How Many are Factorable?

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Abstract: *In this paper, an investigation in quadratic factoring is shared that begins with a straightforward problem that technology enables to evolve into other avenues of investigation of varying complexity and accessibility.*

1. Introduction

In 1995, the National Council of Teachers of Mathematics [1, pp. 1] called for a vision of mathematics (specifically algebra) that "shift emphasis away from symbolic manipulations toward conceptual understanding, symbol sense, and mathematical modeling." While one could argue that we are still far from that goal, the technology that is currently freely and readily available to both teachers and students is making this statement one that is certainly realizable. The use of technology to learn, teach, and do mathematics provides a chance to engage in classroom mathematics beyond what might be considered typical and what has been considered standard for years. Additionally, technology allows access to mathematics and mathematical ideas that might have been previously hidden or inaccessible. In this regard, it is most assuredly the ladder that Kennedy [2] argued for so many years ago as he saw the potential for technology to allow students to learn, play and ultimately discover and engage in meaningful mathematics.

The investigation shared in this paper is made possible because of the use of technology to serve as the ladder to allow students to access mathematics that could be a challenge due to the potential computational burden. The investigation (considering what quadratic polynomials are factorable) makes consistent use of technology as a way to make sense of the problem while attending to mathematical processes expected in any American high school algebra class. Technology further allows the opportunity to engage in study that envelopes many mathematical ideas while still staying grounded in basic processes.

This investigation engages students in several mathematical practices [3] and the process of doing mathematics [4]. Among the mathematical practices, the two most prevalent are 'making sense of mathematics and persevere in solving them' and 'reason abstractly and quantitatively'. As will be demonstrated, the main problem tackled in this work allows students to make sense of factoring while attending to both the process and theory of factoring. This is significant as many students learn the procedure of how to factor but are rarely asked to further consider the purpose and viability of that procedure. In order to accomplish this, abstracting some of the procedural aspects of factoring is required.

2. The Probability of a Quadratic Being Factorable

The process of factoring polynomials, specifically quadratic trinomials, is part of nearly every American high school mathematics curriculum. A significant amount of time is spent on learning techniques and related procedures to enable one to efficiently factor. Using technology tools such as spreadsheets, CAS, and graphing calculators we have the potential to deepen student understanding of factoring polynomials.

The initial problem investigated requires nothing more than the ability to know what it means to factor a basic one variable quadratic polynomial of the form $ax^2 + bx + c$ and to know what it means to be factorable. The initial problem investigated is modified from a high school textbook [5, pp. 738-740] to ask students to do the following-

> There are infinitely many quadratic expressions of the form $f(x) = ax^2 + bx + c$, but what percent are factorable over the rationals when a, b, and c are each nonzero integers from -10 to 10?

For initial investigative purposes, students are first asked to generate at least 30 random polynomials that fit the problem criteria. While this is admittedly a small sample, part of the technology use here is having students think through the process of how to get technology to assist in the randomization and knowing that the results they are getting are correct. Several CAS (*Mathematica* specifically) have built in functions that can automate much of what we initially discuss here. However, most students are unfamiliar with even the most basic functions of CAS and so a spreadsheet is often the desired tool to use.

One way to generate the coefficients for the polynomials is to have a spreadsheet generate random numbers according to the criteria and eliminate those that do not fit. For example, the function RANDBETWEEN() in many spreadsheets will return a random integer in a specified range. With this command, if the range is specified as -10 to 10, then 0 will be a possible result, and so one needs to take care to account for this and not count those polynomials. While there are ways to work around this issue, such as using the CHOOSE() function combined with the RANDBETWEEN() function, most students initially working on the problem find it easiest to just throw out any polynomial not meeting the criteria and generate new examples as needed. If one uses a CAS like *GeoGebra* or *Mathematica*, there are built in commands to generate random polynomials, but those commands can have the same issues.

