Combining Brute Force and IT to Solve Difficult Problems

Jozef Hvorecký, Lilla Korenova, Tomáš Barot

Jozef.Hvorecky@osu.cz, Lilla.Korenova@osu.cz, Tomas.Barot@osu.cz

Department of Mathematics with Didactics
Faculty of Education
University of Ostrava
Czech Republic

Abstract: *The brute force method is a typical approach for solving unknown or non-standard problems: without having a more appropriate tool or method, one tries to examine all relevant ("imaginable") candidates whether they satisfy the problem statement or not. As demonstrated below, it works effectively with many otherwise hardly solvable problems. Despite its undisputable value, it is rarely taught at universities. Time consumption might be the cause – it is its main disadvantage in general. To speed up the process, one should look for a clue or a hypothesis leading to its higher effectiveness. Using three examples which are otherwise unsolvable since the students lack the resources to do so, e.g. other more sophisticated methods which are beyond their current curricula, we explain its two principal steps: the generation of space of relevant candidates and their testing. Depending on the space size and complexity of their evaluation, the testing phase may last long. Then, computational power becomes a value. In our below examples, spreadsheet calculations are effectively exploited. In this way, the students may start understanding how to combine mathematics and IT in solving complex application problems – one of the aims of STEAM education. In Conclusions, we point to the fact that there are several ways to demonstrate interrelations between traditional Maths and IT.*

1. Introduction

Solving a problem by brute force – amongst many possible problem solving heuristics – is seldom taught at schools and universities. The preferred methods of solution often rely on more sophisticated and theoretically-based approaches that would lead to the desired exact solutions. Preferences are done to more sophisticated, theory-based approaches directly leading to their exact outcomes. However, "when everything else falls" the brute force method is the only practical heuristic that often works. Thus the students should be aware of its existence, power, applicability, advantages and disadvantages.

The brute-force search (or exhaustive search) is characterized as a very general problem solving technique that consists of systematically enumerating all possible solution candidates followed by checking whether the tested one satisfies the problem's statement [1]. The generating and testing parts are usually separated: the generation precedes the evaluation but they can be intertwined using a stepwise approximation. In the latter case, the new stage starts with another candidate space reduction.

The method was very popular in early days of Artificial Intelligence. Many problems solved using brute force are complex due to huge numbers of candidates e.g. chess [2] or code decryption [3]. In such cases, the credentials and features learned during the first solutions serve as inputs for building strategies which exclude groups of improbable candidates. The stronger restrictions accelerate the testing [4]. Contemporary information technology often exploits brute-force-based applications in various scientific fields including biology [5], chemistry [6] and astronomy [7]. Often, there might exist various ways to the problem solution depending on its formulation [8] and having different complexity leading to different solution characteristics. The learners should become familiar with as many of these as possible.

All this implies that the brute force method deserves to become part of the weaponry of STEAM students. Among others, it is a way how to demonstrate the omnipotent influence of Mathematics so it is a way to its “marketing” [9]. Authentic problems often do not have present obvious patterns and regularities, nor do they usually come along with hints. Furthermore, when students are not given ample opportunities to practise problem-solving they might be misled to believe that exact solutions are the only ones available and that these always exist. Such a belief system often forbids the student from trying different methods as he or she is trapped in the fear of getting lost.

For this reason, we recommend occasional presentations of brute-force-requesting mathematical problems which are on the edge of student’s knowledge – or slightly beyond. The key idea is to demonstrate problems they are not familiar with and its “analytically pure” solutions. Thus, they are incapable to resolve them quickly. A combination of traditional (analytically fair) approaches with brute force may lead them to the needed outcomes.

The process can be run in the form of dialogue. The students should be asked to describe some features of the solution coming from the problem statement without any deeper mathematical apparatus. In one of the below problems, it is could be a conclusion that the number of legs is likely an even integer.

Depending on the needed number of solutions, there are two general approaches:

- If the goal is finding *a* solution, the process ends with finding the first fitting candidate.
- If the goal is finding *all* solutions, all candidates must be tested.

