Carral's Geometric Proof of a Steiner Ellipse Property and an Attempt at Generalisation

Jean-JacquesDahan
jjdahan@wanadoo.fr
IRES of Toulouse
Paul Sabatier University
Toulouse FRANCE

Abstract: *In this paper, we will first use the solution proposed by Michel Carral to prove that the ellipse inscribed in a triangle of maximum area is its Steiner ellipse. This proof is based on a purely geometric reasoning which greatly simplifies the first complete proof proposed by Minda and Phelps ([6]), in the sense that it uses more basic tools and knowledge. Recall that in a previous paper ([7]), an approach with dynamic geometry had already allowed a very simplified approach even if a step was incomplete insofar as it was only justified experimentally. As said above, this problem was solved geometrically by my colleague Michel Carral to whom I had submitted it. He has included it as an exercise in the area geometry book he is finalizing. I thought that his solution deserved better than an exercise in his geometry book and I decided with his agreement to write it trying to respect the spirit of the author but detailing it enough to be understood by all. I will then propose an attempt at generalization to ellipses inscribed in polygons. In passing I will propose a still purely geometric proof of the fact that polygons of minimum area having a given inscribed circle are regular polygons. Eventually a construction under the New Cabri, using macros, illustrates the sequence of polygons circumscribed about a given ellipse approaching the polygon of minimum area circumscribed about this ellipse.*

1. Minimizing the area of triangles circumscribed about a given circle

1.1. A property related to areas of triangles circumscribed about a given circle

Property1 (P1): the area of any triangle circumscribed about a given circle is greater than the area of any isosceles triangle circumscribed about the same circle and with a common angle with the given triangle.

Proof: *ABC* is the given triangle whose inscribed circle is the circle centered at *O* and with radius *OI*. Consider isosceles triangle *AMN* with vertex *A* and the same inscribed circle (Figure 1 left). We will show that the area of *ABC* is greater than the area of *AMN*. It suffices to establish that the area of *PNC* is less than the area of *PMB* because Area(*ABC*) = Area(*AMN*) + Area(*PMB*) - Area(*PNC*). We construct *Q* so that (*MQ*) is parallel to (*AC*) (Figure 1 right). Triangles *PNC* and *MPQ* are therefore similar. As *PM* > *PN*, the area of *MPQ* is greater than that of *PNC*. As the area of *PMB* is greater than that of *PMQ*, the area of *PMB* is greater than that of *PNC*.

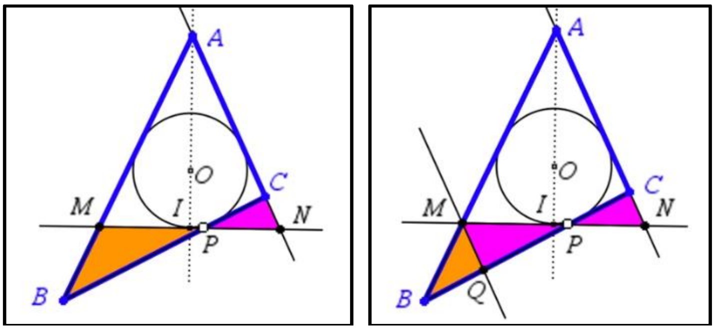

Figure 1: Decreasing areas with isosceles triangles

1.2. A property related to areas of isosceles triangles circumscribed about a given circle

Property 2 (P2): the area of any isosceles triangle 1 (here ABC) with a given inscribed circle is greater than the area of any equilateral triangle 2 (here DEF) with the same inscribed circle.

With the notations of Figure 2, let us show that the area of the triangle *EFG* is always greater than the area of the equilateral triangle *ABC*, with *E* on the perpendicular bisector of [*BC*], or else on [*uA*] (first case: angle $\hat{E}$ greater than 60°) or above *A* (second case: angle $\hat{E}$ less than 60°).

