Incorporating Digital Interactive Figures: Facilitating Student Exploration into Properties of Eigenvalues and Eigenvectors

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Abstract: *Linear algebra is a key topic in mathematics and many other disciplines. In this paper, we consider a set of digital interactive figures (I-figs) using Mathematica created for linear algebra students in introductory and advanced courses, which prioritize pattern-seeking and examples over arithmetic processes. The figures discussed here are a part of digital worksheets designed to facilitate students' ability to visualize and work with eigenvalues and eigenvectors. Minimizing student computation for the benefit of conceptual focus while looking at an unlimited number of examples is an overarching theme. Different design intentions are explored for each of the four example figures. The worksheets provide a foundation for motivating students to participate in a system of observation, conjecture, proof, and theorem. Students were further supported through classroom lessons and additional homework activities.*

1. Introduction

Linear Algebra is a key topic in mathematics, is used in many other disciplines, provides the foundation for many AI and data analytic techniques, and plays a significant role in industry.

In teaching and learning linear algebra, the use of Computer Algebra Systems (CAS) such as MATLAB, Maple, Python, or Mathematica has a valuable place (Lay, Lay, & McDonald [7]). For example, in working with matrices and vectors, solving systems, matrix multiplications, finding inverses, and reducing a matrix to a reduced echelon form, to name a few, technology can help with accurate and fast calculations as well as allow one to concentrate on the deeper conceptual aspects of the discipline that are often difficult for students to grasp. Technology can help to build a visual image of the details, as well as the big picture, and allows for predictions, investigations, and producing conjectures.

Linear algebra students' ways of thinking about abstract notions of linear algebra, and in particular eigenvalues and eigenvectors, have also been a topic of study in some studies (e.g., Thomas & Stewart [12]; Salgada & Trigueros [9]; Wawro, Watson, & Zandieh [13]). In the early 90s, the Linear Algebra Curriculum Study Group (LACSG) recommended that "faculty should be encouraged to utilize technology in the first linear algebra course" [3]. The linear algebra recommendations by LACSG 2.0 also emphasized the importance of technology in teaching linear algebra [11].

Despite national initiatives in the US on teaching linear algebra with MATLAB, for example, the ATLAST project initiated by Steven Leon and his colleagues (Leon et al. [8]) in the early 90s, systematic studies of linear algebra instructors' thought processes and their students' feedback on the effect of technology on their understanding are lacking. In a survey paper by Stewart, Andrews-Larson, and Zandieh [10], the authors looked at linear algebra education papers from 2008-2017 in mathematics education journals and found almost no systematic classroom studies on the effectiveness of the use of Mathematica, Maple, Python, or MATLAB in the classroom, with the exception of a study of using MATLAB in mathematical modeling (Dominiques-Garcia et al. [4]). However, the survey paper found studies on eigentheory using Geometer's Sketchpad (e.g., Gol Tabaghi [5]; Caglayan [2]), and GeoGebra (Beltran-Meneu et al. [1]) focused on the geometric aspects of the topic.

In this paper, we present and discuss the use of interactive figures (I-figs) embedded into a student worksheet and used by the authors in both introductory and advanced linear algebra courses. Additional I-figs can be found in the interactive electronic version of the textbook *Linear Algebra and its Applications* [7]. Our emphasis is on conjecturing and formal understanding and less on geometry, in line with Harel's [6] notion of the shortcomings of using geometry as a foundation to build linear algebraic ideas. Instead, we focus most of our visuals on rectangular arrays of numbers, representing matrices and vectors, and setting up the I-figs to illustrate various patterns in the relationships of the objects presented.

The research team (the authors) regularly teach introductory and advanced courses in linear algebra. The first and second authors have started using the I-figs in teaching their classes, with guidance from the third author on how to collect data in order to research their effectiveness and improve their use.

The introductory course typically starts with representing a system of equations as a matrix and solving the system using row operations, followed by a discussion of linear dependence and independence, span, basis, linear transformations, vector spaces and subspaces, determinants, eigenvalues and eigenvectors, and orthogonality. The predominant visual for the course is rectangular arrays, which are used to illustrate vectors, matrices, and linear transformations. In the introductory course, the homework is built on computation and understanding concepts, with the expectation that students can be asked to do simple proofs on their own. The more advanced course, however, emphasizes the vector spaces and linear transformations at a more rigorous level, leading to a more detailed development of eigenspaces and orthogonality. The homework in the advanced course is predominantly proof based.

