Students’ Cognitive Developments in Learning Basic Differentiation Rules Using the Desmos Classroom Based on the Three Worlds of Mathematics

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Abstract

Investigating how undergraduate students learn derivatives is crucial to supporting them in successfully continuing their studies in integral calculus. This case study investigates students' cognitive development as they learn basic differentiation rules using the Desmos Classroom (DC) based on the Three Worlds of Mathematics (TWM). This research includes 25 students enrolled in Calculus 1 at Sampoerna University during the fall semester of the 2022–2023 academic year. DC is used as a generic organizer to facilitate an embodied operation on a function's graph. DC enables students to drag the tangent line and tangency point along the graph of a function, and it allows them to magnify the screen, which helps them make sense of the tangent line and derivative concepts. Students prove the basic differentiation rules on the DC through graphical exploration, numerical computations for practices, and symbolic manipulations. Based on the TWM, the DC can contribute to the student's cognitive development by helping them learn the basic differentiation rules. All students performed well in the axiomatic formal world by proving the derivative of a trigonometric function. Most students (92%) also solved the tangent line problem, which required them to think in the proceptual world. Many students (64%) also have no limitations on graphical representation. According to this result, students' success in thinking in the axiomatic formal world does not imply their success in the proceptual world. Similarly, success in thinking in the proceptual world does not imply success in graphical representation.

1. Introduction

Several strategies, including the Three Worlds of Mathematics (TWM) framework [1], have been proposed to facilitate students' learning of the derivatives. When students learn the derivative of a function, the TWM framework outlines three distinct worlds. These worlds involve the simultaneous development of conceptual embodiment, which involves the use of human perception and action; proceptual symbolism, which involves the manipulation of symbols derived from operations; and axiomatic formalism, which involves the construction of formal knowledge within an appropriate fundamental framework deduced through mathematical proof [2]. Students should commence TWM-based learning activities by experiencing the dynamic embodiment of change. Then, they should be able to perform sufficient arithmetic to obtain numerical approximations and appropriate symbolic expressions for the derivative of a function.

Tall [1] proposed using a generic organizer in calculus learning activities to actualize the TWM framework. This computer-based "generic organizer" allows students to investigate calculus concepts and generate dynamic function graphs. It enables focusing on function graphs so that every part of the graph can be seen. Tall [3] argues that computer programs are indispensable for demonstrating dynamic graphical visualization and providing precise numerical and symbolic calculations, which facilitate learning calculus derivatives. Students can visualize mathematical concepts through various graphical, numerical, and algebraic representations made possible by dynamic computer programs. Students can generate tables of values, input equations, and plot diverse functions. The dynamic nature of these mathematical objects enables students to visualize and comprehend mathematical problems or procedures in ways that traditional paper-and-pencil methods cannot [4].

Desmos Classroom (DC) is a dynamic web-based application with many advantages over other programs and applications. The program is free, user-friendly, intuitive, and extremely effective for graphic design, and encourages student participation and engagement in mathematics learning [5]. As a generic organizer, DC provides robust graphical and computational capabilities that facilitate an intuitive approach to assist students in acquiring the basic differentiation rules. Based on this justification, the researcher intended to investigate students' cognitive development in learning the basic differentiation rules using Desmos Classroom (DC) based on the TWM framework. Consequently, this case study research aims to address the following research questions:

(1) How is the students' cognitive development process in learning basic differentiation rules using DC based on the TWM framework?
(2) Can the learning activities using DC based on the TWM framework contribute to students' cognitive development?

2. Students’ Cognitive Development According to the TWM Framework

Tall [6] developed the Three Worlds of Mathematics (TWM) framework that comprehensively analyzes the students' cognitive development as they acquire mathematical concepts. These three worlds operate and evolve differently, comprising distinct cognitive domains, as outlined below:

(1) Conceptual embodiment involves the student's perception and actions, the formation of mental images expressed in increasingly sophisticated verbal forms, and the development of an appropriate mental construct within the student's imagination.
(2) Proceptual symbolism evolves from physical actions to mathematical procedures symbolized and conceptualized as both operations to be performed and symbols that can be manipulated and calculated.
(3) Axiomatic formalism involves constructing formal knowledge within axiomatic systems using a suitable foundational framework.

The embodied approach to derivatives prioritizes forming derivative concepts through physical interactions with the function graph. It enables students to experience dynamic changes, arrive at accurate symbolic representations of a function's derivative, and make numerical estimations. According to Tall [7], the cognitive foundation of derivatives is local straightness. Local straightness is a fundamental concept of a small portion of a graph that will appear straight. Using this cognitive root, students can perceive the entire graph's slope. Figure 1 depicts how students can construct the tangent line by dragging the graph as an object and witnessing the slope changes.

Figure 1. An Embodied Approach to the Derivative of a Function [8].

