

Topological Structures Between Two Closed Surfaces Inspired by An Entrance Exam Problem

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Abstract

We shall explore with technological tools on a problem we posed in [8], which originated from a college practice entrance question (see [3]). In this paper, we are investigating whether a certain closed surface – inspired by the aforementioned college exam problem and an affine transformation – is topologically equivalent to a sphere, utilizing the three-dimensional visualization capabilities of computer technology to aid our initial analysis. This technological affordance not only makes our investigation accessible to readers with just an undergraduate mathematics background but also allows us to study this surface in detail, finally leading us to a rigorous solution to the problem. Our work highlights the essential role that technology plays in advancing and communicating mathematical research.

1 Preliminary

We first describe the college practice entrance problem (see [3]):

We are given a fixed circle and a fixed point A in the interior of the circle $(x-a)^2 + (y-b)^2 = r^2$. A line passes through A intersects the circle at C and D respectively, and the point E is the midpoint of CD . Find the locus E when C moves along the circle.

This problem has been discussed using parametric equation approaches in 2D and 3D, see ([4], [5], [6], [7]). The locus E lies on CD and satisfies

$$\overrightarrow{ED} = s \cdot \overrightarrow{CD}, \quad (1)$$

where s is a given real number. For convenience, we sometimes write it as

$$E = sC + (1 - s)D. \quad (2)$$

We need some terminology before we start. We recall that a topological space is a non-empty set together with a topological structure that is given by the open sets of the topology. A non-empty set X can be equipped with different topological structures, and any subset $A \subset X$ can be endowed with an inherited topological structure, namely, the *relative topology*: the open sets in the relative topology on A are exactly the subsets of the form $A \cap O$ for some open sets O of X .

For the spaces $\mathbb{R}^n$, with coordinates $(x_1, \dots, x_n)$, the usual topological structure is given by a base of open balls, which is defined by the metric space structure of $\mathbb{R}^n$. Recall that the *metric function* $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$, for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ makes $(\mathbb{R}^n, d)$ a *metric space*. This metric gives rise to open balls $B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid d(x, y) < \varepsilon\}$, where ε is a positive real number, which are the basic open sets for the usual topology on $\mathbb{R}^n$, i.e., every open set of $(\mathbb{R}^n, d)$ can be written as the union of some open balls.

Thus, any curve or any hyper-surface in $\mathbb{R}^n$ can be viewed as topological spaces with their respective relative topologies, acquired from their environment space. For the cases we are interested in, the environment spaces will be simply $\mathbb{R}^2$ or $\mathbb{R}^3$; thus, we can view these curves and surfaces as relative topological spaces on which we can apply topological tools that are introduced in what follows.

Now, let us recall a fundamental concept in topology, namely that of *continuity*. Let (X, τ_X) and (Y, τ_Y) be two topological spaces, with τ_X and τ_Y their respective topologies. A function $f : X \longrightarrow Y$ is *continuous* if for any open set $O \subset Y$ it holds that $f^{-1}(O) \subset X$ is open in X . Here, the notation $f^{-1}(U)$ denotes the inverse image, in X , of $U \subset Y$.

When f is bijection, then it has an inverse $f^{-1} : Y \longrightarrow X$ defined by: for any $y \in Y$, $f^{-1}(y) = x \in X$ if and only if $f(x) = y$. We say that f is a *homeomorphism* if it is a continuous bijection and with f^{-1} also being continuous. In this situation, X and Y are said to be *homeomorphic* topological spaces.

Homeomorphisms between topological spaces are also called *continuous transformations*, or *topological transformations*. Intuitively, these transformations can be thought of as functions that transforms points of one space, that are arbitrarily close to each other, onto points of another space that are also arbitrarily close. Spaces that are related in this way are said to be *topologically equivalent*. Proving that two spaces are homeomorphic requires one to find a homeomorphism between them. Now proving that two spaces are *not* homeomorphic is much harder since one must establish that no such homeomorphism between them exists. A common technique to achieve this non-existence crucially relies on the concept of connectedness, which we introduce now. Let X be a topological space. Suppose that $X = U \cup V$ for some open sets U and V with $U \cap V = \emptyset$, we say that X is *disconnected*. Otherwise, X is *connected*. This definition of connectedness is not intuitive and hence harder to use. A more natural version of connectedness is to require that any two points in the space be connected with a curve. Let X be a topological space. A continuous function $f : [0, 1] \longrightarrow X$ is called a *path*. We say that

X is *path-connected* if for any two points $x, y \in X$, there exists a path $f : [0, 1] \rightarrow X$ such that $f(0) = x$ and $f(1) = y$. An important and well-known result is that every path-connected space is connected.

Returning to the task of establishing certain two spaces are not homeomorphic, we use two well-known topological spaces of the figure-8, denoted by E , and the circle C as illustration of this topological proof technique. Let p be the point on E where it crosses itself. Suppose there is a homeomorphism $f : E \rightarrow C$. Then the restriction map $\hat{f} : E - \{p\} \rightarrow C - \{f(p)\}$ is again a homeomorphism; whence the spaces $E - \{p\}$ and $C - \{f(p)\}$ would be topologically equivalent. Now, $E - \{p\}$ is disconnected as its two disjoint components are the circles. But $C - \{f(p)\}$ is path-connected and hence connected, thus we arrived at the desired contradiction.

