{for a bicentric quadrilateral .} \tag{1}$$

From these one sees that the sum of products of opposite sides is also a Poncelet invariant as Ptolomy's Theorem for cyclic quadrilaterals states

$$S_4 = d_{12}d_{34} + d_{23}d_{14} = d_{13}\, d_{24} \qquad \text{for a cyclic quadrilateral .}$$

Pitot's Theorem gives

$$(d_{12} + d_{34}) - (d_{23} + d_{14}) = 0 \qquad \text{for a tangential quadrilateral .}$$

There are many invariants already discovered, some of which will be reviewed later in this paper. We prescribe the inradius ρ and the circumradius R. For a bicentric n-gon to exist the distance d between the incentre and the circumcentre has to satisfy a polynomial equation, Fuss's equation: see §1.6. So ρ, R and d are all fixed, invariant, as the bicentric n-gon traverses in its different positions. As noted in the preceding paragraph, there are other quantities which are invariant. Results on bicentric hexagons seem less well known than results for triangles and quadrilaterals. An example of the various items we discovered (and which we have not seen elsewhere - but might well be in nineteenth century literature) is as follows.

Theorem 1 *For bicentric hexagons the sum of the products of the opposite sides, that is* $S = d_{12}d_{45} + d_{23}d_{56} + d_{34}d_{16}$*, is a Poncelet invariant.*

Our (re?-)discoveries include relatively simple formulae for the circumradius of a bicentric hexagon. Perhaps the simplest is equation (2). This, like equation (1), just depends on products of the side-lengths of opposite sides and, when as always henceforth $\rho = 1$, is

$$R^2 = \frac{(8-S)S(4+S)^2}{192(S-2)^2}. \tag{2}$$

The formulae are more elaborate than the quadrilateral case of equation (1) but very much simpler than the elaborate formula for the circumradius of a general cyclic hexagon.

1.4 Radic's formulation and notation

Radic's formulation [8, 10] is here very slightly modified as we take the inradius $\rho = 1$. Let T_j be the tangent length at vertex j. With the angles at the vertices denoted by α_j we have $T_j = \cot(\alpha_j/2)$. For bicentric hexagons Radic gives the vector of tangent lengths as

$$\mathbf{T} = \left[T_1, \frac{(T_1+T_3)t_s}{1-T_1T_3}, T_3, \frac{t_s}{T_1}, \frac{1-T_1T_3}{T_1+T_3}, \frac{t_s}{T_3}\right], \tag{3}$$

$$t_s^2 = \frac{T_1T_3 - T_1^2T_3^2}{1+T_1^2+T_1T_3+T_3^2}, \qquad t_s > 0 \text{ (i.e. } 0 < T_1T_3 < 1). \tag{4}$$

(Radic's lower case t_j corresponds to my ρT_j.) The regular hexagon has $T_1 = T_3 = 1/\sqrt{3}$ and $t_s = 1/3$ and hence all entries in $\mathbf{T}$ have $T_j = 1/\sqrt{3}$. The constraints on T_1 and T_3 include that they are positive and $T_1\,T_3 < 1$, and this yields that $0 < t_s \le 1/3$.

We will not repeat Radic's derivation of (3,4). We observe, as did Radic, that $T_j\,T_{j+3} = t_s$ and this has the immediate consequence that the product of the 6 entries T_j being t_s^3, a fact we record in equation (8). There are several other consequences of equations (3,4), e.g.

$$T_1T_3 + T_3T_5 + T_5T_1 = 1 = T_2T_4 + T_4T_6 + T_6T_2, \quad \frac{T_1T_3T_5}{T_1+T_3+T_5} = t_s^2 = \frac{T_2T_4T_6}{T_2+T_4+T_6},$$

and, perhaps less obviously, equation (7).

If one regards t_s as given, equation (4) is a biquadratic for the pair (T_1, T_3). The biquadratic can be written as $e_2 = 2 + 1/t_s$ with e_k the k-th elementary symmetric polynomial in the T_j.