It is essential to establish a connection between the embodied operation and the symbolic domain, which includes numerical and algebraic computations. Proceptual symbolism is the symbolic representation of functions and their derivatives [9]. According to Gray and Tall [2], a procept is a mental object consisting of a to-be-executed process and the concept that results from that process. In addition, proceptual thinking is defined as the capacity to manipulate symbols as both processes and concepts and switch between various symbols for the same mathematical object [2]. Using the same notation, the flexible and ambiguous use of symbolism to represent the dual nature of processes to be executed and concepts to be comprehended significantly improves proceptual thinking. As students advance, their internal depth, or "interiority," of the number concept grows, resulting in greater manipulation flexibility [11].

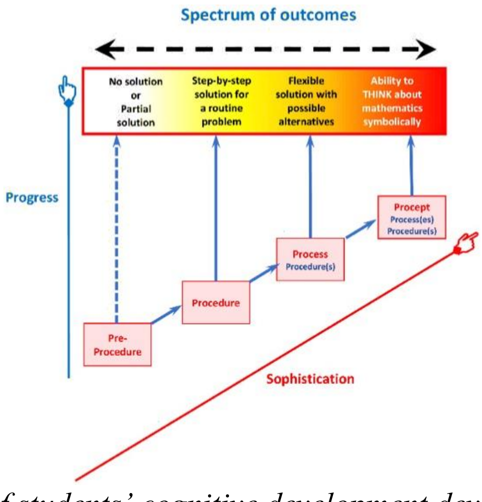


Figure 2. The spectrum of students' cognitive development developed by Tall et al. [12].

The spectrum of students' cognitive development outcomes at various levels of complexity are depicted in Figure 2. There are four levels on the spectrum: pre-procedure, procedure, process, and procept. This suggests that problems requiring only standard procedural responses will only serve to differentiate between students who make the transition successfully and those who do not. Multiple process paths provide alternative methods for identifying potential execution errors, including an innate awareness that something is wrong when an error occurs [13]. The procept level involves interpreting symbols as both processes to be carried out and concepts to ponder, facilitating the execution of dual processes and concepts.

3. Research Methods

In this study, the researcher adopted the constructivist paradigm to investigate the students' cognitive development when learning the basic differentiation rules using DC based on the TWM framework. Based on Stake’s study [14], constructivism emphasizes the idea that knowledge is primarily determined by social interpretation and not by objective reality. The constructivist epistemology was used throughout the study since the researcher was able to closely interact with all research participants. The students' responses to the DC-based learning activities supported this constructivist viewpoint. This research involved 25 students enrolled in the Calculus 1 course at Sampoerna University during the fall semester of the 2022–2023 academic year.

The research design of this study was a case study, which was chosen for its suitability in answering "how" questions [15], [16]. The Desmos Classroom (DC) has been validated by experts in mathematics, mathematics education, and technology in education, and became the main research instrument. DC was used as a generic organizer based on the TWM framework to assist students in making sense of the derivative concept and applying the basic differentiation rules to solve related problems. Students' responses in the DC were analyzed to investigate their cognitive development while learning the basic differentiation rules (see Figure 2).

In this investigation, additional research instruments are utilized. Included here are the observation guidelines, and test questions. The following observation guidelines also developed by referring to the Three Worlds of Mathematics framework:

Table 1: Observation Guidelines

Three Worlds of Mathematics	**Sample of activity**
The conceptual-embodied world	Students move the two points crossed by the secant line so that the two points are very close to each other and then students see the visualization of the tangent line. Students zoom in on a graph near a particular point to see it looks like a straight line.
The proceptual-symbolic world	Determine the derivative of a function using the basic differentiation rules. Find the slope of a tangent line. Determine the equation of a tangent line.

Axiomatic Formal	Prove the basic differentiation rules.

In terms of analysis, an interpretive theoretical approach was used to provide a framework for comprehending students' responses when learning the basic differentiation rules using DC based on the TWM framework. The methodology was inductive, and the results were predominantly descriptive [17]. Data analysis began with a comprehensive examination of the students' responses collected from DC. Each item's responses were thoroughly examined to identify emerging trends or themes. The classification of these themes according to the included descriptions allowed for a systematic organization and interpretation of the data [18].

To ascertain the outcomes of students' cognitive development, their responses to the DC activities and test-problem solutions were evaluated. Students' responses in DC revealed students' cognitive development when learning the basic differentiation rules. The purpose of the test questions (see Table 2) was to evaluate the students' cognitive development after learning the basic differentiation rules which refer to the spectrum of students' cognitive development presented in Figure 2.

Table 2. The Test Questions

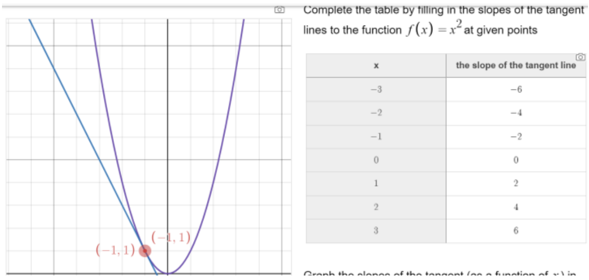
Problem	**Expected Cognitive Outcome**	**Remarks**
Problem 1: Prove that $\frac{d}{dx}[\mathbf{\cos x}] = -\mathbf{\sin x}$	Axiomatic Formal	Students were expected to think in the axiomatic formal world by using formal definition of derivative to prove the derivative of cosine function.
Problem 2: Find an equation of the parabola $\mathbf{y = ax^2 + bx + c}$ that passes through (0,1) and is tangent to the line $\mathbf{y = x - 1}$ at (1,0). Create the graphical representation of the tangent line problem.	Procept	Students were expected to think in the proceptual world by performing the duality of mathematics-related processes and concepts to successfully solve the problem.
	Graphical Representation	Students are expected to create a graphical representation of the tangent line problem.