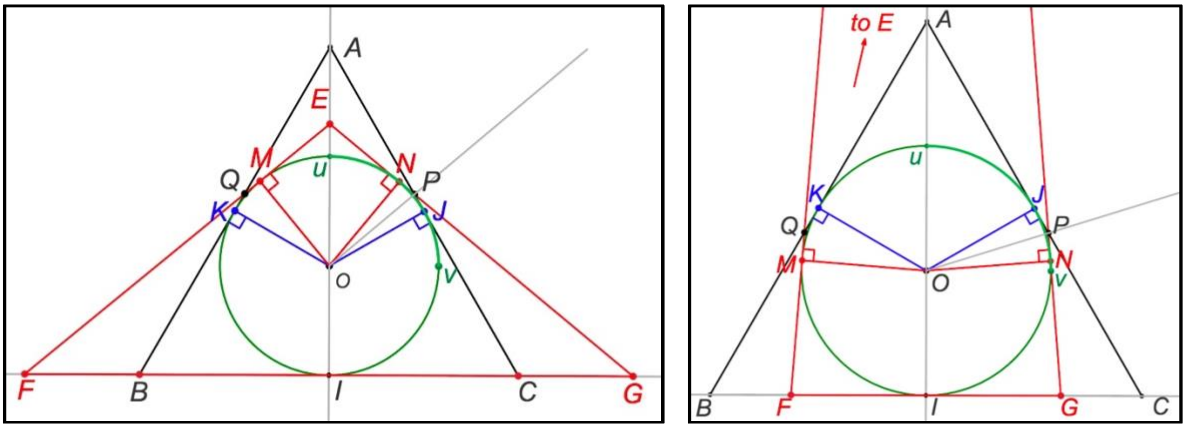

Figure 2: From isosceles to equilateral triangles

First case (Figure 2 left): we know that *AJ* = *JC*; as *E* belongs to [*uA*], *N* precedes J on the green arc *uv*, and therefore *P* is inside the angle $\widehat{NOJ}$. Therefore *PA* < *PC*. As *PE* < *PA* and *PC* < *PG*, we can deduce that *PE* < *PA* < *PC* < *PG*. Since the triangles *NEA* and *NCG* have an equal angle $\widehat{ENA} = \widehat{CNG}$ and since the sides surrounding the angle $\widehat{ENA}$ are respectively less than the sides surrounding the angle $\widehat{CNG}$, we deduce that the area of triangle *ENA* is less than the area of triangle *CNG*. At last, Area(*EIG*) = Area(*AIC*) + (Area(*CNG*) - Area(*ENA*)) > Area(*AIC*) because we have just proved that the difference between parentheses is positive. Finally, triangle *EFG* has an area greater than the area of triangle *ABC*.

Second case: a demonstration analogous in all respects to that of the first case can be carried out based on Figure 2 right and leads to the same result.

Consequence of the previous properties: Any triangle circumscribed about a given circle has an area greater than the area of any equilateral triangle circumscribed about the same circle.

Remark: during our work on this property, we found several other interesting proofs provided at the end of this paper (see paragraph 5.).

2. About areas of triangles circumscribed about a given ellipse

2.1. Reminder (the Steiner ellipse)

Definition: the Steiner ellipse of a triangle is the ellipse inscribed in this triangle tangent to its three sides respectively at their midpoints.

Its existence is easily proved ([7]) by using the fact that any triangle is the image of an equilateral triangle by an affinity. The Steiner ellipse is none other than the image by this affinity of the inscribed circle of the equilateral triangle. The complete justification is based on the fact that any affinity preserves midpoints and tangents.

2.2. Preliminary property

The ratio between the area of the Steiner ellipse of a triangle and the area of this triangle is independent of the triangle and equal to $\frac{\pi\sqrt{3}}{9}$.

Proof: recall that affinities preserve area ratios and contact properties. We also recall that any triangle *ABC* is the image of an equilateral triangle *A'B'C'* by a certain affinity. The inscribed circle of the equilateral triangle *A'B'C'* is therefore transformed by the reciprocal affinity into the Steiner ellipse of *ABC*. As we know that the ratio of the areas of the inscribed circle of an equilateral triangle to the area of this equilateral triangle is equal to $\frac{\pi\sqrt{3}}{9}$, we obtain the result stated above.

2.3. Ellipse of maximum area inscribed in a given circle

Property 3 (P3): for a given triangle and an ellipse inscribed in this triangle, the ratio between the area of the ellipse and the area of the triangle is less than the ratio between the area of the Steiner ellipse of this triangle and the area of this triangle.

Proof: given a triangle *ABC* and an ellipse (*Ell*) inscribed in this triangle (Figure 3). We know from the above that there is an affinity *Aff* transforming this ellipse into a circle (*Cer*) and the triangle *ABC* into a triangle *A'B'C'* circumscribed about (*Cer*). Thanks to the properties of the affinities, we have the following equality of the area ratios: $\frac{Area(Ell)}{Area(ABC)} = \frac{Area(Cer)}{Area(A'B'C')}$

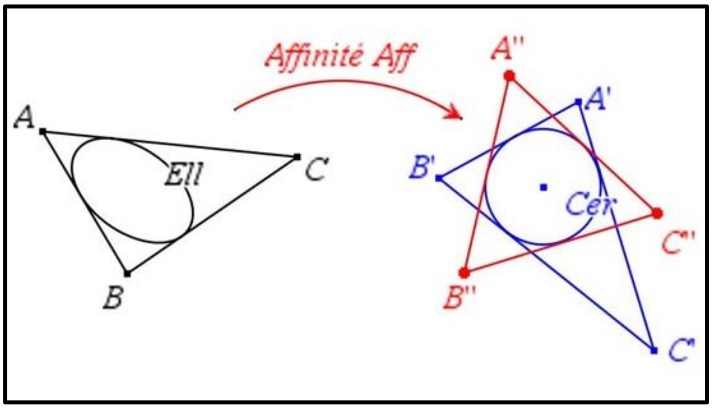

Figure 3: Using affinity

If we consider an equilateral triangle *A''B''C''* circumscribed about the circle (*Cer*) we know, using (P1) that $Area(A'B'C') > Area(A''B''C'')$, from which we deduce that:

$$\frac{Area(Cer)}{Area(A'B'C')} < \frac{Area(Cer)}{Area(A''B''C'')} \text{ and so } \frac{Area(Ell)}{Area(ABC)} < \frac{Area(Cer)}{Area(A''B''C'')} = \frac{\pi.\sqrt{3}}{9} = \frac{Area(Ell\_St)}{Area(ABC)}$$

where *Ell_St* is the Steiner ellipse of triangle *ABC*.