Radic's calculations lead to a simple relation between t_s and the circumradius R (having set $\rho = 1$),

$$R^2 = \frac{(1-t_s^2)(1+t_s)(1+3t_s)}{16t_s^2}, \tag{5}$$

$$= \frac{(e_2-3)(e_2-1)^2(e_2+1)}{16(e_2-2)^2}. \tag{6}$$

Another quantity treated by Radic is the distance $R_j = |z_j|$ of vertex j from the origin, the incentre. We have $R_j = \mathrm{cosec}(\alpha_j/2)$ and $R_j^2 = 1 + T_j^2$. On using this last equation with (3,4) we have

$$\frac{1}{R_1^2} + \frac{1}{R_3^2} + \frac{1}{R_5^2} = \frac{1}{R_2^2} + \frac{1}{R_4^2} + \frac{1}{R_6^2}. \tag{7}$$

1.5 Elementary symmetric polynomials

More can be said about the various e_k. The semiperimeter, or equivalently the area as $\rho = 1$, is e_1. That the sum of the angles of the hexagon is 4π accords with

$$e_1 - e_3 + e_5 = 0.$$

The elementary symmetric polynomials $e_j(\mathbf{T})$, besides satisfying the angle constraint $e_1 - e_3 + e_5 = 0$, satisfy

$$e_6 = t_s^3, \quad e_4 = 1 + 2\,t_s, \quad e_2 = 2 + \frac{1}{t_s}, \text{ and } \ e_5 = t_s^2 e_1, \quad e_3 = (1 + t_s^2)e_1. \tag{8}$$

This enables all the e_j to be expressed in terms of e_1 and e_2:

$$e_3 = \left(1 + \frac{1}{(e_2-2)^2}\right) e_1, \ e_4 = \frac{e_2}{e_2-2}, \ e_5 = \frac{e_1}{(e_2-2)^2}, \ e_6 = \frac{1}{(e_2-2)^3}. \tag{9}$$

We now have as invariants t_s, e_2, e_4, e_6 and e_3/e_1, e_5/e_1.

1.5.1 Palindromic polynomials

As e_j have been, at least since Newton, inextricably linked with questions associated with polynomials, we briefly note here – with the application to our problems being that given e_1 and e_2 (area and circumradius along with $\rho = 1$) one can find the T_j from a palindromic equation. This is very obvious for bicentric quadrilaterals as one has $e_1 = e_3$ and $e_4 = 1$ so the equation satisfied by the T_j is

$$T^4 - e_1 T^3 + e_2 T^2 - e_3 T + 1 = 0 \ \text{ for bicentric quadrilaterals.}$$

For bicentric hexagons the equation satisfied by the T_j is

$$T^6 - e_1 T^5 + e_2 T^4 - \frac{(5 - 4e_2 + e_2^2)e_1}{(e_2-2)^2} T^3 + \frac{e_2}{e_2-2} T^2 - \frac{e_1}{(e_2-2)^2} T + \frac{1}{(e_2-2)^3} = 0.$$

From T define $\hat{T} = T\sqrt{e_2 - 2}$. Then $\hat{T}$ satisfies a palindromic equation, which one solves (reducing the problem to solving a cubic in u) with the substitution $u = \hat{T} + 1/\hat{T}$. This accords with that, if T_j is a tangent length, then so is $T_{j+3} = t_s/T_j$. In any event, specifying the bicentric hexagon with $\rho = 1$ through e_2 giving the circumradius and other invariants, and the area/semi-perimeter e_1 to give particular instances, seems natural with the additional reassurance that (e_1, e_2) gives a solvable sextic for the tangent lengths.

1.5.2 Biquadratic equations

This is an aside from our main aim of presenting invariants. However the readership of ATCM have diverse backgrounds so some mention of the widely applied biquadratic equations is perhaps justified. From equations (4) and (8) we have the symmetric biquadratic equation

$$1 + T_1^2 + T_3^2 - (-3 + e_2)(-1 + e_2)T_1 T_3 + (-2 + e_2)^2 T_1^2 T_3^2 = 0. \tag{10}$$

The Poncelet traverses, geometrically visualised as in the MathWorld animation, map 1-1 and onto to points (T_1, T_3) moving around the level curve (10). There are Jacobian elliptic function parametrisations of the level curve, due to Jacobi and Cayley. (Amongst the many applications of biquadratic equations we mention a couple. They are used in connection with QRT-maps in the theory of discrete integrable systems. They are also used to provide solutions of the Yang-Baxter equation.)