The data are analyzed using the spectrum of students' cognitive development outcomes devised by Tall et al. [12]. The validity of this case study research is achieved by integrating data sources that provide a comprehensive picture of the examined issue [19]. *Triangulation* involves the use of different methods to investigate the research questions and provides a means of checking validity [19]. Students' cognitive development while learning the basic differentiation rules was analyzed based on their responses or answers inputted in the Desmos Classroom. Students' verbal responses to the lecturer's instructions during the class sessions were also recorded and written on the observation sheet. To answer the second research question, the researcher also evaluated the students' solutions to two problems, requiring them to think in the axiomatic formal and proceptual worlds and produce graphical representations of the problems.

4. Results and Discussion

How is the students' cognitive development process in learning basic differentiation rules using DC based on the TWM framework?

In the previous meeting, students have experienced learning through embodied approach by exploring the graph of $f(x) = x^2$ and moving its tangent lines at several points. They plotted the slope values of the tangent line obtained from each value of x (see Figures 3 and 4). The resulting plotting image is the graph of $y = 2x$, so they concluded that the derivative of $f(x) = x^2$ is $f'(x) = 2x$ (*symbolizing embodiment*). In addition, they have proven that $\frac{d}{dx}x^2 = 2x$ by applying the formal definition of the derivative ($f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ and provided the limit exists (see Figure 5).



x	the slope of the tangent line
−3	−6
−2	−4
−1	−2
0	0
1	2
2	4
3	6

Figure 3. Student O's responses regarding the slope of the tangent line of the function $f(x) = x^2$ for $x = -3, -2, -1, 0, 1, 2, 3$.

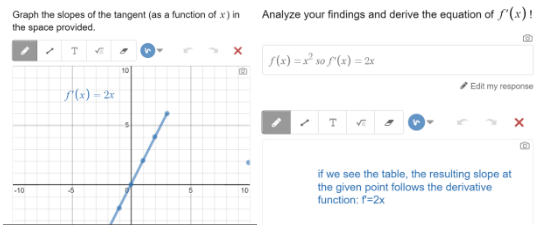


Figure 4. Student H's responses regarding the plotting of the tangent slope values for each x value, and her conclusion.

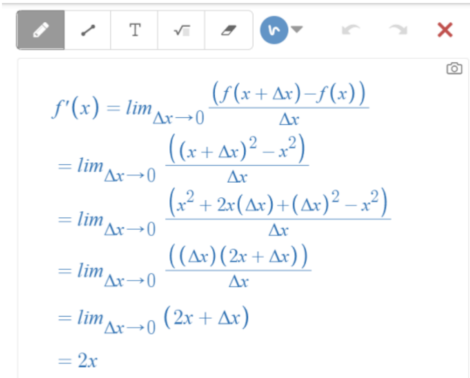


Figure 5. Student B's response (applying the formal definition of derivatives to prove the derivative of $f(x) = x^2$)

Since the beginning of learning activities, students were facilitated to see the visualization of the constant function and its derivative to identify that $f'(x) = 0$ (*symbolizing embodiment*). Next, students were thinking in the axiomatic formal world by proving that the derivative of a constant function $f(x) = c$ that is $f'(x) = 0$ (see Figure 6).

