Sangaku Mathematics Puzzles: A Catalyst for Cultivating Creative Thinking and Problem-Solving Abilities using The Geometer's Sketchpad

Krongthong Khairiree

krongthong.kh@ssru.ac.th or drkrongthong@gmail.com
College of Hospitality Industry Management,
Suan Sunandha Rajabhat University,
Bangkok, Thailand.

Abstract: *The purposes of this study were to examine how Sangaku Mathematics Puzzle serves as a catalyst for cultivating students' creative thinking and problem-solving abilities, aided by the dynamic software: The Geometer's Sketchpad. Action research was conducted in the College of Hospitality Industry Management, Suan Sunandha Rajabhat University, Bangkok, Thailand, in the year 2022. A total of 19 students studying in the second year of their bachelor's degree in education, majoring in mathematics, participated in this study. The duration of the action research project was about two months. A flipped classroom model incorporating cooperative learning and the Geometer's Sketchpad was employed in this study. These methods were used in line with the policy of the Ministry of Education in Thailand during the COVID-19 pandemic situation in the years 2021–2022. The research findings showed that with the combination of Sangaku puzzles and The Geometer's Sketchpad, students are encouraged to think outside the box and approach mathematical problems from different perspectives. In addition, the flipped classroom and cooperative learning model encourages teamwork and communication among students, promoting a deeper understanding of mathematical principles.*

1. Introduction

The Ministry of Education announced the policy of education in Thailand during the COVID-19 pandemic situation in 2021-2022: teaching and learning must carry on to ensure continuity of learning. The Secretary-General of the Office of the Basic Education Commission (OBEC), Mr.Amporn Pinasa, revealed that it is not necessary for the school to be closed and that teachers will teach online only. OBEC gave a policy to all schools to prepare for the five teaching methods [1]. The schools must provide teaching methods based on the context and appropriateness of each school.

The details of the five methods are as follows:

1) On-air refers to teaching via TV, satellite systems, and cable TV systems.
2) Online refers to teaching via the Internet, internet video conference, school, and other systems. Students learn through their computers, smartphone, or tablet.
3) On-hand method refers to teaching without viewing devices. Teachers will prepare learning materials, books, exercises, and worksheets for students to study at home with the help of their parents.
4) On-demand learning refers to teaching and learning through mixed media in electronic formats, such as watching videos, Live teaching notes to review lesson content after live instruction or studying the worksheets through Google Classroom or other applications designed by the teachers. The On-demand method allows students to study at their own pace whenever and wherever they want to, and
5) On-site method refers to in-person face-to-face learning lessons and/or enrichment activities at school or any place.

The students have to adjust the way they learn, and teachers have to change the way they teach, and they must be prepared for any situation that may arise. The Ministry of Education announced that on-demand learning materials are being increased, along with the production and distribution of workbooks that students can use at home. In addition, Dr. Ekachai Kisupan, the chairman of the Basic Education Committee, stated that we need to improve the knowledge and skills of mathematics teachers to deliver integrated mathematics instruction that emphasizes critical thinking. Teachers' competencies must meet the professional standards of teachers and can help students develop their critical thinking and problem-solving abilities.

Most importantly, the student-teacher must increase their skills and competencies in using creativity and modern information technology resources [1]. This is because the student-teacher will become a teacher in a few years. Based on the case study of Vanpetch, Y., and Sattayathamrongthian, M. [2], they found that the education obstacles due to the Covid-19 pandemic impact all sectors, however with every crisis, there is always an opportunity for a pedagogical review and learning something new. There is evidence that learning online can be effective. With the experience gained during the COVID-19 crisis, new digital learning possibilities could be implemented by educational institutions to stimulate the productivity of the lessons. Potential innovations include educational applications, platforms, and resources.

2. Flipped Classroom Model in Mathematics

The flipped classroom model employed in this study is based on Bergmann and Sams' research findings [3], the flipped classroom is a pedagogical model in which the typical lecture and homework elements of a course are reversed. Students watched a short video lecture at home before the class session, while during the in-class time, the students did the exercises, discussions, or projects. The flipped classroom model is a type of blended learning that increases students' engagement in learning mathematics. In the traditional teaching method, students learn mathematics content in the classroom and practice it at home. Mathematics knowledge can be divided into two parts: conceptual and procedural. Learning mathematics is more than computation, it is more than memorizing rules and facts. It is an investigation, exploring, experimenting, posing problems, and solving problems. To understand mathematics, students should be cognitively active while they learn.

