Using virtual reality to teach linear algebra with a focus on affine geometry

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Abstract

We present an innovative experience of the use of virtual reality in a first undergraduate course of mathematics at a Spanish university carried out during the last 3 academic years. This complements the theoretical and practical classes of the subject of linear algebra with notions of affine geometry that will be used in later courses.

1 Introduction

As remarked by Dorier [3], teachers often feel frustrated and disarmed when faced with the inability of their students to cope with ideas that they consider to be so simple. Stewart et al. [11] recommend to take advantage of technology in teaching, motivating concepts with applications.

In this article, we describe a series of activities produced in the video game NeoTrie VR (shortly Neotrie) [1] during the past academic years, which aimed to complement a standard 1st course of linear algebra ([6]).

Video games are a good starting point to get students interested in the subject. For instance, in showing the interface of Unity[2], we can see the (linear) transformation of each object in the scene determining its position, its rotation, and its scale (see Figure 1). Some parts of the codes in C# implemented in Neotrie were also shown to students for didactic and motivating purposes (see a sample at Section 7).

These activities are listed in the website of Neotrie [3] in the same order as they were given to the students the academic course 2021-22. But they came about as a function of several software additions in recent years, as well as the need to innovate in the teaching and learning of linear algebra.

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[1]Neotrie is a virtual reality software developed by the spin-off Virtual Dor of the University of Almería. It is programmed in the Unity videogame engine, under the C# language.

[2]`https://unity.com`

[3]`https://www2.ual.es/neotrie/project/geometria-elemental/`

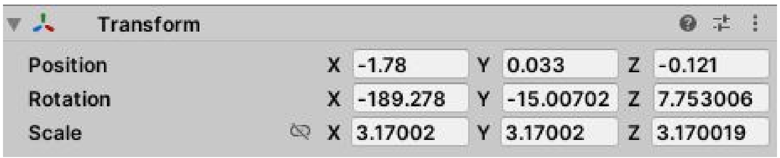

Figure 1: Transform of an object in Unity, with respect to a parent object: position, rotation (around the Z axis, the X axis, and the Y axis, in that order), and scale.

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1. Course 2019-20: Students were introduced in the basics of Neotrie and made in groups the activity described in Section 4. For this, the teacher used an Oculus Rift-S[4] headset in the classroom (see Figure 2), and 50 students volunteers used 4 headsets Lenovo (based on Windows Mixed Reality [5]), each one with its corresponding gaming computer. A graphing

Figure 2: Teacher using Neotrie VR in the classroom in 2019.

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calculator was then developed by Sanchez [10], which allow us to introduce planes from its parametric equations; see Section 3.

2. Course 2020-21: Forced by the COVID-19 pandemic, stereoscopic videos were implemented in the software, and students could see them with cheap VR glasses for mobile phones, also in live video conferences from their homes. These videos are described in Sections 2 and 5.

3. Course 2021-22: Both the teacher and 68 students used 3 devices Meta Quest ([8]) with no gaming computer required, and further perform the activity described in Section 6.

Functions in the software necessary to carry out the activities were developed by Cangas and Rodríguez (see e.g. [2]), as they were essential to produce and use them in the classroom. These developments were also used to design materials for an algebraic topology course, and for distance learning during the COVID-19 pandemic [7].

[4] This device is no longer for sale, and it has been replaced by Meta Quest available at https://store.facebook.com/es/quest/

[5] https://www.microsoft.com/en-us/mixed-reality/windows-mixed-reality

A more detailed study of the actual impact on students through this innovative intervention is still in progress.

2 Planes in STL format and passing through vertices

The study of the relative positions of 3 planes, given by a system of 3 equations, is a standard exercise for high school students, but many of them do not understand the correspondence between the geometric and algebraic counterparts. Most simply follow an algorithm given by the teacher, after solving the system of equations. In general, among the first year mathematics students, a lack of spatial visualization was detected (cf. [1]), which we tried to remediate with the use of virtual reality.

As other STL objects (Standard Triangle Language) [6], planes are inserted in the scene by naming "plane" out loud, thanks to an implemented speech recognition system. They can also be inserted from a virtual file window from our local computer or headset[7]. Objects in Neotrie can be grabbed as if they were real (changing their positions, rotating and scaling, as controlled by the hand). Both the objects and their manipulation require standard products of matrix transformations, which can be used to motivate students in the subject.

