Research on Sangaku and the use of ICT

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Abstract

The author proposes geometry teaching materials based on Sangaku, the mathematical culture of the Edo period, together with utilization of information and communications technology (ICT). As described in this paper, the author proposes the use of computer algebra systems (CAS) and dynamic geometry software (DGS) as ICT supporting mathematics education in junior and senior high schools, and reports the use of CAS for algebraic equations expressed by algebraic equations and the use of DGS for drawing diagrams of Sangaku problems.

1 Introduction

Mathematics teachers must study the history of mathematics. The author considers that understanding the process by which modern mathematics was established is an important skill for teachers when they teach. Fukagawa and Tony Rothman introduce Sangaku in their book *"Sacred Mathematics: Japanese Temple Geometry"*, which presents various contents of Sangaku by people of the time (see [1]). Shimodaira makes a similar reference in *"The Survey of Studying the History of Japanese Mathematics"* (see [8]). Nevertheless, not many universities instruct students in the history of mathematics. There are insufficient teaching materials for the subject, which might be why one product of the history of mathematics is still used in Japanese classrooms.

However, because of the modernization of mathematics education, the contents of elementary geometry in the mathematics curricula of Japanese junior and senior high school have decreased in quality and quantity. These trends have affected not only the geometry problems of today's high school students but also the students' ability to visualize the structure of mathematics. These difficulties also apply to young mathematics teachers and students. Urgent improvements are needed.

The mathematics which developed uniquely during the Edo period is today called Wasan, or old Japanese mathematics. The mathematics developed after Wasan, Western-style arithmetic, presents a contrast to Japanese-style arithmetic. The difference is that an abacus is used mainly for Wasan, whereas written arithmetic is used mainly for Western arithmetic (see [2]).

Based on a collection of Sangaku problems published in Japan, the author started classifying Sangaku problems in 2021. This paper presents some results of the Sangaku study.

2 Usages of Sangaku in Mathematics Education

Sangaku is a custom that arose during the Edo period (1603–1867) to make many people aware of the knowledge gained from the study of mathematics and aware of the creation of mathematical problems. The culture of dedicating mathematical problems such as Ema (votive tablets) to shrines and temples is unique to Japan. Wasan researchers believe that the people of those times attributed the practice to the piety of the Japanese people, who believed that the gods and Buddha had made mathematical research and problems possible.

Along with development of Sangaku as a mathematics teaching material for junior and senior high school students, the author has conducted mathematics classes using Sangaku (see [3]). The main developments are presented below.

Using Sangaku in Educational Activities

(1) Activities to solve Sangaku problems

(2) Activities focusing on Sangaku's technique sentences

(3) Activities to find new mathematics from Sangaku problems

(4) Activities to draw mathematically correct diagrams of Sangaku problems

(5) Activities to create mathematical problems following Sangaku

(6) Activities focusing on the relationship with local culture

The author has found that students become more interested in mathematics when he has used the history and culture of mathematics into account in mathematics classes. As students began to show interest in the history and culture of mathematics, it became clear that such content could significantly affect mathematics education.

Sangaku has many problems that can be handled in junior and senior high school mathematics classes. The author has been using some of these problems in his classes. Students have received them well, and the author believes that if Sangaku's problems were classified and organized, and given to students, they would make excellent teaching materials.

As is the general trend, the author was initially interested in solving Sangaku's problems (see (1) above). This interest led to an interest in grasping the concepts of (2) technique sentences and (3) new mathematics. Concerning problem-making in (5) above, some mathematics teachers conduct this problem-making activity in their classes (see [3]).

Currently, the author is interested in considering the drawing of the figure given in the problem, based on Sangaku's problem-solving methods and the creation of problems related to Sangaku (see [4]).

The author is currently interested in (4) how to draw a diagram of Sangaku's problem, which is this paper's central theme. He will introduce concrete Sangaku in Chapters 3 and 4.