Three distinct examples are provided. The first one shows a seemingly difficult task (including a 4th degree equation) which becomes much easier after identifying its space of candidates. The next one shows an exhaustive search that does not result in a solution. This negative result may then serve as (a) an inspiration for a formal proof of its non-existence, (b) a starting point for creation of more general (and virtually more complex) tasks. The third one addresses a problem with a rather large space of candidates – a set of two Diophantine equations with three unknowns. All three begin by applying a brute force method and then discuss its nature. Then, depending on the type of problem, we next hypothesize some "theoretically-based" solutions, make some generalizations, and propose similar problems that can be solved using brute force. Their statement contradicts the traditional understanding of Mathematics as “a set of formal manipulations from a notation to numerical results”. A big portion of all examples exploits traditional operations to reach their aims. Together they point to the fact that brute force components of Mathematics are useful even if they do not directly lead to the final solution. With a bit of luck, they indicate to the learners how to get nearer to solutions and to encourage them to make next, more successful trials. Unsolved problems in the end of every section send the message: Do not be scared of seemingly difficult problems!

2. The system of two non-linear equations

The problem solver [10] contains the following task: *The United States of America is the union of 50 states and a federal district. It has populated and unpopulated territories in the Pacific and Atlantic. The difference between the fourth powers of the numbers of populated and unpopulated US territories is 5936, while the difference between their squares is 56. How many unpopulated and populated territories does the US have?*

Let us denote the number of populated territories by p and that of unpopulated ones by q. From the formulation of the problem, it is clear that both p and q are natural numbers with $p > q$. It can be derived during discussions with students. Some students are likely to find their relationships expressed by two equations even if they may feel unable to resolve them:

$$p^4 - q^4 = 5936 \quad (2.1)$$
$$p^2 - q^2 = 56 \quad (2.2)$$

Their formulation can repel everyone untrained to solve 4th degree equations. For the same reason, the pair is an apt example for demonstrating the power of Brute Force. Any proposed solutions to this problem can be easily checked by the instructor, which is an advantageous feature of this problem.

2.1. Space of candidates

The first part of the brute force method involves defining the space of all potential candidates. Initially, we accept all pairs of natural numbers (p, q) with $p > q$ because the number of territories must be a non-negative integer. From the formulation of the problem one can conclude that there is at least one such pair. The initial space of candidates is infinite, i.e. it is impossible to test all the candidates completely. We have to look for a further reduction of the search space.

First, one can increase the lower bound of p. As the equation

$$p^4 - q^4 = 5936 \quad (2.3)$$

holds and q is a natural number, $p^4 > 5936$. The calculation

$$\sqrt[4]{5936} \cong 8.777 \quad (2.4)$$

gives 9 as the nearest greater integer. It implies that the value of p is 9 or more.

The first equation can be then transformed to

$$(p^2 - q^2)(p^2 + q^2) = 5936 \quad (2.5)$$

After substitution

$$56.(p^2 + q^2) = 5936 \quad (2.6)$$

we get

$$(p^2 + q^2) = 106 \quad (2.7)$$

The value of p must be lower than 11 because the square of 11 is 121 and q is a natural number. The space of all potential candidates S is thus reduced to the following set:

$$S = \{(p, q) | (9 \le p \le 10; 1 \le q < p\}.$$

2.2. Exhaustive search

Now, the exhaustive search over S can be applied. Due to the substantial reduction of the size of S, it could be done manually. Table 1 shows the calculation of $p^2 + q^2$ made in a spreadsheet. Only the results for the regular candidates are presented – there is no reason to calculate the sum for $p = 9$ and $q = 9$ because this pair is not a member of the space of candidates.

Table 1 *Exhaustive calculation of all candidates*

		q								
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
p	**9**	82	85	90	97	**106**	117	130	145	
	10	101	104	109	116	125	136	149	164	181

The expected sum 106 corresponds to $p = 9$ and $q = 5$, i.e. the USA have 9 populated and 5 unpopulated territories overseas.