Skemp [4] described understanding as the mental experience of a subject by a student relating an object to another object. A personal understanding of a concept involves grasping or acquiring the meaning of the object. According to Johnson and Johnson [5], small group work such as cooperative learning must be used in place of the traditional textbook lecture, whole-class discussion, and individual worksheets method of teaching in order to help students understand mathematics, communicate mathematically, develop confidence in their own mathematical ability, and be able to solve problems solving in mathematics.

3. Cooperative Learning: Group Investigation

Cooperative learning is one of the suggested teaching and learning approaches in student-centered classes and its use is consistent with the theories of learning and how children learn mathematics. There are a variety of cooperative learning methods based on the social psychological principles of cooperative learning. Cooperative learning methods have been adapted from different methods to meet the practical requirements of classrooms. Cooperative learning is a group-learning process built on the belief that students learn better when they talk and work together. Cooperative learning encourages group interaction using assigned roles, with each member sharing responsibility for the group and the work produced. Johnson and Johnson [6] described that the basic principles of cooperative learning must include four principles: positive interdependence, individual accountability, equal participation, and simultaneous interaction. Slavin [7] defined cooperative learning as a teaching method in which students work together in mixed-ability groups toward a common goal. Sharan [8] supported the idea that cooperative learning is a motivational strategy in which students collaborate in small groups to complete specific learning goals. Slavin explained that the essential components of cooperative learning consist of three concepts: team rewards, individual accountability, and equal opportunities for success. Slavin [9] explained in his research findings that cooperative learning can promote intrinsic motivation, which is a key element in successful teaching and learning. Cooperative learning can help students develop positive attitudes, which can generate higher self-esteem, and increased achievement. Through cooperative learning, students can increase their communication skills by interacting with team members. They can become actively involved in the learning process and therefore interested in what they are expected to learn. Research findings by Khairiree [10] show that cooperative learning methods like Math-Jigsaw do motivate learning-resistant students to want to learn and generate higher performance than would have been achieved in traditional classes.

Cooperative Learning: Group Investigation

Cooperative Learning Method: Group-Investigation used in this study was based on the cooperative learning method developed by Sharan and Sharan [11]. Sharan described that the Group-Investigation model involved students working together in small groups with four basic features: interaction, investigation, interpretation, and intrinsic motivation. All these features are incorporated into the six steps of the Group-Investigation model as follows:

1) Class determines subtopics and learning procedures.
2) Students selected their own groups of 4-5 members according to their interesting subtopic. The group plans its investigations.
3) Groups work according to their plans.
4) Groups make their presentations, and
5) Teacher and students evaluate their projects.

Students work together in groups and carry out their plans. The teacher acts as a facilitator follows up and discusses with students in every step until they complete their projects.

4. Sangaku Mathematics Puzzles and the Geometer's Sketchpad

Sangaku Mathematics Puzzles

The Japanese mathematical puzzle known as Sangaku first appeared in the Edo period (1603-1867). The Japanese word "Sangaku" means "mathematical tablet". These tablets presented numerous mathematical issues and geometric theorems in gorgeous calligraphy and elaborate pictures. Japanese people during the Edo period wrote mathematics problems on wooden boards or tablets and hung them under the roofs of Buddhist temples and Shinto shrines [12].

Sangaku looks like a work of art, and the figure is beautifully drawn in color. Each wooden tablet typically contains several illustrated puzzles or theorems, many Sangaku puzzles were based on Euclidian geometry. Sangaku puzzles were not only intricate works of art but also difficult mathematical challenges. They aimed to demonstrate creative problem-solving and comprehension of complex mathematical ideas. The uniqueness of Sangaku is the content of the problems. Ordinary triangle geometry mainly concerns the properties of a triangle. However, Sangaku's problems are concerned with some relationships arising from several mixed geometric figures like circles, triangles, and squares.