It has therefore been established that $\frac{Area(Ell)}{Area(ABC)} < \frac{Area(Ell\_St)}{Area(ABC)}$.

Finally: from this result we deduce that, *Area*(*Ell*) < *Area*(*Ell_St*). This last inequality means that the area of any ellipse inscribed in a triangle is less than the area of the Steiner ellipse of this triangle. It has therefore been proven that **the Steiner ellipse of a triangle is the ellipse of maximum area inscribed in this triangle**.

3. Polygons of minimum area circumscribed about a given circle

3.1. Preliminary property (Figure 4)

Circle C is given with two tangents T_1 and T_2 at t_1 and t_2 with D their axis of symmetry (D is also the bisector of angle $\widehat{t_1Ot_2}$). A variable tangent T_3 at t_3 intersects T_1 and T_2 at v_1 and v_2 respectively. S is the tangent to C perpendicular to D at s.

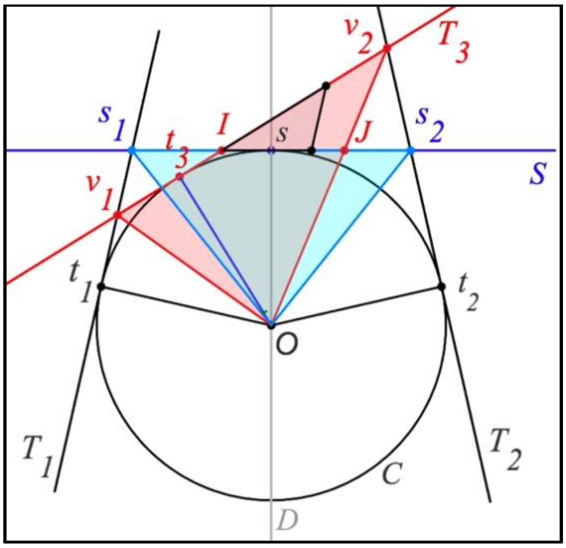

Figure 4: Three tangent lines and a pentagon

Property 4: the area of pentagon $Ot_1v_1v_2t_2$ is minimum when t_3 is in s, i.e. for the pentagon $Ot_1s_1s_2t_2$.

Proof:

$\widehat{v_1Ov_2} = \widehat{v_1Ot_3} + \widehat{t_3Ov_2} = \frac{1}{2}\left(\widehat{t_1Ot_3} + \widehat{t_3Ot_2}\right) = \frac{1}{2}\widehat{t_1Ot_2}$ which is constant, so $\widehat{v_1Ov_2}$ is constant and equal to angle $\widehat{s_1Os_2}$.

Due to the symmetry of the figure we can assume from now that point I belongs to the segment $[s_1s]$.

Using the same reasoning as the previous one, we prove that:

$\widehat{v_1OI} = \frac{1}{2}\widehat{t_1Os} = \frac{1}{4}\widehat{t_1Ot_2} = \frac{1}{4}.2.\widehat{s_1Os_2} = \frac{1}{2}\widehat{v_1Ov_2}$. And so (OI) is the bisector of $\widehat{v_1Ov_2}$.

As a result: $\widehat{v_1Ot_3} = \widehat{v_1OI} - \widehat{t_3OI} = \widehat{IOv_2} - \widehat{IOs} = \widehat{sOJ}$ and therefore the right triangles v_1Ot_3 and sOJ are equal, from which we deduce that $Ov_1 = OJ$ and $Ov_1 \leq Ov_2$ because $OJ \leq Ov_2$.

Finally, as (OI) is the bisector of $\widehat{v_1Ov_2}$ we know that $\frac{Iv_1}{Iv_2} = \frac{Ov_1}{Ov_2}$ and so $\boxed{Iv_1 \leq Iv_2}$.

Moreover, as Is_1 is less than Is_2, the symmetric triangle of triangle Is_1v_1 with respect to I is included in the triangle Is_2v_2 and its area is therefore less than the area of Is_2v_2.

These results will allow us to minimize the area of the pentagon by a judicious choice of the position of t_3. If $\mathcal{A}(F)$ denotes the area of F, we have:

$\mathcal{A}(Ot_1v_1v_2t_2) = \mathcal{A}(Ot_1s_1s_2t_2) + (\mathcal{A}(Is_2v_2) - \mathcal{A}(Is_1v_1)) \geq \mathcal{A}(Ot_1s_1s_2t_2)$ because according to the above, the difference between parentheses is positive (strictly and null in the case where t_3 is in s).