Regardless of the tool used, one will also need a test for factorability. Most students at this stage use a factor command from a CAS and manually type in each polynomial generated. Results from this initial work rarely result in more than 15% of the generated quadratics being factorable. While this will complete the problem, it far from solves the problem. Students are further challenged to make their work more robust and generalizable by being asked to consider what it means for a quadratic to be factorable. This is something most have never thought about before and so the spreadsheet can also provide one way to run this test.

To create a test for factorability, consider a typical quadratic polynomial, $ax^2 + bx + c$, with a, b, and c having the criteria for the original given problem. We know that the discriminate is calculated as $b^2 - 4ac$ and must be a perfect square if the given quadratic is to be factorable. This is a key insight missed by many students and one not usually discussed or encountered explicitly in the high school setting. Usually, this criterion is implied but its meaning is lost and students do not understand that this is truly what it means to be factorable. This abstraction (consideration of the discriminant) is an essential realization for this investigation. To collect data with this in mind, the spreadsheet shown in Figure 2.1 provides one possible setup for the task. This approach uses a way to eliminate potential zero coefficients first and then randomly determine each polynomial coefficient. Columns A, B, and C are the polynomial coefficients as labeled, columns D and E calculate the discriminate value and tests that value to see if it is a perfect square. Finally, column F counts the number of polynomials that are factorable.

	A	B	C	D	E	F
1	a-coefficient	b-coefficient	c-coefficient	Discriminant	Factorable?	Total Factorable:
2	-4	-2	-2	-28	No	3
3	-9	3	8	297	No	Percent Factorable:
4	-4	-10	1	116	No	0.06
5	7	-6	5	-104	No	
6	10	-5	10	-375	No	
7	-7	5	-2	-31	No	
8	4	-4	6	-80	No	
9	-10	4	-3	-104	No	
10	-6	2	7	172	No	
11	1	8	-7	92	No	
12	-3	-5	10	145	No	

Figure 2.1

With this result in mind, an exhaustive investigation of the original problem can be completed. Given that there twenty possible coefficients for each term and the quadratics in question have three terms, there are (20)(20)(20) = 20^3 = 8000 possible quadratic trinomials fitting the given problem. With this list, one can then test for positive perfect square discriminant values. In this problem the maximum value of the discriminate is calculated as $10^2 - 4(10)(-10) = 500$, thus only positive perfect squares less than or equal to 500 need be considered. So, for this class of polynomials, 892 are factorable meaning less than 12% (11.15%) of the total are factorable.

3. Further Investigation

These results provide a catalyst for a deeper investigation by expanding the allowable coefficients on the quadratics. Let $M \in Z$ and let [-*M*, *M*] represent the non-zero integer range for the coefficients of *f(x)*. The initial investigation relied heavily on brute force with a spreadsheet but with a CAS (specifically *Mathematica*) one can use a series of commands to efficiently count polynomials and collect needed information. Program 3.1 from *Mathematica* shows one possible approach.

Program 3.1

```
coefficients = Table[i, {i, 1, M}];
coefficients = Join[coefficients, -1 coefficients]
quadraticList =Flatten[Table[ax2+bx+c,{a,coefficients},{b,coefficients},{c,coefficients}]]

Length[quadraticList]
Discriminant[quadraticList,x]

result= Select[quadraticList,!IrreduciblePolynomialQ[#]&]
Length[result]
Sort[Tally[Discriminant[result,x]]]
```

Using this approach a table is first built to generate the allowable coefficients (the user specifies *M*) then **quadraticList** generates the possible quadratic functions. The length of the list can be found which specifies the number of quadratics generated. Next, the **Discriminate** function provides the list of discriminates for all the quadratics generated. In order to parse out those discriminates that are perfect squares, the built in function **IrreduciblePolynomial** is used. The user defined function **result** filters the list of generated quadratic functions to those that are factorable. The **Length** function returns the number of quadratics that are factorable. Finally, the **Sort** and **Tally** functions provide an ordered list of the perfect square discriminants along with how many times each one occurred.