2. The Objectives and the Design of Interactive Figures

2.1 Student Need for Familiarity with Linear Algebra Objects

For many mathematics students, linear algebra is often their first exposure to mathematics and mathematical objects outside of their experience in traditional algebra and calculus courses. Linear algebra is then potentially their first exposure to variables denoting more than just scalar values or scalar valued functions. A succinct encapsulation of this limitation is in the fundamental eigenvalue/eigenvector equation $A\mathbf{x} = \lambda\mathbf{x}$, where A represents an $n\times n$ matrix, $\mathbf{x}$ is a vector in $\mathbb{R}^n$, and λ is a scalar. This equation requires students to view each of these variables as separate entities equatable because both operations, the transformation of $\mathbf{x}$, represented by $A\mathbf{x}$, and the scaling of $\mathbf{x}$, represented by $\lambda\mathbf{x}$, form vectors in $\mathbb{R}^n$.

Furthermore, when introduced to the notion of a vector in linear algebra, students may tend to rely upon geometric representations of vectors, common in physics or vector calculus. This can become an impediment to student understanding as they work with higher dimensional systems that cannot be geometrically represented. While there is certainly a place for geometric thought and interpretation in linear algebra, we build many of our I-figs to encourage students to see matrices and vectors in the form of rectangular arrays as the primary visual. This approach lifts to any (finite) dimension and is an asset to students learning computational techniques such as row operations and calculating determinants.

2.2 Using Technology to Experience Mathematics Through Observation, Conjecture, Proof, and Theorem

Much of a mathematician's work can be described in the general process of observation, conjecture, proof, and theorem (not ascribing a perfectly sequential process, instead inviting repeated iteration upon these steps). It is worth mentioning that we order a proof before the theorem despite students typically seeing textbooks that display theorems with their proofs following. In this mathematician's system of observation, conjecture, proof, and theorem, we view the theorem as necessarily existing only after the proof validates it, and the final theorem is the product of condensing the result of the proof into a succinct and useable statement.

However, many linear algebra students have little to no experience reading, analyzing, or constructing such formal mathematics. One intention of these worksheets is to help students establish a practice of experimenting mathematically and to form conjectures about the objects they study. In particular, we have developed and will continue to iterate a series of digital worksheets meant to give students a structured environment to experiment with vectors and matrices so that they may begin making observations and, ideally, formulating conjectures.

To facilitate this experimentation and observation, a part of what we wish to accomplish with these worksheets is to eliminate any unnecessary arithmetic that would normally restrict or distract students from pattern-seeking. These I-figs do not seek nor advise to eliminate the arithmetic practice of processes like matrix multiplication or eigenvalue calculation. Instead, these actions are performed by the digital worksheet to remove them as a cognitive load for the student so that they might focus on more general structural patterns in the algebra. Other homework and coursework should provide sufficient opportunity for these rote practices.

2.3 Role of Digital Worksheets

For content delivery, the purpose of this series of I-figs is to provide students a venue to encounter concepts and numerical or algebraic patterns in linear algebra that are common and will be expanded upon in future lessons in a more rigorous fashion. As inherently introductory (or even pre-introductory) activities, these digital worksheets were aimed to be fairly succinct, requiring only 15 to 20 minutes to allow students to meaningfully engage with the presented ideas. The design of these worksheets was guided by two user-oriented principles: minimal investment and ease of use, and two content delivery principles: example generation and encouragement to conjecture.

A central philosophy for these worksheets is that they may be approached by students with minimal investment on the students' part. In computational I-figs, the worksheet performs operations such as scaling or multiplying matrices, or as in this worksheet, more complex actions such as computing eigenvalues, reducing student cognitive load. As such, students do not necessarily need to know how to calculate eigenvalues in hopes that they may focus only on the relationship between the scalar matrix exponent and its effect on eigenvalues. This also serves to allow students to view any desired number of examples in a reasonable and accessible amount of time.

Each activity is designed so that example vectors or matrices (as appropriate) are generated by the worksheet rather than by manual student input, though students are typically allowed to control the size of the matrix generated (usually 2×2 up to 4×4) and to generate new examples. By intentionally avoiding incorporating manual input of matrix values, the worksheet aims to minimize the amount of effort and time required of students to begin pattern-seeking.