Sangaku usually presented their problems in three different sections:

1) *Problem text*: An introductory section describing the diagram and indicating which figure from the diagram the observer needs to find the value.
2) *Answer*: Section, either providing the numerical value of the sought figure or advising the reader to the following section.
3) *Formula*: Section containing a formula for calculating the solution.

Some Sangaku tablets were preserved from the many more produced in the country at the beginning of the Edo period. Today, these Sangaku tablets are preserved in museums and temples in Japan [13]. Geometry is one of the core contents in the syllabus of mathematics in Thailand. The contents and topics are similar to Sangaku puzzles, such as circles, triangles, squares, and Pythagoras's theorem. Teachers can use the Sangaku puzzle as a mathematics problem to design challenge worksheets for their students to solve the Sangaku puzzles. However, some Sangaku puzzles are not easy to draw. Majewski, Chuan, and Hitoshi [14] supported the idea that many Sangaku puzzles are not as simple to construct as they appear in the figures. Especially when circles are inscribed into something more complicated than a triangle, this can lead to very elaborate constructions. They explained that we can examine Sangaku mathematics puzzles using dynamic geometry software, try to construct some of them, and validate them using a computer software program.

The Geometer's Sketchpad and Sangaku Puzzle Construction

The Geometer's Sketchpad (GSP) is one of the dynamic mathematics software programs that provide opportunities for students to investigate and discover mathematics concepts, in particular geometric patterns [15]. Since 2004, the Ministry of Education Thailand has purchased a National license for the Geometer's Sketchpad and distributed it to schools and universities. GSP was translated into Thai language and widely used in Thailand. More than 1,000 mathematics teachers were trained to use GSP as a tool in their mathematics classes. After attending the workshops, many teachers consequently conducted action research in mathematics and incorporated the use of GSP as a tool in their classes. In using GSP, students learn by exploring, investigating, and discovering. GSP helps students visualize abstract mathematical relationships and various problem structures using pictorial representations [16]. The Institute for the Promotion of Teaching Science and Technology (IPST), Ministry of Education, Thailand announced in the mathematics curriculum that the Geometer's Sketchpad is one of the indicators of software in mathematics. The teachers have to use GSP as a tool to teach mathematics subjects [17]. The teacher should enable students' abilities to use GSP to construct mathematics puzzle problems. These will enhance students' learning of mathematics with creative thinking skills, thinking carefully, and having fun. The teachers can include Sangaku puzzle topics to teach in any one of the five delivery formats that the Ministry of Education, Thailand has designated: on-air, online, on-hand, on-demand, or on-site.

5. Action Research in Mathematics Class

Action research was conducted in the College of Hospitality Industry Management, Suan Sunandha Rajabhat University, Bangkok, Thailand, in June-July 2022. A total of 19 students studying in the second year of their Bachelor of Education, majoring in mathematics, participated in this study. The students enrolled in the course BMA 3301: Blended Learning in Mathematics. The duration of the action research project was about two months.

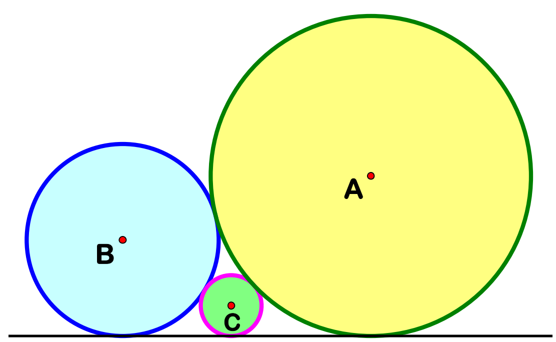
A flipped classroom model incorporating cooperative learning: Group Investigation and the Geometer's Sketchpad were employed in this study. These methods were used in line with the policy of the Ministry of Education in Thailand during the COVID-19 pandemic situation in the years 2021-2022. The action research questions were:

1) How can a flipped classroom model be implemented using Group-Investigations and the Geometer's Sketchpad on Sangaku mathematics puzzles?
2) How does the Geometer's Sketchpad impact the students' creative thinking of mathematics and assist them in constructing Sangaku puzzles?

Research Question 1:
How can a flipped classroom model be implemented using Group-Investigation and the Geometer's Sketchpad on Sangaku mathematics puzzles?