In this way, as we can see in the video `https://youtu.be/YbfATKVFE78` (see Figure 3) the teacher can manipulate 3 planes and place them in several positions, while explaining in real time, through a video live class given by videoconferencing. At this point, students know from high school how to solve a system of linear equations in $\mathbb{R}^3$, with interpretation in terms of number of solutions; the rank of a system of vectors given by the matrix of its coordinates and the Rouché-Frobenious theorem. As we have said earlier, the students have not fully grasped the connection between the algebraic and geometric aspects.

During the VR explanation, one can use the 3D chalk to draw in the space, mark the normal vector on each plane. The normal vector corresponds to the coefficients, a, b, c, of the equation of the plane $ax + by + cz = d$. One can draw with the 3D chalk the intersection line of two planes and the intersection point of the 3 planes, if it exists. The drawings can also be grabbed and saved as a STL object in a Neotrie file (this is an ASCII file storing information of all dynamic elements of the scene).

In the video edition, we have included textual indications on the Rouché-Frobenius theorem, which relate the solutions of systems of equations to the ranks of the corresponding matrices involved. What is really important is that students develop a good understanding of the role of the normal vector of the planes to see the geometric-algebraic connection of the solutions of the system.

If we want more precision in the positions of the planes, we will use the hand construction actions in Neotrie, which allow us to build vertices, edges and faces. These elements are dynamic and we can place them wherever we want. In addition, we can use the intersection tool to find the exact intersections of each plane position.

The labeling tool allows adding dynamic texts in the scene and associated to each geometrical element: coordinates of points, directions of vectors in a line, normal vector on a face, as

[6] The format STL stores the information through triangular tessellations of a figure. It was created by The Albert Consulting Group for 3D Systems in the 1980s to transmit information from 3D CAD (Computer Aided Design) models to early 3D printers, and that has remained to this day.

[7] Tools and functions are listed in the guide available at `https://www2.ual.es/neotrie/guide/`

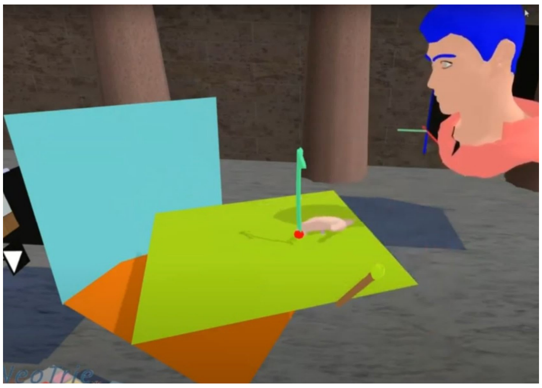

Figure 3: Capture of a video manipulating 3 planes for intersecting. Drawing the normal vector on one of the planes.

well as its equation. These are considered with respect to a 3-axis system as shown in Figure 4.

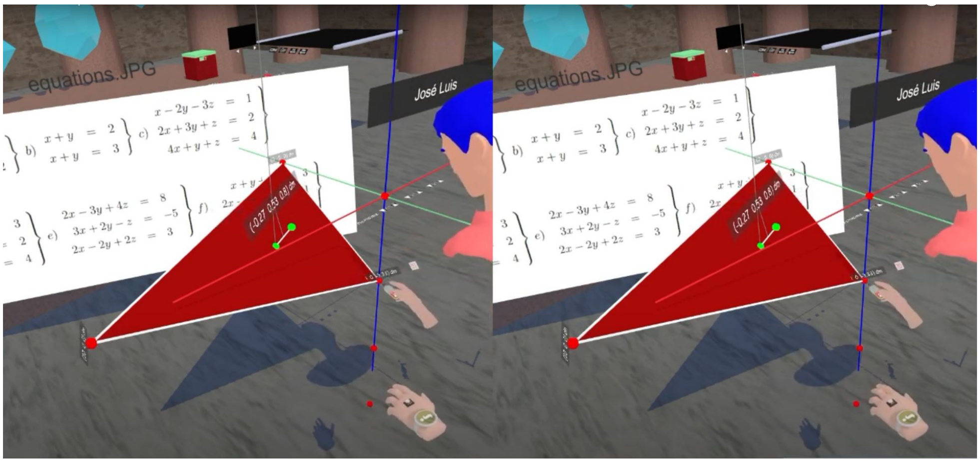

Figure 4: Capture of the stereoscopic video `https://youtu.be/hKgfEz1Ey0c` showing the construction of a plane with integers coefficients by using the division tool. The labeling tool allows the user to insert coordinates of points, among other useful information.