In piloting this worksheet, we see students making valuable observations but without the tools or confidence to pose their observations as conjectures or to attempt to prove a result. To address this, our current lesson system involves providing a digital worksheet either before or directly after the first lesson in a supported content area. Students are invited to experiment with the worksheet, make any observations they can, and record their thoughts and questions as a reflection assignment. In subsequent class time, the worksheet can be revisited together in order to collect and organize the various conjectures that can be made from student observations, paying attention to carefully modeling how an observation can be framed as a conjecture. For those conjectures with accessible proofs, the next step would be to collectively build such a proof or proof outline.

Naturally, some students will be inclined to investigate beyond the scope of the worksheet and would like to experiment with their own input objects. The I-figs are intentionally restricted in the amount of student input required and hence allowed, and so do not permit students to design their own experiments. This is in line with the goal for these worksheets to foster and encourage student conjecturing. In addition, students can be encouraged to perform calculations by hand themselves or via other resources such as Wolfram matrix labs or MATLAB.

2.4 Mathematics Software Requirements

The digital interactive figures (I-figs) being used were constructed using Mathematica, however, another CAS could be used. Important goals for the instructors in creating the I-figs and worksheets include less investment of time and coding ability by students, the ability to create worksheets containing the I-figs with commentary and questions, and no additional cost for students to use the figures and worksheets.

The robust Manipulate command enables these I-figs to be fully self-encapsulated. Interacting with the figure only requires students to manipulate slide bars, click buttons, and select from drop-downs. This is important for the ease of use for students of various educational backgrounds who may or may not have any coding experience.

Mathematica is structured to allow various cell types to both coexist and nest compatibly. This allows the I-figs to be narratively structured with instructional or descriptive text blocks and images to be neatly arranged, creating a conversational worksheet for students.

The I-figs and worksheets run on Wolfram Digital Player, which is an application developed and supported by Wolfram and is available to download for free. While programming in Mathematica does require a license (which a given university or college may or may not possess), the accessibility of Wolfram Digital Player removes cost as a barrier for classroom and student use of the I-figs and worksheets once they are created.

3. The Eigenvalue/Eigenvector Worksheet and Instructors' Reflections

In this section, we will show examples of I-figs ranging from illustrating one idea per figure to more sophisticated I-figs that display multiple patterns and invite several possible conjectures or counterexamples. In all examples to follow, different algebraic objects are displayed as appropriate, allowing students to visualize and contrast the different structural forms of scalars, vectors, and matrices that coexist in the linear algebraic equation $A\mathbf{x} = \lambda\mathbf{x}$. Furthermore, we expect this experience to facilitate students' handling of these objects in subsequent lessons, assignments, and settings beyond the course.

3.1 Focused I-Figs: Displaying the Effect of a Single Algebraic Operation on Eigenvalues

Consider the below example of two worksheet modules (Figure 1) used in the introductory course. The participants are able to dictate the desired size of the matrix within the range of

2×2 to 4×4, and to choose key scalars as appropriate. A "new matrix" button allows the student to generate new examples. In each of these two figures, there is a single pattern intended to be emphasized. Broadly, this compartmentalization is meant to eliminate distraction and allow students to reasonably quickly make observations, keeping time investment minimal and pattern recognition as their primary focus.