1.6 Distance d between the incentre and circumcentre

With the inradius ρ set to one, Fuss's equation is

$$3(R^2 - d^2)^4 - 4(R^2 + d^2)\,(R^2 - d^2)^2 - 16R^2d^2 = 0.$$

See, for example, [1, 4, 5]. A short calculation verifies that with R^2 given by (5) or by (6)

$$d^2 = \frac{(-5+e_2)(-3+e_2)^2(-1+e_2)}{16(-2+e_2)^2} = \frac{(1-t_s)^2(1+t_s)(1-3t_s)}{16t_s^2},$$

satisfies Fuss's equation.

1.7 First invariants involving the sides

Our main results later in this paper concern invariants related to geometric quantities, distances and angles associated with the bicentric hexagon. Those in this subsection are well-known.

The distance matrix DM is the matrix with entries $d_{i,j}$ and for any cyclic n-gon, at least with $3 \le n \le 6$, the determinant of DM is a numerical factor times the product of the side lengths $d_{j,j+1}$. See Theorem 2. With the inradius $\rho = 1$ we have $d_{j,j+1} = T_j + T_{j+1}$ and using (3,4) we have, concerning the sides,

$$d_{12} + d_{34} + d_{56} = e_1 = d_{23} + d_{45} + d_{16} \quad \text{and} \quad d_{12}\,d_{34}\,d_{56} = t_s\,(1+t_s)e_1 = d_{23}\,d_{45}\,d_{16}. \tag{11}$$

Thus the quantity $d_{12}\,d_{34}\,d_{56}/(d_{12} + d_{34} + d_{56})$ is invariant. In other words, the product of the sides divided by the square of the perimeter is invariant.

A classical result concerning the principal diagonals of cyclic hexagons is Fuhrmann's Theorem which we have used to check some calculations.

$$d_{12}\,d_{45}\,d_{36} + d_{23}\,d_{56}\,d_{14} + d_{34}\,d_{16}\,d_{25} + d_{12}\,d_{34}\,d_{56} + d_{23}\,d_{45}\,d_{16} - d_{14}\,d_{25}\,d_{36} = 0. \tag{12}$$

We will revisit this in §3.1:

1.8 Examples of bicentric hexagons

Once one has fixed ($\rho = 1$ and) R, equivalently e_2, if one wishes to find numeric values of some invariant, it suffices to evaluate it at some particular bicentric hexagon with that e_2. Favourite examples for this are shown in Figure 1.

For these bicentric hexagons at given $e_2 \ge 5$ (i.e. given R), as shown by Radic [9] (see his Figures 3.1a and 3.1b), the area (semiperimeter) e_1 takes extreme values for the hexagons shown in Figure 1. We can also plot how these bounds on e_1 change as e_2 changes, and note, in particular, as $e_2 \to 5 = e_{2*}$ that the allowed interval gets smaller and both bounds tend to $2\sqrt{3} = e_{1*}$. Thus, for bicentric hexagons,

$$12 + 4(e_2 - 5) = 4e_2 - 8 \le e_1^2 \le \frac{(e_2-3)(e_2+1)^3}{4(e_2-2)^2} = 12 + (e_2 - 5) + \frac{(e_2-5)^3(e_2-1)}{4(e_2-2)^2}. \tag{13}$$

The two formulae for the areas of these two bicentric hexagons, presented as functions of R, ρ and d, are given in [7].