The following activities showed how the author who acts as a teacher implemented the Cooperative learning method: Group-Investigation and the Geometer's Sketchpad on Sangaku mathematics puzzles in this research study. The steps of the cooperative learning method: Group-Investigation based on Sharan [15] were as follows:

Step 1: The class determines subtopics and learning procedures. The Sangaku mathematics puzzle was identified into four Sangaku puzzle subtopics, which are as follows:

Sangaku Puzzle 1: Satimiya Three Tangent Circles. Given three circles tangent to each other and to a straight line as shown in the diagram. The radius of the large circle is $\mathbf{r_1}$, and the radius of the medium circle is $\mathbf{r_2}$. What is the radius ($\mathbf{r}$) of the small circle?	 Figure 1: Three Tangent Circles

Sangaku Puzzle 2: Yoshifuji Mishima Problem

As shown in the diagram, there is a right-angle triangle which contains a circle, and an equilateral triangle.
What is the side length of the equilateral triangle in terms of x and $\mathbf{y}$?

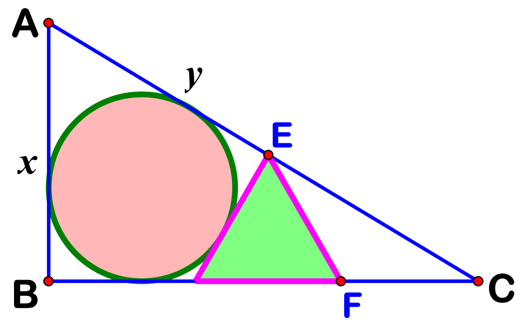


Figure 2: Sangaku Yoshifuji Mishima Problem

Sangaku Puzzle 3: Sangaku Problem 47

As shown in the diagram: Maximize $\mathbf{y}$ as a function of x assuming BC = a is constant.

(Sangaku Problem 47: Sacred Mathematics: Japanese Temple Geometry/Fukagawa Hidetoshi, Tony Rothman)

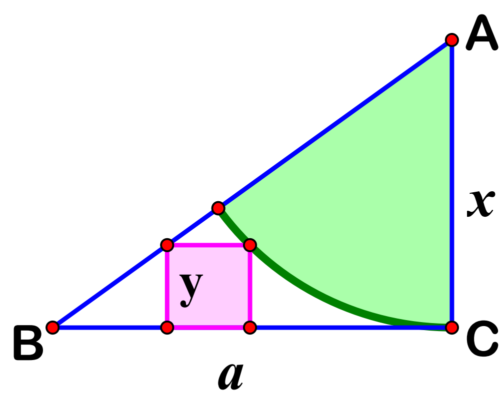


Figure 3: Sangaku Problem 47

Sangaku Puzzle 4: A Square-Three Arcs and a Circle

As shown in the diagram, Sangaku in a square-Three arcs and a circle.

Find a relationship between the radius of the circle and the side of a square.

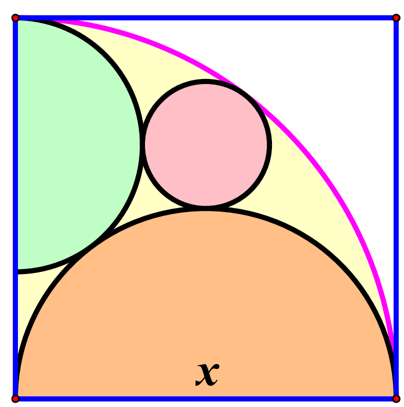


Figure 4: Sangaku in a square-3 arcs and a circle

Step 2: Students selected their own groups of 4-5 members according to their interest in the Sangaku puzzle. Each group is a mixed-ability group.

Step 3: Students plan their investigation and discuss it in their own groups. They plan to use the Geometer's Sketchpad to construct the Sangaku puzzle on their selected puzzle.

Step 4: Using a flipped classroom model. Students work in groups according to their plans inside the classroom and through an online platform. They constructed a Sangaku puzzle using the Geometer's Sketchpad. The students had a whole class discussion, and the teacher was a facilitator, helping them when needed.

Step 5: Each group made the presentation. Students explained to the whole class how they came up with their solutions and how to use the Geometer's Sketchpad to construct the Sangaku puzzle.

Step 6: The teacher and students evaluate the group projects based on the preset rubric.