2.3. Discussion

The initial system of two non-linear equation appears difficult so solve. At the same time, using the principles of brute force method, it was reduced to the format which could be even resolved manually using exhaustive search of 19 candidates. In fact, starting with 9 would produce the expected result (i.e. 106) in the fifth calculation. At the same time, no one could be aware of these facts in advance. Thus, the application of spreadsheet has been justified.

Using the manual approach, one might then try to find a solution for $p = 10$. The third value (109) surpasses 106 and indicates its non-existence of any q.

The pair $p = 9$ and $q = 5$ is the only solution of the pair of equations.

Similar problems

- *Let x, y are natural numbers and $3^x - 3^y = 19602$. Find the value $x^y + y^x$.*
- *Let x, y are natural numbers and $2^x - 2^y = 19602$. Find the value x + y.*

3. Divisibility by 3

We now demonstrate a different approach using this next problem as an illustration:
Given three distinct digits A, B, C, where A>B. Find three-digit numbers ABC and BCA for which their difference ABC – BCA is not divisible by 3.

3.1. Space of candidates

The problem formulation suggests the learners to search an appropriate triple. The random choice of three digits soon shows that they are uneasy to find – a more systematic approach is necessary. As *A*, *B* and *C* are digits and we work with three-digit numbers, their limits are clear: *C* can be any digit from 0 to 9; *B* is the first digit of a three-digit number i.e., must be at least 1; *A* is bigger than *B* for it starts from 2. All digits must be different. The candidate space is

$$S = \{(A, B, C) \,\big|\, (0 \leq C \leq 9; 1 \leq B < 9;\ \ 2 \leq A \leq 9;\ A > B;\ A \neq C; B \neq C)\ \}$$

3.2. Exhaustive search

The candidates from the set *S* will be tested in a spreadsheet table; Table 2 shows their distribution: The values of A and B are located in their equally named columns (starting from A3 and B3). Notice the presence only these pairs for which $A > B$. The next columns in the given row correspond to the values of *C*. The inner fields (both empty and crossed) represent the numbers *ABC*.

The table is partially filled in – when its value of *C* equals to *A* or *B*, the particular cell is crossed. These candidates will not be tested because their triple *A*, *B*, and *C* does not belong to the space of candidates *S*. Only the empty cells of Table 2 form the candidate space. Due to its size, the calculation in a spreadsheet makes sense.

Testing the candidates means to evaluate the divisibility of *ABC* – *BCA* by 3 in all non-crossed cells. The values of *A* and *B* are located in their equally named columns (starting from A3 and B3 down). The values of *C* are placed in the second row in the cells C2 to L2.

Into the cell C3 we place the following formula:

=MOD(($A3*100 + $B3*10 + C$2) – ($B3*100 + C$2*10 + $A3); 3) (2.7)

The function MOD(*m*; *n*) calculates the remainder of *m* after its division by *n*. The $ signs guarantee that the formula will be appropriately copied and its content will refer to the corresponding *A*. *B*, and *C*. The formula is then copied into all uncrossed cells. In the end, all of them contain 0. It means that all differences are divisible by 3. The problem which initialized our discussion does not have a solution.

Table 2 *Organization of candidates in the spreadsheet*

		C									
A	B	0	1	2	3	4	5	6	7	8	9
2	1		x	x							
3	1		x		x						
3	2			x	x						
4	1		x			x					
4	2			x		x					
4	3				x	x					
5	1		x				x				
5	2			x			x				
5	3				x		x				
5	4					x	x				
6	1		x					x			
6	2			x				x			
6	3				x			x			
6	4					x		x			
6	5						x	x			
7	1		x						x		
7	2			x					x		
7	3				x				x		
7	4					x			x		
7	5						x		x		
7	6							x	x		
8	1		x							x	
8	2			x						x	
8	3				x					x	
8	4					x				x	
8	5						x			x	
8	6							x		x	
8	7								x	x	
9	1		x								x
9	2			x							x
9	3				x						x
9	4					x					x
9	5						x				x
9	6							x			x
9	7								x		x
9	8									x	x