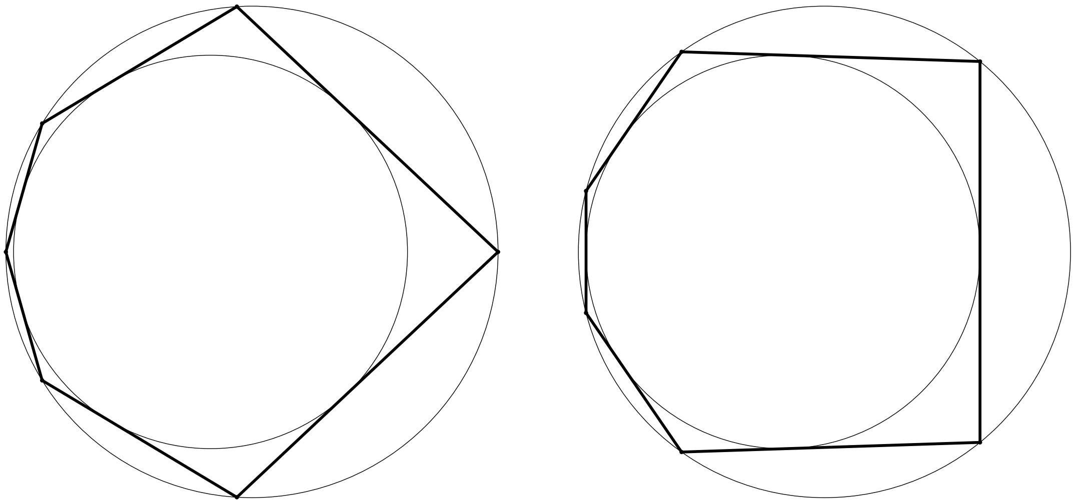

Figure 1: Bicentric hexagons at left with maximum area $e_1 = 3.65198$; at right with minimum area, 3.65148. Here $\rho = 1$ and $t_s = 3/10$, $e_2 = 16/3$.

2 Distance geometry and distance matrices

We do not need any of the various properties of distance matrices, but have used them, via Theorem 2, as checks in our code. As regards work in this paper, consider the distance matrices as convenient record keeping for distances used in the various applications.

2.1 An observation concerning cyclic polygons

Theorem 2 *(i) The determinant of the distance matrix $DM(n)$ for a set of n points on a circle with $3 \leq n \leq 6$ satisfies*

$$\det(DM(n)) = -(-2)^{n-2} \prod_{j=1}^{n} d_{j,j+1}. \tag{14}$$

For a cyclic n-gon the product on the right hand side is the product of the side lengths of the n-gon.
(ii) The Euclidean distance matrix $EDM(n)$, the matrix with entries the squares of those in $DM(n)$, for a cyclic n-gon with $3 \leq n \leq 6$ has rank 3.

At $n = 4$, items (i) and (ii) are equivalent to Ptolomy's Theorem. We have no reason to believe that the restriction $n \leq 6$ is necessary. However the proof we have requires n to be bounded (and within the memory and time limits of our computer and algebra packages). Our proof starts from the z_j on a circle, and continues with brute calculation.

2.2 The distance matrix of a bicentric hexagon

The side lengths are given (when $\rho = 1$) by $d_{j,j+1} = T_j + T_{j+1}$. The "2-diagonals" for a bicentric n-gon with circumradius R are given by $d_{j-1,j+1} = 4RT_j/(1 + T_j^2)$. The principal diagonal d_{14} is found by applying Ptolomy's Theorem to the cyclic quadrilateral with vertices 1, 2, 3 and 4. The other principal diagonals are found in the same way. We have

$$\begin{aligned} d_{14} &= \frac{d_{13}d_{24} - d_{12}d_{34}}{d_{23}} = \frac{d_{15}d_{46} - d_{16}d_{45}}{d_{56}}, \\ d_{25} &= \frac{d_{15}d_{26} - d_{12}d_{56}}{d_{16}} = \frac{d_{24}d_{35} - d_{45}d_{23}}{d_{34}}, \\ d_{36} &= \frac{d_{13}d_{26} - d_{23}d_{16}}{d_{12}} = \frac{d_{35}d_{46} - d_{34}d_{56}}{d_{45}}. \end{aligned}$$

$$DMbi(6) = \begin{bmatrix} 0 & T_1 + T_2 & \frac{4RT_2}{1+T_2^2} & d_{14} & \frac{4RT_6}{1+T_6^2} & T_1 + T_6 \\ T_1 + T_2 & 0 & T_2 + T_3 & \frac{4RT_3}{1+T_3^2} & d_{25} & \frac{4RT_1}{1+T_1^2} \\ \frac{4RT_2}{1+T_2^2} & T_2 + T_3 & 0 & T_3 + T_4 & \frac{4RT_4}{1+T_4^2} & d_{36} \\ d_{14} & \frac{4RT_3}{1+T_2^3} & T_3 + T_4 & 0 & T_4 + T_5 & \frac{4RT_5}{1+T_2^5} \\ \frac{4RT_6}{1+T_6^2} & d_{25} & \frac{4RT_4}{1+T_4^2} & T_4 + T_5 & 0 & T_5 + T_6 \\ T_1 + T_6 & \frac{4RT_1}{1+T_1^2} & d_{36} & \frac{4RT_5}{1+T_2^5} & T_5 + T_6 & 0 \end{bmatrix}.$$

One check we have performed is substituting the values we have, on using (3,4) in $d_{j,j+1}$ and $d_{j-1,j+1}$ in the above, and then using these values, and (5), in Fuhrmann's equation (12) we find the Fuhrmann equation is satisfied.

