Teaching to high school students the intimate relation between definite integrals, piecewise quadrature and areas

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Abstract

Numerical integration is an important topic for modern analysis and differential equations. From an educational point of view, in the high school mathematical curriculum, the definite integral is often understood as an application of indefinite integrals, and area computation as a further application. High school students enjoy calculating definite integral sound area by using the Fundamental Theorem of Calculus without understanding the essence of definite integrals as limits of Riemann sums, a natural idea leading to the more advanced notions of piecewise quadrature and measure.

We propose a unified teaching approach of definite integrals for high school students which allow not only a mathematical understanding of the notions of definite integrals and area computations through the notion of piecewise quadrature but also the relation with the different figures of area known already and prepare the ground for the notions of numerical integration and theory of measure to be learned at undergraduate level. Another important application is the calculus of limit of the sum of terms of an infinite sequence that can be represented as a finite area through piecewise quadrature.

To facilitate an authentic comprehension, the concepts are accompanied by GeoGebra scripts to visually depict their application in mathematical education, thus highlighting the direct integration of Information Technology within this domain.

1 Introduction

High school mathematical education is very important for acquiring essential knowledge and skills, for preparing students for university undergraduate level as well as later career, for making them able to live in a modern world packed with information.

In particular, we consider the limit to be a fundamental concept in calculus, an indispensable skill for any student interested in fields of pure or applied mathematics, physics, engineering, computer science, etc. Beside providing a rigorous way to define continuity of functions, derivation and so on, understanding limits is a necessary step for obtaining mathematical maturity, for building intuition and opening the way to modeling and solving phenomena of the real world.

Even though concrete problems related to limits can be found in the countless number of calculus textbooks, we point out that limits of the following type are challenging for many high school students.

$$\begin{aligned} &\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right]=\log 2\\ &\lim_{n\to\infty}\frac{n}{\sqrt[n]{n(n+1)(n+2)\ldots(n+n)}}=\frac{e}{4}. \end{aligned} \tag{1}$$

A related notion is the area which is introduced in elementary education to develop a sense of measuring and comparing shapes. Almost any teenager can tell you that the area of circle radius r is πr^2, but how many of them understand why it is?

The unit of area is usually defined as the area of a circle with sides of unit length being equal to 1×1, and from here the area of any square or rectangle can be computed by decomposing it in one unit and simply counting the squares.

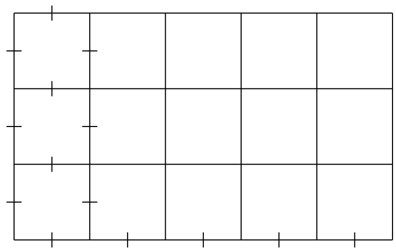

Figure 1: The notion of area.

The extension to triangle's area is quite intuitive.

However, when moving on to the circle area, things get more complicated. The irrational number π can be defined as the length of a circle of unit diameter. To prove that $\pi = 3.14\ldots$ is firstly suggested by direct measurements, but to provide a rigorous mathematical proof of this fact, one needs to apply the squeeze theorem for limits to the perimeters double inequality

$$\text{inscribed cyclic polygon} \leq \text{circle length} \leq \text{perimeter of the circumscribed.}$$

The circle area formula is now proved by decomposing the circle into sectors and rearrange them.

Knowing circle length and rectangle area, it is trivial to obtain the area of the circle as πr^2.

The area mystery goes further into high school mathematics ([7], [8]). If $f : [a, b) \to \mathbb{R}$ is a continuous function, then after defining the primitive F, i.e. $F'(x) = f(x)$, the definite integral