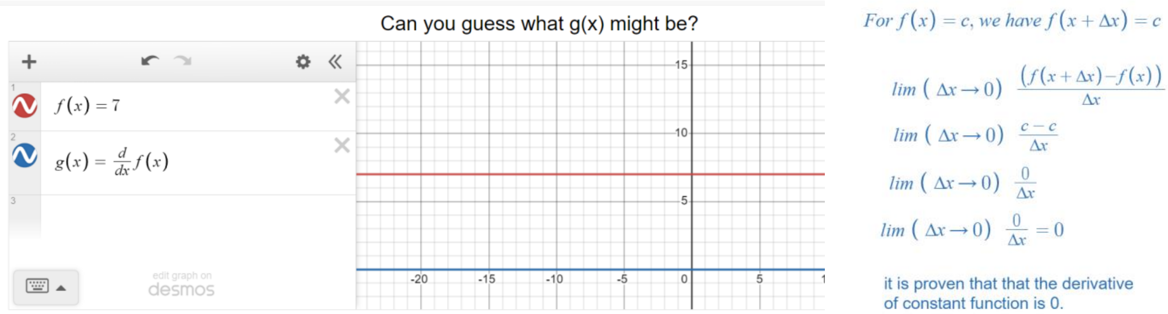
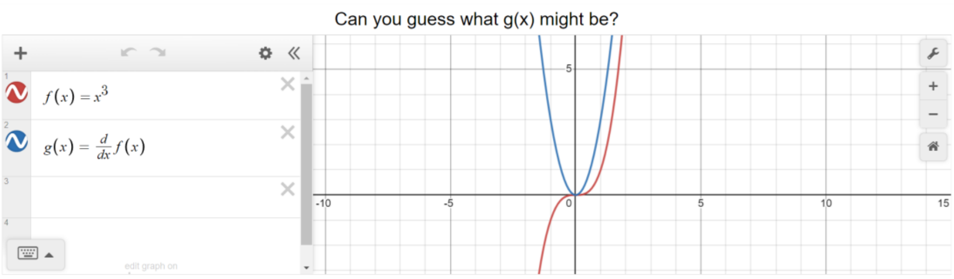
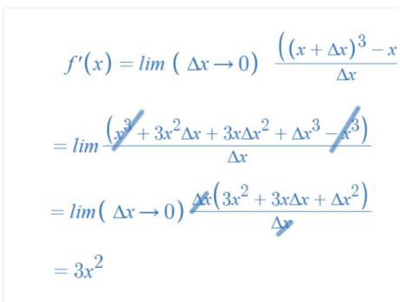
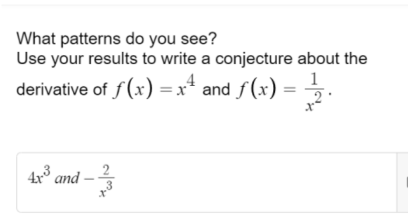


Figure 6. Student O's response

Similarly, students identify the derivative of the polynomial function $f(x) = x^3$ by visualizing the graph of $f'(x)$ (*symbolizing embodiment*). Next, they were rethinking in the axiomatic formal world by proving the derivative of $f(x) = x^3$, that is $f'(x) = 3x^2$ (see Figure 7). Students also prove the derivatives of polynomial functions: $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$. Students see graphical representations of the five polynomial functions and their derivatives (*embodying symbolism*). Then, they determine the derivative of the function $f(x) = x^4$ based on the pattern that emerges from the derivatives of the five polynomial functions (see Figure 8). Some students were able to generalize that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$ (see Figure 9).







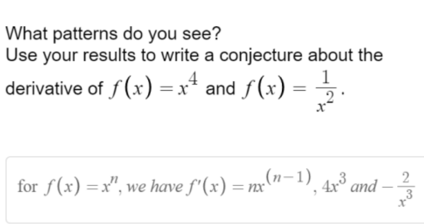


Figure 7. Student BA's response

Figure 8. Student A's response

Figure 9. Student O's conclusion

Students were rethinking in the proceptual world by determining the binomial expansion $(x + \Delta x)^3$ and they used the binomial expansion to the power of n, to verify the binomial expansion for n = 2, 3, and 4. Furthermore, students were rethinking in the axiomatic formal world by proving that the derivative of the function $f(x) = x^n$ is $f'(x) = nx^{n-1}$ (*the power rule*). Then, they apply the formal definition of the derivative of a function and the binomial to the power of n (see Figure 10).

$$\frac{d}{dx}\left[x^n\right] = lim\ (\ \Delta x \rightarrow 0)\ \frac{\left(\left(x+\Delta x\right)^n - x^n\right)}{\Delta x}.$$

$$lim\ (\ \Delta x \rightarrow 0)\ \frac{\left(x^n + nx^{(n-1)}\Delta x + .. + \Delta x^n\right) - x^n}{\Delta x}$$

$$lim\ (\ \Delta x \rightarrow 0)\ \frac{\Delta x\left(nx^{(n-1)} + \frac{n(n-1)x^{(n-2)}}{2}\Delta x\right.}{\Delta x}$$

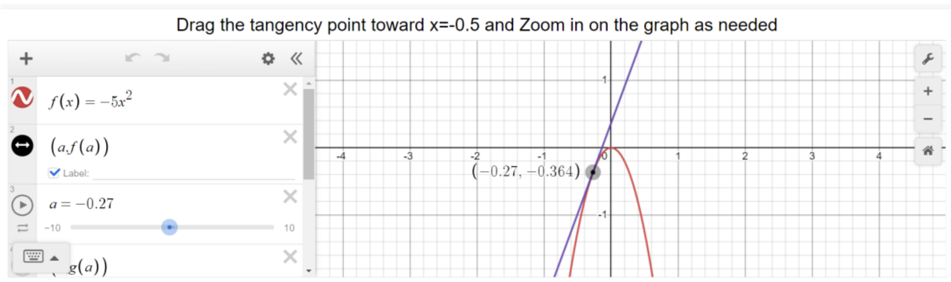
$$lim\ (\ \Delta x \rightarrow 0)\ nx^{(n-1)} + \frac{n(n-1)x^{(n-2)}}{2}\Delta x$$

$$= nx^{(n-1)} + 0 + 0 + \ldots$$

$$= nx^{(n-1)}$$

Figure 10. Student N's response

Students learn through the embodied approach by enlarging the graph of $f(x) = -5x^2$, moving the tangent line so that it touches $f(x)$ at $x = -0.5$. Students think in the proceptual world by obtaining $f'(x) = -10x$. Then they determine the tangent line's slope and eventually determine the equation of the tangent line (see Figure 11). Students also think in the axiomatic formal world by proving the sum rule, $\frac{d[f+g]}{dx} = f'(x) + g'(x)$ (see Figure 12).