Conclusion, the area of pentagon $Ot_1v_1v_2t_2$ is minimum when $[v_1v_2]$ is superimposed on $[s_1s_2]$, i.e., when (v_1v_2) is the tangent to C at s.

As the area of triangle Ov_1v_2 is equal to half the area of the pentagon $Ot_1v_1v_2t_2$, this one has minimum area when it merges with the triangle Os_1s_2.

3.2. Polygon of minimum area circumscribed about a given circle

Theorem: Any polygon with *n* sides circumscribed to a given circle has minimum area when the polygon is regular.

Let us first prove that necessarily such a polygon has its contact points with its inscribed circle at the middles of its sides.
We are going to conduct a reasoning by absurd, by demonstrating that if one of the sides is not tangent at its midpoint, we can find another circumscribed polygon with *n* sides of lesser area.

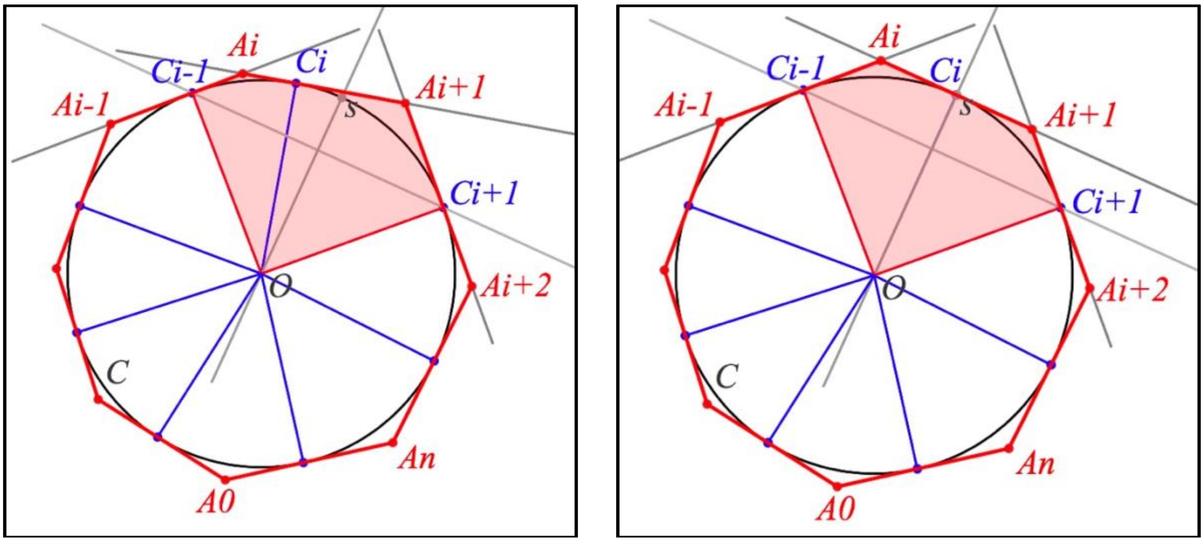

Figure 5: Circumscribed polygons of minimum area

Let us consider a polygon $A_0....A_n$ circumscribed about a circle C (Figure 5). Suppose that the contact point C_i of the side A_iA_{i+1} with C is not the midpoint of this side. Let us show that we can deform this polygon and obtain another of lesser area. For this, we keep the white polygonal part $OC_{i+1}A_{i+2}....A_nA_0......A_{i-1}C_{i-1}O$ and distort the pentagonal part $OC_{i-1}A_iA_{i+1}C_{i+1}O$ by moving the point C_i up to s (as done previously), which also minimizes the area of this pentagon and therefore makes it possible to obtain a new polygon with *n* sides of lesser area (see Figure 5 right), which results in a contradiction.
Moreover, as $C_{i-1}A_i$ is equal to A_iC_i which is half of A_iA_{i+1}, two consecutive sides are equal (similarly for A_iA_{i+1} and $A_{i+1}A_{i+2}$). Triangles OA_iA_{i+1} are all isosceles and equal because with the same vertex O, with heights equal to the radius of C and equal bases. Such a polygon is therefore regular, which had to be demonstrated.