These series of commands allow for efficient collection of data and more importantly, allow the determination of the percentage of quadratic polynomials that factor as *M* increases. Without technology, this avenue of exploration could prove nearly impossible to pursue. Table 3.1 shows results for different M values.

Table 3.1

M	Total Quadratics	Number Factorable	Percent Factorable
10	8000	892	11.15
11	10,648	1084	10.18
12	13,824	1404	10.16
13	17,576	1640	9.33
14	21,952	1972	8.98
15	27,000	2344	8.68
20	64,000	4628	7.23
25	125,000	7800	6.24
30	216,000	12,076	5.59

It may seem initially surprising or counter intuitive that the percentage of factorable polynomials decreases as the *M* value increase for the polynomial coefficients. Notice that the number of possible polynomials for each case can be calculated in general as $(2M)(2M)(2M) = 8M^3$ and the data show that the number of factorable polynomials is potentially modeled by a function that will always remain less than $8M^3$. The graph in Figure 3.1 shows how the total quadratic polynomials in each case compares to the number of factorable polynomials in each case. The curve represents the number of quadratics polynomials, while the individual data points are the number of factorable quadratic polynomials.

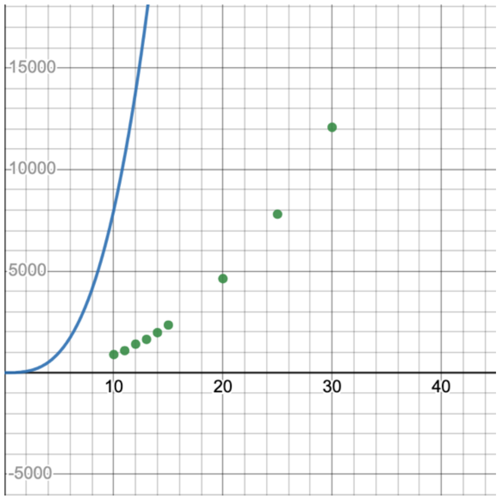

Figure 3.1

Using the discriminate one can calculate

$$M^2 - 4(-M)(M) = M^2 + 4M^2 = 5M^2$$

as the upper limit on the largest possible positive discriminate value. This also determines the entire range of discriminate values as $[1 - 4M^2, 5M^2]$. Thus, the largest number of perfect square candidates that can be obtained is bound by $(\sqrt{5})M$. If we wanted to be more precise, we can put the number $(\sqrt{5})M$ in the floor function. For example, when $M = 10$, the discriminant range is [-399,500] and there are 22 (bounded by $(\sqrt{5})10$ which is ~22.36) possible positive perfect square candidates. The exhaustive list in Table 3.2 shows that 441 (bounded by 500) is the largest square number that can appear in this range. There are no combination of numbers when $M = 10$ to yield a discriminate of $22^2 = 484$.

Table 3.2

Discriminate	0	1	4	9	16	25	36	49	64	81	100	121	144	169
Number of times occurred	44	56	48	52	40	52	44	64	44	56	48	72	40	52

Discriminate	196	225	256	289	324	361	441
Number of times occurred	48	36	24	24	24	16	8

This upper limit increases to 24 possible perfect square candidates when $M = 11$, but like $M = 10$, the actual largest perfect square is 441. In this way, the range of the discriminate is growing faster than the upper limit on the actual perfect squares candidates. Graphically in Figures 3.2 and 3.3, one can see that the distribution of all discriminants is relatively stable and normal like and thus most likely will not significantly change for increasing values of M.