$$f'(x) = -10x$$
$$f'(-0.5) = -10(-0.5) = 5$$
$$x_1 = -0.5 \quad y = -1.25 \qquad then\ y - y_1 = m\left(x - x_1\right)$$
$$y + 1.25 = 5(x + 0.5)$$
$$y + 1.25 = 5x + 2.5$$
$$y = 5x + 1.25$$

Figure 11. Student H's responses

$$lim\ (\ \Delta x \rightarrow 0)\ \frac{\left(f(x+\Delta x) + g(x+\Delta x)\right) - \left(f(x) + g(x)\right)}{\Delta x}$$

$$= lim\ (\ \Delta x \rightarrow 0)\ \frac{\left(f(x+\Delta x) - f(x)\right) + \left(g(x+\Delta x) - g(x)\right)}{\Delta x}$$

$$= lim\ (\ \Delta x \rightarrow 0)\ \frac{\left(f(x+\Delta x) - f(x)\right)}{\Delta x} +$$

$$lim\ (\ \Delta x \rightarrow 0)\ \frac{\left(g(x+\Delta x) - g(x)\right)}{\Delta x}$$

$$= f'(x) + g'(x)$$

Figure 12. Student BA's responses

Students learned through an embodied approach by moving the tangent line along the graph of the function $f(x) = \frac{\left(3x^7 - 5\sqrt{x} + 1\right)}{x}$, enlarging the visualization of $f(x)$ and its tangent line, and positioning the tangent line so that it touches the function at $x = 0.5$ (see Figure 13). Students were thinking in the proceptual world by using the power rule to obtain $f^{'}(x) = 18x^5 + \frac{5}{2}x^{\frac{-3}{2}} - x^{-2}$. Students calculate the slope of the tangent line $m = f^{'}(0.5) = 3.6335678$, and finally, they got the equation of the tangent line, which is $y = 3.6335678x - 6.8407839$.

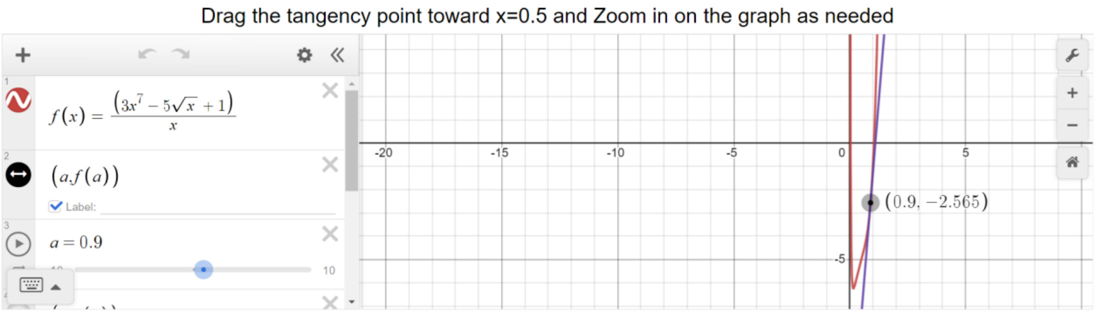


Figure 13. Student BA's graphical exploration

Students were facilitated to identify the derivatives of the sine and cosine functions by seeing the graphical visualization of each function and its derivative (*symbolizing embodiment*). Students were rethinking in the axiomatic formal world by proving that $\frac{d}{dx}[\sin x] = \cos x$. The proof uses the formal definition of the derivative of the function $f^{'}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, the trigonometric identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$, and special limits, $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ (see Figure 14). Furthermore, students were rethinking in the proceptual world by applying the derivative rules of sine, cosine, and polynomial functions with degree n to solve related problems.