Research Question 2:

How does the Geometer's Sketchpad impact the student's creative thinking of mathematics and assist them in constructing Sangaku puzzles?

The example of constructing the Sangaku Problem that impacts students' creative thinking in this paper is the Sangaku Puzzle 1: Samiya Three Tangent Circles.

*Sangaku puzzle-1***:** Satimiya Three Tangent Circles

Given three circles tangent to each other and to a straight line as shown in the diagram. The radius of the large circle is r_1, and the radius of the medium circle is r_2.

a) What is the radius of the small circle?
b) How to construct the Three Tangent Circles.

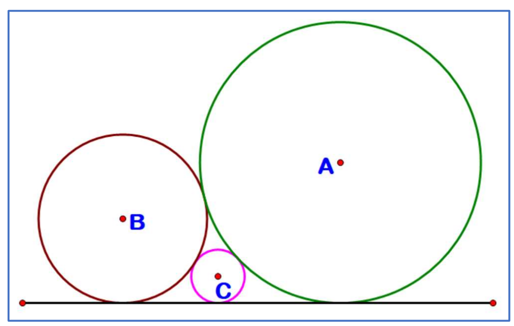


Figure 5 Sangaku Puzzle 1

a) What is the radius of the small circle?

(i) Find FE:

The right triangle ABH. From Pythagoras Theorem:

$AH^2 + BH^2 = AB^2$

$BH = FE$

$(r_1-r_2)^2 + FE^2 = r_1^2 + r_2^2$

After a few calculation steps:

$FE = \sqrt{4r_1r_2}$

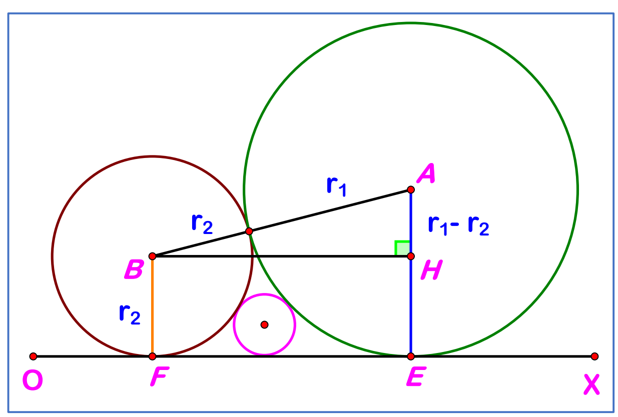


Figure 6 Find Distance FE

(ii) Find the radius of the small circle?
From the diagram: BCM and ACN are right angles triangles:

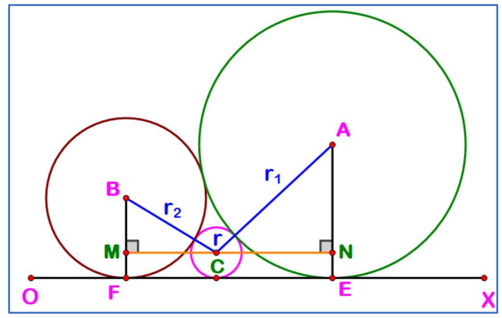


Figure 7 Find Radius of Small Circle C

From Pythagoras Theorem:

$MC^2 = BC^2 - BM^2$; $MC = \sqrt{BC^2 - BM^2}$

$CN^2 = AC^2 - AN^2$; $CN = \sqrt{AC^2 - AN^2}$

$MC + CN = \sqrt{BC^2 - BM^2} + \sqrt{AC^2 - AN^2}$

From (i): $MC + CN = MN = FE = \sqrt{4r_1r_2}$

From the diagram:

$BC = r_2 + r$; $BM = r_2 - r$; $AC = r_1 + r$; $AN = r_1 - r$

$$\sqrt{(r_2 + r)^2 - (r_2 - r)^2} + \sqrt{(r_1 + r)^2 - (r_1 - r)^2} = MN = FE = \sqrt{4r_1r_2}$$

We have:

$[(r_2 + r)^2 - (r_2 - r)^2] + [(r_1 + r)^2 - (r_1 - r)^2]$

$= [\{r_2^2 + 2rr_2 + r^2\} - \{r_2^2 - 2rr_2 + r^2\}] + [\{r_1^2 + 2rr_1 + r^2\} - \{r_1^2 - 2rr_1 + r^2\}]$