It is intuitively evident that all simple closed curves in the plane and all closed polygons are topologically equivalent to a circle. Similarly, all ellipsoids, closed cylinders, closed cones, convex polyhedra, and other simple closed surfaces are equivalent to a sphere. We assume all these aforementioned topological equivalences in the ensuing development.

Recall that any surface is a topological space whose topological structure is inherited from the ambient space $\mathbb{R}^3$ it lives in. Also, two surfaces are topologically equivalent if there exists a homeomorphism f between them, where these surfaces are viewed as two topological spaces. Such a homeomorphism f is a bijection with the property that both f and its inverse f^{-1} preserve open sets. If a certain geometric object is transformed into another topologically equivalent by bending, stretching, etc., this process entails a special type of topological transformation called a *continuous deformation*.

In fact, we could refer to the surfaces we are working with in this paper as *topological manifolds* since, as subsets of $\mathbb{R}^3$, they comply with all the conditions for that kind of manifolds: being Hausdorff topological spaces, with a covering of open sets, each one homeomorphic to an open set in $\mathbb{R}^3$ and in the intersections of the sets of the covering, the *transition functions* are also homeomorphisms from a subset of the surface onto another subset of the surface. Thus, in what follows we shall be dealing with surfaces that are topological manifolds and we will refer to them with such a terminology.

1.1 Surfaces of perturbing the longitude and latitude affect the topological structures

Suppose S is a given parametrized surface, say

$$S = \{ (f(u, v), g(u, v), h(u, v)) \mid u, v \in R \}, \quad (3)$$

for a rectangular domain $R \subset \mathbb{R}^2$, and real-valued continuous functions f, g , and h defined on R . Usually the values of the parameters will be $0 \leq u \leq 2\pi$, and $0 \leq v \leq \pi$, hence we will refer to them as *longitude* and *latitude*, respectively.

Now, with the same parametrization (3) for S we can consider some other surfaces, namely

$$S^1 = \{ (f(u + \alpha_1, v), g(u + \alpha_2, v), h(u + \alpha_3, v)) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \}, \quad (4)$$

or

$$S^2 = \{ (f(u, v + \beta_1), g(u, v + \beta_2), h(u, v + \beta_3)) \mid \beta_1, \beta_2, \beta_3 \in \mathbb{R} \}, \quad (5)$$

Recall that Cartesian coordinates (x, y, z) are related with spherical coordinates (ρ, u, v) by the rule

$$\begin{bmatrix} \rho \\ u \\ v \end{bmatrix} \longrightarrow \begin{bmatrix} x(\rho, u, v) \\ y(\rho, u, v) \\ z(\rho, u, v) \end{bmatrix} = \begin{bmatrix} \rho \cos u \sin v \\ \rho \sin u \sin v \\ \rho \cos v \end{bmatrix} \quad (6)$$

Throughout the paper, points in $\mathbb{R}^3$ are often expressed in column vector form. We first note how the new parameters α_i and β_i change the longitude u and latitude v . This *perturbation* of the parameters turns out to be quite noticeable.

Exercises:

We consider the ellipsoid S_0 , which is represented by $\begin{bmatrix} f(u, v) \\ g(u, v) \\ h(u, v) \end{bmatrix} = \begin{bmatrix} 5 \cos u \sin v \\ 4 \sin u \sin v \\ 3 \cos v \end{bmatrix}$, where $u \in [0, 2\pi]$ and $v \in [0, \pi]$, with parameters of $\alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{\pi}{3}, \alpha_3 = \frac{\pi}{6}, \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{3}$, and $\beta_3 = \frac{\pi}{4}$. We invite readers to verify the followings:

1. The surface $\begin{bmatrix} f(u + \alpha_1, v) \\ g(u + \alpha_2, v) \\ h(u + \alpha_3, v) \end{bmatrix}$ is topologically equivalent to the ellipsoid S_0 .
2. The surface $\begin{bmatrix} f(u, v + \beta_1) \\ g(u, v + \beta_2) \\ h(u, v + \beta_3) \end{bmatrix}$ is not topologically equivalent to the ellipsoid S_0 .
3. We include a Maple worksheet for readers to explore in [S1].

2 Topologically Equivalent or Non-equivalent Surfaces

We recall that a surface with a boundary is not topologically equivalent to a sphere, e.g., a cylinder with an open top, which may be stretched into a disk (a circle plus its interior). In addition, a closed surface such as a torus (a doughnut) is not equivalent to a sphere, since no amount of bending or stretching can get rid of the hole in the torus to make it into a sphere. We now explore a separate affine transformation that is inspired by (2) and applied on a surface. We ask if the image of the transformation is topologically equivalent to the original surface. We remind readers that the discussion here is from layman's point of view, we invite interested readers to establish concrete homeomorphism between two spaces. In view of (2), we replace the antipodal points C and D by two respective surfaces. Considering s as an scaling factor, we can ask for the properties of a new surface defined as the locus, when spherical coordinates are used. Consider the surface S given by (3). Let the locus surface $\{(F(u, v, \alpha_1, \beta_1, s), G(u, v, \alpha_2, \beta_2, s), H(u, v, \alpha_3, \beta_3, s))\}$, generated by the points in $\mathbb{R}^3$ defined by:

$$\begin{bmatrix} F(u, v, \alpha_1, \beta_1, s) \\ G(u, v, \alpha_2, \beta_2, s) \\ H(u, v, \alpha_3, \beta_3, s) \end{bmatrix} = s \begin{bmatrix} f(u, v) \\ g(u, v) \\ h(u, v) \end{bmatrix} + (1 - s) \begin{bmatrix} f(u + \alpha_1, v + \beta_1) \\ g(u + \alpha_2, v + \beta_2) \\ h(u + \alpha_3, v + \beta_3) \end{bmatrix}, \quad (7)$$

where $u, v \in \mathbb{R}$, α_1 and β_1 are given angles, and $s \in \mathbb{R}$. If we start with a surface S , which is transformed, according to the transformation rule (7) when s is specified, into the surface $\tilde{S} = \{(F(u, v, \alpha_1, \beta_1, s), G(u, v, \alpha_2, \beta_2, s), H(u, v, \alpha_3, \beta_3, s))\}$.

2.1 Our ellipsoid

If $\begin{bmatrix} f(u, v) \\ g(u, v) \\ h(u, v) \end{bmatrix} = \begin{bmatrix} 5 \sin(v) \cos(u) \\ 4 \sin(v) \sin(u) \\ 3 \cos(v) \end{bmatrix}$, then we consider the ellipsoid $\begin{bmatrix} f(u, \frac{v}{2}) \\ g(u, \frac{v}{2}) \\ h(u, \frac{v}{2}) \end{bmatrix} = \begin{bmatrix} 5 \sin(\frac{v}{2}) \cos(u) \\ 4 \sin(\frac{v}{2}) \sin(u) \\ 3 \cos(\frac{v}{2}) \end{bmatrix}$ and denote it as S^* , where $u, v \in [0, 2\pi]$. We see from Figure 1(b) that S^* is a union of two top and bottom halves of the ellipsoid S^* , which are denoted by S_t^* and S_b^* and depicted in black and red respectively.

Next, we set $\alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{\pi}{3}, \alpha_3 = \frac{\pi}{6}, \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{3}, \beta_3 = \frac{\pi}{4}$, and $s = \frac{1}{6}$, we obtain the surface of our interest

$$S_1 = \{[F(u, v, \alpha_1, \beta_1, s), G(u, v, \alpha_2, \beta_2, s), H(u, v, \alpha_3, \beta_3, s)] : u, v \in [0, 2\pi]\}. \quad (8)$$

More specifically, we have

$$\begin{aligned} F(u, v, \alpha_1, \beta_1, \frac{1}{6}) &= \frac{5 \cos u \sin v + 25 \cos(u + \frac{\pi}{4}) \cos v}{6}, \\ G(u, v, \alpha_2, \beta_2, \frac{1}{6}) &= \frac{2 \sin u \sin v + 10 \sin(u + \frac{\pi}{3}) \sin(v + \frac{\pi}{3})}{3}, \\ H(u, v, \alpha_3, \beta_3, \frac{1}{6}) &= \frac{\cos v + 5 \cos(v + \frac{\pi}{4})}{2}. \end{aligned} \quad (9)$$

We see S_1 is a *closed surface* from Figure 1(a). On the other hand, when $s = \frac{1}{3}$, the surface

$$[[F(u, v, \alpha_1, \beta_1, s), G(u, v, \alpha_2, \beta_2, s), H(u, v, \alpha_3, \beta_3, s)]]$$

is an *open surface* (see Figure 1(c)) and hence is not topologically equivalent to a sphere or an ellipsoid. Since S_1 and S^* are both closed surfaces, we naturally pose the following that was asked in [8].

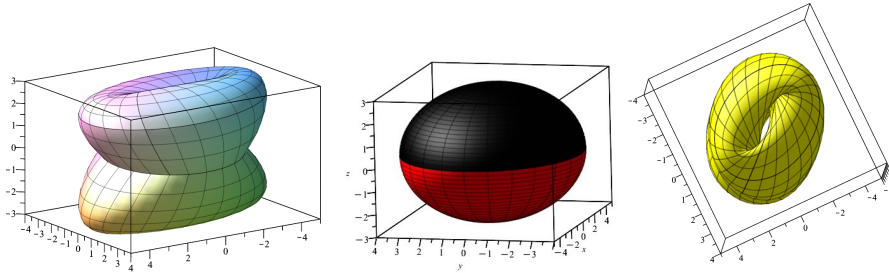


Figure 1(a). The closed surface S_1

Figure 1(b). Ellipsoid S^*

Figure 1(c). Not topologically equivalent to S^* for $s = \frac{1}{3}$.

2.2 Are the closed surfaces S_1 and S^* topologically equivalent?

Before we answer the question, we need to understand how the surface S_1 is formed, we thus proceed further as follows:

1. If we restrict the domain of S_1 to $u \in [0, 2\pi], v \in [0, \pi]$, we obtain the surface

$$S_1^t = \left\{ \left[\left[F(u, v, \alpha_1, \beta_1, \frac{1}{6}), G(u, v, \alpha_2, \beta_2, \frac{1}{6}), H(u, v, \alpha_3, \beta_3, \frac{1}{6}) \right] : u \in [0, 2\pi], v \in [0, \pi] \right\}, \quad (10)$$

which is shown in Figure 2(a). On the other hand, if we restrict the domain of S_1 to $u \in [0, 2\pi], v \in [\pi, 2\pi]$, we obtain

$$S_1^b = \left\{ \left[F(u, v, \alpha_1, \beta_1, \frac{1}{6}), G(u, v, \alpha_2, \beta_2, \frac{1}{6}), H(u, v, \alpha_3, \beta_3, \frac{1}{6}) \right] : u \in [0, 2\pi], v \in [\pi, 2\pi] \right\}, \quad (11)$$

which is shown on Figure 2(b).