3 Invariants

3.1 Preliminary notes on invariants

We begin with a brief review of the setting. The inradius $\rho = 1$ and circumradius $R \geq 2/\sqrt{3}$ are given, and we suppose the incentre is at the origin of coordinates and the circumcentre is at $(d, 0)$ where the invariant d is given by Fuss's equation. The circumradius R of a bicentric hexagon is related to e_2 or t_s by equation (5).

We already have e_4 and e_6 and e_3/e_1 and e_5/e_1 as invariants.

Points with stay fixed during a Poncelet traverse are of interest. Results, obtained by others, include the following.

For an even-sided [bicentric] polygon, the diagonals are concurrent at the limiting point of the two circles. This is treated at the following pages.
`https://math.stackexchange.com/questions/3604700/the-concurrency-property-of-the-diagonals-sem`
`https://mathworld.wolfram.com/LimitingPoint.html`
The concurrency point of the diagonals is invariant, and is on the line joining the incentre and circumcentre. The stackexchange page gives references for proofs:
Lachlan, Modern Pure Geometry, 1893, pg 217, Ex. 4.
Halbeisen and Norbert, A Simple Proof of Poncelet's Theorem, 2014, Theorems 4.1,4.2.

The mathworld page gives the distance x' of the limiting point from the circumcentre as

$$x' = \frac{d^2 - \rho^2 + R^2 \pm \sqrt{(d^2 - \rho^2 + R^2)^2 - 4d^2R^2}}{2d}.$$

Special case of Weill's Theorem. *Let ζ_j denote the tangency points. Then the centre of mass $(\sum \zeta_j)/n$ is fixed during a Poncelet traverse.*

The centres of mass of the vertices $(\sum z_j)/n$ traverse a circle, and its centre remains fixed during a Poncelet traverse. See [12]

3.2 Invariants involving $d_{i,j}$ and $\cos(\alpha_j)$

We remark that angle information can be found from distances. With θ_{ikj} denoting the angle between (i,j) and (j,k),

$$\cos(\theta_{ikj}) = \frac{d_{ik}^2 + d_{kj}^2 - d_{ij}^2}{2\, d_{ik}\, d_{kj}}.$$

In particular

$$\cos(\alpha_k) = \frac{d_{k-1,k}^2 + d_{k,k+1}^2 - d_{k-1,k+1}^2}{2\, d_{k-1,k}\, d_{k,k+1}}.$$

3.2.1 Involving just the sides $d_{j,j+1}$

Several results have been stated before. See equation (11) for results in Radic's papers. Here we add the observation that

$$\frac{d_{12}d_{34}d_{56}}{d_{12}+d_{34}+d_{56}} = t_s(1+t_s), \tag{15}$$

is a Poncelet invariant. The proof of the invariance stated in Theorem 1 is just a calculation verifying

$$S = d_{12}d_{45} + d_{23}d_{56} + d_{34}d_{16} = 2(1+3t_s). \tag{16}$$

Let $\mathbf{d} = [d_{j,j+1}]$ and also write $\mathbf{d}^2 = [d_{j,j+1}^2]$/ Consider first elementary symmetric functions of $\mathbf{d} = [d_{j,j+1}]$. We have, from equation (15),

$$e_1(\mathbf{d}) = 2e_1, \qquad \frac{e_6(\mathbf{d})}{e_1^2} = t_s^2(1+t_s)^2.$$

Some other invariants are $e_3(\mathbf{d})/e_1$ and $e_5(\mathbf{d})/e_1$. There are polynomial relations between the $e_j(\mathbf{d})$ but these seem more complicated than those between the $e_j(\mathbf{T})$ given at equations (9).

The quantities $\sigma_j(\mathbf{d}) = e_j(\mathbf{d}^2)$ are used in connection with formulae for the area of cyclic hexagons. See `https://mathworld.wolfram.com/CyclicHexagon.html`
Clearly $\sigma_6(\mathbf{d}) = e_6(\mathbf{d})^2$.

There are easier results coming from not restricting attention to just sides, but considering diagonals too.

