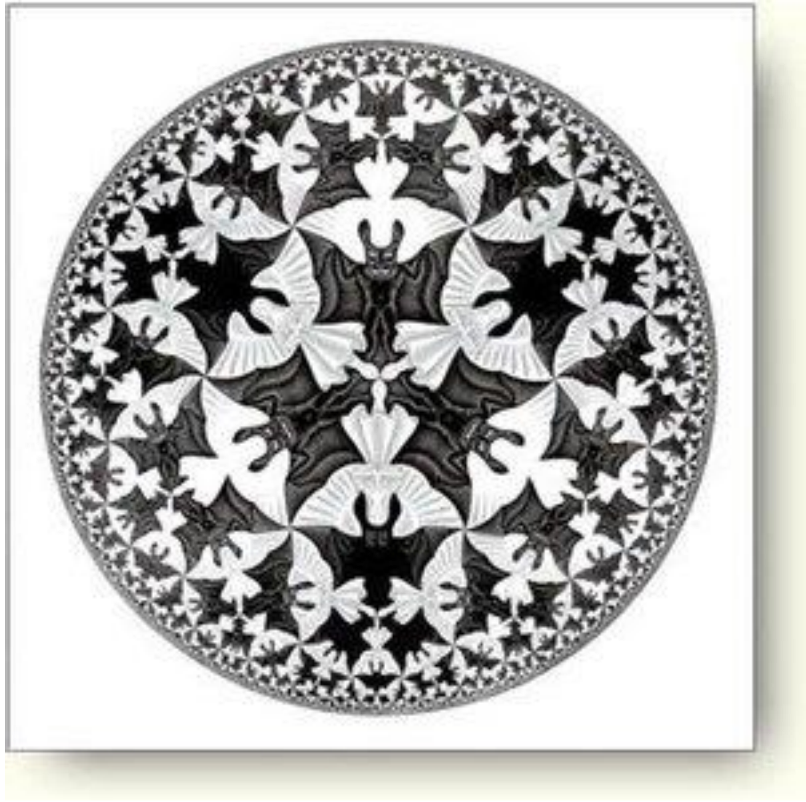
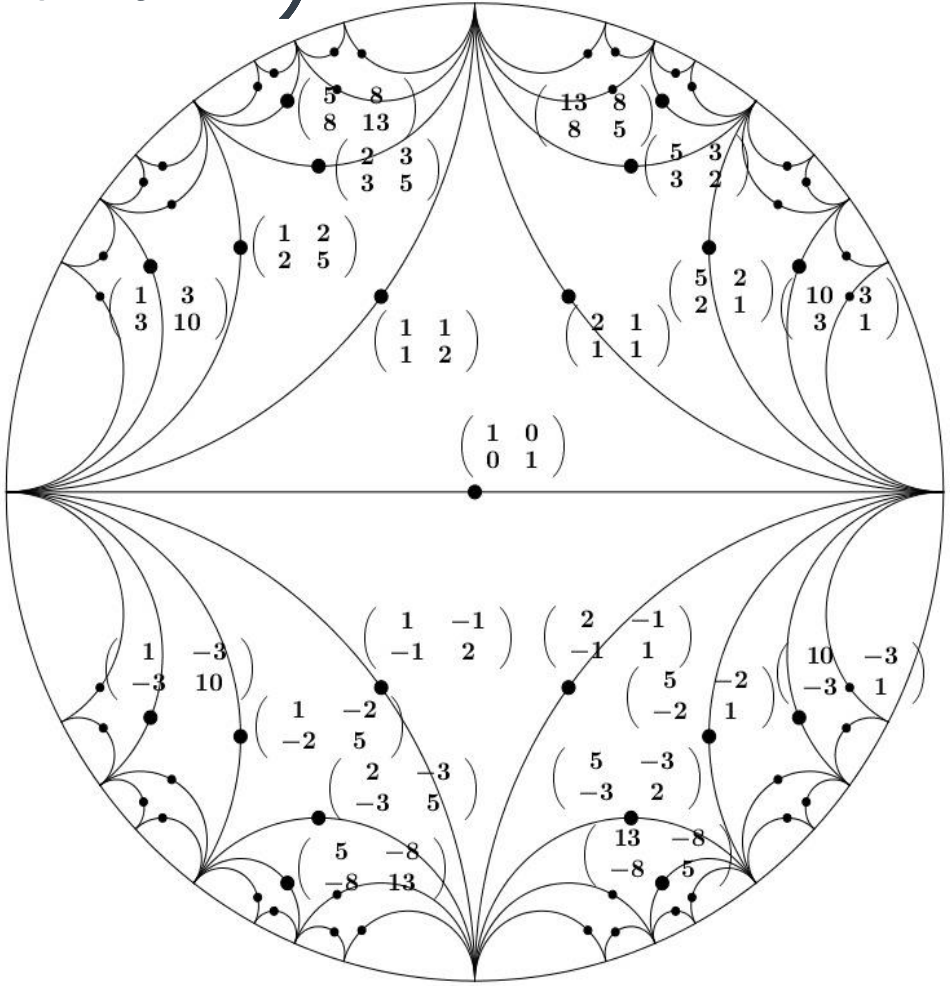
⓪ Self-introduction (as chair)



Circle Limit IV (Heaven and Hell)
by Escher (July 1960)



$$SL(2,R) \ni \begin{pmatrix} a & b \\ b & d \end{pmatrix} \to \frac{(\, a-d, 2b\,)}{a+d+2} \in D^2$$

Locus of viewpoints from which a conic appears circular

Makoto Kishine, St.Viator Rakusei Junior and Senior High School
Yoichi Maeda, Tokai University