$= [r_2^2 + 2rr_2 + r^2 - r_2^2 + 2rr_2 - r^2] + [r_1^2 + 2rr_1 + r^2 - r_1^2 + 2rr_1 - r^2]$

$= [4\ rr_2] + [4\ rr_1]$

Therefore:

$$\sqrt{(r_2 + r)^2 - (r_2 - r)^2} + \sqrt{(r_1 + r)^2 - (r_1 - r)^2} = \sqrt{4r_1r_2}$$

$$\sqrt{4rr_2} + \sqrt{4rr_1} = \sqrt{4r_1r_2}$$

$$\sqrt{rr_2} + \sqrt{rr_1} = \sqrt{r_1r_2}$$

After a few calculation steps, we will obtain:

⇨ $$\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{r}}$$

and; $$r = \frac{r_1r_2}{(\sqrt{r_1} + \sqrt{r_2})^2}$$

The example of students' group work on constructing *Sangaku puzzle-1*: Satimiya Three Tangent Circles using the Geometer's Sketchpad is as follows.

- Open the Geometer's Sketchpad, in the File Menu, choose New Sketch, and follow the instruction step-by-step.
- Used the *Number menu* to construct parameters: $r_1 = 8$ cm, and $r_2 = 5$ cm.
- Construct a circle A of radius 8 cm, tangent to a straight line OX at point E;
- AP is a diameter length = $r_1 + r_2$
- The circle with diameter AP intercepts line OX at point D.

We have the length $ED = \sqrt{r_1r_2}$

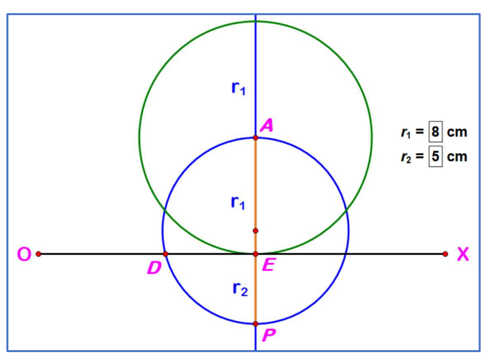


Figure 8: Construct a Large Circle A

- Construct a circle at D with a radius DE, intersect OX at point F
- Construct a line PD;
- Construct a perpendicular at point F, and intersect a line PD at a point B
- BF is a radius of circle B,

  Therefore B is the medium circle with a radius r_2.

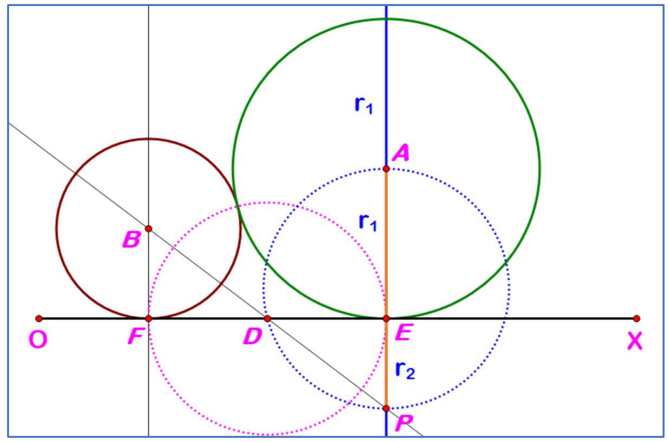


Figure 9: Construct a Medium Circle B

Construction a circle C with radius r = $\frac{r_1 r_2}{(\sqrt{r_1} + \sqrt{r_2})^2}$

- Construct a circle at point A with a radius $r_1 + \mathbf{r}$ or $r_1 + \frac{r_1 r_2}{(\sqrt{r_1} + \sqrt{r_2})^2}$ intersect AP at point K
- Construct a circle at E with a radius EK, intersect AP at point T
- Construct a perpendicular at point T, and intersect a circle at point C