3.2.2 Involving also the diagonals $d_{j,j+2}$ and $d_{j,j+3}$

Theorem 3 *The 2-diagonals of a bicentric hexagon satisfy:*

$$\frac{\sum d_{j-1,j+1}}{\sum d_{j,j+1}} = \frac{\sum d_{j-1,j+1}}{2e_1} = \frac{4t_sR}{1-t_s^2}, \tag{17}$$

$$\prod d_{j-1,j+1} = (4R)^6 \frac{t_s^5}{(1-t_s^2)^4}. \tag{18}$$

Theorem 4 *The principal diagonals of a bicentric hexagon satisfy*

$$\frac{\sum_{j=1}^3 d_{j,j+3}}{\sum_{j=1}^6 d_{j,j+1}} = \frac{\sum_{j=1}^3 d_{j,j+3}}{2e_1} = \frac{1+t_s}{2(1-t_s)}, \tag{19}$$

$$X = \frac{d_{14}\, d_{25}\, d_{36}}{e_1} = \frac{(1+t_s)^3}{1-t_s}. \tag{20}$$

Use Fuhrmann's Theorem (12) and then equation (15) and (20) to give

$$\frac{d_{12}\, d_{45}\, d_{36} + d_{23}\, d_{56}\, d_{14} + d_{34}\, d_{16}\, d_{25}}{e_1} = X - 2t_s(1+t_s) = \frac{(1+t_s)(1+3t_s^2)}{1-t_s}.$$

3.3 Invariants involving R_j and $\cos(\alpha_j)$

We have, for any tangential n-gon,

$$\cos(\alpha_j) = 1 - \frac{2}{R_j^2}, \qquad R_j^2 = \frac{2}{1-\cos(\alpha_j)} = \frac{1}{\sin(\alpha_j/2)^2} = 1 + T_j^2.$$

Specializing to $n = 6$,

$$\sum_{j=1}^{6} \frac{1}{R_j^2} = \frac{e_5(\mathbf{R}^2)}{e_6(\mathbf{R}^2)} = \frac{2(-3+2e_2-e_4)}{-1+e_2-e_4+e_6}. \tag{21}$$

From this and equations (7) and (9) we have, for a bicentric hexagon,

$$\frac{1}{R_1^2} + \frac{1}{R_3^2} + \frac{1}{R_5^2} = \frac{2}{1-t_s^2} = \frac{1}{R_2^2} + \frac{1}{R_4^2} + \frac{1}{R_6^2}. \tag{22}$$

3.4 The circumradius in terms of $d_{i,j}$

Just as in the case of the bicentric quadrilateral, the circumradius-squared can be expressed in terms of the sum of products of opposite sides, S as in equation (16). This equation gives $t_s = (S-2)/6$ and this in equation (5) gives, with $2 < S \le 4$, the equation (2) stated in the Introduction.

One can find other formulae for R^2 using different combinations of the d_{ij}, e.g. from equations (5,15,18).

3.5 More invariants involving vertex angles α_j

$$\begin{aligned}\sum \cos(\alpha_j) &= 6 - \frac{8}{1-t_s^2} = -\frac{2(1+3t_s^2)}{1-t_s^2}, \\ &= -3 + \frac{(e_2-5)(e_2+1)}{(e_2-3)(e_2-1)}.\end{aligned}$$

$$\sum \cos(\alpha_j + \alpha_{j+1}) = 6 - \frac{4(1+3t_s)}{1-t_s^2}.$$

$$\sum \cos(\alpha_j + \alpha_{j+1} + \alpha_{j+2}) = 6 + \frac{8((-1+5t_s^2) + t_s(1+3t_s^2))}{(1-t_s^2)^2}.$$

More general results, for bicentric n-gons, not merely hexagons, are given in [11].

4 Conclusion

Our conclusion is completely consistent with one of ATCMs themes: that Computer Algebra is immensely useful in both checking the impressive algebraic results of past generations and also for discovering, possibly new, results, for example, our theorems. Questions involving elliptic billiards, and rays, sometimes are similar to, or related to, the bicentric polygon work, e.g. [15, 14] and [6, 11, 3].

There are, of course, many open questions. There are results for triangles and bicentric quadrilaterals aligning with those presented here for bicentric hexagons. However we do not have them for bicentric pentagons, geometries presented, at least in special cases, at earlier ATCMs, see [13].

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