12th, December, 2023 at ATCM 2023 Pattaya

π

⓪ Reference for this talk

https://php.radford.edu/~ejmt/

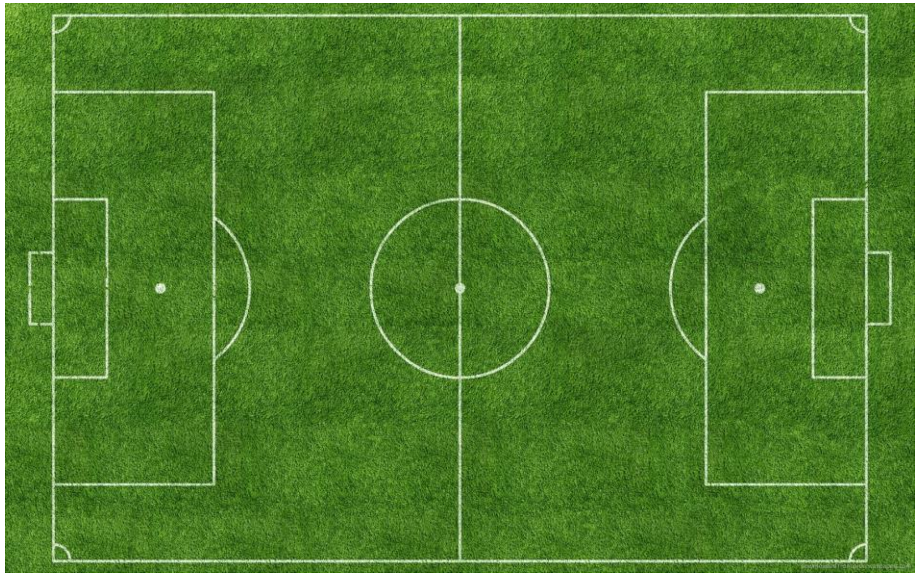
[The Electronic Journal of Mathematics & Technology (radford.edu)](https://php.radford.edu/~ejmt/)

Number 2 (Jun 2023)

Papers

Title	***Another Topological View of Curves and Surfaces Inspired by 2D and 3D Locus Problems***	Abstract
Author	**Wei-Chi YANG, Guillermo DÁVILA, Weng Kin HO**	Log in
Title	***Exploration of envelopes of parameterized families of surfaces in a technology-rich environment***	Abstract
Author	**Thierry DANA-PICARD**	Log in
Title	***Exact Orbits of Light Rays Reflected Inside Ellipses, Traced by Means of Rational Formulae with the Help of the CAS Derive™ 6***	Abstract
Author	**Aldo BOITI**	Log in
Title	***Locus of viewpoints from which a conic appears circular***	Abstract
Author	**Makoto KISHINE, Yoichi MAEDA**	Log in

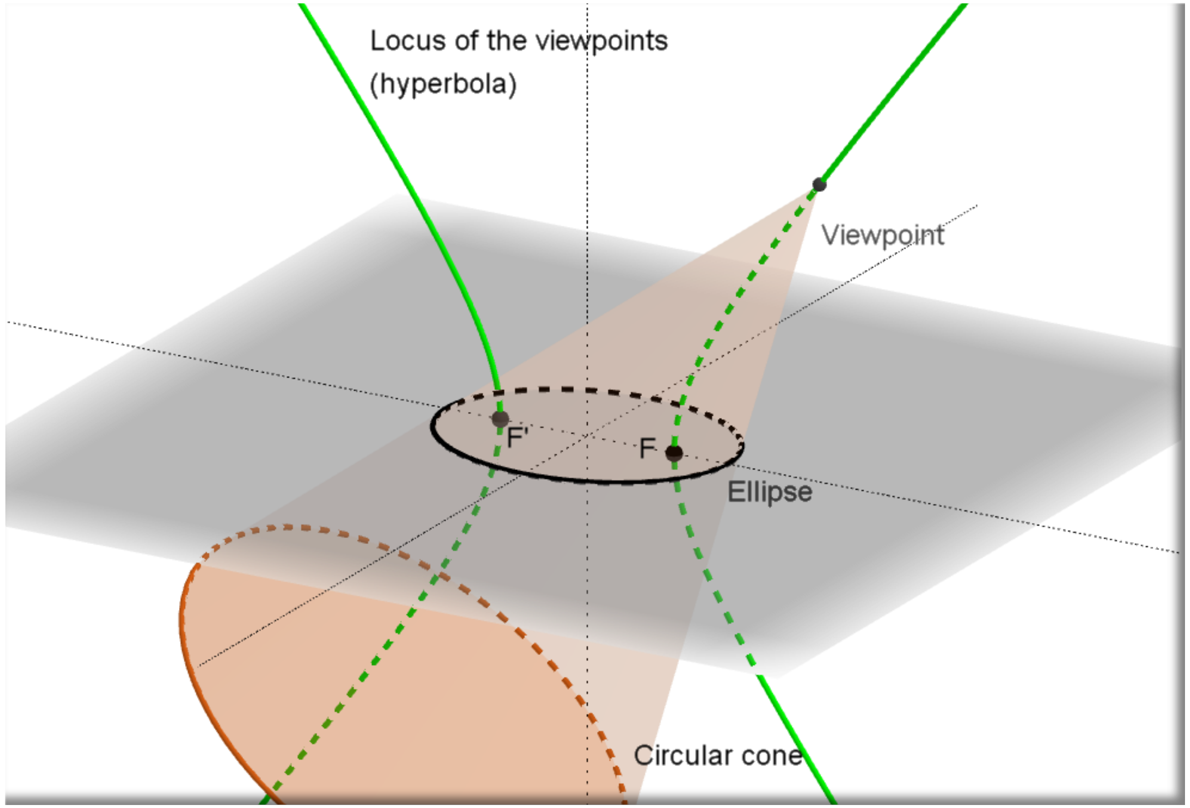
① Introduction (Ellipse is everywhere !)



Soccer Field

Circle looks like an ellipse.

② Reverse problem (Ellipse appears circle)



From where does the ellipse looks like circular?