4. Attempt to generalize Steiner's initial property

4.1. *n*-sided polygon of minimum area circumscribed about a given ellipse

Let us prove that if an *n*-sided polygon is circumscribed about a given ellipse, the area of this polygon is minimum if the contact points are the midpoints of the sides.
Let us reason by absurd: if a polygon $\mathcal{P}$ with *n* sides, circumscribed to a given ellipse $\mathcal{E}$, does not have all its contact points at the midpoints of its sides, let us examine its image by an affinity $\mathcal{Aff}$ transforming the circumscribed ellipse $\mathcal{E}$ into a circle $\mathcal{C} = \mathcal{Aff}(\mathcal{E})$.
The polygon $\mathcal{Aff}(\mathcal{P})$ is therefore a polygon circumscribed to a circle $\mathcal{C}$. We know that we have the following equality of area ratios:

$$\frac{\mathcal{A}(\mathcal{P})}{\mathcal{A}(\mathcal{E})} = \frac{\mathcal{A}(\mathcal{A}ff(\mathcal{P}))}{\mathcal{A}(\mathcal{C})}$$

Moreover, the contact points of $\mathcal{A}ff(\mathcal{P})$ with $\mathcal{C}$ are not all located at the midpoints of its sides: it is therefore not the polygon of minimum area circumscribed to $\mathcal{C}$. Let us consider one of those minimizing this area, $\mathcal{P}'$ (which is a regular polygon). We have:

$$\frac{\mathcal{A}(\mathcal{P}')}{\mathcal{A}(\mathcal{C})} < \frac{\mathcal{A}(\mathcal{A}ff(\mathcal{P}))}{\mathcal{A}(\mathcal{C})}$$

Its image by $\mathcal{A}ff^{-1}$ is a polygon circumscribed to the starting ellipse which verifies:

$$\frac{\mathcal{A}(\mathcal{A}ff^{-1}(\mathcal{P}'))}{\mathcal{A}(\mathcal{E})} = \frac{\mathcal{A}(\mathcal{P}')}{\mathcal{A}(\mathcal{C})}$$

So, the area of $\mathcal{A}ff^{-1}(\mathcal{P}')$ is less than the area of $\mathcal{P}$. The area of $\mathcal{P}$ is therefore not minimal, which completes the proof.
Note that $\mathcal{A}ff^{-1}(\mathcal{P}')$ is a polygon circumscribed to $\mathcal{E}$ with minimal area.

4.2. Iterations for the construction of such an *n*-sided polygon when *n* = 6

We construct first an ellipse and six tangent lines at six given points of this ellipse. These six tangent lines define a hexagon (H_1) circumscribed about this ellipse. This hexagon is not necessarily the hexagon of minimum area circumscribed about the ellipse. The following macros will allow us to construct a sequence of hexagons (H_n) circumscribed about the given ellipse whose areas are decreasing and so approaching the hexagon of minimum area.
Construction of an ellipse (*E*): we use two points *F* and *F'* (foci) and one its point *B* (on the perpendicular bisector of [*FF'*]). We record this construction as a macro (Initial objects: *F*, *F'* and *B*. Final object: ellipse (*E*)). Name of this macro: **Ellipse**. See Figure 6 left.
Construction of the tangent line to an ellipse at a given point: first we construct the director circle (*C*) associated with focus *F*; it is the circle centered at *F* with radius equal to *Fb* where *b* is the symmetric point of *F* with respect to *B*. If *c* is a given point on the ellipse (*E*) and *c'* the intersection point between (*C*) and ray [*Fc*), we know that the perpendicular bisector of [*F'c'*] is the tangent line *Tc* to the ellipse at *c*. We record this construction as a macro (Initial objects: *F*, *F'*, *B* and *c*. Final object: tangent line (*Tc*)). Name of this macro: **tangent ellipse**. See Figure 6 left.

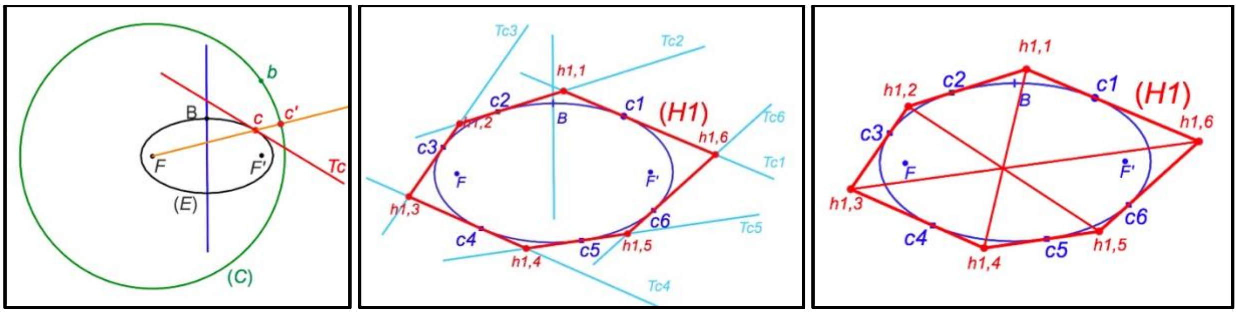

Figure 6: Approaching the hexagon of minimum area circumscribed to a given ellipse

Construction of a hexagon circumscribed about a given ellipse (Figure 6 middle): we use the macro ellipse to construct the ellipse with foci at *F* and *F'* and passing through *B*. We use then the macro **tangent ellipse** to construct the six tangent lines T_{c1} to T_{c6} to the ellipse at c_1 to c_6. Eventually, we construct the hexagon (H_1) supported by these tangent lines, passing through the points $h_{1,1}$ to $h_{1,6}$.