The concepts of vectors, vector addition, scalar multiplication, linear generation, linearly independence, base, etc. were given in theoretical lessons. In the next video (`https://youtu.be/NWK7d-TDBxE`), it is shown how to create a plane using the Neotrie tools. Restriction moves are set to a grid of 1 dm on each axis, and how to use the divider tool to get a fractional point. The teacher made the plane $x - 2y - 3z = 0$ which can be generated by the vectors $(2, 1, 0)$ and $(1, -1, 1)$ (see Figure 5). In this case, it is a subspace, since the plane passes through the origin. In the end, any vector of the plane is obtained as the sum of two vectors, multiples of the generators (applying the parallelogram law). If $v = r(2, 1, 0) + s(1, -1, 1)$, the sides of the parallelogram are $r(2, 1, 0)$ and $s(1, -1, 1)$. To find the coordinates r and s of v with respect to the basis $\{(2, 1, 0), (1, -1, 1)\}$, one would have to solve the corresponding system of equations (this is not done by Neotrie). What one can do in Neotrie is find r by dividing the module of the vector $r(2, 1, 0)$ by that of the vector $(2, 1, 0)$. Length of vectors can be measured directly with the measurement tape, or the label tool.

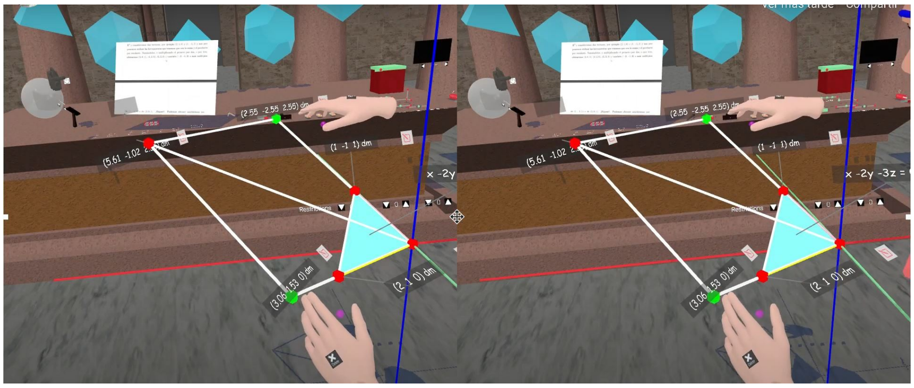

Figure 5: Capture of a stereoscopic video explaining generation of a plane by two linearly independent vectors, and how to find the coordinates of any point with respect to those vectors.

3 Planes with the 3D graphing calculator

The graphing calculator [10] enables us to represent curves and surfaces by means of their parametric equations. Given a system of equations, students were asked to parametrize each of the equations to introduce it in the calculator. Students know well how to go from a Cartesian to a parametric equation, and viceversa, although as we have mentioned, they sometimes fail in the graphical representation and its relation to the equation.

In two consecutive videos (see Figure 6), it is explained how to introduce the equation of a plane $ax + by + cz = d$ in the graphing calculator, a, b, c, d, b are replaced by α, β, γ and δ, to use the sliders in the graphing calculator. Any plane with equation $\alpha x + \beta y + \gamma z = \delta$ can then be introduced in the graphing calculator as $x = \delta/\alpha - \beta y/\alpha - \gamma z/\alpha$, $y = U$, $z = V$ (assuming $\alpha \neq 0$).

Restricting the sliders bars to change through the integers (marking 1 below each slider) one could then easily introduce the 3 planes of the system of equations (with $\alpha \neq 0$).