③ Several questions for this talk

Q0: on from which a conic appears circular

Q1: on conic section

Q2: on trigonometric function

Q3: on watching an equilateral triangle

④ Q1 : Makoto’s naïve question

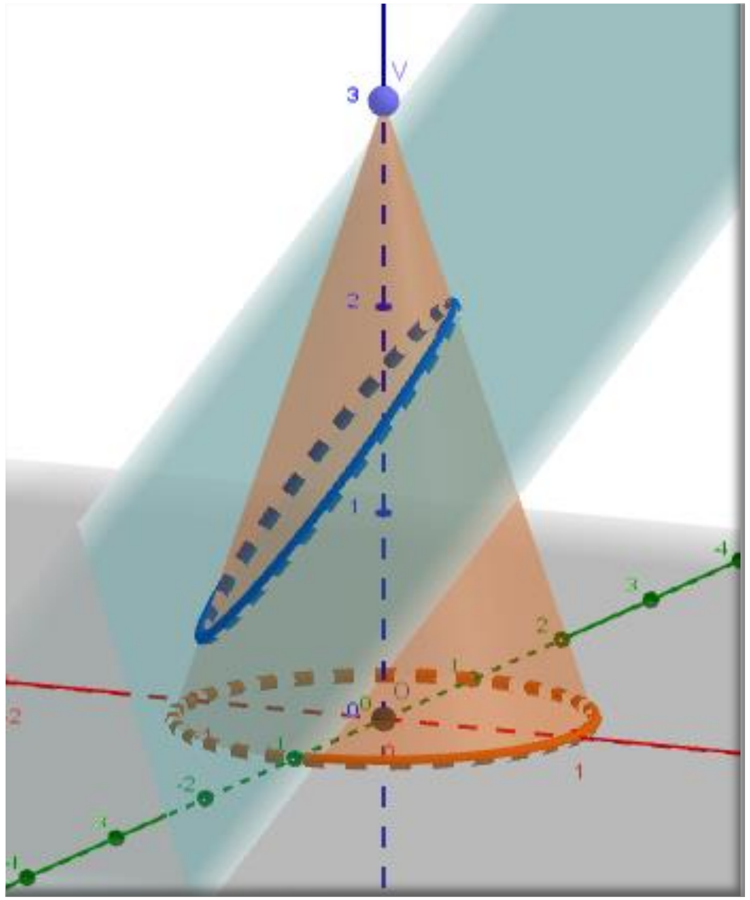
Makoto: “We know that if we cut a cone horizontally, it will become a circle.

Why does it become an ellipse when we cut it obliquely?”

(egg like oval shape ?)

Yoichi: “Well,⋯, intuitively, it is not trivial, I think. But, ellipse looks like a circle.”

(suspicious, unconvincing)



⑤ Q2 : Entrance exam. for medical school

Let a, b, c, d be acute angles.

If

$\cos a = \cos b * \cos c$, and

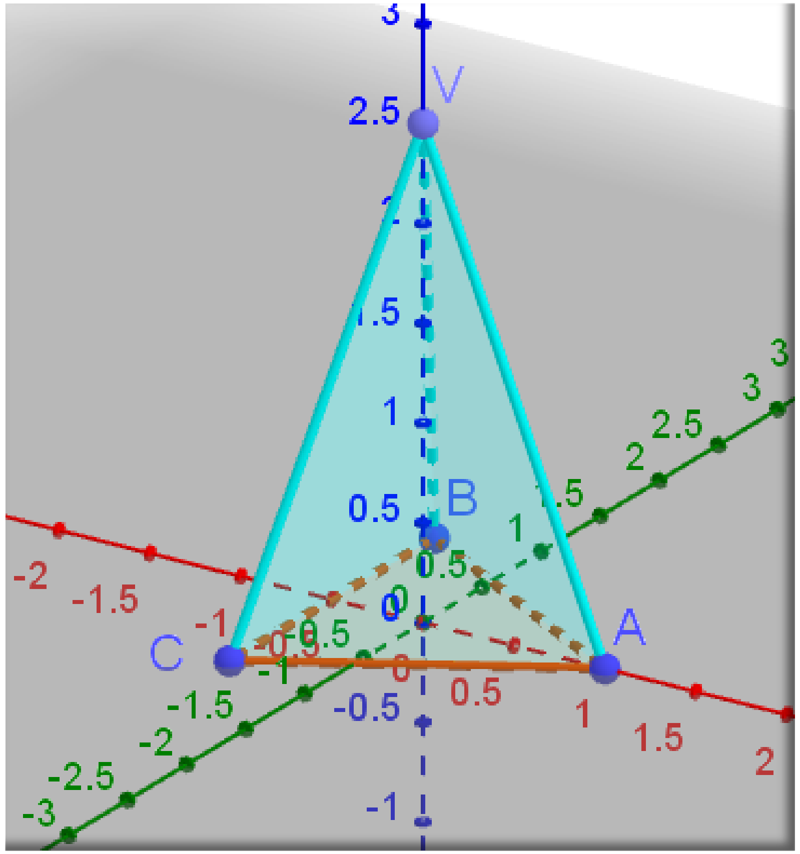
$\sin b = \sin a * \sin d$, then,

$\boxed{} = \sin c * \tan d.$

Fill the blank as a function of b.