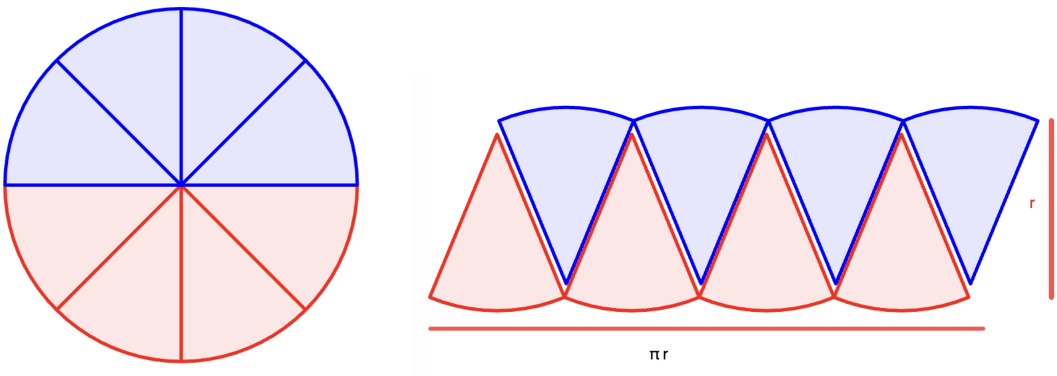

Figure 2: The area of the circle.

is defined by the formula

$$\int_a^b f(x)dx := F(b) - F(a) \tag{2}$$

and further, as application the area under the graph is defined by

$$S := \int_a^b f(x)dx \tag{3}$$

allowing in this way to students to easily to compute definite integrals and areas.

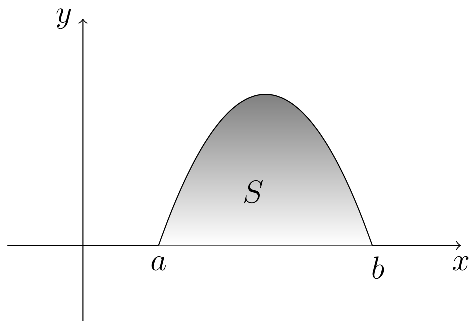

Figure 3: The area $Area\left(f\big|_{[a,b]}\right)$ under the graph of the function $y = f(x)$.

However, very few of them understand and can explain why the definite integral in (2) gives the area by formula (3).

In the present paper we purpose a unified teaching approach to definite integrals and areas in high school mathematics through the notion of limit that bring not only a deeper understanding of these notions, but also relation with circle area known already, computation of limits like the area in (1). In fact, formulas (2) and (3) are not definitions, but consequence of the piecewise quadrature definition (see for instance [1], [3], [4], [5], [6] and other similar textbooks).

Moreover, this kind of teaching prepare the background for more advanced topics like measure theory and numerical integration to be learned at the university level.

2 The piecewise quadrature

The piecewise quadrature, or approximating the area under the curve, and then passing to the limit, is the solution to the problems presented in the Introduction.

Indeed, for a function $f : [a, b] \to \mathbb{R}$, a partition $\Delta = (a = x_0, x_1, \dots, x_n = b)$ of the interval $[a, b]$ and a system of n points $t_i \in [x_{i-1}, x_i]$, $i \in \{1, \dots, n\}$, we define the real number

$$\sigma_\Delta(f, t) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

called the **Riemann sum** associated to f, Δ and $\{t_i\}$.

Obviously, $t_i = x_{i-1}$, $t_i = x_i$ or $t_i = \frac{1}{2}(x_{i-1} + x_i)$ are obvious choices for t_i.

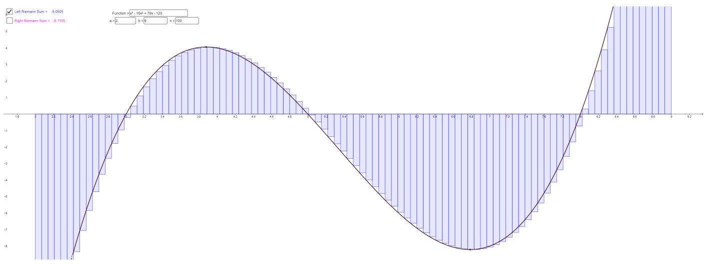

Figure 4: The Riemann sum.