This hexagon is one hexagon circumscribed to the ellipse. We have the opportunity to check the Brianchon theorem: as displayed in Figure 6 right, we can check that the three diagonals of this hexagon intersect at the same point. We record this construction as a macro (Initial objects: *F*, *F'*, *B* and the six contact points c_1 to c_6. Final object: hexagon (H_1) tangent to the ellipse at c_1 to c_6). Name of this macro: **hexagon circumscribed**.

Using an affinity transforming the ellipse onto a circle: the ellipse is transformed onto its principal circle (circle centered at O, midpoint of [*FF'*] and passing through *A*, horizontal right vertex of the ellipse) by an affinity with axis (*FF'*) and with direction (*OB*). So, use this affinity to transform hexagon (H_1) circumscribed to the ellipse onto another hexagon (H'_1) circumscribed to the principal circle. We transform each point c_n onto a point c'_n in using the following construction: line (L_1) perpendicular to (*FF'*) passing through c_1 intersecting (*FF'*) at e_1, ray [e_1c_1) intersecting the principal circle at c'_1 which is the image of c_1 by the affinity. Therefore, the image of Tc_1 by this affinity is the tangent line to the principal circle at c'_1 which is the perpendicular line to ($O\ c'_1$) at c'_1. If we iterate this process, we can obtain the six tangent lines to the principal circle supporting the black hexagon (H'_1) image of the red hexagon (H_1) by our affinity. See Figure 7 left.

Using the technique presented in 3.2.: as the side of hexagon (H_1) supported by T_{c2} has not its contact point c_2 at its midpoint, we are sure that we can state the same thing for the side of (*H'1*) supported by $T_{c'2}$. So, to construct a hexagon (*H''1*) circumscribed to the principal circle whose area is less that the area of (H'_1), we replace the polygon $Oc'_1k_1k_2c'_3$ by the polygon $Oc'_1k'_1k'_2c'_3$ where *i* is the midpoint of [$c'_1c'_3$], where *j* is the intersection point between ray [*Oi*) and the principal circle, where k'_1 and k'_2 are respectively the intersection points between the tangent line to the circle at *j* with $T_{c'1}$ and $T_{c'3}$. See Figure 7 middle. We can see (H_1) in red, (*H'1*) in black and (*H''1*) in blue in Figure 7 right.

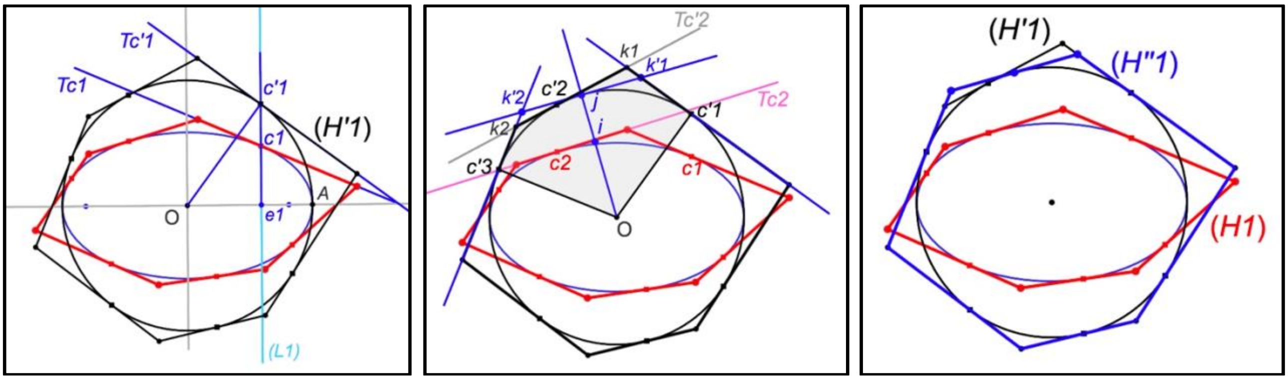

Figure 7: Using an affinity

Obtaining hexagon (*H₂*) whose area is less than that of (*H₁*) (also circumscribed to the given ellipse): (H_2) is simply the image by our affinity of (H''_2). The constructions are the same are those illustrated in Figure 7 left: we must only remember that the image of the principal circle by this affinity is our ellipse. We transform each contact point of (H''_2) with the principal circle by the affinity and we construct at each of these point the tangent line to the ellipse with the previous macro **tangent ellipse**. These six tangent lines define **(H_2)**. See these constructions in Figure 8 left and middle. In Figure 8 left, the construction of the first tangent green line of (H_2) is displayed. In Figure 8 middle, three of these tangent lines are displayed to see better that only one tangent line has been changed from (H_1) to (H_2). Eventually Figure 8 right displays (H_2). We can notice that the side modified in changing its supporting tangent line has its midpoint as a contact point with the ellipse.