2.1 Composition of Sangaku

In general, Sangaku consists of a problem statement, an answer, a technique sentence, and a diagram. Figure 2.1 below shows a Sangaku from Io shrine in Osaka Prefecture, Japan, in which nine problems are depicted on a single Ema (see [9]; [10]).

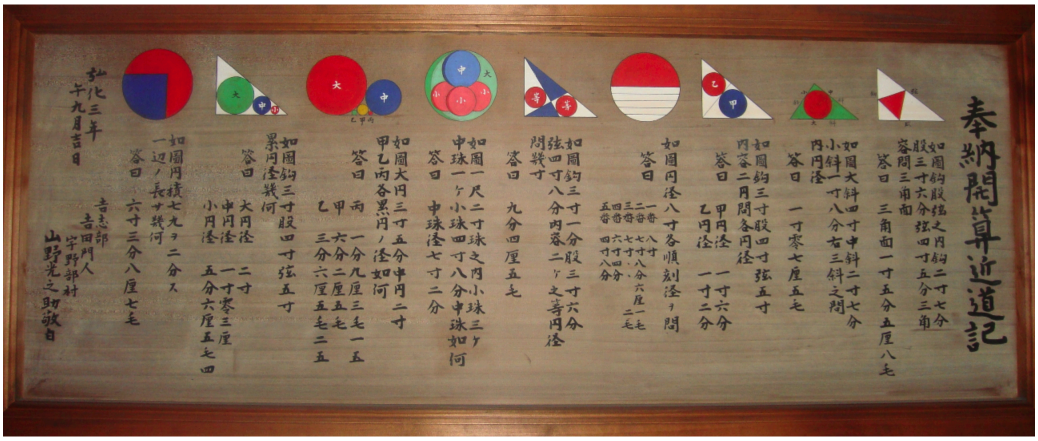


Figure 2.1 Restored Sangaku of Io shrine, Osaka, Photo: Wasan no Yakata (see [10])

The author introduces a composition of Sangaku which is third from the right in Figure 2.1.

Problem: A right triangle has three sides, 3, 4, and 5 (sun)[1], as shown in Figure 2.2.

Find the figure's diameters of the Kou circle L and the Otsu circle S.

Answer: the diameter of Kou circle L is 1.6 (sun).

the diameter of Otsu circle S is 1.2 (sun).

Technique: This Sangaku has no technique sentence.

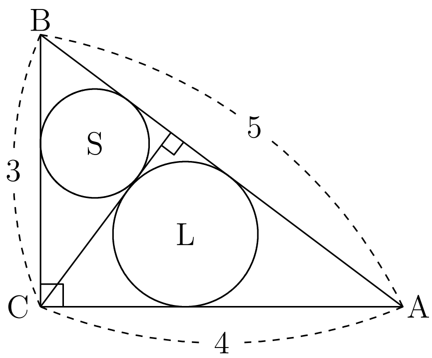


Figure 2.2 Sangaku of Io shrine, Osaka

Note1: Generally, Sangaku has a technique sentence. Sometimes, Sangaku has no technique sentence, as in this Sangaku.

Note2: Kou and Otsu represent the order. For this Sangaku problem, we can consider Kou and Otsu circles as Large and Small circles, respectively.

[1] "Sun" is the Edo period unit of length, 1 (sun) ≒ 3.03 (cm)