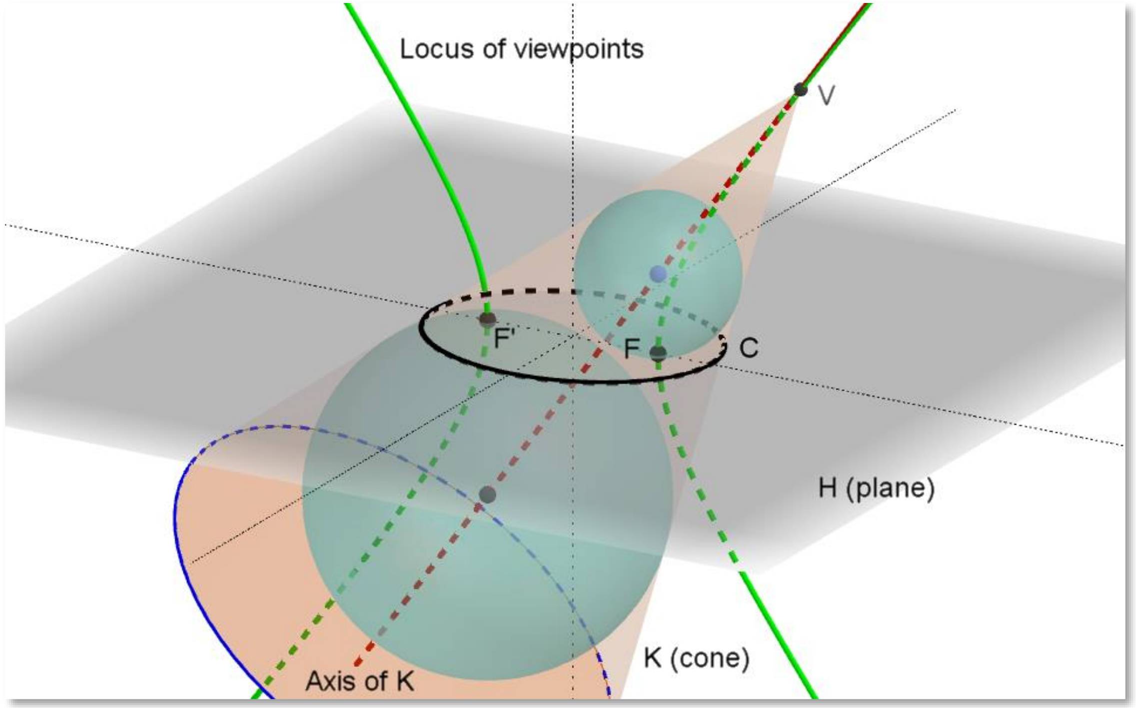
(It looks easy at first glance, but not so easy)

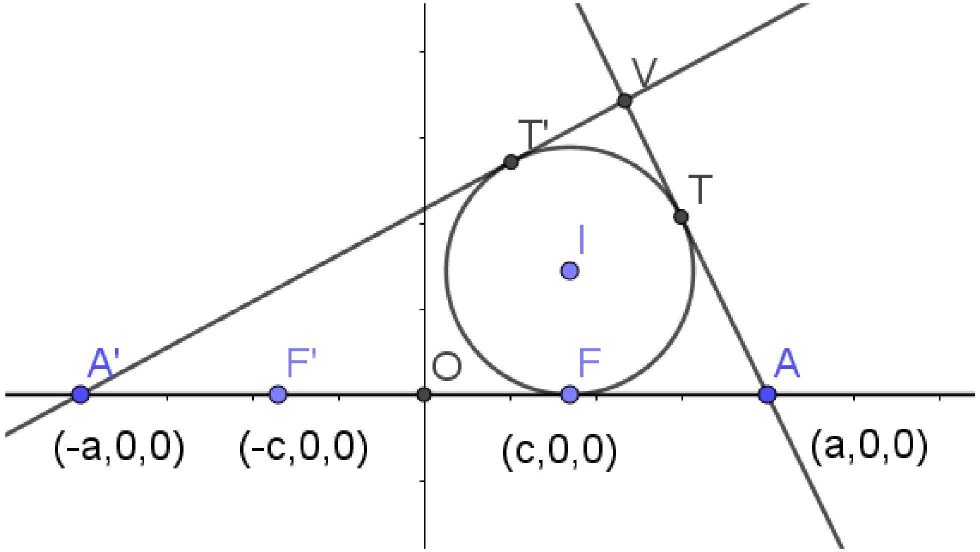
⑥ Q3 : Watching regular triangle problem



From where does the regular triangle looks like regular triangle?　(only directly above?)

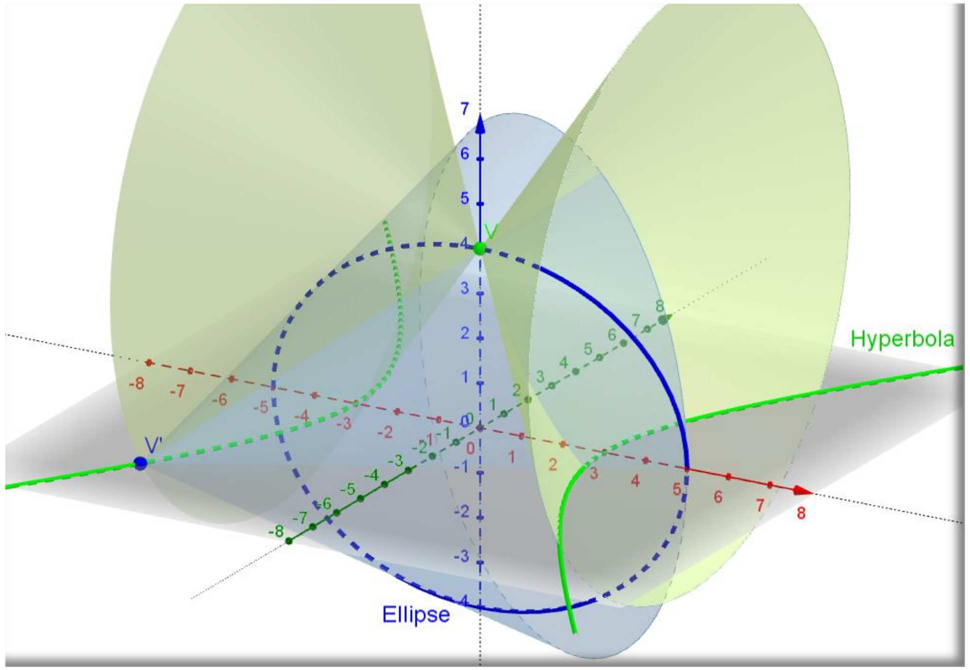
⑦ Famous Dandelin's construction





$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \Longrightarrow \quad \frac{x^2}{c^2}-\frac{z^2}{b^2}=1 \quad \left(c = \sqrt{a^2-b^2} \right)$$

