① Self-introduction (as chair)

Circle Limit IV (Heaven and Hell) by Escher (July 1960)

\[ SL(2, R) \ni \begin{pmatrix} a & b \\ b & d \end{pmatrix} \rightarrow \frac{a-d, 2b}{a+d+2} \in D^2 \]
Locus of viewpoints from which a conic appears circular

Makoto Kishine, St.Viator Rakusei Junior and Senior High School
Yoichi Maeda, Tokai University

12\textsuperscript{th}, December, 2023 at ATCM 2023 Pattaya
Reference for this talk

https://php.radford.edu/~ejmt/
The Electronic Journal of Mathematics & Technology (radford.edu)

<table>
<thead>
<tr>
<th>Number 2 (Jun 2023)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Papers</strong></td>
</tr>
<tr>
<td><strong>Title</strong></td>
</tr>
<tr>
<td>Another Topological View of Curves and Surfaces Inspired by 2D and 3D Locus Problems</td>
</tr>
<tr>
<td>Wei-Chi YANG, Guillermo DÁVILA, Weng Kin HO</td>
</tr>
<tr>
<td>Exploration of envelopes of parameterized families of surfaces in a technology-rich environment</td>
</tr>
<tr>
<td>Thierry DANA-PICARD</td>
</tr>
<tr>
<td>Exact Orbits of Light Rays Reflected Inside Ellipses, Traced by Means of Rational Formulae with the Help of the CAS Derive™ 6</td>
</tr>
<tr>
<td>Aldo BOITI</td>
</tr>
<tr>
<td>Locus of viewpoints from which a conic appears circular</td>
</tr>
<tr>
<td>Makoto KISHINE, Yoichi MAEDA</td>
</tr>
</tbody>
</table>
① Introduction ( Ellipse is everywhere ! )

Soccer Field

Circle looks like an ellipse.
② Reverse problem (Ellipse appears circle)

From where does the ellipse looks like circular?
Several questions for this talk

Q0: on from which a conic appears circular
Q1: on conic section
Q2: on trigonometric function
Q3: on watching an equilateral triangle
④ Q1 : Makoto’s naïve question

Makoto: “We know that if we cut a cone horizontally, it will become a circle. Why does it become an ellipse when we cut it obliquely?” (egg like oval shape?)

Yoichi: “Well, ..., intuitively, it is not trivial, I think. But, ellipse looks like a circle.” (suspicious, unconvincing)
Let $a$, $b$, $c$, $d$ be acute angles.
If
\[
\cos a = \cos b \cdot \cos c, \quad \text{and} \\
\sin b = \sin a \cdot \sin d,
\]
then,
\[
\text{[ ] } = \sin c \cdot \tan d.
\]

Fill the blank as a function of $b$.

(It looks easy at first glance, but not so easy)
⑥ Q3 : Watching regular triangle problem

From where does the regular triangle looks like regular triangle? (only directly above? )
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{x^2}{c^2} - \frac{z^2}{b^2} = 1 \quad (c = \sqrt{a^2 - b^2}) \]
⑧ Duality (Ellipse ↔ Hyperbola)

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{x^2}{a^2 + b^2} + \frac{z^2}{b^2} = 1
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{x^2}{a^2 - b^2} - \frac{z^2}{b^2} = 1
\]
9 Self duality (Parabola $\mathcal{P}$)

$$x = \frac{y^2}{8c} - c \quad \Rightarrow \quad x = -\frac{z^2}{8c} + c$$
What it means to see an ellipse?

To look like a circle \iff A small circle on $S^2$

To look like an ellipse \iff A spherical ellipse on $S^2$
11  Spherical ellipse (the fourth parameter)

major axis: $a = \angle \text{NOA}$, minor axis: $b = \angle \text{NOB}$, focus: $c = \angle \text{NOC}$.