3 Geometry Construction using DGS

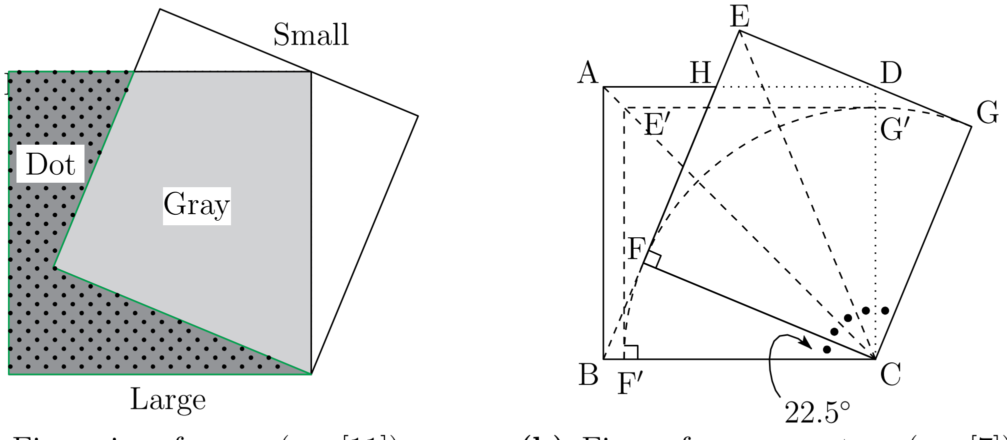
Around 1990, dynamic geometry software (DGS) came into wider use. The first software was *Cabri*, developed by Laborde and his colleagues at the University of Grenoble, France, in 1988. When the author worked in the field of geometry, he actively used DGS. Several other DGS products became available. The author has used and currently uses various DGS frequently: *Cinderella* and *KETCindy* (see [5]). The author is also familiar with *Cinderella* and finds *CindyScript*, which comes with *Cinderella*, to be smooth in terms of using mathematics. High compatibility with LaTeX improves the quality of handouts given to students.

Because Sangaku was written on wooden boards about 350 years ago, many characters and figures are difficult to read. In addition, some figures in question are not correct. For this reason, the author asks students to reproduce the diagram of the problem to ascertain whether the problem is valid, or not. By drawing the diagram of the problem, drawing can play a part in the utilization of mathematics. As one example, how can a diagram like that in Figure 3.1a be constructed? These are the activities described in (4) of Chapter 2.

This chapter presents consideration of the Sangaku problem using a drawing by *Cinderella* and *KETCindy*.

3.1 Problem: Find the common area of two pieces of origami

The author introduces the problem of Sangaku from Kibitsuhiko Shrine in Okayama prefecture (see [11]). This problem is not a matter of folding origami paper. It is a problem to find the common area of two origami papers (one large and one small) when we rotate and stack them around a vertex as shown in Figure 3.1a. After examining the solution, the author thought the conditions might be insufficient. Therefore, 22.5° was added to the text of the Sangaku problem in Figure 3.1b (see [7]). The author chose the following problem.



(a) Figure in reference (see [11]) **(b)** Figure for commentary (see [7])

Figure 3.1 Sangaku of Kibitsuhiko Shrine, Okayama

Problem: As Figure 3.1a shows, two squares of origami are intersected: one large and one small. Insert the areas of the dark dotted area named "Dot" and Gray portions. If the area of Dot is 50, then find the area of Gray interest. However, let a small origami be rotated 22.5° clockwise at a lower right vertex of a large origami.

Answer: The area of Gray is 80 and the bottom end.

Solution: As shown in Figure 3.1b, intersect two squares origami, letting one large and one small be a square ABCD and a square EFCG. Furthermore, let it be $\angle\mathrm{FCB} = 22.5°$. Let x and y respectively denote the sides of large and small origami pieces.

Letting S_1 be an area of Pentagon ABCFH, then the following equation holds.

$$S_1 = \frac{5\sqrt{2}-4}{8}x^2 \tag{3.1}$$

Letting S be an area of Quadrilateral EFCG, then the following equation holds.