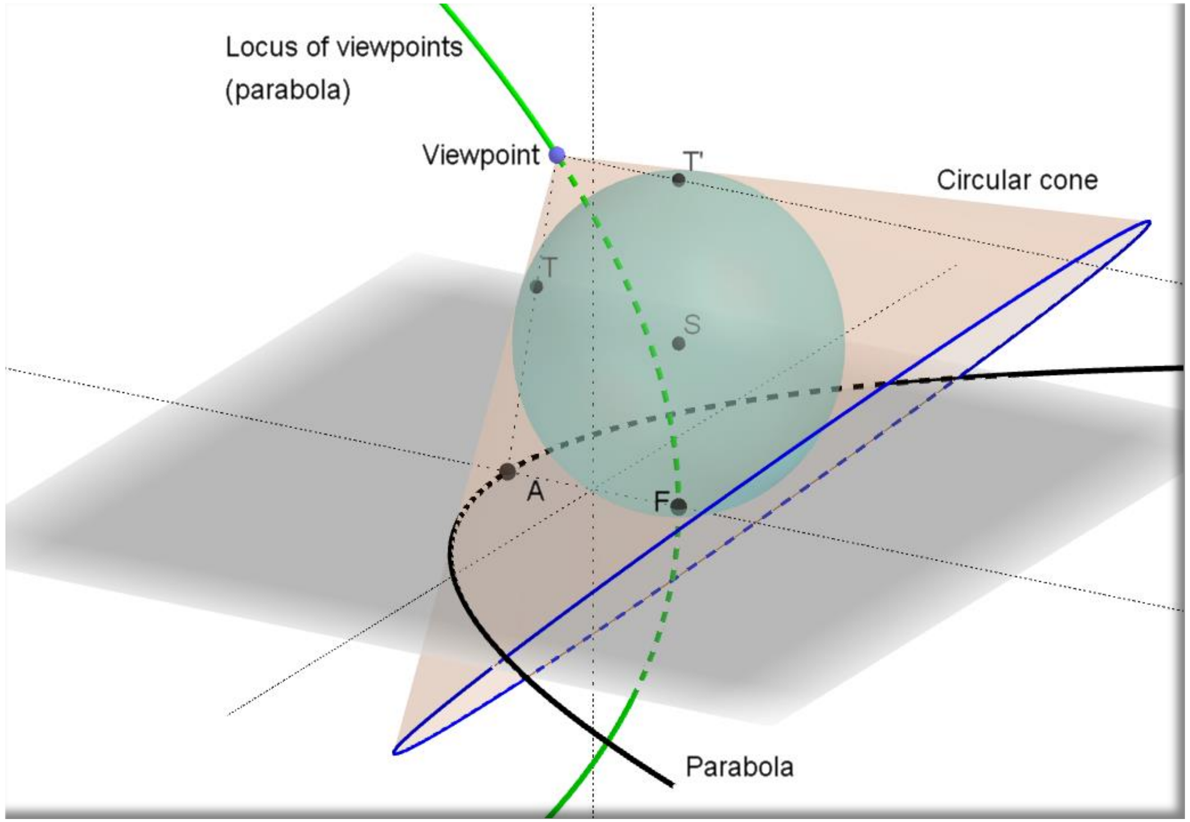
⑧ Duality (Ellipse ↔ Hyperbola)



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{x^2}{a^2+b^2} + \frac{z^2}{b^2} = 1$$

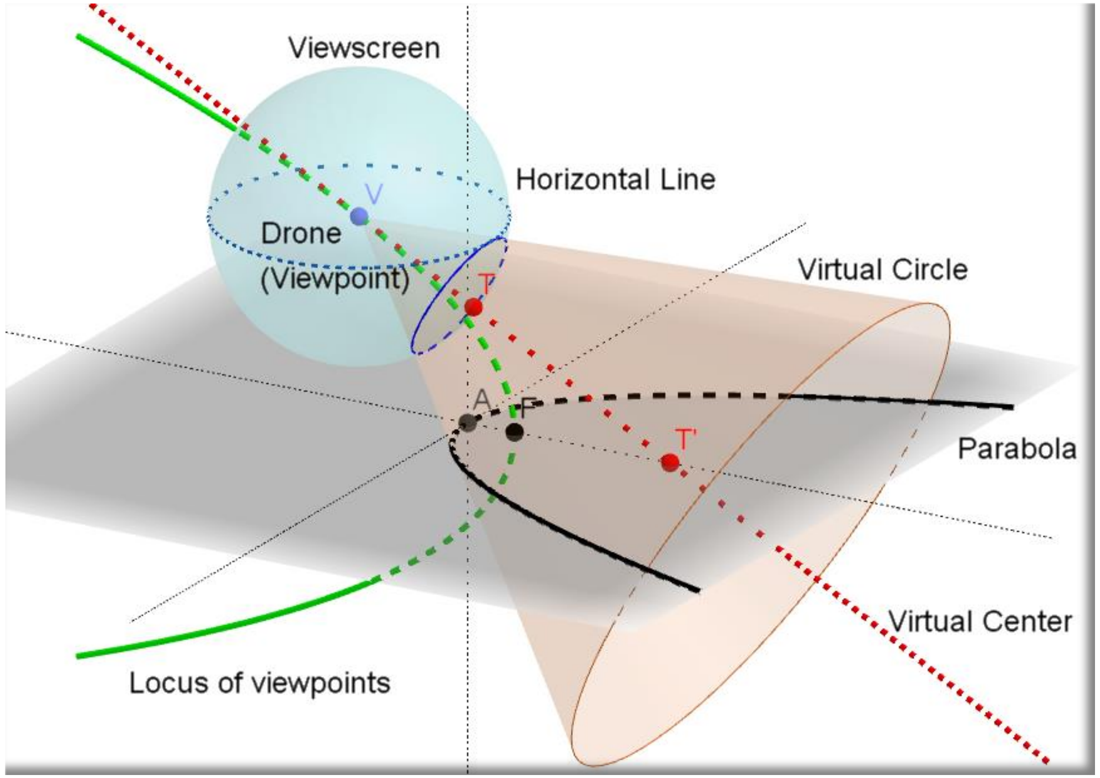
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{x^2}{a^2-b^2} - \frac{z^2}{b^2} = 1$$