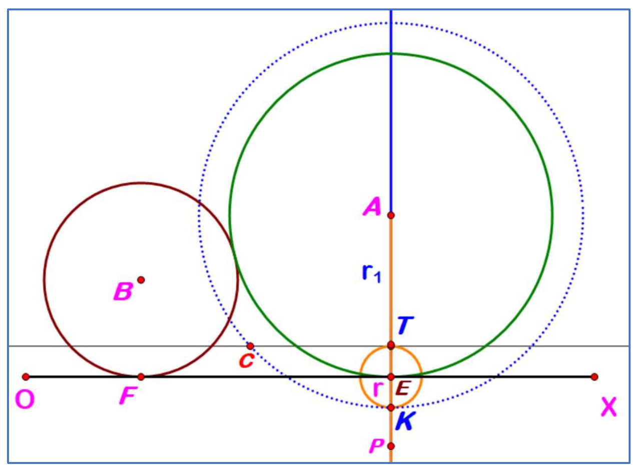


Figure 10: Find Center of Small Circle C

- Construct a perpendicular at point C, and intersect OX at point Q
- CQ is a radius of circle C,
- Therefore, C is the small circle with a radius ***r*** or

  $\frac{r_1 r_2}{(\sqrt{r_1} + \sqrt{r_2})^2}$.

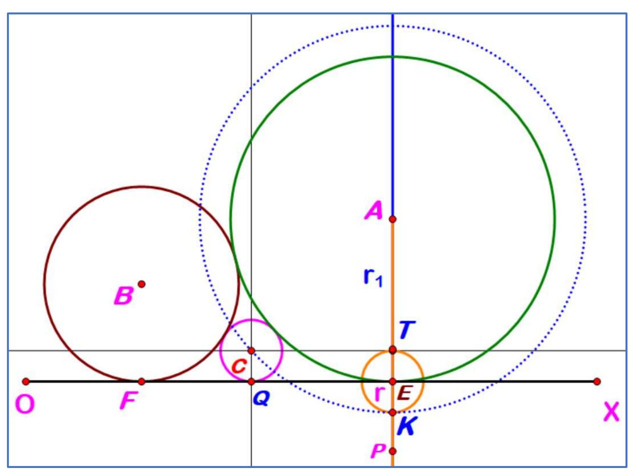


Figure 11; Construct a Small Circle C

- Construct circle interior of the three circles, select color as needed.
- Hide the circles and lines, we do not want to show.
- Completed Sangaku Puzzle 1 Satimiya Sangaku:Three Tangent Circles was shown on the right.

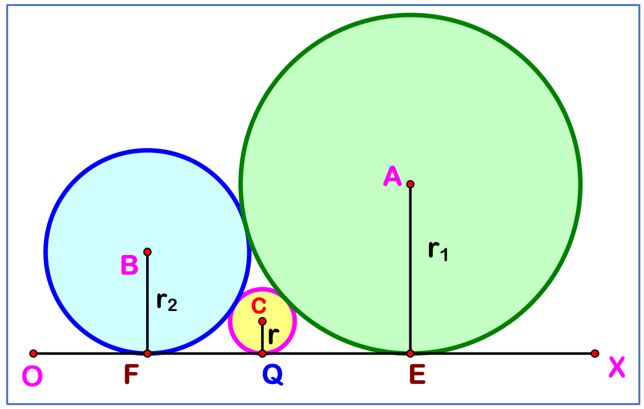


Figure 12: Satimiya Sangaku:Three Tangent Circles

Students' Investigation

With the use of the Geometer's Sketchpad, students animated the value of parameter r_1 and r_2 to investigate the relation of the "Sangaku: Three Tangent Circles". They found that no matter what the size of the two radii r_1, and r_2, the radius (r) of the small circle is always equal to $\frac{r_1 r_2}{(\sqrt{r_1}+\sqrt{r_2})^2}$ and the three circles tangent to each other and to a straight line.

6. Research findings

The research findings revealed that at the beginning of the research study during the Covid-9 pandemic the students had problems with communication within their group through online platforms. Some group members did not know how to solve the problem and could not use GSP to develop their Sangaku puzzles. Sangaku takes up a lot of problems about the figure, such as the properties of circles, and it is not easy for students to solve such problems. The students learned through a combination of pedagogical methods, using dynamic technology in mathematics, traditional face-to-face in-class, and online learning platforms. That was used in cooperative learning: Group Investigation, the Geometer's Sketchpad in constructing Sangaku mathematics puzzles. This knowledge and experience impact students' confidence in learning and teaching mathematics in the future.