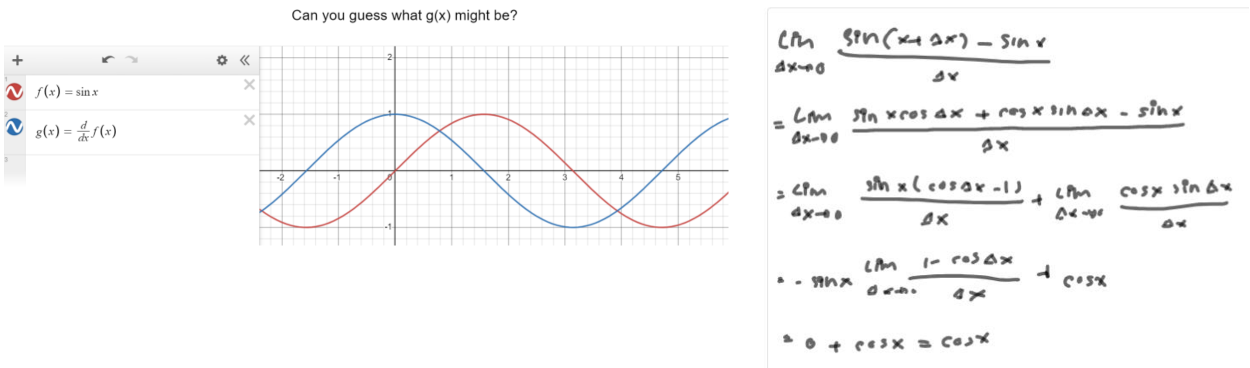


Figure 14. Student N's responses

Students were learning through an embodied approach by moving the tangent line to the function $f(x) = a \sin x$ to exactly touch the function at the point (0,0) and enlarging the display of the function and the tangent line (see Figure 15). Based on this activity, students concluded that the slope of the tangent line to $f(x) = a \sin x$ at (0,0) is equal to $f'(0)$ (see Figure 16).

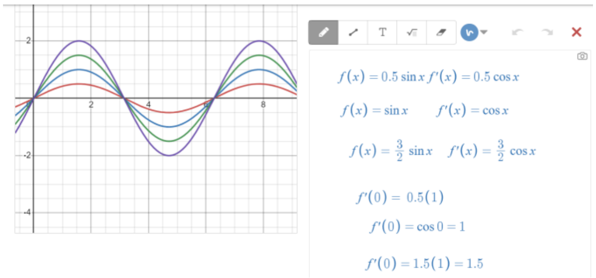


Figure 15. Student H's Response

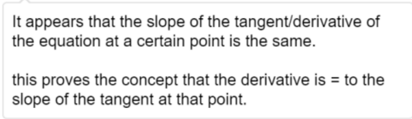


Figure 16. Student V's Findings

The learning process using DC based on the TWM framework focuses on developing students' perceptions of the basic differentiation rules and the derivatives as the slope of the tangent line. DC provides a graphical representation to assist students in identifying the function's derivative. After that, students think in the axiomatic formal world by proving the basic differentiation rules. They were also thinking in the proceptual world by applying the basic differentiation rules to solve related problems, facilitated through DC numerical and symbolic manipulation features. Additionally, the embodied learning activities were reflected by moving the tangent line along the function's domain until it was exactly tangent to the associated tangent point. At the same time, students also magnified the function and the tangent line (*embodying symbolism*). It reveals that DC can visualize the tangent line and compute its slope. These exploration activities assist students in finding the tangent point, the slope of the tangent line, and the tangent line's equation. According to Tall [1], learning activities that promote embodied calculus approaches, followed by numerical and symbolic manipulations, can aid students in cognitive development processes. In this study, students engaged in a series of iterative learning activities through DC that facilitated thinking in the embodied world through graphical exploration, thinking in the proceptual world by applying basic differentiation rules to solve related problems, and thinking in the axiomatic formal world by proving basic differentiation rules.

Can the learning activities using DC based on the TWM framework contribute to students' cognitive development?

In the first problem, it was found that all students succeeded in proving that the derivative of $f(x) = \cos x$ is $f'(x) = -\sin x$. Thus, students showed their success in thinking in the formal axiomatic world. As can be seen from one of the following student solutions:

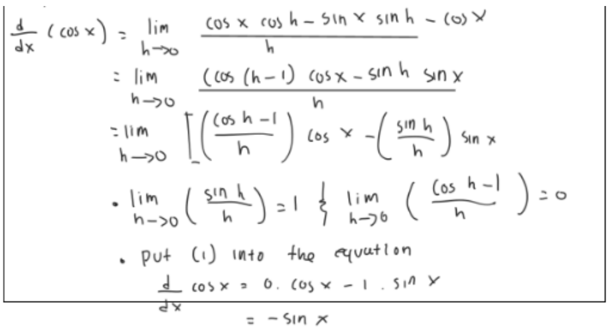


Figure 17. Student Y's solutions for problem 1

In the first step, Student Y used trigonometric identity to expand $\cos(x + \Delta x)$ into $\cos x \cos \Delta x - \sin x \sin \Delta x$ (see Figure 17). Next, she identified that there was a common factor, $\cos x$ which can be used to simplify the numerator to become $\cos x\,(\cos\Delta x - 1) - \sin x \sin \Delta x$cos. Then Student Y used the limit operation property to obtained $\lim\limits_{\Delta x \to 0} \frac{\cos(x+\Delta x)-\cos x}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{\cos x\,(\cos\Delta x - 1)}{\Delta x} - \lim\limits_{\Delta x \to 0} \frac{\sin x \sin\Delta x}{\Delta x}$. In the end, she used two special limits $\lim\limits_{\Delta x \to 0} \frac{\cos\Delta x - 1}{\Delta x} = 0$ and $\lim\limits_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1$, and she proved that $\frac{d}{dx}[\cos x\,] = -\sin x$.