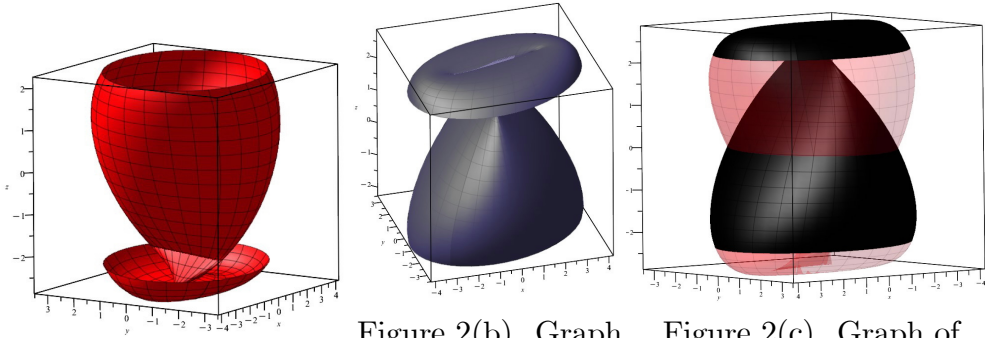


Figure 2(a). Graph of S_1^t

Figure 2(b). Graph of S_1^b

Figure 2(c). Graph of $S_1^t \cup S_1^b$

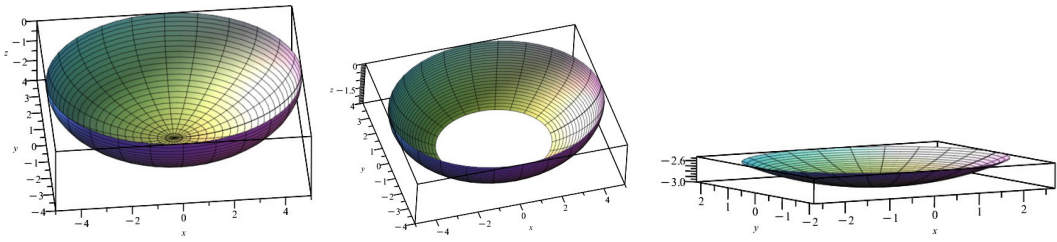
2. We included a Maple worksheet for readers to explore in [S2] on the closed surface shown in Figure 2(c) and the ellipsoid shown in Figure 1(b) .
3. With $\alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{\pi}{3}, \alpha_3 = \frac{\pi}{6}, \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{3}, \beta_3 = \frac{\pi}{4}$, and $s = \frac{1}{6}$, the animations of the space curves, using ([1]) for the longitudes (when $u \in [0, 2\pi]$), and latitudes for S_1 (when $v \in [0, 2\pi]$) respectively provide us some interesting graphical aspects of how S_1 is generated:
- (a) `animate(plot3d, [[F(u, v, 1/6), G(u, v, 1/6), H(u, v, 1/6)], u = 0 .. 2*Pi, v = 0 .. t], t = 0 .. 2*Pi, thickness = 3, color = yellow);`
 - (b) `animate(plot3d, [[F(u, v, 1/6), G(u, v, 1/6), H(u, v, 1/6)], u = 0 .. t, v = 0 .. 2*Pi], t = 0 .. 2*Pi, thickness = 3, color = yellow);`
 - (c) See [S3] and [S4] respectively for the animations.
4. We remind readers that $S_1 = S_1^t \cup S_1^b$, shown in either Figures 1(a) or 2(c), resembles the union of two open tea-cups as seen in Figure 2(c) or its transparent 3D printing of Figure 5(b). If we look at the 3D printings for S_1^t (red) and S_1^b (black) in Figure 5(a), and if S_1 were topologically equivalent to the ellipsoid S^* , it is equivalent of asking the following

Sub-question: Is S_1^t or S_1^b is topologically equivalent to an open hemisphere with a boundary of a flat circular base? We proceed as follows:

1. First we set E_L to be the lower half of the ellipsoid S^* (see Figure 3(a)). In other words,

$$E_L \text{ is } \begin{bmatrix} f(u, \frac{v}{2}) \\ g(u, \frac{v}{2}) \\ h(u, \frac{v}{2}) \end{bmatrix} = \begin{bmatrix} 5 \sin(\frac{v}{2}) \cos(u) \\ 4 \sin(\frac{v}{2}) \sin(u) \\ 3 \cos(\frac{v}{2}) \end{bmatrix}, \text{ where } u \in [0, 2\pi] \text{ and } v \in [\pi, 2\pi]. \text{ We can}$$

decompose E_L into two sub-surfaces by cutting it with a horizontal plane $z = \pi + 1$. We see $E_L = E_{L_1} \cup E_{L_2}$, with $E_{L_1} = [f(u, \frac{v}{2}), g(u, \frac{v}{2}), h(u, \frac{v}{2})]$, $u \in [0, 2\pi]$ and $v \in [\pi, \pi + 1]$ (see Figure 3(b)), and $E_{L_2} = [f(u, \frac{v}{2}), g(u, \frac{v}{2}), h(u, \frac{v}{2})]$, where $u \in [0, 2\pi]$ and $v \in [\pi + 1, 2\pi]$ (see Figure 3(c)). We see that E_{L_1} is an open surface with two boundaries, and E_{L_2} is another open surface with only one boundary, which is topologically equivalent to E_L or a hemisphere. In fact, any horizontal plane $z = v_0 \in (\pi, 2\pi)$ will result in similar topological decompositions.