⑨ Self duality (Parabola ↺)



$$x = \frac{y^2}{8c} - c \quad \Rightarrow \quad x = -\frac{z^2}{8c} + c$$

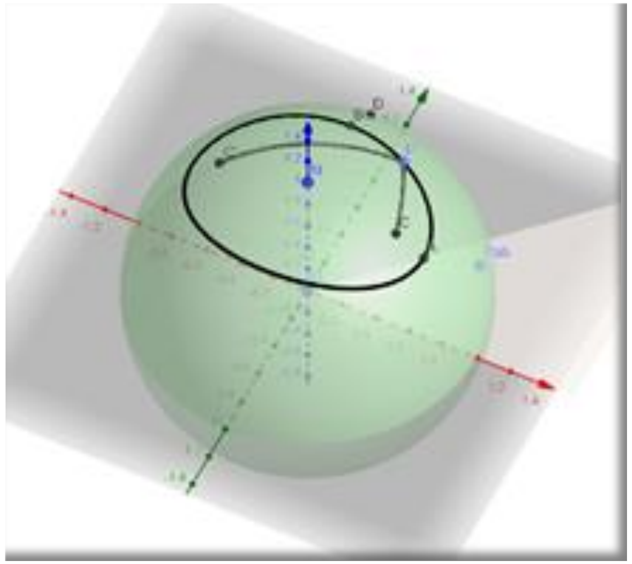
⑩ What it means to see an ellipse ?

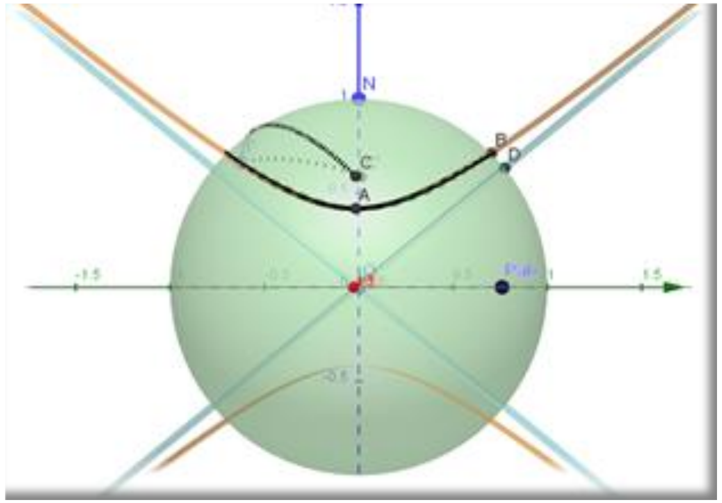


To look like a circle ⇔ A small circle on S^2

To look like an ellipse ⇔ A spherical ellipse on S^2

⑪ Spherical ellipse (the fourth parameter)





$$\begin{cases} \dfrac{x^2}{\tan^2 a} + \dfrac{y^2}{\tan^2 b} = z^2 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

$$-\frac{y^2}{\cos^2 a * \tan^2 d} + \frac{z^2}{\cos^2 a} = 1$$

major axis : a = ∠NOA , minor axis : b = ∠NOB , focus : c = ∠NOC.

2 foci : C=(cos c, 0, sin c) , C′ = (-cos c, 0, sin c),

string length = 2a. ⇒ cos a = cos b * cos c (Pythagorean in S^2)

asymptotic angle : d = ∠NOD ⇒ sin b = sin a * sin d

⑫ Difficult calculations

Question:

$\cos a = \cos b * \cos c$, $\sin b = \sin a * \sin d$ $\Rightarrow$ $\boxed{\quad ? \quad}$ $= \sin c * \tan d$

Answer: (Eliminate *a* !)

$$\cos^2 a + \sin^2 a = 1 \quad \Rightarrow \quad \cos^2 b \cos^2 c + \frac{\sin^2 b}{\sin^2 d} = 1,$$

$$\Rightarrow \quad \cos^2 c + \frac{\tan^2 b}{\sin^2 d} = \frac{1}{\cos^2 b} = 1 + \tan^2 b \,, \qquad \left(\leftarrow \frac{1}{\cos^2 b} = 1 + \tan^2 b \right)$$

$$\Rightarrow \quad \tan^2 b \left(\frac{1}{\sin^2 d} - 1 \right) = 1 - \cos^2 c = \sin^2 c \,, \qquad (\leftarrow \cos^2 c + \sin^2 c = 1)$$

$$\Rightarrow \quad \tan^2 b \, \frac{\cos^2 d}{\sin^2 d} = \sin^2 c \,, \qquad (\leftarrow \cos^2 d + \sin^2 d = 1)$$

$$\Rightarrow \quad \tan^2 b = \sin^2 c \, \tan^2 d \,,$$

$$\Rightarrow \quad \tan b = \sin c \, \tan d \,. \qquad (\leftarrow \tan b, \ \sin c, \ \tan d > 0)$$