We record this construction as a macro (Initial objects: *F*, *F'*, *B* and the six contact points c_1 to c_6, the first of which is the one on the tangent line to be modified. Final object: hexagon (H_2) and the new contact point). Name of this macro: **following hexagon**.

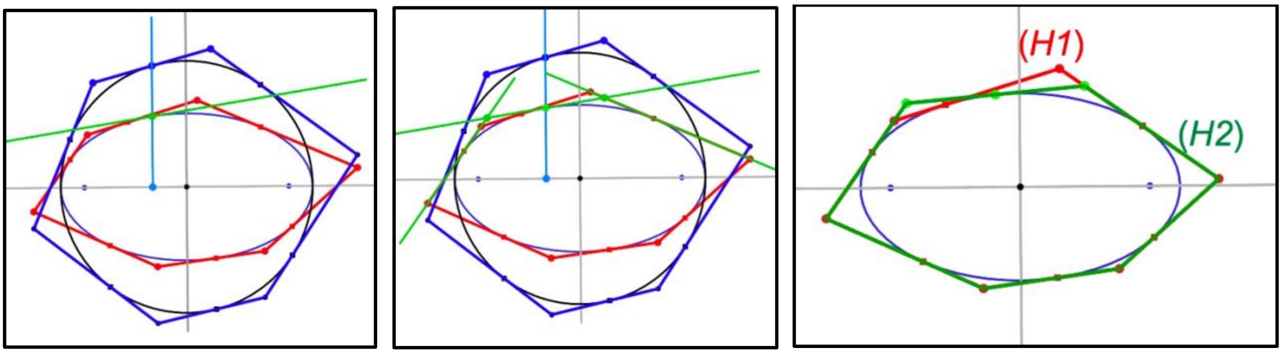

Figure 8: From (H_1) to (H_2)

The interest of this macro is to generate easily the sequence (H_n) avoiding all the intermediate constructions we had to do to obtain (H_2) from (H_1). We have applied it to (H_2) to obtain (H_3) (Figure 9 left) and then to (H_3) to obtain (H_4) (Figure 9 middle). We can state in Figure 9 right that the perpendicular bisectors of each side of (H_4) seem to pass through the contact points of (H_4) (or very close) with the ellipse. Eventually (H_4) can be considered as a good approximation of a hexagon of minimum area circumscribed about our ellipse. Note that we needed to iterate our construction only three times.

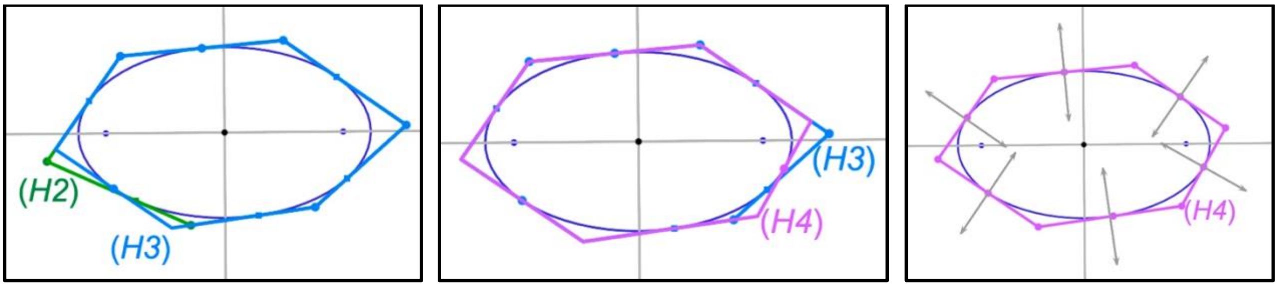

Figure 9: Reaching (H_4) close to the solution

Remark: to optimize our process, the first contact point chosen in order to apply the macro must be as far as possible from the midpoint of the side containing this contact point.

5. Two other proofs of property 2

5.1. Analytical proof (Figure 10 left)

We choose a circle of radius 1 (centered at *J*) in the system of axes (*O*, *I*, *J*). The area of the equilateral triangle *DEF* circumscribed about this circle is equal to $3\sqrt{3}$. In (*O*, *I*, *J*), the ordinate of point *A* is greater than 2 which is the ordinate of *K*. Let us evaluate the area of an isosceles triangle *ABC* circumscribed about the same circle as a function of the ordinate β of *A*, that is to say *OD.OF*. We start by evaluating the abscissa α of *C* as a function of β. For this, let us express that the distance from *J* (0,1) to the line (*AC*) is equal to 1.

An equation of (AC) is $\frac{x}{\alpha}+\frac{y}{\beta}-1=0$ or $\beta.x+\alpha.y-\alpha.\beta=0$, so the previous condition is expressed as follows: $\frac{|\alpha-\alpha.\beta|}{\sqrt{\alpha^2+\beta^2}}=1$ or $\frac{\alpha.(\beta-1)}{\sqrt{\alpha^2+\beta^2}}=1$ because $\beta>2$ or :

$\alpha.(\beta-1)=\sqrt{\alpha^2+\beta^2}$ or $\alpha^2.(\beta-1)^2=\alpha^2+\beta^2$ that is $\alpha^2=\frac{\beta^2}{(\beta-1)^2-1}$.

