Mathematical Problem-Solving with GeoGebra: An Analysis of Students' Processes

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Abstract: *This paper reports the students' processes when tasked to solve mathematical problems in Geometry using GeoGebra. Twelve Grade 10 students divided into 6 dyads participated in the study. Each dyad solved 3 problems using a laptop and their verbal conversations, social interactions, and screen activity were recorded and videotaped. The conversations were transcribed word for word and the transcripts were supplemented by the description of their GeoGebra activities. The qualitative analysis focused on the mathematical problem-solving with technology (MPST) processes. The results revealed that grasping, analyzing, exploration, planning, creating, verification, and dissemination were the MPST processes which were usually observed. Challenging problems were characterized by a series of explorations and verifications which were made possible by the features of GeoGebra.*

1. Introduction

The problem-solving process in mathematics is usually characterized by four stages. Understanding the problem comes first, followed by devising and implementing a strategy, and then looking back to see whether the correct answer is obtained [1]. [2] expanded Polya's stages into six problem-solving processes which he called episodes namely: reading, analysis, exploration, planning and implementation and verification. Since then, a plethora of research has been written to investigate and improve students' problem-solving skills. Problem-solving has evolved from just merely a pen and paper activity into a more sophisticated activity that uses technology.

The emergence of computers, laptops and handheld devices led to the creation of dynamic geometry software (DGS) which allows someone to perform geometric constructions, sketch graphs, explore, and create mathematical models in a dynamic environment. An example of a DGS commonly used nowadays is GeoGebra. In GeoGebra, dynamic aspects from DGS are combined with the capabilities of a computer algebra system (CAS). The software allows users to input equations, functions, and coordinates; to keep record of the details in the algebra part of the computer screen; and to visualize everything in the geometry part of the computer screen. GeoGebra enables a user to work dynamically with points, lines, segments, or rays by clicking and or dragging them or by directly changing an equation [3]. Because of these features, GeoGebra became a widely used software for teachers' instruction, and students' modelling and problem-solving activities. It has been utilized to aid teaching, to understand difficult concepts, to visualize mathematical relationships, to provide proofs and to solve problems.

With the increasing demand on the use of computer software such as GeoGebra and its incipient application in the teaching and learning process, the need to investigate the procedure involved is of great value. This paper discusses the micro-level analysis of the problem-solving process followed by students with the use of a GeoGebra.

2. Conceptual Framework

[4] proposed a framework for describing techno-mathematical fluency to investigate the mathematical problem-solving with digital tools in a beyond-school mathematical competition. The techno-mathematical fluency emerged from 'problem-solving with technologies' exercises which were characterized by the interplay between mathematical knowledge and technological fluency. Techno-mathematical fluency is based on [2] episodes of mathematical problem-solving and processes of digital literacy framework [5]. The processes of digital literacy were analyzed whilst problem-solving episodes were happening. Each digital literacy process was carefully placed and analyzed in each of the five problem-solving episodes. The analytical tool that describes the techno-mathematical fluency of the students is collectively termed as MPST or mathematical problem-solving with technology. The processes underlying MPST are grasping, noticing, interpreting, integrating, exploring, planning, creating, verifying, and disseminating. The table below shows how [4] categorized the problem-solving processes of the student by merging the problem-solving episodes and the digital literacy processes.

Table 1

Processes underlying mathematical problem-solving with technology (MPST)

Processes	Descriptions
Grasp	Appropriation of the situation and the conditions in the problem, and early ideas on what it involves (Read[a], Statement[b])
Notice	Initial attempt to comprehend what is at stake, namely the mathematics that may be relevant and the digital tools that may be necessary (Analysis[a], Identification[b], Accession[b])
Interpret	Placing affordances in the technological resources on pondering mathematical ways of approaching the solution. (Analysis[a] ; Evaluation[b] , Interpretation[b])
Integrate	Combining technological and mathematical resources within an exploratory approach. (Exploration[a] ; Organisation[b], Integration[b])
Explore	Using technological and mathematical resources to explore conceptual models that may enable the solution. (Exploration[a] ;Analysis[b])
Plan	Outlining an approach to achieve the solution based on the analysis of the conjectures explored. (Planning and Implementation[a] ; Synthesis[b])

Create	Carrying out the outlined approach, recombining resources in new ways which will enable the solution and create new knowledge objects, units of information or other outputs which will contribute to solve-and-express the problem. (Planning and Implementation[a] ; Creation[b])
Verify	Engaging in activities to explain or justify the solution achieved based on the mathematical and technological resources. (Verification[a])
Disseminate	Present the solutions or outputs to relevant others and consider the success of the problem-solving process. (Verification[a] ; Reflection[b] , Dissemination[b])

[a] Phase of mathematical problem-solving as proposed by [2]
[b] Process of digital technology problem-solving as proposed by [5]

The MPST framework was used in shedding light on the problem-solving processes of the subjects in a collaborative setting.