3.3. Discussion

The negative tests of candidates demonstrate that they – with no exception – are divisible by 3. Now we are aware of the fact. Our next question is: *Is there a deeper reason behind it?* The next consideration allows the teacher to demonstrate a metamorphosis of a fact (no solution) into the concept of proof. In it, we exploit the notation using *A*, *B*, and *C* as digits representing hundreds, tens and units:

$$ABC - BCA = (100*A + 10*B + C) - (100*B + 10*C + A) =$$
$$= 99*A - 90*B - 9*C = 9*(11*A - 10*B - C) \quad (2.8)$$

The difference *ABC – BCA* can always be expressed as a product of two numbers in which one of the factors is 9. Thus, not only is it divisible by 3, it is divisible by 9, too. This piece of knowledge is used in the first of next problems.

3.4. Similar problems

- *Given three digits A, B, C, where A>B. Find a three-digit numbers ABC and BCA for which their difference ABC – BCA is not divisible by 9.*
- *ABC and BCA are 3 digit numbers. Find a three-digit numbers ABC and BCA for which their difference ABC-CBA is not divisible by eleven.*

4. Two equations with three unknowns

Let us solve the problem: *In Fairy Tale Kingdom, the king breeds 40-leg centipedes and 3-head multi-leg dragons. In total, they have got 638 legs and 53 heads. How many legs do the dragons have?*

4.1. Space of candidates

Presuming that every centipede has one head and all dragons have the same number of legs, there are three unknowns: the number of centipedes (*C*), the number of dragons (*D*), and the number of dragon's legs (*X*). The relations can be expressed by the following equations:

$$C*40 + D*X = 638 \quad (2.9)$$
$$C*1 + D*3 = 53 \quad (2.10)$$

The equation (2.10) can be expressed as

$$D*3 = 53 - C. \quad (2.11)$$

The number of centipedes is at least 1, one can derive that the maximum number of dragons is restricted by the formula

$$D \leq \frac{52}{3} \quad (2.12)$$

Its result is 17.333… The king has got no more than 17 dragons.

The total of legs is 638. If there would only be centipedes, there maximum would be

$$D \leq \frac{638}{40} \quad (2.13)$$

Its result is $D \leq 15.95$. The number of centipedes is at most 15. It limits the space of centipede and dragon candidates to

$$S = \{(C, D) | 1 \leq C \leq 15; 1 \leq D \leq 17\} \quad (2.14)$$

The equation (2) can also be transformed to

$$C = 53 - D*3 \quad (2.15)$$

The upper bound of dragon's leg's number can be calculated from (1) assuming the presence of 1 centipede and of 1 dragon. It implies that

$$C*40 + D*X = 638 \quad (2.16)$$
$$40 + X = 638 \quad (2.17)$$

Thus, the upper limit is 598 legs.

4.2. Exhaustive search

The number of centipedes can be calculated using the number of dragons. Let us substitute the formula (3) into (1). We get an equation with two unknowns:

$$(53 - D*3)*40 + D*X = 638 \quad (2.18)$$

$$2120 - 120*D + D*X = 638 \quad (2.19)$$

$$D*(X\text{-}120) = -1482 \quad (2.20)$$

The evaluation will be done by a mass searching for the values of *D* and *X* that satisfy the formula (4).

Again, the execution will be done using spreadsheet calculations. In addition, as legs are paired organs, their minimum is two ($X \geq 2$) and one can consider even numbers only.

The leftmost column will contain the full range of *D* from 1 to 17. The first row contains the number of legs. As this number can potentially be quite high – the upper bound is 398, we start with small even integers (2 to 10). If there will be no solution, the formula can be copied to the next columns.

The calculation is shown in Table 3.