Then the correct definition of a **Riemann integrable function** $f : [a, b] \to \mathbb{R}$ is that there exists a real number I_f such that for any $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that for any partition $\Delta = (x_0, x_1 \dots, x_n)$ of $[a, b]$ with $\|\Delta\| < \eta_\varepsilon$ and any points $x_{i-1} \leq t_i \leq x_i$, $1 \leq i \leq n$, the inequality

$$|\sigma_\Delta(f, \xi) - I_f| < \varepsilon$$

holds good. The number I_f will call the **definite integral** of f on $[a, b]$ and denoted by

$$\int_a^b f(x)dx.$$

In other words, if Δ_n is a sequence of partitions such that $\lim_{n\to\infty} \|\Delta_n\| = 0$ for any points system sequence $x_{i-1}^n \leqq t_i^n \leqq x_i^n$, then we have

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sigma_{\Delta_n}(f, \xi^n). \tag{4}$$

Using this definition, it is easy to check the celebrated **Leibniz-Newton formula**

$$\int_a^b f(x)dx = F(b) - F(a), \tag{5}$$

where F is a primitive function of f, as well as to understand why the definite integral gives the area under the graph.

Indeed, if we consider the partitions sequence

$$\Delta_n = (x_0^n, x_1^n, \ldots, x_{k_n}^n) \text{ of the interval } [a,b],$$

such that $\lim_{n\to\infty} \|\Delta_n\| = 0$, then by the mean theorem applied to F on $[x_{i-1}^n, x_i^n]$, it results that there exists $t_i^n \in (x_{i-1}^n, x_i^n)$ having the property

$$F(x_i^n) - F(x_{i-1}^n) = F'(t_i^n)(x_i^n - x_{i-1}^n).$$

However, since $F'(x) = f(x)$ on $[a,b]$, we get

$$F(x_i^n) - F(x_{i-1}^n) = f(t_i^n)(x_i^n - x_{i-1}^n)$$

and by summing up we obtain

$$\sigma_{\Delta_n}(f, t^n) = \sum_{i=1}^{k_n} f(t_i^n)(x_i^n - x_{i-1}^n) = \sum_{i=1}^{k_n} [F(x_i^n) - F(x_{i-1}^n)] = F(b) - F(a)$$

for any positive integer n. This gives the Leibniz-Newton formula (5).

The intuitive idea behind formula (5) is therefore quite easy to understand. The mean theorem for the primitive $F(x)$ provides the sequence of intermediate points $\{t_i^n\}$ where the slope of the tangent to the graph of $F(x)$ is parallel to the chord connecting the end points of the interval. By using these intermediate points for $f(x)$, then it results that the area $f(t_i^n)(x_i^n - x_{i-1}^n)$ coincide with the difference $F(x_i^n) - F(x_{i-1}^n)$, hence the conclusion.