$$S = \frac{17-11\sqrt{2}}{2}y^2 \tag{3.2}$$

Because $\triangle\mathrm{CFB} = \triangle\mathrm{CGD}$,

$$\frac{\sqrt{2}-1}{2}y^2 = \frac{\sqrt{2}}{8}x^2 \tag{3.3}$$

$$y^2 = \frac{18+14\sqrt{2}}{17}S_1 \tag{3.4}$$

$$S = \frac{\sqrt{800}-1}{17}S_1 \tag{3.5}$$

Because $S_1 = 50$,

$$S = \frac{\sqrt{800}-1}{17} \times 50 = 80.2478\cdots \tag{3.6}$$

3.2 Consideration of three students' drawings

In Figure 3.1b of this problem, small origami E′F′CG′ is placed on the large origami ABCD so that the vertices C mutually overlap. The vertex D is on edge GE. Then, given area S_1 of pentagon ABCFH, the problem is to ascertain area S of quadrilateral HFCD (see [7]).

This section presents the procedure for drawing the diagram in Figure 3.1b, as considered by the three students: A, B, and C.

Student A: She will consider using the condition, $\angle\mathrm{GCD} = \angle\mathrm{FCB} = 22.5°$. Therefore, she will get $\angle\mathrm{ECD} = \angle\mathrm{ACF} = \angle\mathrm{ECA} = 22.5°$. Furthermore, she can draw a small origami EFCG from $\angle\mathrm{CFE} = 90°$.

From the procedure described above, square EFCG can be constructed.

Student B: He will consider that $\triangle\mathrm{CGD} \equiv \triangle\mathrm{CFB}$, with side EG passing through vertex D. Next, he will draw a circle of radius CF (dashed arc) centered at C. Then the intersection with the bisector of the angle ACB is the point F to be sought. Therefore, line EF passes through vertex B.

From the procedure described above, square EFCG can be constructed.

Student C: He will consider the following: vertex F of the small square is the intersection of the arc with center at C and radius CF′ and the bisector of ∠ACB. Then, because EF and CF are perpendicular, EF is a part of the tangent line of circle CF at F. Furthermore, vertex E is the intersection of the perpendicular lines from D to line EF. Finally, vertex G intersects vertical lines from C to line EG.

From the procedure described above, square EFCG can be constructed.

3.3 Consideration

This Sangaku is about finding the area of a region. The author demonstrates that it is useful to create a diagram of the problem, i.e., a geometric construction.

In general, the author uses various mathematical contents in his drawings. For this reason, the author considers that drawing is a representative example of the use of mathematics in mathematics and has a fundamentally important role in inquiry-based learning. Development of such examples is a meaningful activity (see [4]).

4 Geometry Construction using CAS and DGS

As explained in this chapter, the author solves the problem of Sangaku at Hayachine Shrine in Iwate prefecture (see [12]). Then the author introduces an example of drawing a given figure. Nevertheless, it is a very difficult problem to solve using written arithmetic. The content of junior and senior high school classes is too difficult for this problem. Therefore, we chose to use ICT to solve the arithmetic problem.

Problem: As shown in Figure 4.1, an equilateral triangle includes one large circle, two medium circles, and two small circles inside.

Find the respective diameters of the large, medium, and small circles when the sides of the equilateral triangle are 3.

Answer: The large circle diameter is $1.44910\cdots$,

the middle circle diameter is $0.78413\cdots$,

and the small circle diameter is $0.451643\cdots$.

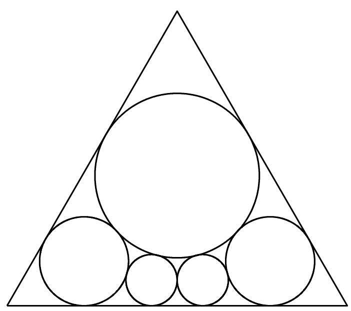


Figure 4.1 Sangaku of Hayachine Shrine, Iwate

Solution: Let x, y, and z respectively denote the radii of the large circle, the medium circle, and the small circle inside equilateral triangle ABC with side 3, as shown in Figure 4.2. Let x, y, and z respectively denote the radii of the great circle, the medium circle, and the small circle inside equilateral triangle ABC with side 3, as shown in Figure 4.2. In addition, let D be the midpoint of BC.