Table 3 *Searching for the numbers of dragons and legs*

	Dragon's legs (*X*)				
Dragons (*D*)	**2**	**4**	**6**	**8**	**10**
1	-118	-116	-114	-112	-110
2	-236	-232	-228	-224	-220
3	-354	-348	-342	-336	-330
4	-472	-464	-456	-448	-440
5	-590	-580	-570	-560	-550
6	-708	-696	-684	-672	-660
7	-826	-812	-798	-784	-770
8	-944	-928	-912	-896	-880
9	-1062	-1044	-1026	-1008	-990
10	-1180	-1160	-1140	-1120	-1100
11	-1298	-1276	-1254	-1232	-1210
12	-1416	-1392	-1368	-1344	-1320
13	-1534	-1508	**-1482**	-1456	-1430
14	-1652	-1624	-1596	-1568	-1540
15	-1770	-1740	-1710	-1680	-1650
16	-1888	-1856	-1824	-1792	-1760
17	-2006	-1972	-1938	-1904	-1870

The highlighted cell indicates that the king breeds 13 dragons with 3 heads and 6 legs. From (3) we get the number of centipedes

$$C = 53 - 13*3 = 14 \quad (2.21)$$

4.3. Discussion

The above way to solution can be also used a problem generator. Table 4 contains the solutions of the problem for the given number of dragons, centipedes, and the constant number of their heads (53). The inner values contain the total of their legs for dragons with 2, 4, 6, 8, and 10 legs, respectively. The row referring to 13 dragons and 14 centipedes contains the value –1482, i.e. the one corresponding to dragons with 6 legs.

The base problem would looks like:

The king of Fairy Tale Kingdom breeds 40-leg centipedes and 3-head multi-leg dragons. In total, they have N legs and 53 heads. How many legs the dragons have?

To generate a particular task, one can consider any value from the inner section of Table 4. Every selection of *N* leads to a new problem. For example, selecting 1336 as the total of legs forms the task with 32 centipedes and 7 eight-leg dragons.

Table 4 *The problem generator for the dragons and centipedes problem*

		Dragon's legs (X)				
Dragons (*D*)	Centipedes (*C*)	**2**	**4**	**6**	**8**	**10**
1	50	2002	2004	2006	2008	2010
2	47	1884	1888	1892	1896	1900
3	44	1766	1772	1778	1784	1790
4	41	1648	1656	1664	1672	1680
5	38	1530	1540	1550	1560	1570
6	35	1412	1424	1436	1448	1460
7	32	1294	1308	1322	1336	1350
8	29	1176	1192	1208	1224	1240
9	26	1058	1076	1094	1112	1130
10	23	940	960	980	1000	1020
11	20	822	844	866	888	910
12	17	704	728	752	776	800
13	14	586	612	**638**	664	690
14	11	468	496	524	552	580
15	8	350	380	410	440	470
16	5	232	264	296	328	360
17	2	114	148	182	216	250

4.4. Similar problems

- *In Fairy Tale Kingdom, the king breeds 40-leg centipedes and 3-head multi-leg dragons. In total, they have 1112 legs and 53 heads. How many legs the dragons have?*
- *A frog named Vilma eats 5 spiders a day. When Vilma is very hungry, it eats 10 of them. During 10 last days, it ate 60 spiders. How many days was Vilma very hungry?*
- *On Mars, there are Triads that have 3 eyes and Pentiads that have 5 eyes. 26 aliens having 86 eyes live in a Martian village. How many of them are Pentiads?*

5. Conclusion

When Mathematics is taught properly, it offers tools which help its learners to solve a larger scale of problems than just those in their textbooks. Authentic problems present real challenge to students who may not have enough mathematical weapons, i.e., just those single-purposed procedures to solve textbook exercises. The students should be trained to test their courage by solving problems crossing the frontiers of their knowledge. As we showed, the Brute Force is a multipurpose tool for struggling new problems.

To elucidate its principles to learners, we exemplified its application to solving difficult problems. Their solutions have the two following two characteristics:

- The formal mathematics-based considerations helped reduce the number of candidates. They demonstrated to the learners that their knowledge of traditional mathematics supports progress in tackling the task, even if they cannot resolve it completely.
- The testing stage effectively used the computational power of information technology. In the problem of centipedes and dragons the presumed number of candidates was overwhelming. We decided to test first 288 of them – and succeeded. At the same time, in the same spreadsheet, the test could be easily extended. The implementation into a spreadsheet led to substantial time savings – not mentioning possible human errors arising from tedious computations.