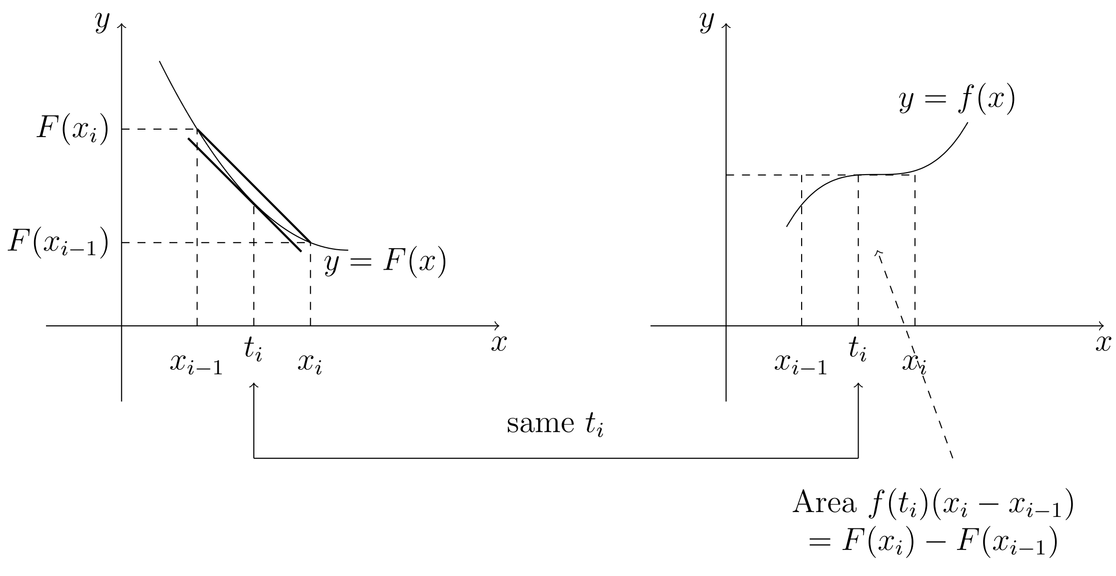

Figure 5: The relation of $F(x_i) - F(x_{i-1})$ and area of $f(t_i)(x_i - x_{i-1})$.

The correct statement about the area is as follows.

If $f : [a,b] \to \mathbb{R}$ is a continuous function, then the domain under the graph has finite area and this area is given by $\int_a^b f(x)dx$.

The proof is now easy.

Let us consider again the partitions sequence

$$\Delta_n = (a = x_0^n < x_1^n < \ldots < x_{p_n}^n = b)$$

such that $\lim_{n\to\infty} \|\Delta_n\| = 0$, and let us denote by m_i^n and M_i^n the lower and upper bounded of f on the closed, bounded interval $[x_{i-1}^n, x_i^n]$, respectively.

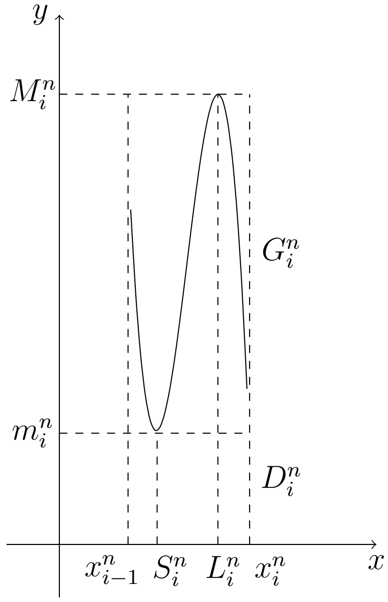

Figure 6: Computing area by piecewise quadrature.

The students should know (or it can be recalled) that any continuous function on a closed, bounded interval attains its bounds, i.e. in our case, there exists $S_i^n \in [x_{i-1}^n, x_i^n]$ and $L_i^n \in [x_{i-1}^n, x_i^n]$ (S for small and L for Large) such that

$$f(S_i^n) = m_i^n \text{ and } f(L_i^n) = M_i^n.$$

Let us denote by D_i^n and G_i^n the rectangles $[x_{i-1}^n, x_i^n] \times [0, m_i^n]$ and $[x_{i-1}^n, x_i^n] \times [0, M_i^n]$, respectively. Their union gives

$$E_n := \bigcup_{i=1}^{p_n} D_i^n,\ F_n : L = \bigcup_{i=1}^{p_n} G_i^n$$

with area

$$Area(E_n) = \sum_{i=1}^{p_n} m_i^n (x_i^n - x_{i-1}^n) = \sum_{i=1}^{p_n} f(S_i^n)(x_i^n - x_{i-1}^n) = \sigma_{\Delta_n}(f, S_i^n)$$

and

$$Area(F_n) = \sigma_{\Delta_n}(f, L_i^n).$$

The function f is continuous, hence integrable, and its definite integral is given by (4). If we choose any sequence points $x_{i-1}^n \leq t_i^n \leq x_i^n$, then clearly

$$Area(E_n) = \sigma_{\Delta_n}(f, S_i^n) \leq \sigma_{\Delta_n}(f, t_i^n) \leq \sigma_{\Delta_n}(f, L_i^n) = Area(F_n). \tag{6}$$

Taking now into account that

$$\lim_{n\to\infty} Area(E_n) = \lim_{n\to\infty} Area(F_n) = Area\left(f\big|_{[a,b]}\right)$$

by passing to limit in (6) it follows

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sigma_{\Delta_n}(f, S_i^n) = \lim_{n\to\infty} Area(E_n) = \lim_{n\to\infty} \sigma_{\Delta_n}(f, L_i^n) = Area(F_n),$$

hence

$$Area\left(f\big|_{[a,b]}\right) = \int_a^b f(x)dx.$$

where $Area\left(f\big|_{[a,b]}\right)$ is the area of the domain under the graph of f on $[a, b]$, see Figure 3.