⑬ Amazing 7 relations among a,b,c,d

$$\cos a = \cos b * \cos c$$

$$\sin b = \sin a * \sin d$$

$$\tan b = \sin c * \tan d$$

$$\tan c = \cos d * \tan a$$

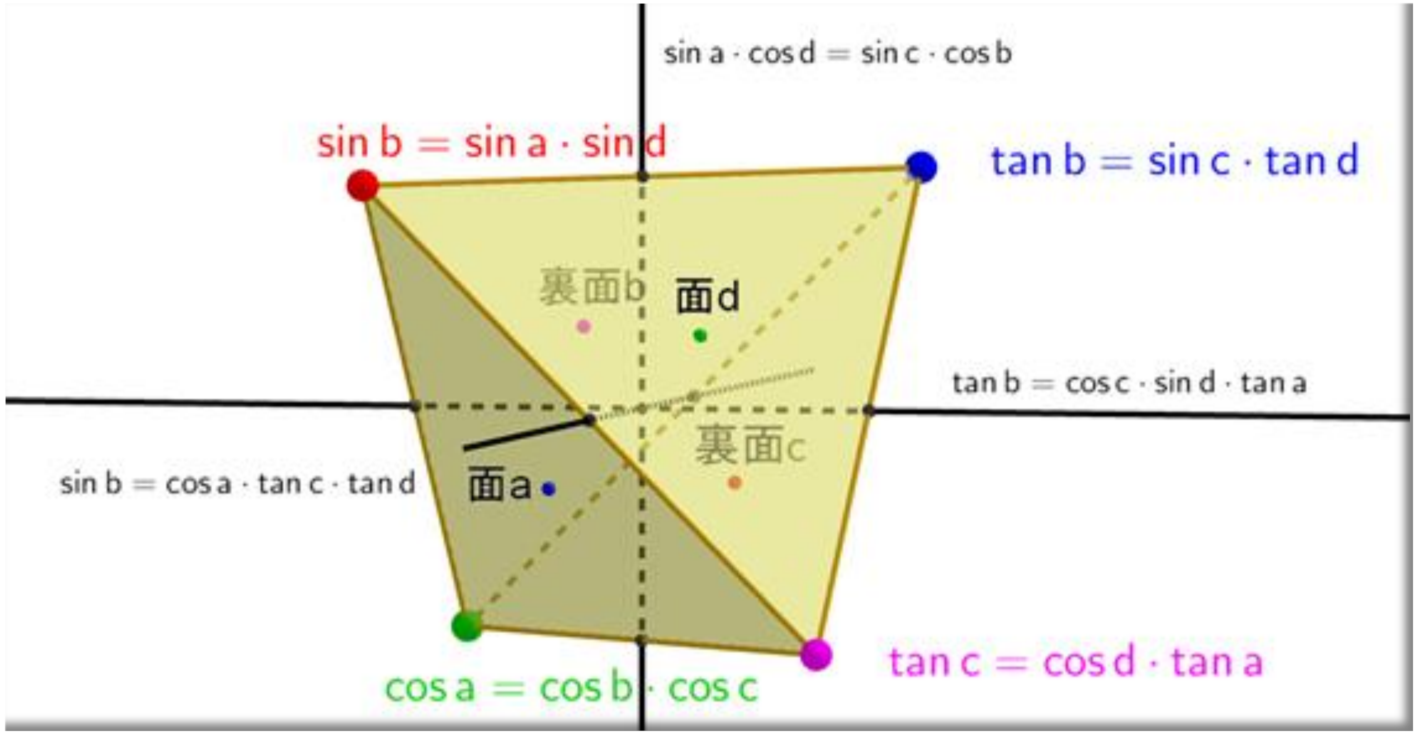
$$\tan b = \cos c * \sin d * \tan a$$

$$\sin a * \cos d = \sin c * \cos b$$

$$\sin b = \cos a * \tan c * \tan d$$

⑭ Duality revisited

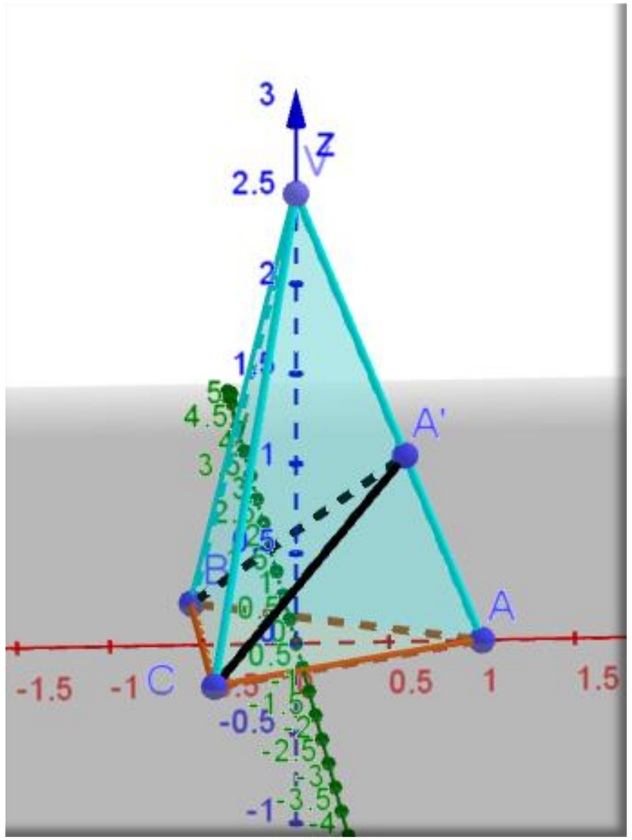
configuration of 7 relations with tetrahedron