2 foci: $C=(\cos c, 0, \sin c)$, $C' = (-\cos c, 0, \sin c)$,

string length = 2a.  \[ \cos a = \cos b \cdot \cos c \] (Pythagorean in $S^2$)

asymptotic angle: $d = \angle \text{NOD} \Rightarrow \sin b = \sin a \cdot \sin d$

\[
\begin{align*}
\frac{x^2}{\tan^2 a} + \frac{y^2}{\tan^2 b} &= z^2 \\
x^2 + y^2 + z^2 &= 1
\end{align*}
\]

\[
-\frac{y^2}{\cos^2 a \cdot \tan^2 d} + \frac{z^2}{\cos^2 a} = 1
\]
Question: 
\[ \cos a = \cos b \cos c, \ \sin b = \sin a \sin d \ \Rightarrow \text{?} = \sin c \tan d \]

Answer: (Eliminate \( a \) !)

\[ \cos^2 a + \sin^2 a = 1 \ \Rightarrow \ \cos^2 b \cos^2 c + \frac{\sin^2 b}{\sin^2 d} = 1, \]

\[ \Rightarrow \ \cos^2 c + \frac{\tan^2 b}{\sin^2 d} = \frac{1}{\cos^2 b} = 1 + \tan^2 b, \quad ( \leftarrow \ \frac{1}{\cos^2 b} = 1 + \tan^2 b ) \]

\[ \Rightarrow \ \tan^2 b \left( \frac{1}{\sin^2 d} - 1 \right) = 1 - \cos^2 c = \sin^2 c, \quad ( \leftarrow \ \cos^2 c + \sin^2 c = 1 ) \]

\[ \Rightarrow \ \tan^2 b \frac{\cos^2 d}{\sin^2 d} = \sin^2 c, \quad ( \leftarrow \ \cos^2 d + \sin^2 d = 1 ) \]

\[ \Rightarrow \ \tan^2 b = \sin^2 c \tan^2 d, \]

\[ \Rightarrow \ \tan b = \sin c \tan d. \quad ( \leftarrow \ \tan b, \ \sin c, \ \tan d > 0 ) \]
Amazing 7 relations among a, b, c, d

\[
\begin{align*}
\cos a &= \cos b \times \cos c \\
\sin b &= \sin a \times \sin d \\
\tan b &= \sin c \times \tan d \\
\tan c &= \cos d \times \tan a \\
\tan b &= \cos c \times \sin d \times \tan a \\
\sin a \times \cos d &= \sin c \times \cos b \\
\sin b &= \cos a \times \tan c \times \tan d
\end{align*}
\]
Duality revisited
configuration of 7 relations with tetrahedron

Duality: half rotation with “\( \tan b = \cos c \cdot \sin d \cdot \tan a \)”

a ↔ b, c ↔ d, sin ↔ cos, tan ↔ cot.
⑲ Simple observation leads the solution.

If the vertex angle is less than 60°, there are two ways to fit the equilateral triangle to the isosceles triangle cone.
A3 : Equilateral triangle problem

There are many viewpoints more than expected.
Jiddu Krishnamurti (1895-1986)

K:

“When we are aware of it and come into contact with it directly, the observer is the observed. There is no difference between the observer and the thing observed. When fear is observed without the observer, there is action, but not the action of the observer acting upon fear.”

Peace is the best for us!!
Bonus question (Hachijo-fuji 854m)

This photo tells us the altitude where we are.
References (GeoGebra file)

https://www.geogebra.org/m/dbqkabee  (Ellipse on the ground)
https://www.geogebra.org/m/n8ecaqea  (Hyperbola on the ground)
https://www.geogebra.org/m/wkbg2hnv  (Dandelin’s construction)
https://www.geogebra.org/m/epdxgqz2    (Duality of Viewpoints)
https://www.geogebra.org/m/ytxan4vs  (Parabola on the ground)
https://www.geogebra.org/m/mqy6k5vh  (Viewscreen)

Enjoy your GeoGebra life !!