Finally, we consider the limits of the type (1). It can be easily computed if regarded as the limit of a Riemann sum. Indeed, the sum under the limit can be transformed as follows

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} = \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \ldots + \frac{1}{1+\frac{n}{n}}\right].$$

If we consider the function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \frac{1}{1+x}$ and for any $n \geq 1$, we consider the equidistant partition

$$\Delta_n = \left(x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \ldots < x_n = \frac{n}{n} = 1\right)$$

with norm $\|\Delta_n\| = \frac{1}{n}$, and points system

$$t_i = x_i = \frac{i}{n},$$

then

$$\sigma_{\Delta_n} = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{1}{1+\frac{i}{n}}\left(\frac{i}{n} - \frac{i-1}{n}\right)$$
$$= \sum_{i=1}^{n} \frac{1}{i+n} = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}.$$

Therefore, we obtain

$$\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right] = \lim_{n\to\infty} \sigma_{\Delta_n}(f, t)$$
$$= \int_0^1 \frac{dx}{1+x} = [\log(1+x)]_0^1 = \log 2.$$

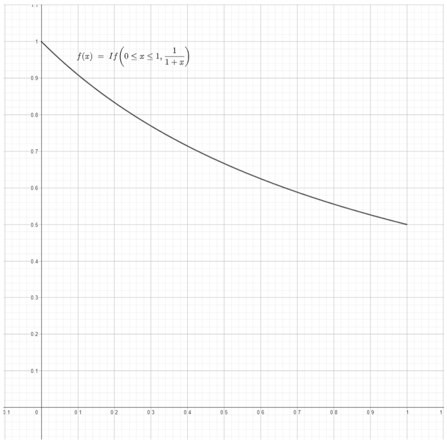

Figure 7: Computing limits by piecewise quadrature.

A further problem is the second limit (1). Indeed, we need to transform the product in sum by using logarithm.

We have

$$\log \frac{\sqrt[n]{n(n+1)(n+2)\dots(n+n)}}{n} = -\log n + \frac{1}{n}\left[\log n + \log(n+1) + \dots + \log(n+n)\right]$$

and, denoting

$$\log(n+1) = \log n\left(1+\frac{1}{n}\right) = \log n + \log\left(1+\frac{1}{n}\right)$$

we obtain

$$\log \frac{\sqrt[n]{n(n+1)(n+2)\dots(n+n)}}{n} = \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \dots + \log\left(1+\frac{n}{n}\right)\right].$$

Therefore,

$$\lim_{n\to\infty} \log \frac{\sqrt[n]{n(n+1)(n+2)\dots(n+n)}}{n} = \int_0^1 \log(1+x)dx = \log 4 - 1$$
$$\lim_{n\to\infty} \frac{\sqrt[n]{n(n+1)(n+2)\dots(n+n)}}{n} = e^{\log 4 - 1} = \frac{4}{e},$$

hence

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n(n+1)(n+2)\dots(n+n)}} = \frac{e}{4}.$$

This is an important technique that high school student should master.

3 Materials

Computer technology plays a significant role in mathematical education, visually enhancing and adding new value to the traditional mathematical learning.

In the present project, we have used the computer in order to have the students visualizing mathematical concepts as area and piecewise quadrature and explore them through interactive scripts to better understand geometrical shapes and functions.

By integrating computer science into classical mathematical learning, we were able to create a more engaging, relevant, and enriching educational experience for our students.

In particular, we have used GeoGebra ([2]), developed by Markus Hohenwarter in 2001, which is a dynamic mathematics software that combines geometry, calculus and other fields of mathematics in an interactive and visual environment. It is a powerful tool for teaching and learning mathematics, as well as for exploring mathematical concepts and conducting research. It is available on various platforms, including Windows, macOS, Linux, and as mobile apps for Android and iOS.