For line BD, because BC = 2BD, the following holds.

$$2(\sqrt{3}y + 2\sqrt{yz} + z) = 3 \tag{4.1}$$

Similarly, for side AB, the following holds:

$$\sqrt{3}(x+y)+2\sqrt{xy} = 3 \tag{4.2}$$

Furthermore, for line segment AD,

$$2x + \sqrt{x^2 + 2xz} + z = \frac{3\sqrt{3}}{2} \tag{4.3}$$

The simultaneous equations can be solved using (4.1), (4.2), and (4.3), as

$$\begin{cases} 2(\sqrt{3}y + 2\sqrt{yz} + z) & = 3 \\ \sqrt{3}(x + y) + 2\sqrt{xy} & = 3 \\ 2x + \sqrt{x^2 + 2xz} + z & = \dfrac{3\sqrt{3}}{2} \end{cases} \tag{4.4}$$

It is difficult to solve simultaneous equations (4.4). Therefore, the author considered computer algebra systems (CAS).

Here, *Mathematica* was used as the CAS, which yielded the following output.

$$\begin{cases} x & = \dfrac{3}{104}\left(-27 + 23\sqrt{3} + \sqrt{-12 + 94\sqrt{3}}\right) \\ y & = \dfrac{1}{104}\left(63 + 33\sqrt{3} - 3\sqrt{2(-6 + 47\sqrt{3})} - 2\sqrt{6(-6 + 47\sqrt{3})}\right) \\ z & = \dfrac{3}{104}\left(45 + 5\sqrt{3} - 2\sqrt{2(-6 + 47\sqrt{3})} - \sqrt{6(-6 + 47\sqrt{3})}\right) \end{cases}$$

It is noteworthy that the approximate values of x, y, and z are, respectively, $x \fallingdotseq 0.724551$, $y \fallingdotseq 0.392065$, and $z \fallingdotseq 0.225822$.

4.1 Geometric construction with the loci of quadratic curves

It is possible to plot the centers of large circle P, medium circle Q, and small circle R from the properties of quadratic curves.

Here, radius z of circle R is obtained from the *Mathematica* calculation results. The positions of the centers of circles P and Q were determined as in Step 2 and Step 3, using parabolic and hyperbolic trajectories, using only the values of z.

The specific procedure is described below.

Step 1. As Figure 4.2 shows, we draw the bisector a of angle A and let D be its intersection with BC. Point D is the midpoint of BC. Center R of the small circle lies on the bisector d of angle ADB. From the value of the radius z of the circle R, we can also draw a point E on BC. And, RE $\perp$ BC, RE $=$ ED $= z$, we can also draw a point R on the bisector d of angle ADB.

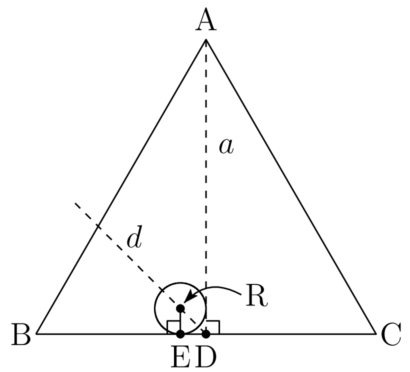


Figure 4.2 Center point R of small circle lies on bisector d of angle ADB

Step 2. As shown in Figure 4.3, center Q of the medium circle lies on the bisector b of angle B. Furthermore, Q lies on the parabola p with R as its focus. However, line ℓ is its directrix, passing through F. Line *ell* is parallel to BC and satisfies EF $=$ RE.

Consequently, Q is the intersection of bisector b of angle B and parabola p.