3. Methodology

The population of this study is the set of Grade 10 students currently enrolled in a Science, Technology and Engineering (STE) high school institution. One section was randomly chosen and students from this group were ranked according to their third quarter academic performance. Three sets of students were identified: high performing (top 4 students), average performing (the middle 4 students) and low performing (bottom 4 students). Hence, a total of 6 dyads participated in the study. The pairings of students in each group were done randomly. The pseudonyms assigned to each student in presented in Table 2.

Table 2
Pseudonyms assigned to each student

High Performing Group	Average Performing Group	Low Performing Group
Kobe and Liz	Anna and Marie	Franz and Topher
Ian and Grace	Jed and Kat	Rox and Sab

Each group answered three mathematical problems using GeoGebra. The three problems of varying degrees of difficulty (easy, average, difficult) were created by the researcher.

The following were the problems given to each group.

1. A line passes through the center of the circle and intersects it at points $(-1,2)$ and $(7,4)$. Find the equation of the circle.
2. Find the equation of the circle inscribed in the triangle enclosed by the line $y=-2x+5$ and the coordinate axes.
3. Construct a rhombus whose one side lies on the line $y=-2x+5$. Check all properties that will prove that the figure you constructed is really a rhombus.

To ensure that the participants' skills or lack of skills in using GeoGebra will not affect that problem-solving attempts, a 3-session training on GeoGebra was conducted outside the students' regular classroom schedule. The sessions were conducted in the school's computer laboratory. All necessary procedures and permissions (from the students' guardians and the school head) were secured before conducting the orientation. The training materials used were based on the manual published by GeoGebra on their website.

One computer unit (laptop) was used by each dyad. The latest version of GeoGebra was installed in the laptop. The sequence of the problem-solving sessions was as follows: first 2 dyads of the high performing group, 2 dyads of the average performing group, and lastly, the 2 dyads of the low performing group. Each dyad was given a maximum of 45 minutes to answer each problem. As the groups answered the problem, they were asked to talk with each other and verbalize whatever they were thinking and the processes they employed. Each of the students' problem-solving activity was videotaped and was supplemented by the researcher's field notes.

As the students worked in pairs, their actions, dialogue, and gestures toward each other and toward the computer were videotaped. The screen activity was recorded using the Free Screen Recorder application. The final outputs of the students were the GeoGebra files which contained their answers to the problems and the construction protocol.

4. Results

4.1 Problem-solving outcomes

The outcomes of the problem-solving activities of the students highlight their score, number of steps in the construction protocol and the time spent. The score given to the final answer are as follows: '0' when there's no answer; '1' if the answer was incorrect; '2' when there was an estimated correct answer but the construction failed the drag test; '3' if the answer was correct but the construction failed the drag test; '4' when the answer was correct, the construction passed the drag test but was not able to present all possible solutions; and '5' when the answer and the construction in GeoGebra were both correct.

Table 3 shows the results of the problem-solving episodes. The first number on the table indicates the score received by the dyads, the second number pertains to the number of steps in the construction protocol and the third is the problem-solving time. For instance, Kobe and Liz tailed '5, 5, 2 min' on problem 1 which means that they were able to obtain the correct solution with a valid construction using 5 GeoGebra steps within 2 minutes.

Table 3

Problem-solving outcomes of all dyads

Dyad	Names	Problem 1	Problem 2	Problem 3
1	Kobe and Liz	5, 5, 2.0 min	2,15, 23.9 min	4, 15, 5.9 min
2	Ian and Grace	5, 5, 2.9 min	2,14, 28.8 min	2, 27, 9.5 min
3	Maria and Anna	5, 5, 3.8 min	1,12, 15.7 min	1, 26, 31.6 min
4	Jed and Kat	5, 5, 5.7 min	1,21, 22.2 min	2, 39, 14.05 min
5	Franz and Topher	1, 4, 1.8 min	3,11, 10.4 min	2, 14, 22.03 min
6	Sab and Rox	2, 7, 10.3 min	0, - , 13.9 min	4, 26, 22.2 min