Figures 3(a),(b) and (c). Plots of E_L , E_{L_1} and E_{L_2} respectively

2. We now decompose S_1^t into two sub-surfaces using the horizontal plan $z = \frac{\pi}{2}$. We see $S_1^t = S_{11}^t \cup S_{12}^t$, where S_{11}^t if $u \in [0, 2\pi], v \in [0, \frac{\pi}{2}]$, shown in Figure 4(a) below. On the other hands, if $u \in [0, 2\pi], v \in [\frac{\pi}{2}, \pi]$, we obtain the surface S_{12}^t , which is shown in Figure 4(b). We see S_{11}^t is topologically equivalent to E_{L_1} . However, since S_{12}^t is a surface which has two disjoint elliptical rims (indicated in red in Figure 4(b)), it cannot be topologically equivalent to E_{L_2} which only has one (circular) rim.

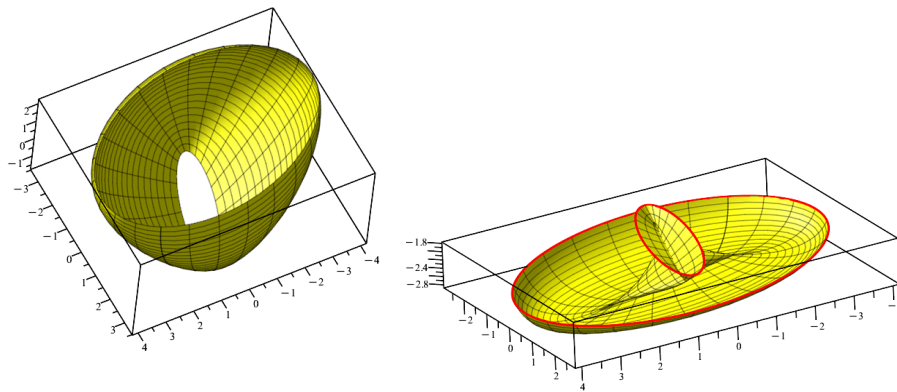


Figure 4(a). The plot of S_{11}^t . Figure 4(b). The plot of S_{12}^t .

3. We encourage readers to find the precise value v so that $(S_{11}^t)^*$ resembles the Figure 4(c), and $(S_{12}^t)^*$ is shown in Figure 4(d), which both contain a *pinched* red line. For example,

one may consider picking a proper value for v in the equation for S_1 (9) (when $\alpha_1 = \frac{\pi}{4}$ and $\beta_1 = \frac{\pi}{2}$).

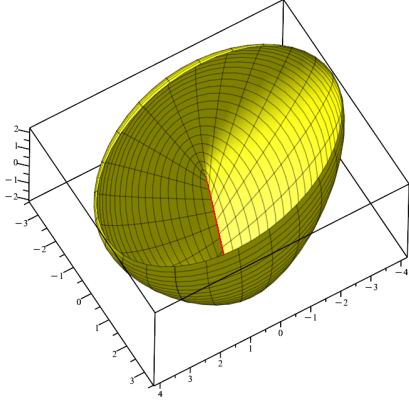


Figure 4(c). The red pinch-line for $(S_{11}^t)^*$, when $v \approx \frac{\pi}{2} + 0.15167$

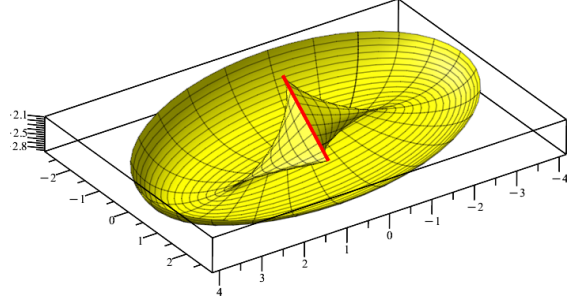


Figure 4(d). The red pinch-line for $(S_{12}^t)^*$, when $v \approx \frac{\pi}{2} + 0.15167$

Alternatively, we can find the places where we have pinch-lines as follows:

- (a) For fixed t and $u \in [0, 2\pi]$, the graphs of the function

$$F^* : (u, t) \rightarrow \left[F(u, t, \alpha_1, \beta_1, \frac{1}{6}), G(u, t, \alpha_2, \beta_2, \frac{1}{6}), H(u, t, \alpha_3, \beta_3, \frac{1}{6}) \right]$$

represents set of ellipses, and we conjecture through the animation in [S5] that there exist t such that ellipses degenerate to a pinched line. We shall show graphically how such t can be found such that the ellipses $F^*(u, t)$ degenerate to a line segment.