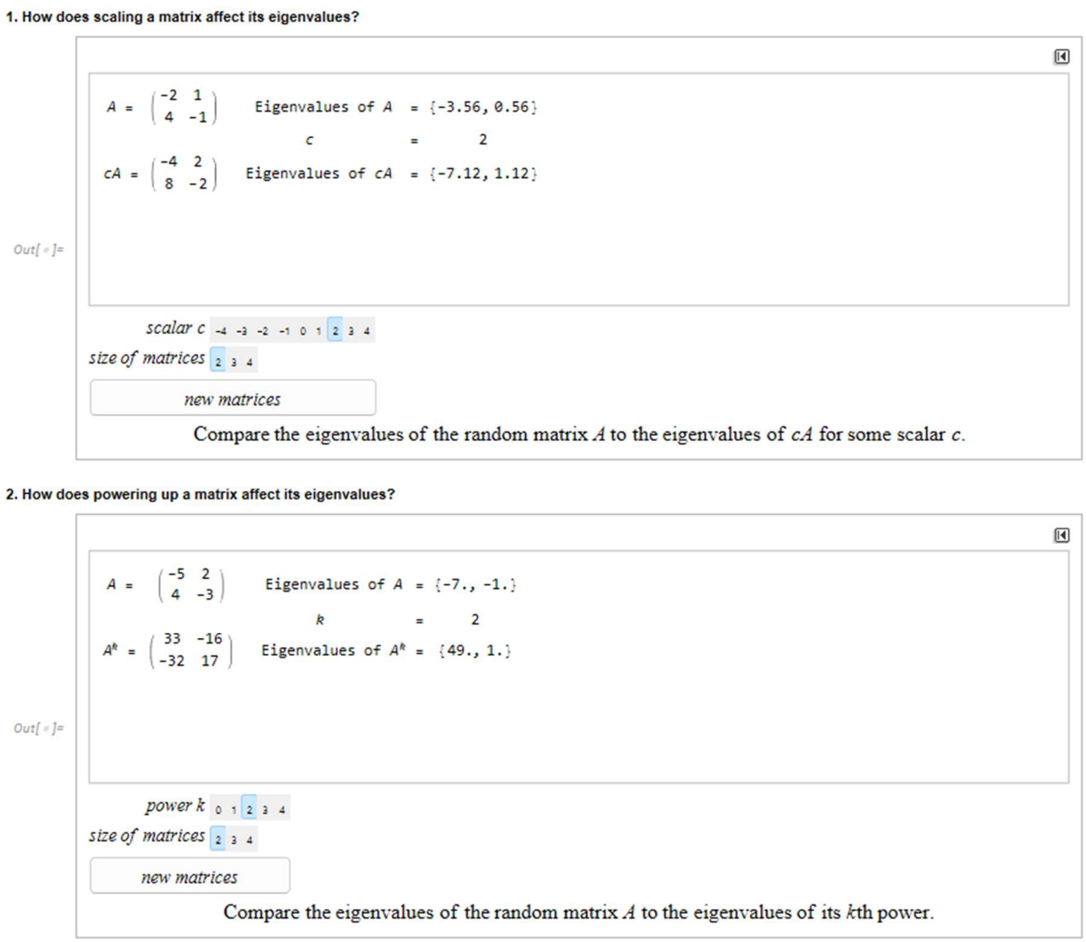

Figure 1 First Modules in Eigenvalue Worksheet

3.2 Building some Geometric-Algebraic Connections

Other activities have been made with the intention of emphasizing a connection between geometric and algebraic ideas. Consider the following example (Figure 2.1). The vector **v** is displayed, both geometrically and as a rectangular array, along with the transformed A**v**. Students are given control of the second component of **v** via a slider (which naturally changes A**v** as well). While the transformation matrix A is not specified, students at this point in their linear algebra education could potentially calculate A if curious. However, the salient interest of this figure is to motivate students to connect the linear algebraic equation $A\mathbf{v} = \lambda\mathbf{v}$ with both geometric and algebraic representations of that scalar relationship. These two eigenvalue-eigenvector relations can be found by experimenting with the slider for **v** or using the two buttons provided to move directly to those values (Figure 2.2).