Legend (for final answer):
0 – no answer
1 – incorrect answer
2 – with an estimated correct answer but fails the drag test
3 – with a correct answer but fails the drag test
4 – with correct answer, passed the drag test but failed to present all possible solutions
5 - Correct answer and correct construction

Table 3 shows that, for nearly all three problems, the high performing group outperformed the other two groups in terms of scores, number of construction protocol steps, and solution time. Problem 1 was the easiest of the three problems. Only the low performing dyads were not able to get the correct answer. All dyads had difficulty understanding Problem 2, particularly on finding the meaning of the words enclosed and coordinate axes. Only Dyad 5 were able to arrive at the correct answer, but the construction failed the drag test, that is, changing a movable component of a construction and still the mathematical properties were maintained. Two dyads (1 and 6) were able to solve Problem 3 successfully. However, the number of steps in the construction protocol and the amount of time needed to finish the problem varied greatly.

Figure 1 shows a final output of a dyad. The left pane is called the algebra view which contains the coordinates of the points and the equations of the lines and the circle. The right most area is the construction protocol which highlights the steps and tools used by the student. The middle area is the graphics view. This is where students do the actual construction.

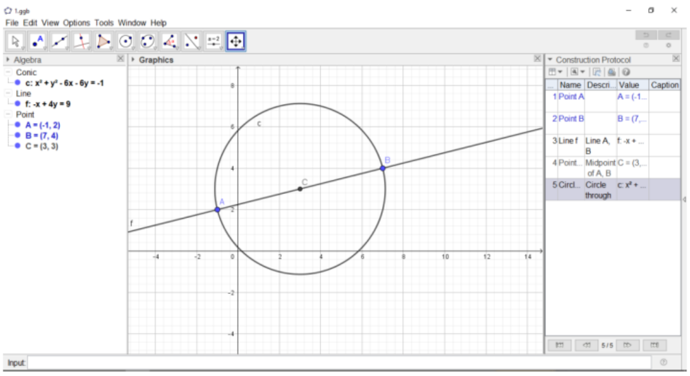


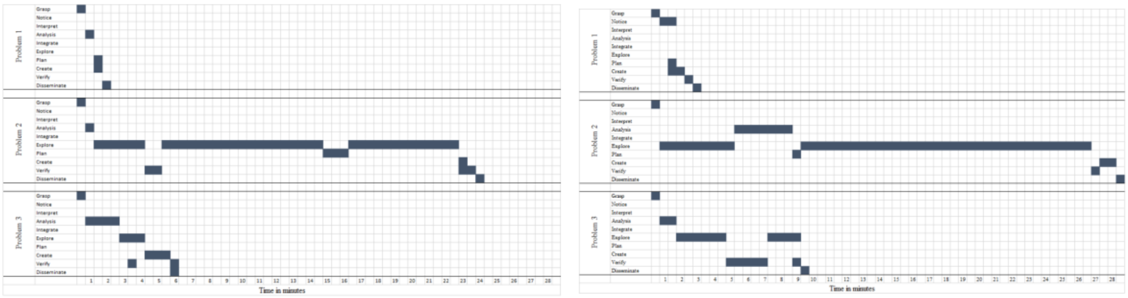
Figure 1. A final GeoGebra output for Problem 1.

4.2 Mathematical problem-solving processes

The time-line presentation of each problem-solving session by each dyad is presented below. The vertical axis lists the processes, and the horizontal axis has the time in minutes. Each grid shows a 30-second time interval.

The summary of the MPST processes of Dyad 1, one of the two dyads under the high performing group, is shown in Figure 4.1(a). It can be gleaned from the figure that, for Problem 1, the dyad had an easy time answering the question. After the *grasp* phase, they *analyzed* the question quickly and moved to the *plan* and *create* phase and finally the *disseminate* stage. The second problem, however, took them almost 24 minutes to finish. The processes involved were *grasp, analysis, explore, verify, plan, create* and *disseminate*. It can be noticed that they spent most of their time exploring for the right solution. The dyad had a hard time looking for the center of the inscribed circle. In the last problem, the pair spent nearly twice the time it took to finish Problem 1. The processes involved are *grasp, analysis, explore, create, verify* and *disseminate*. The analysis phase was filled with discussion of what a rhombus is and the explore part was spent on trying to create a rhombus by constructing first an equilateral triangle.