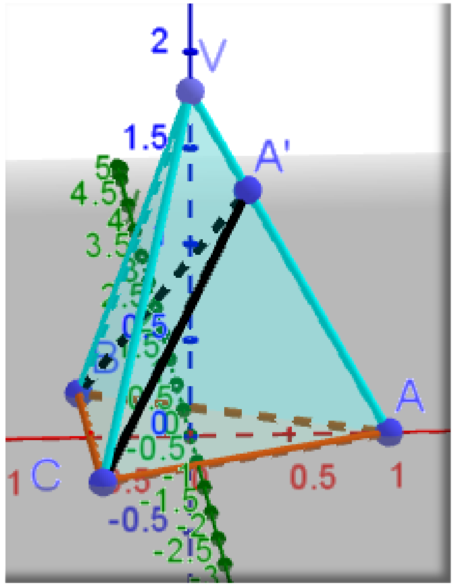


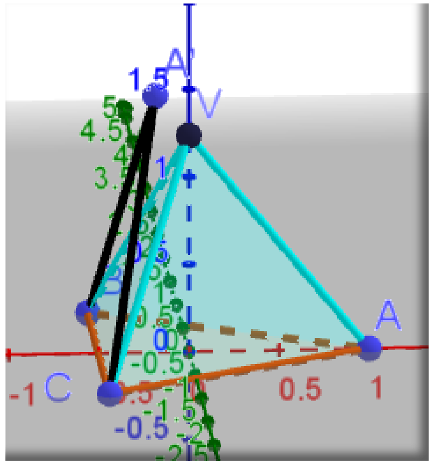
Duality : half rotation with “tan b = cos c*sin d * tan a”

a ↔ b, c ↔ d, sin ↔ cos, tan ↔ cot.

⑮ Simple observation leads the solution.

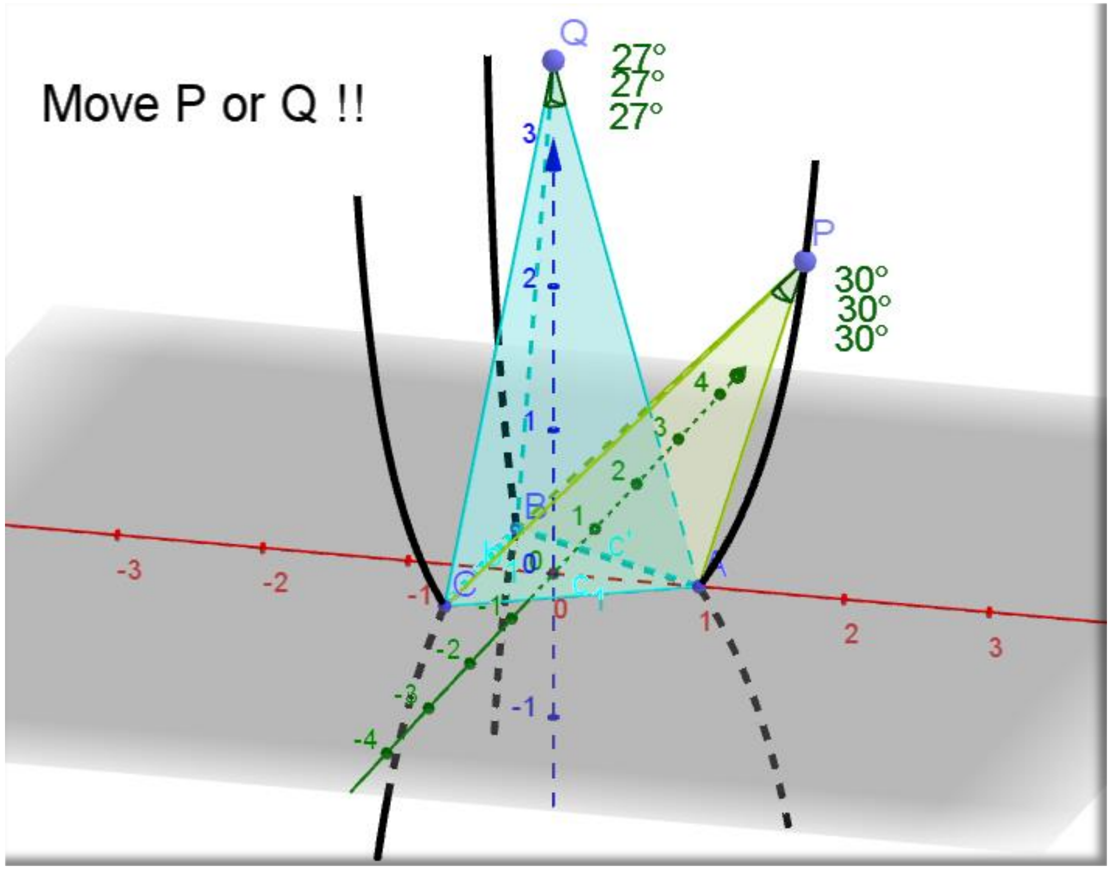






If the vertex angle is less than 60°, there are two ways to fit the equilateral triangle to the isosceles triangle cone.

⑯ A3： Equilateral triangle problem



There are many viewpoints more than expected.

⑰ Jiddu Krishnamurti (1895-1986)

K :

“When we are aware of it and come into contact with it directly, the observer is the observed.

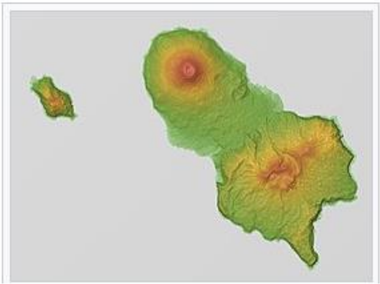
<u>There is no difference between the observer and the thing observed.</u>

When fear is observed without the observer, there is action,

but not the action of the observer acting upon fear.”

Peace is the best for us !!

⑱ Bonus question (Hachijo-fuji 854m)



This photo tells us the altitude where we are.

⑲ References (GeoGebra file)

https://www.geogebra.org/m/dbqkabee (Ellipse on the ground)
https://www.geogebra.org/m/n8ecaqea (Hyperbola on the ground)
https://www.geogebra.org/m/wkbg2hnv (Dandelin’s construction)
https://www.geogebra.org/m/epdxgqz2 (Duality of Viewpoints)
https://www.geogebra.org/m/ytxan4vs (Parabola on the ground)
https://www.geogebra.org/m/mqy6k5vh (Viewscreen)

Enjoy your GeoGebra life !!