7. Conclusions

In conclusion, the integration of Sangaku Mathematics Puzzle with The Geometer's Sketchpad presents an exciting and effective way to foster creative thinking and problem-solving skills in students. It empowers them to tackle intricate mathematical challenges, encourages collaborative learning, and ultimately enhances their overall mathematical proficiency. Overall, this integration ignites a passion for mathematics, making learning a dynamic and enjoyable process. Students build confidence in their abilities and become adept at tackling mathematical problems, not merely as exercises, but as opportunities for growth and enrichment. The combined power of the Sangaku Mathematics Puzzle and the Geometer's Sketchpad paves the way for a new generation of innovative thinkers, equipped with the essential skills needed to excel in various academic and professional endeavors. The "Sangaku Mathematics Puzzle" serves as a catalyst for cultivating creative thinking and problem-solving abilities, aided by the innovative tool, "The Geometer's Sketchpad."

8. References

[1] Office of the Basic Education Commission, "Policy of five teaching styles during COVID-19 pandemic". January 28, 2021, https://www.obec.go.th/archives/377135

[2] Vanpetch,Y. and Sattaayathamrongthian, M. *The challenge and opportunities of Thailand education due to the covid-19 pandemic: Case study of Nakhon Pathom, Thailand.* ITSE-2020 E3S Web of Conference 210, 18058 (2020). https://doi.org/10.1051/e3sconf/202021018058

[3] Bergmann, J. and Sams, A. (2012). Flip your classroom: reach every student in every class every day. Printed in the United States of America.

[4] Skemp, R. R. (1978). Relational Understanding and Instrumental Understanding. *Arithmetic Teacher,* 26(3), 9-15.

[5] Johnson, D.W. and Johnson, R.T. (1991). *Learning Mathematics and Cooperative Learning: Lesson Plans for Teachers*. MN: Interaction Book Company.

[6] Johnson, D. W., & Johnson, R. T. (2014). Cooperative learning in the 21st century. *Anales de Psicología, 30*(3), 841–851.

[7] Slavin,R.E. (1991). Student team learning: A practical guide to cooperative learning 3rd". Washington, D.C: National Education Association of the United States.

[8] Sharan,S. (1980). Cooperative learning in small groups: Recent methods and effects on achievement, attitudes, and ethnic relations. *Review of Education Research Summer*, *1980, Vol.50, N0.2*, pp.241-271.

[9] Slavin,R.E. (1990). "Cooperative learning: Theory, research and practice". Massachusetts: Allyn and Bacon.

[10] Khairiree, K. (2015). Classroom Management: Cooperative Learning and the Geometer's Sketchpad in Algebra. *International Academic Conference on Law, Politics, and Management.* IACLPM 2015 Vilnius. http://www.lawpoliticsconference.com/

[11] Shran,Y. & Sharan,S. (1992). "Expanding cooperative learning through group investigation". NY: Teachers College Press.

[12] H. Fukagawa & T. Rothman. (2008). *Sacred mathematics- Japanese Temple Geometry*. Princeton University Press.

[13] Hosking, R.J. (2017). Solving Sangaku: A tradition solution to a Nineteenth Century Japanese Temple problem. *Journal for History of mathematics. Vol.30. No.2 (Apr.2017), 53-69.* http://dx.doi.org/10.14477/jhm.2017.30.2.05

[14] Majewski, M., Chuan, J.C. & Hitoshi, N. *The New Temple Geometry Problems in Hirotaka's Ebisui Files.* Mirek jen Chung GSP 3052010_18118.pdf

[15] Jakiw, N. (1992). Geometer's Sketchpad. User guide and reference manual. Key Curriculum Press, Berkeley 1992.

[16] Khairiree, K. (2019). Augmented Reality and Blended Learning: Engaging Students Learn Word Problems with Bar Model and the Geometer's Sketchpad. In *Proceeding of the 24th Asian Technology Conference in Mathematics,*. Leshan, China, 2019.

[17] Ministry of Education (2017). *"Basic Education Core Curriculum B.E.2551, (Revised B.E.2650, A.D. 2017).* http://academic.obec.go.th/newsdetail.php?id=75