In the second problem, students are expected to think in the proceptual world by obtaining the equation of a parabola $y = ax^2 + bx + c$ which passes through the point (0,1) and tangent to the line $y = x - 1$ at the point (1,0). It was found that 23 out of 25 students (92%) could think in the procept spectrum when solving the tangent problem. As seen in Student J's solution (see Figure 18), he obtained the slope of the tangent line $y = x - 1$, which is $m = 1$. Next, he substituted $x = 0$ and $y = 1$ into the parabolic equation, and then he obtained $c = 1$. In the next step, he looked for the derivative of the parabolic function, which is $y' = 2ax + b$. Student J utilized the value of the parabolic's derivative at $x = 1$, which is equal to $m = 1$, and he got $2a + b = 1$. After that, Student J substituted $c = 1, x = 1$ and $y = 0$ into the parabolic equation which resulting the second equation, $a + b = 0$. He eliminated the two equations and obtained $a = 2$ and $b = -3$. In the last step, Student J successfully obtained the parabolic equation $y = 2x^2 - 3x + 1$, and created a graphical representation of the tangent line problem.

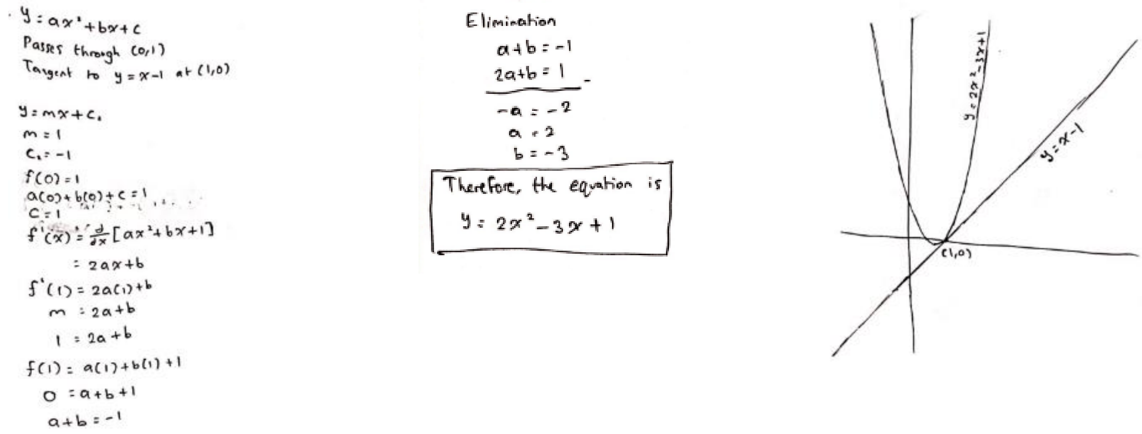


Figure 18. Student J's solution for problem 2

Student JA and Student J are two students who did not reach the procept spectrum when answering the second question. Student JA incorrectly interpreted the value of b in the parabolic equation $y = ax^2 + bx + c$. He thought that the slope of the tangent line $y = x - 1$ is equal to $m = 1 = b$. As a result, he obtained incorrect results, such as the equation$1 = 2a(1) + b$, the values of a and b, and even the parabola equation. Subsequently, he failed to draw the graph of the parabola. In addition, the error made by Student M was that he thought that the tangent line also passing through the point (0,1), so he substituted $x = 0$ and $y = 1$ into $y' = 2ax + b$. This led him to conclude that $y' = m = 1 = b$. This incorrect result made him fail to obtain the parabola equation and create a graphical representation of the problem (see Figure 19).

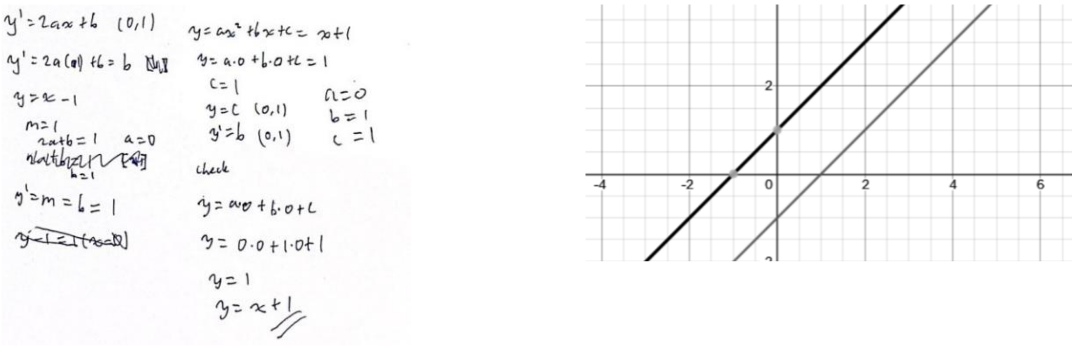


Figure 19. Student M's answer

Besides Student JA and Student M, 7 other students also did not produce the graphical representation of the tangent line problem (problem 2). Unlike Student JA and Student M, the seven students have obtained the parabolic equation. Thus success in thinking in the proceptual world does not imply success in the graphical representation. The students made various mistakes, and it was indicated Student S did not even try to produce the graphical representation, Student R did not succeed in drawing the graph of the parabola $y = 2x^2 - 3x + 1$, and the remaining five students (Student F, Student Y, Student A, Student MA, Student AU) could draw the graph of the parabola but did not draw the tangent line $y = x - 1$.