An interesting possibility for our students was to visualize the planes and all the spaces between them and the temple, and could have the sensation of infinity `https://youtu.be/eecWlRZikVO`.

The graphing calculator itself does not make any operation between surfaces yet. To avoid that, as described in the video, planes with editable vertices, edges and faces can be created in the 3-axis system and then one can use the intersection tool to calculate the intersection of the 3 planes. First, one touches two planes to get a line, and then the third plane and the line to get the intersecting point.

4 Training students with some affine geometry exercises

The following voluntary practice was designed to be done in groups of 4 or 5 people, for 1 hour session. Members had already to know how to use the basic modes and general handling of the Neotrie tools. Here it was sought that they learnt to place and interpret the elements correctly, and to improve their spatial visualization. They had to solve, discuss and compare the solutions to exercises both on paper and in the software directly, also leaving the turn to the next member, so that everyone is involved in the practice. It was not evaluated and those

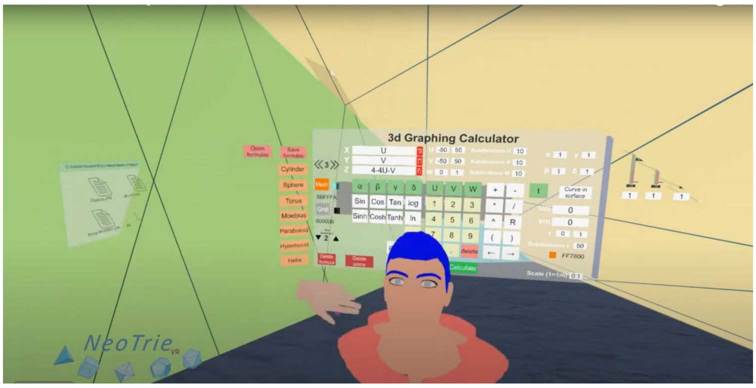

Figure 6: Capture of the video `https://youtu.be/bFZqS3BLbNA` showing the use of the graphing calculator to represent 3 planes. See the stereoscopic video `https://youtu.be/8T9_0-Hcl04` to represent any linear equation, and find intersections.

who had obtained poor results in the previous practice or had spatial vision difficulties were encouraged to do it. It also served to test the software and improve some functions.

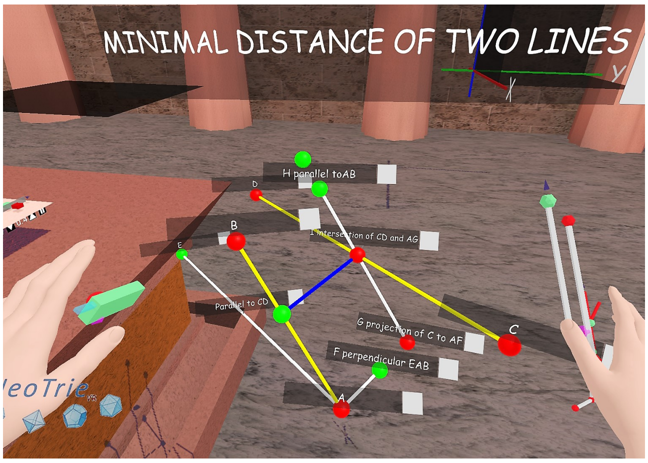

Figure 7: Finding the minimal distance between two lines or segments.

1. *Plane and line intersection:* Make a plane U of equation $x + y + 2z = 0$. Make the segment between $(1, 1, 0)$ and $(1, 2, 1)$, and find the parametric equation of the line r which determines it. Find the intersection between the plane U and the line r. (See Figure 8).

2. *Volume of the tetrahedron:* Calculate the volume of the tetrahedron generated by the plane $x + y + 2z = 1$, and the coordinate planes.

3. *Minimal distance between two lines:* Find the points P_0 and P_1 of minimal distance between the line passing through $(1, 0, 1)$ and $(2, -2, 4)$, and the line passing through $(0, -2, 4)$ and $(2, -2, 1)$. Calculate the minimal distance between these lines. (See Figure 7).