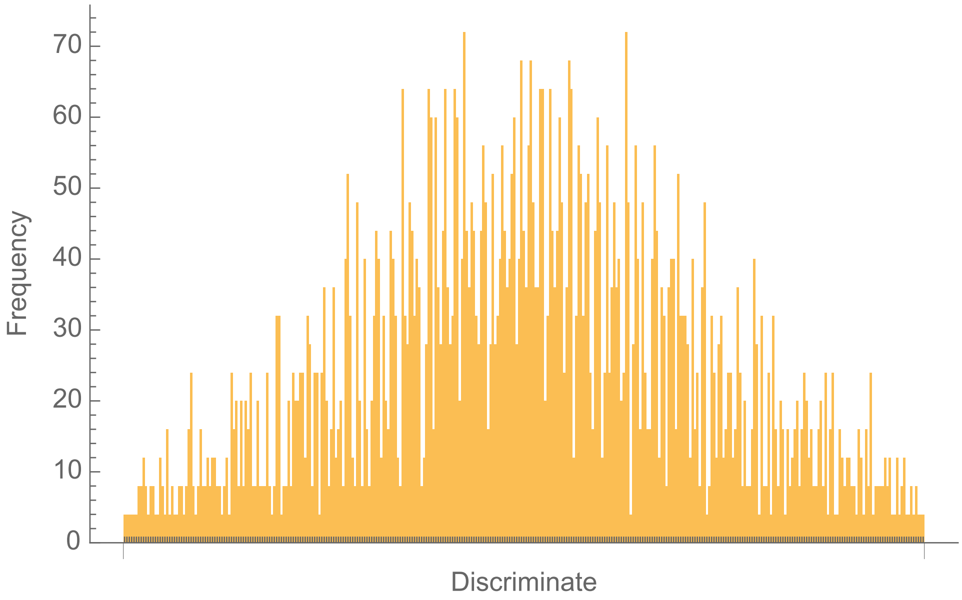

Figure 3.2 ($M = 10$)

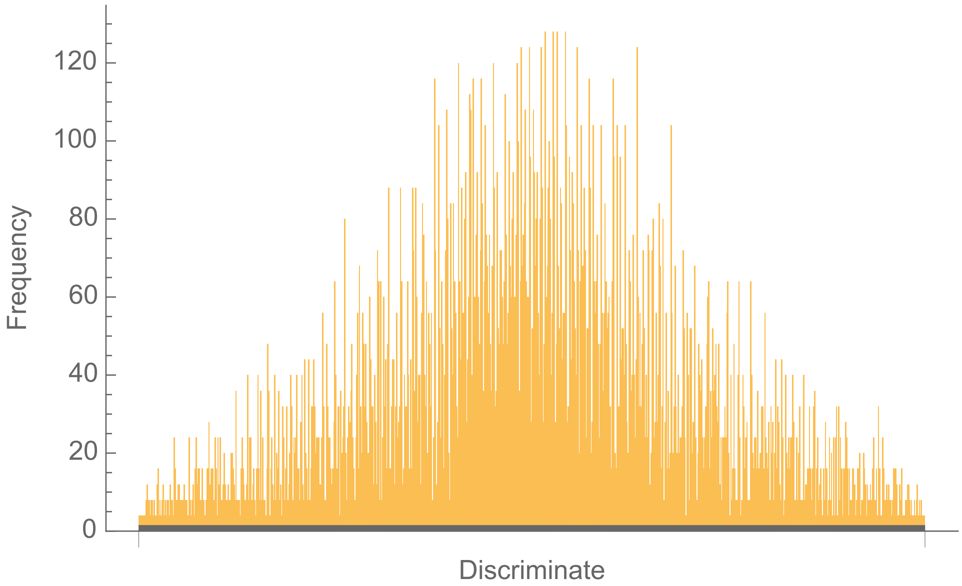

Figure 3.3 ($M = 15$)

There are still avenues of exploration and analysis left, but the work done here shows there is exploration and investigation in even the most routine of topics. The use of technology here was essential in gathering data, understanding the problem, and providing a playground for investigation. Technology provides the conduit for this investigation.

References

[1] Heid, M. K. et al. (1995). *Algebra in a Technological World*. Reston, VA: NCTM.

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[3] National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Authors.

[4] National Council of Teachers of Mathematics (2014). *Principles to Actions*. Reston, VA: National Council of Teachers of Mathematics.

[5] McConnell, J. W. et al. *UCSMP Algebra, 2nd ed*. Chicago, IL: McGraw Hill Wright Group.