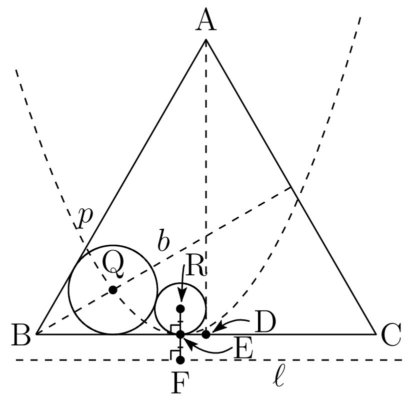


Figure 4.3 Center point Q of medium circle lies on bisector b of angle B and parabola p

Step 3. As shown in Figure 4.4, center P of the large circle lies on the bisector a of angle A. A point P also lies on the hyperbola h with foci Q and R. However, the difference in distance from the two foci Q and R is the difference in radii of the circles Q and R. Therefore, point P is the intersection of bisector a of angle A and hyperbola h.

From the discussion presented above, circle P is tangent to circles Q and R, respectively, at points S and T.

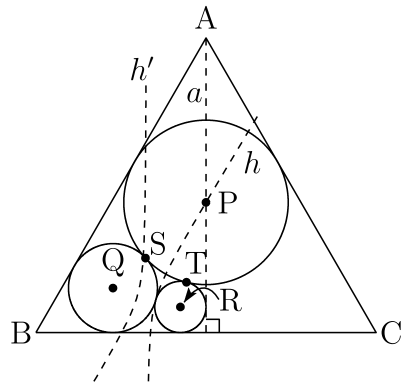


Figure 4.4 Center point P of large circle lies on bisector a of angle A and hyperbola h

Step 4. Similarly, we can draw the medium circle and the small circle on the right side of the figure. Therefore, we can construct the diagram presented in Figure 4.1.

4.2 Consideration

The results presented above show that square roots and four arithmetic operations can express the radius values of x, y, and z. We can construct this diagram using a ruler and compass (see [6]). In this case, one can consider the range of a drawing by a ruler and compass. Such activities can be expected to engender the inquiry-based learning promoted in Japanese secondary education. Furthermore, the author proposed that a large, medium, and small circle can be constructed by adding the loci of a parabola and a hyperbola to the drawing lines obtained using a ruler and compass. The author designates this method as "the Quadratic Curves Addition Method" (see [4]). Quadratic curves cannot be drawn with a ruler and compass. Therefore, the author considered description of the locus of the quadratic curve in the script of *Cinderella* (*CindyScript*). However, instead of expressing the quadratic curve as a mathematical expression, the author attempted to clarify it in elementary geometry. As shown in Figure 4.2, if it is a parabola, then it finds the focal point and the directrix. If it is a hyperbola, as in Figure 4.3, then it is based on detection of the difference between the two foci and their distance.

However, some students wonder if the point on hyperbola h' cannot be the circle they seek. Therefore, they drew a circle centered on point P′ of the hyperbola h'. Figure 4.5 shows that they can draw a circle tangent to the circles Q and R at points S′ and T′. Through this experience, it is important for students to learn how the circle P touches the circles Q and R.

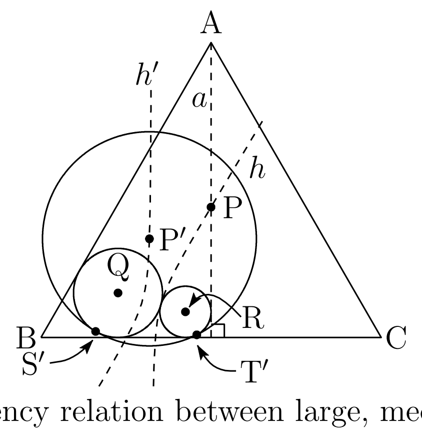


Figure 4.5 Another tangency relation between large, medium, and small circles; Center point P′ lies on the other hyperbola h'

The author used *Cinderella* and *KETCindy* to produce this drawing. One cannot draw the locus of a quadratic curve using a ruler and compass.