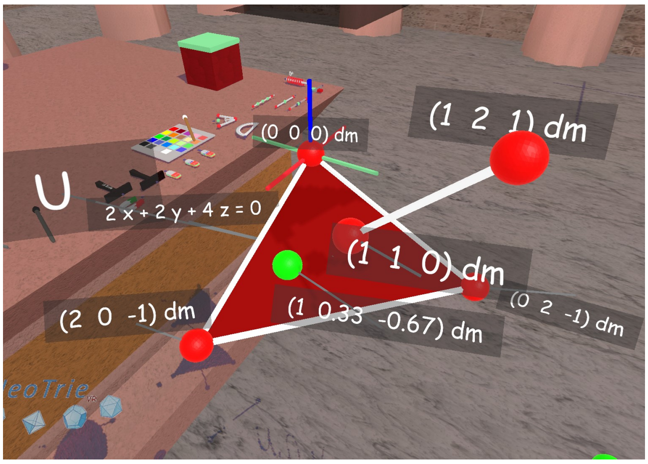

Figure 8: Intersection of a line and a plane.

4. *Parallel line that passes through an outside point:* Find the line parallel to P_0P_1 that passes through the point $(3, 2, -5)$.

5. *Plane perpendicular to a line:* Draw the line s that pass through the points $(2, 3, 4)$ and $(4, 6, 8)$. Find the plane V perpendicular to the line s that passes through the point $(2, 3, 4)$.

6. *Planes intersection:* Find the intersection of the planes U and V.

   Varied Parallels and perpendiculars:

7. Draw the plane W parallel to the plane U which contains the point $(-1, 0, 3)$.

8. Find the parallel line to W that passes through the point $(5, 0, 0)$.

9. Find the perpendicular line to W that passes through the point $(5, 0, 0)$.

10. Find all perpendicular planes to W which contain $(5, 0, 0)$.

11. *Intersection of 3 planes:* Find the intersection of the planes $-3x+y-z = 5$, $x+2y+z = 0$, $2x + z = 3$.

12. Discover all possible solutions of the previous system of equations, by moving the vertices.

5 Using the hypercube for systems of 4-variable equations

After introducing equations and subspaces in $\mathbb{R}^n$ in general, and seeing various calculations in 4 variables, one can get into 4th-dimensional vector spaces, with the geometric vision provided by the hypercube. This is the first time the students have worked in 4 dimensions (algebraically), hence it deems necessary for them to know how to find and represent points and subspaces geometrically. To do this, the teacher recorded a stereoscopic video in Neotrie, defining the coordinates of the vertices of the hypercube, giving a method for reaching these points by following the trajectory on each axis indicated by the coordinate. Here, it was used the standard model of the hypercube as a cube moving in a "perpendicular" axis w, whose projection in $\mathbb{R}^3$ is any other than the 3 coordinate axes x, y, z. This model allows to represent geometrically the intersection of different subspaces of 1, 2, and 3 dimensions.

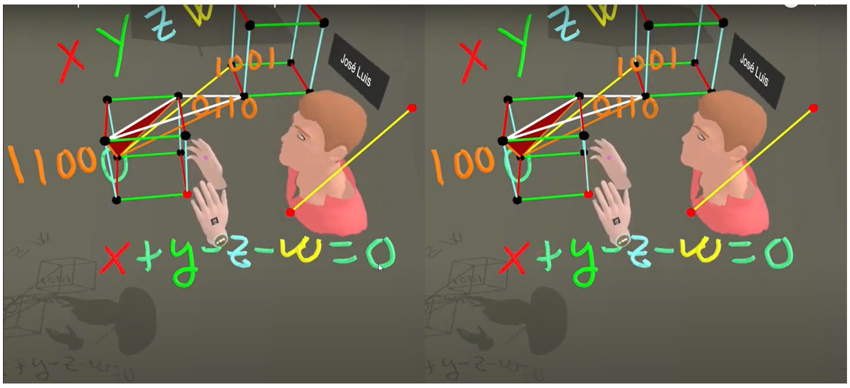

Figure 9: Coordinates and subspaces in a 4D cube. See videos https://youtu.be/C2RLnLt-orA and https://youtu.be/QCivzFngzdg

A stereoscopic video that goes one step further https://youtu.be/IvFZY_gky0Q was made later to better understand the 4th dimension, by visualizing 4D objects, either by their projections or sections, although it was not shown to the students, in the hope of showing it in the next course, as it may be more appropriate for a more advanced calculus or topology course. Related to this topic, an interesting VR software to visualize and play with 4th dimensional objects can be reached at https://4dtoys.com.