- (b) It is clear that we may project the ellipses onto $z = 0$ without altering the respective areas. We set up the area function for the ellipses as

$$J(t) = \int_0^{2\pi} G(u, t, \alpha_2, \beta_2, \frac{1}{6}) \left(\frac{d}{dt} F(u, t, \alpha_1, \beta_1, \frac{1}{6}) \right) du.$$

(c) We plot $y = J(t)$, $t \in [0, 2\pi]$ using ([1]) as in Figure 4(e).

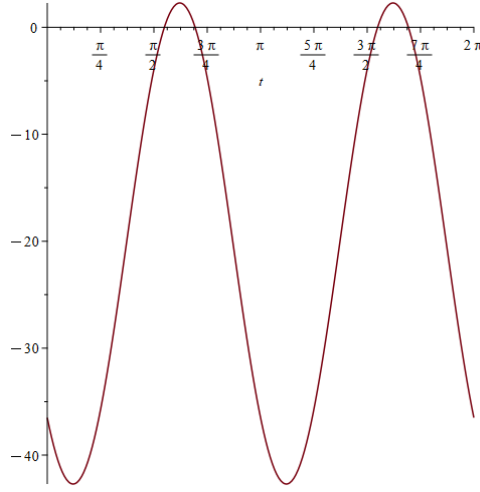


Figure 4(e). Plot of $y = J(t)$

(d) Now we set the area of ellipses $J(t) = 0$, and find the first

$$t = t_1 = 1.724344291. \quad (12)$$

We invite the readers to find the remaining places where ellipses are degenerated to pinch-lines, they are

$$t_2 = 2.178274393, t_3 = 4.865936945, t_4 = 5.319867047. \quad (13)$$

These pinch-lines will subsequently prove useful as we present the formal proofs that the surface S_1 is not homeomorphic to the ellipsoid or the sphere (see Section 3).

4. Therefore, we see that $(S_{11}^t)^*$ can not be topologically equivalent to E_{L_1} since there does not exist a homeomorphism between $(S_{11}^t)^*$ and E_{L_1} . Similar reasons can be drawn to show that $(S_{12}^t)^*$ cannot be topologically equivalent to E_{L_2} .

Hence, S_1^t cannot be topologically equivalent to a hemisphere with a boundary of a flat circular base. We conclude that neither S_1^t nor S_1^b is topologically equivalent to a hemisphere

with a boundary. Therefore, S_1 cannot be topologically equivalent to the ellipsoid S^* .

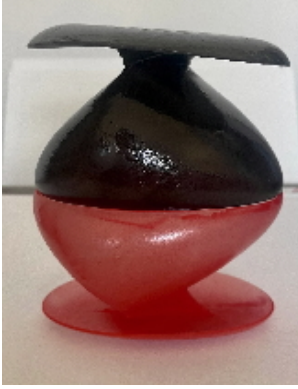


Figure 5(a) 3D printings of S_1^b (black) and S_1^t (red).



Figure 5(b) 3D printing of 2(c).

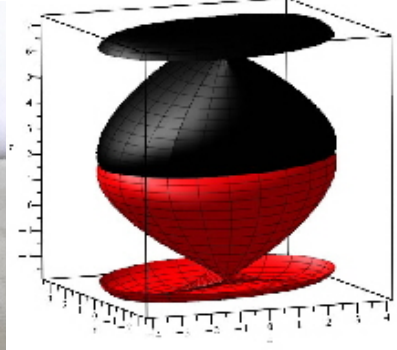


Figure 5(c). Mathematical plot of the 3D printing 5(a)

Remarks:

1. Since both S_1 and S^* are both closed surfaces, it is non-trivial at a glance, if two are topological equivalent. It is only after exploration with technological tools and 3D printing by knowing the 3D printing of Figure 5(b) is exactly the copy of $S_1 = S_1^t \cup S_1^b$ (or Figure 2(c)), do we realize that S_1 is a union of two open tea cups of S_1^t and S_1^b .
2. We also remark that we printed the 3D surfaces S_1^t (in red) and S_1^b (in black) individually, and stack S_1^b on top of S_1^t to get the graph of Figure 5(a). By using simple vertical shifting along the z -axis, we leave it to readers to explore how we can *generate the equation for the surface* that look like Figure 5(c).
3. Intuitively, if we start with the mathematical equation for S_2 (or Figure 5(c)), then it is much easier to visualize that S_2 is not topological to the ellipsoid S^* or a sphere.

3 Proofs: S_1 Is Not Homeomorphic To A Sphere

We recall a manifold in mathematics is a topological space that locally resembles Euclidean space near each point, see [S6]. We use the following notation throughout this section:

$$S_1(u, v) := (F(u, v), G(u, v), H(u, v)),$$

where

$$\begin{aligned} F(u, v) &= \frac{5 \cos u \sin v + 25 \cos(u + \frac{\pi}{4}) \cos v}{6}, \\ G(u, v) &= \frac{2 \sin u \sin v + 10 \sin(u + \frac{\pi}{3}) \sin(v + \frac{\pi}{3})}{3}, \\ H(u, v) &= \frac{\cos v + 5 \cos(v + \frac{\pi}{4})}{2}. \end{aligned} \tag{14}$$

We shall use 3D Desmos to help us visualize the manifold and the accompanying details that we will disclose subsequently. Readers are encouraged to visit the graphical animation available at [S7] or go to <https://www.desmos.com/3d/1cr28pthtr>. We remind readers that the following numerical values are simply approximations for demonstration purposes only.