The general description of the students' cognitive spectrum outcomes in solving problems related to the basic differentiation rules is depicted by the following Venn diagram:

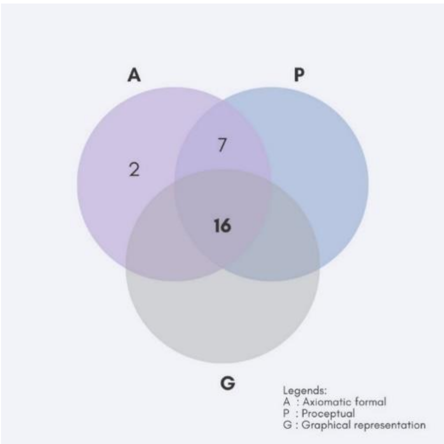


Figure 20. Students' Cognitive Development

- All students could think in the axiomatic formal world by proving the derivative of the function $f(x) = \cos x$.
- Two students succeeded in thinking in the axiomatic formal world but failed to think in the proceptual world. They also had difficulty creating a graphical representation of the given tangent line problem. It is because they failed to obtain the parabolic equation $y = ax^2 + bx + c$. Thus, success in thinking in the formal axiomatic world does not imply success in the proceptual world.

- Most students (23 out of 25 students, or about 92%) succeeded in thinking in the proceptual world by determining the equation of the parabola $y = ax^2 + bx + c$ through the point (0,1) and tangent to the line $y = x - 1$ at the point (1,0).
- 7 students succeeded in thinking in the proceptual world but failed to draw the graph of the parabola and the tangent line $y = x - 1$. In addition, there were 2 other students (Student JA and Student M) who were unsuccessful both in thinking in the proceptual world and in producing the graphical representation. It can be concluded that success in thinking in the proceptual world does not imply success in making the graphical representation.
- Some types of errors that were identified when students tried to create the graphical representation of the tangent line problem were: (1) Student S did not produce a graphical representation; (2) Student R did not succeed in drawing the graph of the parabola $y = 2x^2 - 3x + 1$ correctly, and (3) five others (Student F, Student Y, Student A, Student MA, Student AU) did not draw the tangent line $y = x - 1$.
- About 64% of the students (16 people) could draw the graphical representation of the parabola and the tangent line $y = x - 1$. They were also consistently successful in thinking in the formal axiomatic world and the proceptual world.

This study revealed that DC based on the TWM as a generic organizer can contribute to students' cognitive development while learning the basic differentiation rules. Although very few students struggle to think in certain cognitive worlds. This suggests that success in one cognitive world does not guarantee success in another. However, all students in this study could think in the axiomatic formal world by proving the derivative of the function. Most students (92%) succeeded in thinking in the proceptual world when solving the tangent line problem. Also, many students (64%) had no limitations in the graphical representations of the given problem. This further supports the idea that the learning activities using DC based on the TWM framework contribute to students' cognitive development. These findings were consistent with those reported in Tall's study [10],[20], which also discovered that students' capacity to sketch gradients of supplied graphs considerably increased and that their conceptualizations were extended while learning through a computer program.

5. Conclusion

This study has shown that the DC based on the TWM framework can be used as a generic organizer to build students' perceptual meaning of the derivatives and the basic differentiation rules. The DC based on the TWM provides recurrent activity cycles that include graphical exploration for derivative conceptual embodiment, numerical computations, and symbolic manipulations. All these activities can contribute to students' cognitive development. About 64% of the students were consistently successful in thinking in the axiomatic formal world and the proceptual world and could draw the graphical representation of the line problem. All students in this study were successfully thinking in the axiomatic formal world by proving the derivative of $\cos x$. Most (92%) also succeeded in thinking in the proceptual world when solving the tangent line problem. However, some students struggled with creating graphical representations of the tangent line problem. This suggests that success in one world does not guarantee success in another world. Therefore, lecturers should incorporate activities that target each cognitive world to support students' cognitive development.

This case study involved 25 students as research participants. The researchers suggest further research with a larger sample size to generalize the findings of this study. Utilizing DC based on the TWM framework has the potential to be implemented and studied further in other calculus topics. One is by graphically exploring the problem of higher-order derivatives of transcendental functions, which is not easy for students to guess. The extent to which DC as a generic organizer based on the TWM framework can help students make sense of the concepts of various calculus topics (embodied), gradually explore the proceptual world, and finally understand these topics symbolically (procept) can be studied holistically.

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