Even though we are using it here merely for visual integration, GeoGebra includes all necessary tools for calculus, such as finding derivatives, integrals, and tangents to curves. It can plot a large variety of functions graphs and enhance the study of their behavior and limits.

3.1 The area of the circle

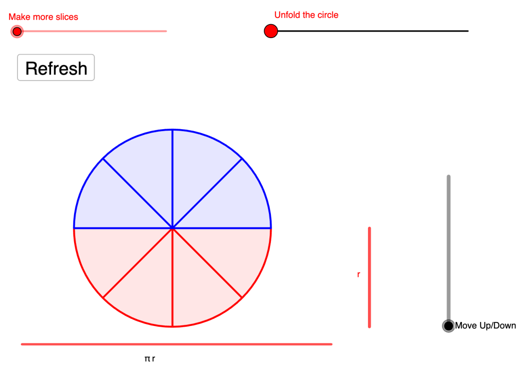

Figure 8: https://www.geogebra.org/classic/jzsxxnhq

3.2 The Riemann sum

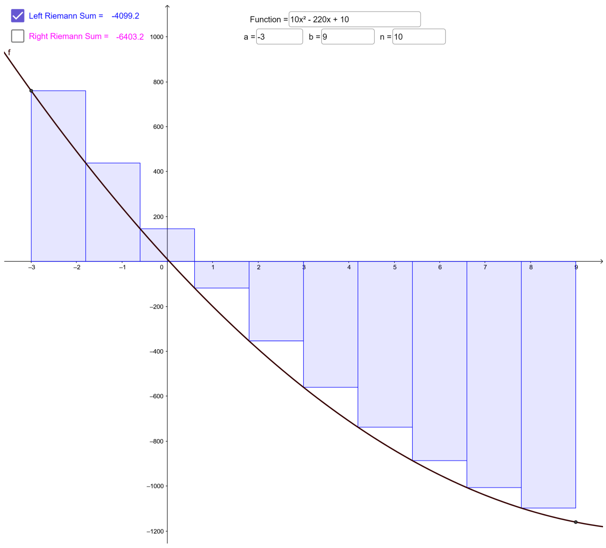

Figure 9: https://www.geogebra.org/classic/h4P4cjsT

4 Acknowledgements

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5 Conclusion

The concept of piecewise quadrature holds significant importance in high school mathematics, as it allows students to establish connections between the notions of limits and area.

Introducing piecewise quadrature prior to covering topics such as area and definite integrals can prove to be highly beneficial. It serves as a foundational motivation and model for these advanced concepts, enabling students to grasp the fundamental principles that underlie calculus. By employing this teaching model, students not only gain proficiency in numerical integration but also develop the ability to apply these mathematical concepts to real-world problems. Additionally, they learn to evaluate the accuracy and precision of approximations, which is a critical skill in various scientific disciplines.

By nurturing critical thinking and problem-solving skills through the study of piecewise quadrature, students are better prepared to become successful scientists. These skills become particularly valuable when they later encounter advanced numerical integration methods used extensively in engineering, physics, computer science, and other fields.

Through a comprehensive understanding of piecewise quadrature, students are empowered to tackle complex problems, analyze real-world scenarios, and make informed decisions. This knowledge lays a strong foundation for their academic and professional pursuits, equipping them with the necessary tools to excel in the STEM fields.

Moreover, the introduction of piecewise quadrature provides a bridge that connects seemingly abstract mathematical concepts to practical applications. Students can explore how these mathematical tools are utilized in diverse industries to solve problems, design solutions, and optimize processes. This real-world relevance fosters a deeper appreciation for mathematics and its practical implications.

In conclusion, integrating piecewise quadrature into the high school mathematics curriculum is a strategic move to enhance students' learning experience. By introducing this concept early on, students can build a solid understanding of numerical integration, critical thinking, and problem-solving skills. This preparation will undoubtedly pave the way for their future success as they pursue further education and careers in scientific and technical fields.

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