We decompose the manifold S_1 into four parts:

- f_1 represents the part where $0 \leq v \leq 2.18$. (See Figure 6). We observe that there are two pinch-lines when $v \in [0, 2.18]$. We have discussed the first pinched line from Equation (12). Here $v \approx 2.18$ tallies with the location of the second pinch-line (see below).

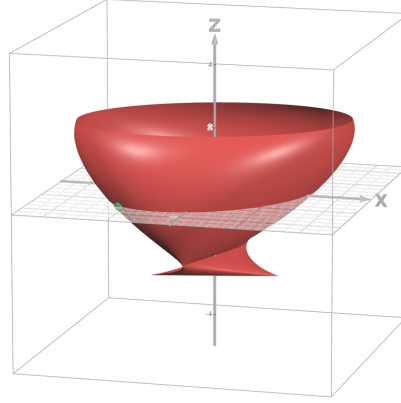


Figure 6. Surface for f_1

- f_2 represents the part where $2.18 \leq v \leq 4.0503$ which is the part ensuing from the second pinch-line and that lies in the lower plane $z \leq 0$.
- f_3 represents the part where $4.0503 \leq v \leq 4.87$. The value $v = 4.87$ tallies with the third pinch-line.
- f_4 represents the part where $4.78 \leq v \leq 2\pi$, which is the remaining portion ensuing from the third pinch-line and that lies in the upper plane, i.e., $z \geq 0$.

The reader can check on the corresponding circular icon next to the various f_i 's to make the corresponding surface appear or disappear. One may also use the mouse-click to hold and rotate the manifold S_1 so as to view it from different angles.

3.1 Pinch-lines make S_1 not being topologically equivalent to a sphere

In this section, we shall supply the readers with the graphical details that lead to the following two major observations:

1. There are exactly *four* distinct so-called pinch-lines on the manifold S_1 ; the removal of which would result in a disconnected space with disjoint components.

2. The manifold S_1 is *self-intersecting* in the sense that there exists two distinct pairs of parameters (u, v) and (u', v') such that

$$S_1(u, v) = S_1(u'v')$$

but the tangential planes to the curve (that passes through this point) corresponding to these two pairs of parameters are non-parallel.

Using these results obtained graphically, we observe two different arguments that justify the result that S_1 is not homeomorphic to the sphere.

We begin by establishing the locations of the four pinch-lines that lie on S_1 . To do this, we make use of the latitudes, given by

$$L_t = S_1(u, t), \quad u \in [0, 2\pi].$$

From the graphical display, it can be seen that the latitude L_t is an ellipse centred at $(0, 0, H(u, t))$. The latitude L_t is given by the equation labelled by f_6 in the 3D Desmos display.

Because the analytical approach is way too tedious, we rely on a graphical, whence qualitative, approach.

Since $H(u, t)$ is independent of u it follows that this ellipse is parallel to the xy plane. As t increases over the interval $[0, 2\pi]$, the latitude L_t goes through the following vertical motion:

1. It begins its descent from the initial position at the rim of the open cup S_1^t (at $t = 0$) to the first pinch-line (at approximately $t = 1.7225$, see (12)).
2. It then continues its descent to the second pinch-line (at approximately $t = t_2 = 2.178274393$, see (13)).
3. It continues its descent till it reaches its lowest height at approximately $t = 2.48$, and then starts its journey of ascent. Note that this is the value of t that minimizes $H(u, t)$ over the domain $[0, 2\pi]$.
4. It ascends till it reaches the third pinch-line (at approximately $t = t_3 = 4.865936945$, see (13), which is exactly π after the first pinch-line).
5. It then continues its ascent and reaches the fourth pinch-line (at approximately $t = t_4 = 5.319867047$ see (13), which is exactly π after the second pinch-line).

We now turn to our second observation. This is inspired by examining the movement of the longitude and animation from [S4] that the surface S_1 is self-intersecting at several places. We observe the first place of such self-intersecting occurs at the junction where two cups meet (see Figures 2(c) and 5(b)), whose equation is given by

$$L^t = S_1(t, v), \quad v \in [0, 2\pi].$$

This longitude is given the label f_5 in the 3D Desmos display. As t increases over the interval $[0, 2\pi]$, the longitude L^t gyrates counterclockwise (when viewed from above) relative to the xy -plane. It is worth-noting that at any value of $t \in [0, 2\pi]$ the longitude L^t cuts each of the pinch-lines at exactly one point. In particular, the direction of motion of these intersection

points with the two pinch-lines in Figure 6, for instance, are in *different* directions – hence, indicating the presence of a twist from the surface only up to the first pinch-line to the other surface shown in Figure 6, say f_1 .