The other dyad under the high performing group, also did well in Problem 1 solving the item in less than 3 minutes. The session includes *grasp, notice, plan, create, verify* and *disseminate* processes. Similarly, the pair spent a lot of time answering Problem 2. This session was filled with a lot of *exploration*, and *analysis* activities. The other processes for session 2 were *grasp, plan, create,* *verify* and *disseminate*. They had difficulty in finding the center of the enclosed circle. Also, it took them a while arguing on what enclosed meant and the role of the coordinate axes. The third problem session included *grasp, analysis, explore, verify* and *disseminate* actions. The whole problem-solving session of the dyad is presented in Figure 4.1(b).

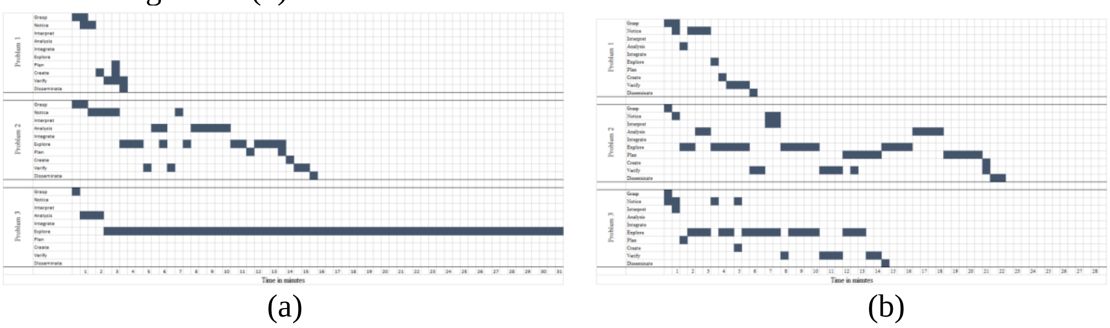


(a) (b)

Figure 4.1. Timeline representations of the high performing group

Figure 4.2(a) summarizes the processes Dyad 3 underwent in solving the three problems. For Problem 1, after reading (*grasp*) and analyzing the tools to be used (*notice*), they immediately constructed (*create*) the circle. But after the verification (*verify*), they found some errors and jumped into planning(*plan*) and then quickly constructed (*create*) the circle, checked its appropriateness and finally reported (*disseminate*) the answer. They were able to finish this in less than 4 minutes. The second problem-solving session included almost all the processes. Even if they were not able to solve the problem, the session was filled with processes such as *grasp, notice, explore, verify, analysis, create*, and *disseminate*. They were not able to understand what it meant to be enclosed. Up to the latter part of the session, their output showed a triangle inscribed in a circle instead of the other way. Problem 3 only included *grasp, analysis* and *explore* processes. The dyad was not able to come up with the correct construction of a rhombus, although there were traces of verification along the activity, the researcher still put the whole process into exploration primarily because of the utterances of the students.

The dyad of Jed and Kat also obtained the correct answer to the first problem. They progressed through the processes in almost a linear manner from *grasp* to *disseminate*. As for problem 2, the tandem struggled to understand on meaning of “enclosed” . They did not take into account the coordinate axes. They had a hard time incorporating the given line and finally ended with a wrong answer. The session was filled with varied processes with almost no recognizable pattern. The processes noted for session 3 were *grasp, notice, interpret*, a lot of *exploration, plan, create, verify*, and *disseminate*. The dyad spent a lot of time working with parallel lines and adjusting the position of the lines so that the parallelogram will have equal sides. The problem-solving processes is summarized in Figure 4.2(b).

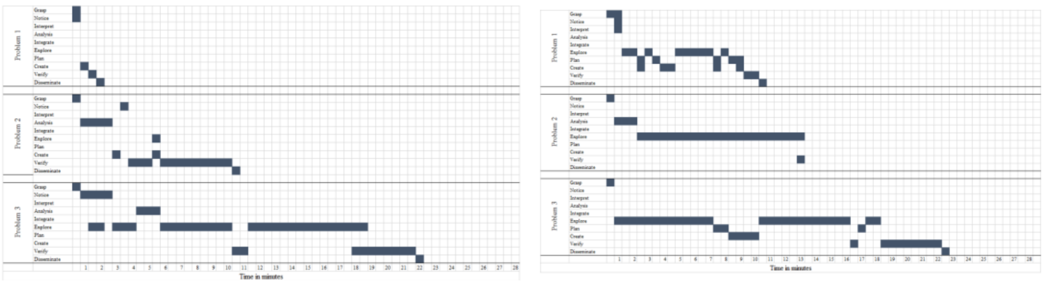


(a) (b)