Information technology simplifies using Brute Force in solving mathematical problems. It allows to test larger numbers of candidates i.e. it makes the verification more extensive than anyone could do in the past. There are other ways of automated testing in addition to spreadsheet calculations, e.g. writing corresponding computer programs. This approach was for example exploited in solving four-color problem, one of the most popular problem of Graph theory [11]. It was done in collaboration between two mathematicians and a programmer.

It shows that some complex Brute Force problems can be solved as team projects, possibly online [12]. In the Division by 3 problem, the candidate space could be cut to portions which would be tested manually by team members.

The Brute Force method also facilitates better comprehension of selected parts of Mathematics for example inequalities or Diophantine equations. As such, it has also its propaedeutic value and can be exploited in introductory courses as a gate into more specific analytical solutions.

There are many other ways of using IT for solving difficult problems by exploiting appropriate software. (See for example [13], [14] and [15]). In all cases, their application requires not only exact design and development but also an evaluation of the outcome(s).

Let us exemplify it on the US Territories problem. Figure 1 shows the functions corresponding the pair of equations (2.1) and (2.2) in Geogebra. There are four intersections – our potential solutions. Not all of them reflect reality. An aftermath discussion with learners is needed to exclude inappropriate pairs of solutions containing negative numbers. In many ways this discussion will resemble the candidate space' reduction.

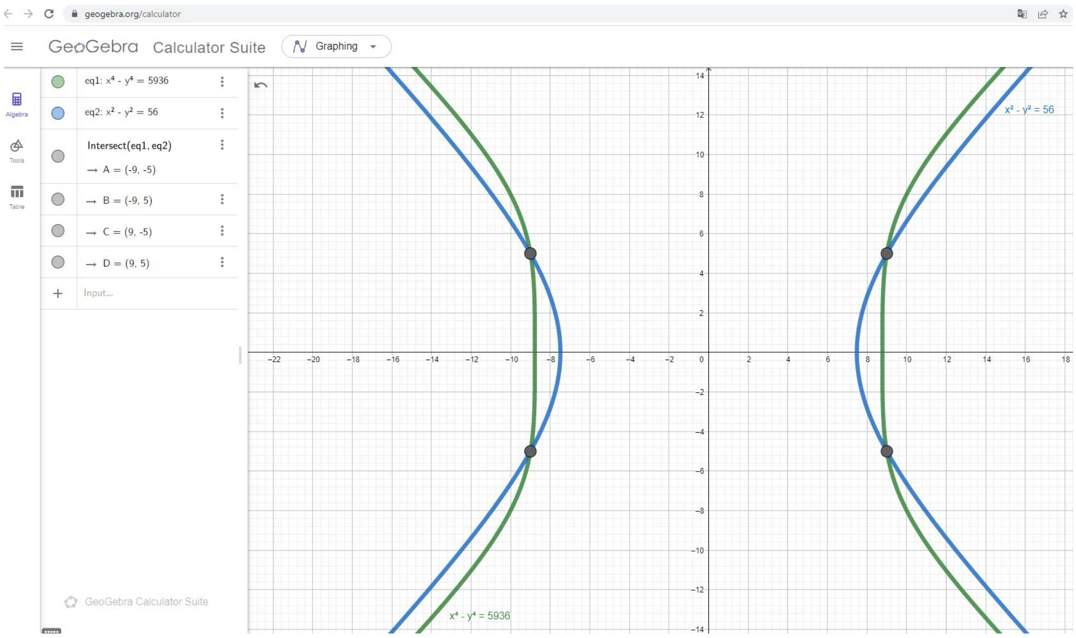

Figure 1 Solutions in Geogebra

As we see, there are many interconnections between seemingly distant concepts of problem solving. To teach Mathematics in its full variety, one has to become skilled in disclosing them. Then, the Brute Force method becomes a glue of different approaches with trials and errors leading to desired result.

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