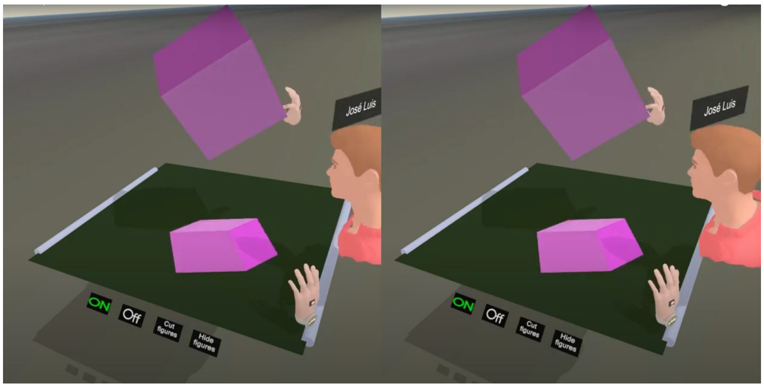

Figure 10: Capture of a stereoscopic video showing how projections or sections allows us to visualize the 4th dimension.

6 Affine transformations with the fractal tool

The following activity was intended for the student to visualize a linear transformation. Over 68 students, the global mean was 7,01 from 10 with a standard deviation of 1,83.

Students were asked to perform this practice in groups of 4 in out-of-class hours (see Figure 12). They were asked to build several linear transformations in $\mathbb{R}^3$ from invented matrices

3×3, with kernels of dimensions 1 and 2. The matrix should be expressed in canonical basis B_c and a non-canonical one B. They had to compute the image and kernel in each case, find the images of $(1, 1, 1)_B$, $(1, 1, 1)_{B_C}$. The results together with a paper and pencil drawing of the image of the unit cube (images of its corners), and a screenshot made in Neotrie had to be sent to the teacher for evaluation.

The goal of this session was to overcome common problems in the subject. First, some students don't see clear difference between affine space and vector space as reported by Geuudet-Chartier [4]. In this practice, students could see the linear part and the translation part of the affine transformation. Another known persistent mistake is students' reading the values of a linear transformation given by a matrix in a basis (see [5]).

The fractal tool was previously designed to iterate several similarities (affinities in general) to build fractals. The way this tool inputs the transformations is simple and intuitive: just touch the origins of all involved frames and finally the figure to transform, as shown in the video `https://youtu.be/X-rMnjZp1Js` (see Figure 11). The label (named as [Transformation]) is added to a vertex of the transformed figure, to see the matrix of the linear transformation and the corresponding translation vector. With this new option, students can experiment with different linear applications and see in real time the effect they produce on a figure. Thus, they can visualise the image, also when the application has a kernel, and whether it is of dimension 1 or 2.

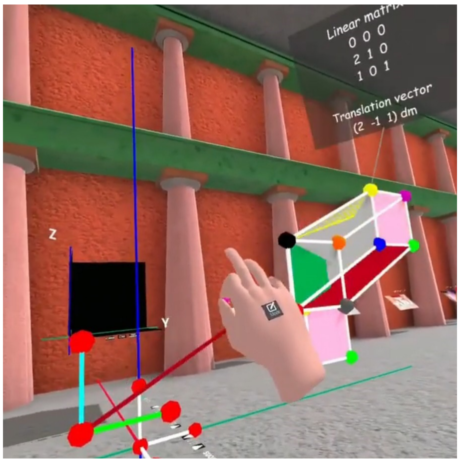

Figure 11: Manipulation of an affine transformation and its corresponding matrix.

Another interesting option of this fractal tool was to find the composition of two transformations (which corresponds to the product of matrices, as it was proved in theoretical sessions). This could be obtained by touching 3 frames: the first one indicates, the original base, and each of the other two, indicates the final frame of the first and second affine transformations. At this point, we remark that our subject of linear algebra only considers frames based on the origin, while in the 2nd course students see more contents on Affine Geometry. Geometric representation of the eigenvectors and eigenvalues of an endomorphism should be also implemented in the software.