Figure 4.2. Timeline representations of the average performing group

The summary of the problem-solving processes transpired in the sessions with Franz and Topher is presented in Figure 4.3(a). It can be gleaned from the figure that, for Problem 1, the session included *grasp, notice, create, verify,* and *disseminate*. The processes were very straight forward but the dyad was not able to answer the problem correctly. The processes observed for Problem 2 were *grasp, analysis, create,* a lot of *verification, explore* and *disseminate*. Franz was quick to recognize that the intersection of the angle bisectors is the location of the center of the inscribed circle. Also, the analysis part included argumentation on the meaning of "enclosed". For Problem 3, the session was filled with a lot of exploration. The dyad spent a lot of time working with parallel lines. They were able to construct a parallelogram with equal sides but failed to have the opposite angles congruent.

Finally, the problem-solving undertakings of Dyad 6 is summarized in Figure 4.3(b). The analysis of data revealed that the first session was characterized by grasp, notice, interpret, and a couple of exchanges between explore, plan, create, verify and disseminate. The dyad's first answer used the point (-1, 2) as the center. But they realized that they should find the midpoint and they did but on the final construction of the circle, The tool "circle with a center through a point" was used and correctly clicked on the midpoint but did not use any of the given points to set the radius. Session 2 included only grasp, analysis, explore and verify processes. They gave up after several explorations. The processes observed during the last problem-solving session included *grasp*, a long chain of *exploration, plan, create, verify* and *disseminate*.



(a) (b)

Figure 4.3. Timeline representations of the low performing group

5. Discussion

The timeline for each group showsthat not all processes were recognized during the students' problem-solving exercise. One factor contributing to this is the difficulty in categorizing some student acts, particularly during the phases of notice and interpretation. Most of the time, students proceeded right away in using GeoGebra rather than discussing first what tools to be used and their function before utilizing them. Additionally, it is very challenging to separate the integrate process from explore since, once again, students will do the activity without mentioning the tools that should be coupled with and for what purposes. The process of categorizing the problem-solving activity is deemed challenging because there was typically no explicit indication that a process has changed.

In this study the timelines of the problem-solving activity suggest that the processes may not occur linearly. This observation is different from the result in [4] and [6]. This variation could be explained by the collaborative nature of the task.

Finally, students were able to spend a lot of time in exploration because the features of GeoGebra allowed them to do so. This is specifically true for the challenging problems. However, if students do not have the mathematical skills that complement the technological skills, obtaining the correct solution may prove to be difficult still.

6. Summary

The MPST framework provides a potent analytical tool for comprehending the microlevel processes that underlie the usage of technology in a mathematical problem-solving activity. In this study, results showed that grasping, analyzing, exploration, planning, creating, verification, and dissemination were the frequently observed processes. In most cases, these processes were not seen in a linear way. The affordances of GeoGebra helped the students to explore and verify when solving difficult problems. However, these features can be maximized if a student has the mathematical skills to interpret the visual representations during these processes.

7. Recommendations

Analyzing student work completed in a classroom context using the MPST framework is possible. If student work is graded, results may be different because students tend to work harder and pay closer attention if they are aware that their performance will have an impact on their marks. The activities can be done individually or in a group. A comparison of the problem-solving processes between the two set-ups can also be made

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References

[1] Polya, G. (1945). *How to Solve it, A New Aspect of Mathematical Method.* Princeton, New Jersey.

[2] Schoenfeld, A. H. (1985). *Mathematical Problem-solving*. Orlando, Florida: Academic Press, Inc.

[3] Preiner, J. (2008). *Introducing Dynamic Mathematics Software to Mathematics Teachers: The Case of GeoGebra*. Dissertation in Mathematics Education Faculty of Natural Sciences. Salzburg.

[4] Jacinto, H., & Carreira, S. (2017). Mathematical problem solving with technology: The technomathematical fluency of a student-with-GeoGebra. *International Journal of Science and Mathematics Education*, 15(6), 1115–1136. https://doi.org/10.1007/s10763-016-9728-8.

[5] Martin, A., & Grudziecki, J. (2006). *DigEuLit: Concepts and Tools for Digital Literacy Development. Innovation in Teaching and Learning Information and Computer Sciences*, 249-267.

[6] Carreira, Susana & Jacinto, Helia. (2019). *A Model of Mathematical Problem Solving with Technology: The Case of Marco Solving-and-Expressing Two Geometry Problems.* 10.1007/978-3-030-10472-6_3