Eventually like $\alpha=\frac{\beta}{\sqrt{(\beta-1)^2-1}}$, the area of ABC is equal to $\frac{\beta^2}{\sqrt{(\beta-1)^2-1}}$. To prove that this area is greater than $3\sqrt{3}$ for $\beta>2$, let us study the variations of the numerical function $f(x)=\frac{x^2}{\sqrt{(x-1)^2-1}}$ for $x>2$.

$f'(x)=\frac{x^2.(x-3)}{((x-1)^2-1).\sqrt{((x-1)^2-1}}$ is therefore of the sign of x-3. Eventually:

$f'(x)<0$ for $2<x<3$ and $f'(x)>0$ for $x>3$ and therefore f admits a minimum for $x=3$, which is $3\sqrt{3}$. This means that any given isosceles triangle with given inscribed circle has minimum area when A is at D.

Conclusion: any isosceles triangle ABC with a given incircle has an area greater than the area of any equilateral triangle with the same inscribed circle.

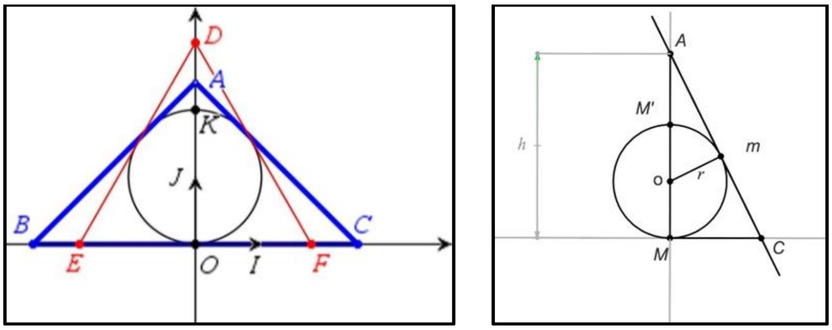

Figure 10: Other proofs of property 2

5.2. Proof using similar triangles (Figure 10 right)

Consider the triangle mAo where (AC) is tangent at m to the circle centered at o with radius r.

Triangles mAo and MAC are similar with a ratio equal to $\frac{AM}{Am}$. Let's evaluate this ratio:

$Ao=h-r$, $om=r$ and then $Am=\sqrt{(h-r)^2-r^2}$. Eventually the ratio is equal to:

$$\frac{h}{\sqrt{(h-r)^2-r^2}}$$

As the area of mAo is equal to $\frac{1}{2}r\sqrt{(h-r)^2-r^2}$, the area of MAC is obtained by multiplying the previous area by the square of the previous ratio, i.e.:

$Area(MAC)=\frac{1}{2}r\sqrt{(h-r)^2-r^2}.\left(\frac{h}{\sqrt{(h-r)^2-r^2}}\right)^2$.

$Area(MAC)=\frac{1}{2}\frac{rh^2}{\sqrt{(h-r)^2-r^2}}$. Or this area is minimum when $h=3r$ that is to say when triangle MAC is half an equilateral triangle.

For the proof, we study the variations of the function $f(h) = \frac{rh^2}{\sqrt{(h-r)^2-r^2}}$. Its derivative is of the sign of (h-$3r$). In fact: $f'(h) = \frac{h.r.(h-3r)}{(h-2r).\sqrt{h.(h-2r)}}$.

Conclusion

The need to find a simplified proof (of the Steiner ellipse) of that of Minda and Phelps ([6]) first led me to propose a proof inspired by an approach with the support of dynamic geometry ([7]). Michel Carral to whom I invited to help me finalize my proof decided on a completely different approach. His purely geometric approach ultimately led to the proof I presented in this paper. Moreover, he extended his work by a study concerning the polygons which I also presented. His method for minimizing the area of a polygon circumscribed to a given ellipse gave me the opportunity to create a set of macros in the New Cabri which allows users to generate very simply a sequence of polygons with decreasing and rapidly converging areas to the desired minimum area polygon. Finally, my initial approach with the tool of dynamic geometry not having succeeded completely, it is the classic approach of pure geometry of Michel Carral which provided a proof using tools and knowledge simpler than those of Minda and Phelps. His complementary work on polygons gave me the opportunity to reuse the dynamic geometry tool to visualize a convergence of a sequence of polygons which had previously remained very theoretical.

Note

Figures 1,3 and 10 were created with TI-Nspire^{TM} CX CAS Premium Teacher Software and the other ones with the New Cabri.

Acknowledgements

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Software

The New Cabri by Cabrilog at http://www.cabri.com
TI-Nspire^{TM} CX CAS Premium Teacher Software at https://www.ti.com