From the 3D Desmos display of the manifold S_1 , one sees that the manifold self-intersects at the plane $z = 0$. More precisely, this happens when $H(t, v) = 0$, i.e.,

$$\begin{aligned}\cos(v) + 5 \cos\left(v + \frac{\pi}{4}\right) &= 0 \\ \sqrt{5\sqrt{2} + 26 \arctan\left(\frac{5}{23}(5 - \sqrt{2})\right)} &= 0 \\ v &\approx 0.9087, 4.0503.\end{aligned}$$

This then yields the two points of intersection of the longitude L^t with the plane $z = 0$:

$$P(t) = S_1(t, 0.9087), \quad t \in [0, 2\pi],$$

and

$$Q(t) = S_1(t, 0.40503), \quad t \in [0, 2\pi].$$

The equation for $P(t)$ is labelled f_8 , while $Q(t)$ has been labelled f_7 in the 3D Desmos display.

In particular, for any $t \in [0, 2\pi]$ it always holds that

$$Q(t) = P(t + \pi).$$

Consequently, at any instant t , the tangential plane passing through $P(t)$ is always tangential to the lower half of the manifold, i.e., where $z \leq 0$, and the tangential plane passing through $Q(t)$ is always tangential to the upper half of the manifold, i.e., where $z \geq 0$. Hence at any instant t , the gradient vectors ∇S_1 at the points, $P(t)$ and $Q(t) = P(t + \pi)$ respectively, share the same x, y -components but have opposite signs for the z -component. We encourage the keen readers to calculate the explicit expression for ∇S_1 at the points $P(t)$ and $Q(t)$.

Note. We invite readers to explore that the surface S_1 is also self-intersecting at those four pinch-lines.

It follows from those two observations mentioned above, we are now ready to reap the fruits of our hard labor.

Proof 1.

We now argue that S_1 is not homeomorphic to the ellipsoid S^* . For the sake of contradiction, let us suppose that S_1 is homeomorphic to the ellipsoid S^* , say via the homeomorphism $f : S_1 \longrightarrow S^*$. Now, we remove the four pinch-lines $\ell_1, \dots, \ell_4$ from the manifold S_1 . By so doing, the resulting manifold $S_1 - \bigcup_{k=1}^4 \ell_k$ becomes a disconnected space that comprises of four disjoint components. Since f is a homeomorphism, $f(\ell_1), \dots, f(\ell_4)$ represents four disjoint simple paths on the S^* . Even after removing these four simple paths, the resulting manifold still remains path-connected and hence connected (just like S^*). Now the restriction of f on the space $S_1 - \bigcup_{k=1}^4 \ell_k$ is still a homeomorphism between the two resultant spaces. But the space $S_1 - \bigcup_{k=1}^4 \ell_k$ is disconnected while the space $S^* - \bigcup_{k=1}^4 f(\ell_k)$ is connected, which is the desired contradiction we hope to arrive at. In conclusion, S_1 is not homeomorphic to the ellipsoid S^* and hence S_1 is not homeomorphic to a sphere.

We now give the second proof that S_1 is not homeomorphic to S^* .

Proof 2.

Our second observation informs us that S_1 is self-intersecting, i.e., the manifold cuts itself at the plane $z = v_k, k = 1, 2, 3, 4$, where v_k 's are places for those four pinch-lines. Had it been that S_1 is homeomorphic to the sphere S^* , then the S^* would also be self-intersecting, which we know is not for a fact! Thus, we may conclude that S_1 is not homeomorphic to S^* or a sphere, as desired.

4 Acknowledgements

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5 Conclusion

In this paper, we employ the affordance of technological tools – three-dimensional visualization – to provide us crucial intuitions as we work towards more rigorous solutions to the topological problem at hand. More specifically, we have gained graphical insights by relying on graphing tools like [1] and [2]. Dynamic geometry software, together with computer algebra, points us to many valid conjectures, which otherwise would have evaded us. While it has been frequently reported that technological tools make learning mathematics fun and accessible, our partnership with computer technology shows that such tools also facilitate explorations into more challenging and theoretical mathematics, such as lower dimension topology in this case. It is hoped that this present technology-aided mathematical exploration convinces our readers that mathematical investigations – no matter simple or advanced – are accessible to students at all levels. Notably, 3D printing makes abstract and complicated objects *concrete*; for instance, students can now physically touch and feel the four pinch-lines, hence reinforcing their topological understanding.

Ultimately, inculcating a greater interest in mathematics for students by providing them accessible technological tools and concrete manipulatives goes a long way towards encouraging students to take on more challenging and intricate problems, which otherwise would remain beyond the reach of traditional pencil-and-paper approach. This we believe forms an important step towards cultivating mathematical creativity and innovation in students.

6 Supplementary Electronic Materials

[S1] Maple worksheet for Exercises:

<https://atcm.mathandtech.org/EP2024/invited/22174/Exercise.mw>.

[S2] Maple worksheet for two closed surfaces:

https://atcm.mathandtech.org/EP2024/invited/22174/two_closed_surfaces.mw.

[S3] Longitude animation for S_1 : https://atcm.mathandtech.org/images/S1_longitude.gif.

- [S4] Latitude animation for S_1 : https://atcm.mathandtech.org/images/S1_latitude.gif.
- [S5] The pinched line: <https://atcm.mathandtech.org/EP2024/invited/22174/pinchline.gif>.
- [S6] Manifold from Wikipedia: <https://en.wikipedia.org/wiki/Manifold>.
- [S7] Desmos Approach: <https://www.desmos.com/3d/lcr28pthtr>.

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