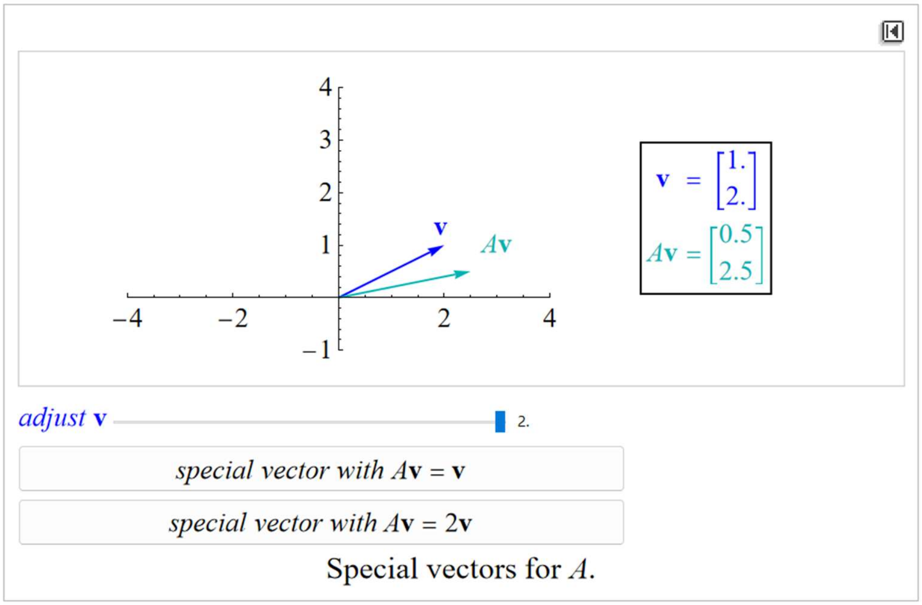

Figure 2.1 Geometric and Algebraic Connection of Eigenvalues and Eigenvectors

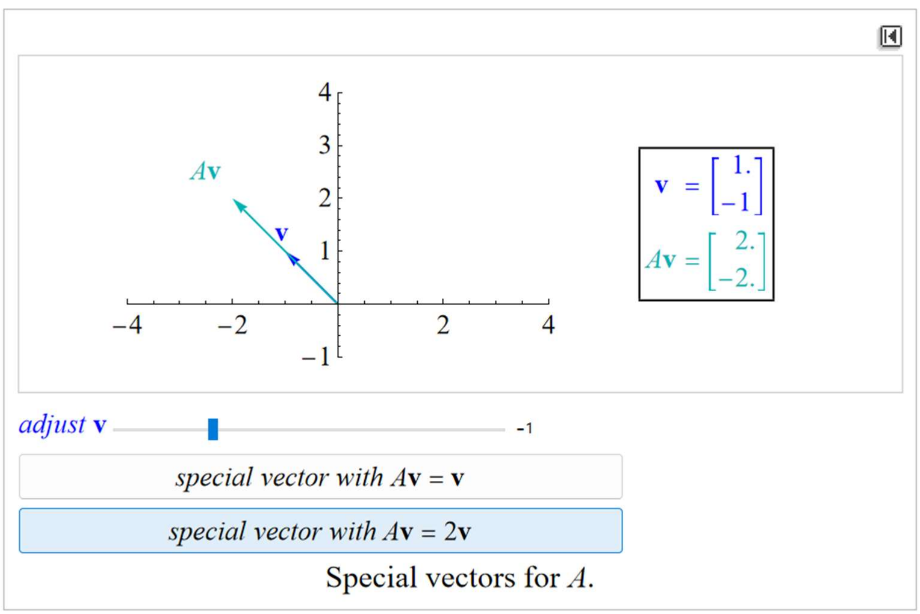

Figure 2.2 Displaying Eigenvalue 2 Geometrically

3.3 Increasing Sophistication: Multiple Potential Conjectures or Counterexamples

Next, we will look at a more complex I-fig (Figure 3) which possesses more than one distinct pattern for students to observe and conjecture. Students may observe that the eigenvalues of AB and BA match, or A and $B^{-1}AB$ share eigenvalues.

This figure also contains several common patterns that students might expect to observe that do not hold, such as the eigenvalues of AB or $A+B$ being directly related to the eigenvalues of A and B separately. An equally important skill alongside conjecturing and proving conjectures is the ability to appraise and counter false conjectures. Our hope for this particular I-fig is to allow students to question a reasonable-looking hypothesis (for example, a direct relationship between A, B, and $A+B$ eigenvalues) and to generate examples via the worksheet for the purpose of finding convincing counterexamples.