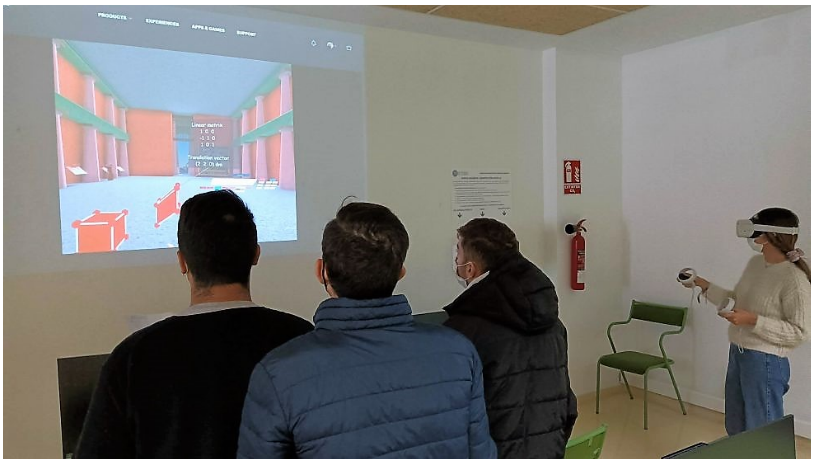

Figure 12: Group of students doing a task in Neotrie about linear maps.

7 Use of codes in lessons

Along the linear algebra course, it was shown the usefulness of linear algebra applied to programming and coding in C# in the Neotrie software. This complements the programming subject they follow in the first year of their mathematics degree, while also serving as a motivation for our subject. For example, in the following implemented method of Cramer's formula, students were able to see how to state and use some of the most commonly used terms in the subject.

```
public Vector3 CramerSolution(Vector3 A, Vector3 B, Vector3 C,
  Vector3 D)
{
    Vector3 solution = Vector3.zero;
    Matrix4x4 M = new Matrix4x4(new Vector4(A.x, B.x, C.x, 0), new
       Vector4(A.y, B.y, C.y, 0), new Vector4(A.z, B.z, C.z, 0), new
        Vector4(0, 0, 0, 1));
    Matrix4x4 Mx = new Matrix4x4(new Vector4(D.x, B.x, C.x, 0), new
       Vector4(D.y, B.y, C.y, 0), new Vector4(D.z, B.z, C.z, 0), new
        Vector4(0, 0, 0, 1));
    Matrix4x4 My = new Matrix4x4(new Vector4(A.x, D.x, C.x, 0), new
       Vector4(A.y, D.y, C.y, 0), new Vector4(A.z, D.z, C.z, 0), new
        Vector4(0, 0, 0, 1));
    Matrix4x4 Mz = new Matrix4x4(new Vector4(A.x, B.x, D.x, 0), new
       Vector4(A.y, B.y, D.y, 0), new Vector4(A.z, B.z, D.z, 0), new
        Vector4(0, 0, 0, 1));
    float d = M.determinant;
    solution = new Vector3(Mx.determinant, My.determinant, Mz.
       determinant);
    if (d != 0)
        return solution / d;
    else return new Vector3(-1000, -1000, -1000);
```

```
}
```

One can apply the Cramer's formula to solve a system of 3 equations with 3 unknwons, as usual. One can also intruduce students to affine geometry with the computation of the center of a circle in the space, which permits to see the usefulness of the scalar and cross product:

```
public Vector3 CenterCircle(Vector3 D, Vector3 E, Vector3 F)
{
    Vector3 N1 = E - D;
    Vector3 N2 = F - E;
    Vector3 normal = Vector3.Cross(N1, N2);
    Vector3 D1 = new Vector3(Vector3.Dot(N1, (D + E) / 2), Vector3.
       Dot(N2, (E + F) / 2), Vector3.Dot(normal, D));
    Vector3 A1 = new Vector3(N1.x, N2.x, normal.x);
    Vector3 B1 = new Vector3(N1.y, N2.y, normal.y);
    Vector3 C1 = new Vector3(N1.z, N2.z, normal.z);
    return CramerSolution(A1, B1, C1, D1);
}
```