Additionally, students might do so by trial and error when they perform complex drawings. It can be good pedagogically for students to make repeated mistakes. However, some students might lose heart because they cannot deduce the correct answer. In such cases, it is educationally meaningful to use ICT for drawing. The author believes that identifying the focal point and directrix, which are characteristics of this quadratic curve, as the figure shows, has educational importance and value from the viewpoint of mathematical utilization. The author will continue to develop teaching materials so that drawing by the quadratic curve addition method can be a teaching material for inquiry-based learning.

5 Conclusion

This report describes a case study using CAS and DGS to solve a problem of mathematics education at junior and senior high school, using Sangaku as the subject matter. Further inquiry-based learning is necessary for Japanese school education. Therefore, it is important to conduct research to explore the use of technology, including the utilization of CAS and DGS.

Chapter 3 introduced one aspect of geometric construction with DGS. Compared to a ruler and compass, the DGS has superior reproducibility. The DGS function can help realize the mathematics' essence. In this sense, the possibility exists that a new mathematics education will open up by placing drawing at the center of mathematics utilization.

In Chapter 4, the author first obtained the values of the radii of the large, medium, and small circles necessary for the plotting using CAS (*Mathematica*) for the set of simultaneous equations. Next, the author found the centers of the small circles using the Script function of DGS (*Cinderella*). Then, the center of the medium circle was ascertained using the quadratic addition method (parabola).

Finally, the author finds the center of the large circle by the quadratic addition method (hyperbola). The centers of the remaining small and medium circles can also be found using the quadratic addition method. The quadratic addition method plays a fundamentally important role in inquiry-based classes. The author would like to continue this avenue of research.

The author is on a mission to train teachers of mathematics at universities. When interviewing students who want to become teachers, he finds that not many have experience using ICT during their high school years. This might be true because high school classes emphasize the solution of university entrance exam questions rather than deepening and exploring the content of mathematics. For this reason, developing teaching materials using Sangaku as subject matter is important.

As a future challenge, we would like to develop drawing materials using the quadratic curve addition method from Wasan and Sangaku.

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References

[1] Fukagawa, H., and Tony, Rothman. (2010). *Sacred Mathematics: Japanese Temple Geometry*, Morikita Publishing.

[2] Kuroda, Y. (2010). *Elementary Arithmetic Education Methods*, Minerva Shobo.

[3] Makishita, H. (2011). *An Empirical Study on the Problem Formulation and Students' Emotional Aspects in Mathematics*, The Bulletin of the Graduate School of Education of Waseda University, Separate 19(1), pp. 263-274.

[4] Makishita, H. (2019). *The problem of Apollonius which was solved by Geometric Construction and Loci of Quadratic Curves*, Transactions of Mathematical Education for Colleges and Universities 25(1), pp. 65-86.

[5] *Samples of KeTCindy*, https://s-takato.github.io/ketcindysample/. 15 Sep. 2022.

[6] Seyama, S. (2014). *Geometry of Compass and Ruler*, Kyoritsu Shuppan.

[7] Shimizu, Y. (1942). *Collection of Sangaku Works Dedicated to Temples and Shrines*, Tohoku University Library.

Source: https://www.i-repository.net/contents/tohoku/wasan/l/q049/14/q049140069l.png. 15 Sep. 2022.

[8] Shimodaira, K. (1988). *The Survey of Studying the History of Japanese Mathematics*, Journal of History of Science, Japan Vol. 27, No. 166, pp. 65-74.

[9] The Kinki Society for the History of Mathematics. (1992). *Sangaku in the Kinki Region*, Osaka Kyoiku tosho.

[10] Wasan no Yakata. *Sangaku of Io shrine*, Source: http://www.wasan.jp/osaka/iyo.html. 12 Sep. 2022.

[11] Yamakawa, Y. (1997). *Sangaku in Okayama prefecture*, self-publication.

[12] Yasutomi, Y. (1987). *Wasan: All of Iwate's existing Sangaku*, Seijisha.