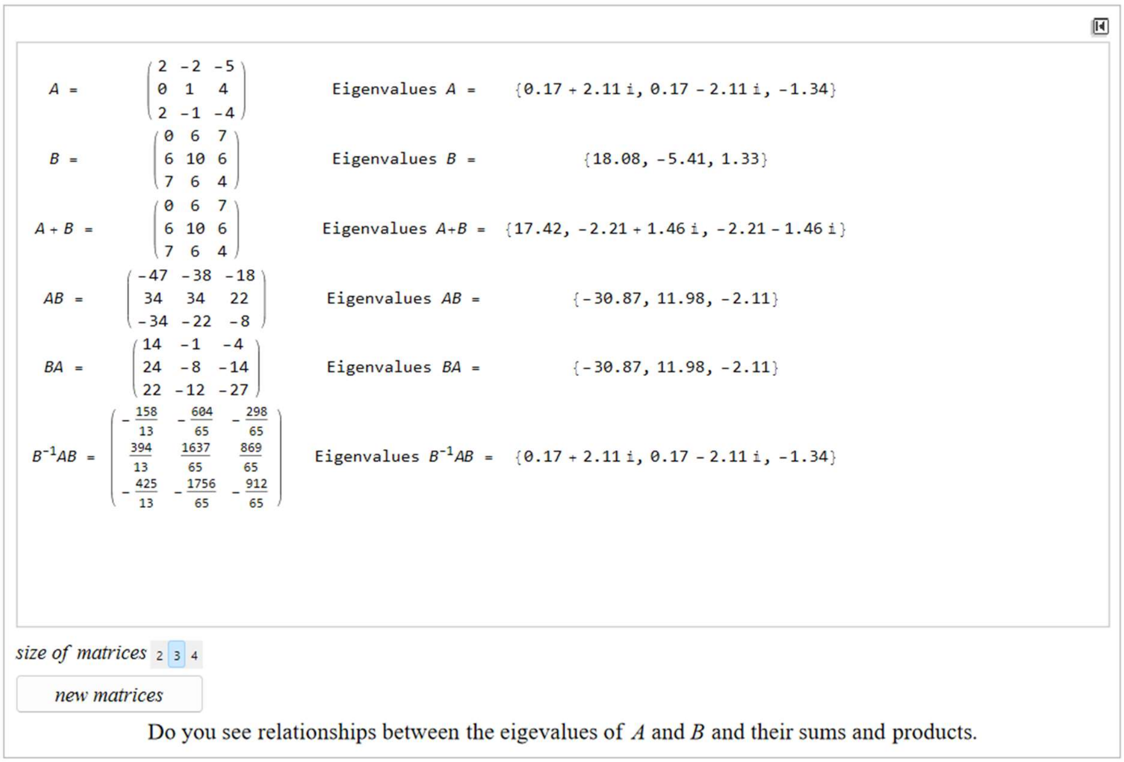

Figure 3 Digital I-fig with Multiple Points of Interest

3.4 Progressive Example: Advanced Students are Expected to Formally Justify Their Observations.

The last figure (Figure 4) is an example of how I-figs can be constructed to curate. In Figures 1 through 3, used in the introductory course, operations on the matrix are the primary focus. The eigenvalues produced from the generated integer matrices could be complex and were rounded to two decimal places. In Figure 4, used in the more advanced course, the focus of the I-fig is on the eigenvalues and eigenvectors. A random set of integer eigenvalues and integer linearly independent eigenvectors are created first and used to construct each matrix. Such a construction creates messy rational matrices, as seen, however, allows the students easier access to assessing and pattern-seeking among the eigenvalues and eigenvectors. In particular, for this I-fig, if the matrix were generated first and yielded irrational and complex eigenvalues, it may be exceedingly difficult for students to recognize the connection between the eigenvalues of A and A^{-1}.

Since this figure was used with a more advanced group of students, in the homework that followed, they were asked to prove that for any invertible matrix A, the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A. Students also observed that the two matrices have the same eigenvectors, helping them see a possible route to proving that the observed relationship between the eigenvalues of the two matrices must always exist.