The affine transformation that sends the frame $P+\{u,v,w\}$ to $Q+\{u_1,v_1,w_1\}$ is done in the following method, which is shown to students

```
public Matrix4x4 MatrixTransformation(Vector3 u, Vector3 v, Vector3
   w, Vector3 P, Vector3 u2, Vector3 v2, Vector3 w2, Vector3 Q)
    {
        Matrix4x4 Matrix1 = new Matrix4x4();
        Matrix1.SetColumn(0, new Vector4(u.x, u.y, u.z, 0));
        Matrix1.SetColumn(1, new Vector4(v.x, v.y, v.z, 0));
        Matrix1.SetColumn(2, new Vector4(w.x, w.y, w.z, 0));
        Matrix2.SetColumn(3, new Vector4(P.x, P.y, P.z, 1));

        Matrix4x4 Matrix2 = new Matrix4x4();
        Matrix2.SetColumn(0, new Vector4(u2.x, u2.y, u2.z, 0));
        Matrix2.SetColumn(1, new Vector4(v2.x, v2.y, v2.z, 0));
        Matrix2.SetColumn(2, new Vector4(w2.x, w2.y, w2.z, 0));
        Matrix2.SetColumn(3, new Vector4(Q.x, Q.y, Q.z, 1));
        return Matrix2 * Matrix1.inverse;
    }
```

8 Some feedback from students

After the activity in Section 2, in the academic course 2020-21, 44 students gave anonymous responses to a questionnaire.

To the question: "Is this the first time you have tried mobile VR (virtual reality) glasses?" 73% answered 'yes'.

To the question "Have you visualized the $x-2y-3z=1$ plane well, with the corresponding cuts on the coordinate axes?" 62% said 'yes' and 34% said 'almost yes'.

To the question “Do you think this system will help you better visualize and understand 3D geometry concepts in the mathematics degree?”, the 41% answered ‘yes’ and 54% believed ‘it would be’.

More than 90% would like to see more 3D Geometry videos with this system, and more than 84% of the respondents were very positive that Neotrie’s videos can be viewed in 3D on cell phones.

These responses, like those obtained in the Algebraic Topology subject [7], encouraged the teacher to continue using Neotrie VR in the course.

A more detailed study of the real impact on students through this innovative intervention along the last 3 academic years is still in progress.

9 Conclusions and future work

This article tries to remedy the lack of spatial vision detected in students of 1st year of mathematics. The intervention we have described is well-received by most of the students, who demanded more of such activities using virtual reality.

We take advantage of the opportunities provided by the Neotrie software to motivate students to study linear algebra and other subjects in the mathematics degree.

In this regard, Neotrie allows the teacher to explain 3D geometry more easily. The functions implemented in the Neotrie software allow the manipulation and construction of planes, and other geometric elements, as well as explanations in real time.

The 3D scene provided in Neotrie is available to work more easily with 4 dimensional objects, such as the hypercube, which yields explanations to the intersection of subspaces in $\mathbb{R}^4$.

The linear transformations of a figure are easily handled and visualized in virtual reality.

This teaching methodology is not complete and we hope in the next academic course to continue in the same line, implementing new features that help both to improve the software and the teaching of the subject.

Thanks to the arrival of Meta Quest, VR headsets without gaming computer, the incorporation of VR in the classroom is greatly affordable [8].

The incorporation of the multiplayer mode, which is currently being developed from versions 3.8.3 of the software[8], will greatly facilitate the teaching and learning of geometry in a collaborative virtual environment, and other subjects of the degree that require 3D vision and 3D reasoning.

In addition, the possibility of importing GeoGebra files into Neotrie, which is currently being implemented, will also be very useful for the educational community.

We finally highly recommend instructors of linear algebra to implement the use of virtual reality in earlier stages of education, to acquire visual and spatial reasoning, not only in mathematics [9], but also in other subjects areas.

Acknowledgement. The author would like to thank the Department of Mathematics of the University of Almería for providing the space and equipment necessary to bring this project to fruition. He was supported by the Agencia Estatal de Investigación grant PID2020-117971GB-C22 and the Programa Operativo Feder (REF. UAL2020-SEJ-B2086).

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