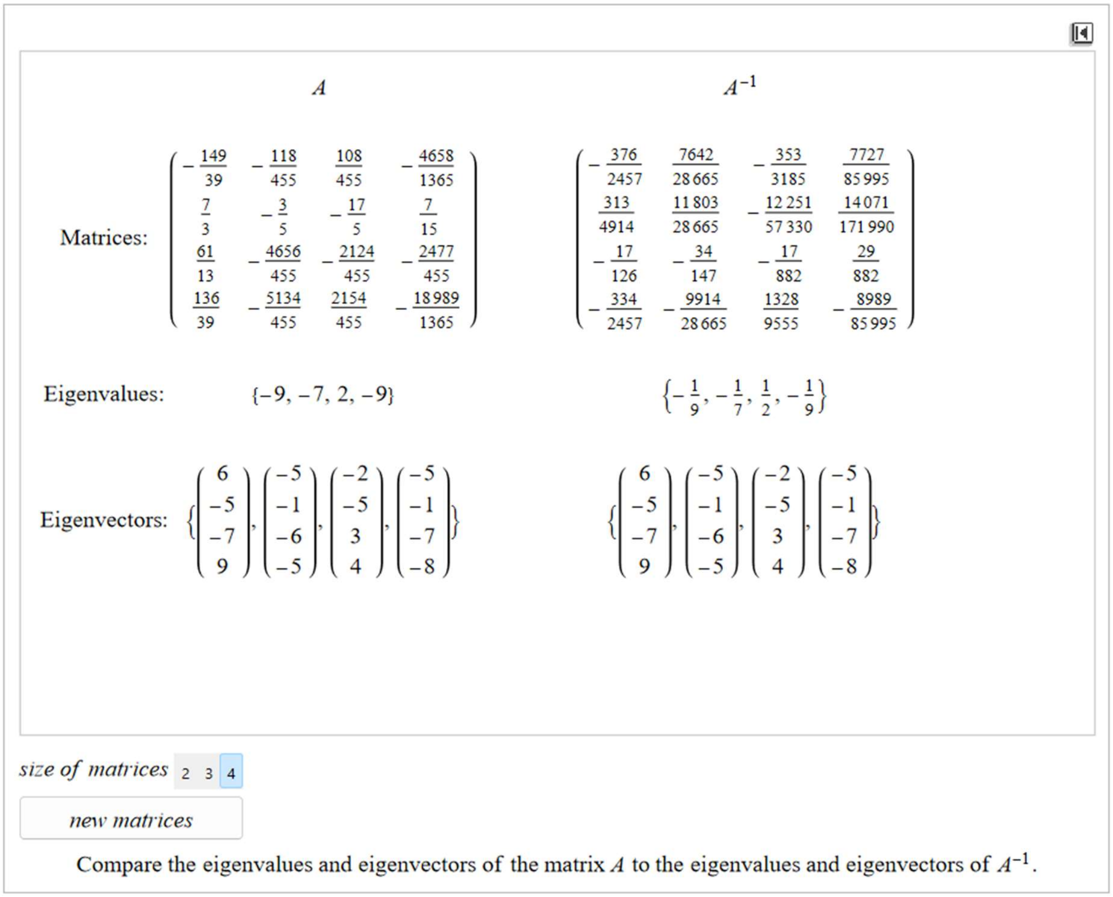

Figure 4 Digital I-fig with Curated Eigenvalues

4. Concluding Remarks and Future Work

This paper discusses a way to operationalize interactive figures, such as those developed in the textbook *Linear Algebra and its Applications* [7], into student worksheets, and study the effectiveness of the proposed use. Our I-figs and worksheets were designed to provide students with the opportunity to work through the mathematical process of observation, conjecture, proof, and theorem.

In using the eigenvalue and eigenvector worksheet in teaching, the authors provided students with an opportunity to look at an unlimited number of examples quickly and effortlessly. This leverages the computational benefits of CAS to allow students to focus and think about linear algebraic properties without the arithmetic burden common to linear algebraic calculations, hence enhancing their knowledge base.

As was found in the linear algebra education survey paper [10], studies about the effectiveness of employing common linear algebra CAS, such as MATLAB, that alleviate the cognitive load of performing arithmetic operations and interacting with linear algebraic objects, were rare. To fill this gap in the literature, careful research studies about effective ways to integrate technology into the classroom in ways that improve students' understanding of linear algebra are needed.

To evaluate the effectiveness of this worksheet, we conducted a research study in the first author's introductory linear algebra course. As part of this study, the worksheet on eigenvalues and eigenvectors was administered to students around the first lecture covering these concepts. Student-centric data was collected, consisting of written reflections following their participation in the worksheet assignment, as well as follow-up surveys in subsequent meetings, including a post-chapter review. Currently, we are in the process of analyzing this data. For a more effective evaluation of students' thought processes, based on our experiences and analysis of the data, we will modify the I-figs and worksheets, as well as our research designs, as needed. In any review scenario, we aim to maintain a mentality of persistent improvement rather than any attempt to confirm a final form for these digital worksheets.

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