

Exploration of parameterized families of surfaces: envelopes, offsets and canal surfaces

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Abstract

The study of envelopes and offsets requires a strong dialog between algebraic computations and graphic representations. Both are available using a Computer Algebra System, and sometimes accompanied by a Dynamic Geometry System. The two kinds of software have different affordances, whence different characteristics of the animations. Visualization raises specific problems. We illustrate this with examples of 1-parameter and 2-parameter families of surfaces in 3D space, in particular canal surfaces and pipe surfaces. In certain cases, the envelopes have non-isolated singularities; we study them switching between parametric and implicit presentations. Finally, networking between them and between visualization and symbolic computations is discussed.

1 Motivations coming from the real world: a STEAM approach

It is now common understanding that STEAM Education is the educative framework suitable for 21st century, no matter the age of the students. STEAM is an educational discipline that aims to spark an interest and lifelong love of the arts and sciences in children from an early age. Science, Technology, Engineering, the Arts and Mathematics are similar fields of study in that they all involve creative processes and none uses just one method for inquiry and investigation¹. In all the components of STEAM, the name of the game is given by the 4 C's of 21st education, namely **C**ollaboration, **C**ommunication, **C**ritical Thinking and **C**reativity [29, 30]. If most of the time, people think about the first two C's as between humans, in fact they are

¹We quote here J. Lathan at <https://onlinedegrees.sandiego.edu/steam-education-in-schools/#STEAM>

important also between machines and between humans and machines; see [40]. Of course, the 2 last C's are purely humans, and the insertion of Arts in STEM education is important to develop creativity.

Computer Algebra Systems (CAS), Dynamic Geometry Systems (DGS) and other technologies, are under constant development. Their new possibilities induce always the need for new tasks [24]. In particular, the possibilities offered by dragging points directly on the screen is a powerful tool for exploration and expressing conjectures; see [3]. An analysis of the construction of such new tasks, enhancing a new working scheme based on exploration, is developed in [22]. Exploration leads to conjectures, and then to proofs.

The new tasks may be the discovery of new properties or new proofs of classical topics, such as in [17]. In this paper we relate to the same old-new topic: envelopes of parameterized families of surfaces. In the early 1960's, Thom [35] complained that the topic has almost disappeared from the syllabus. The reasons invoked were that the classical theory is not rigorous enough, has too few theorems, too many particular cases (fixed points, singular points, stationary curves, etc.), and nothing ensures that all the "pathological" cases have been included in a catalogue. Actually, the topic has never been abandoned, because of its centrality in numerous fields, such as robotics and safety zones in industrial plants or entertainment parks [12], pollution and depollution of soils [39]. The introduction of mathematical software, both for research and for teaching, and other technologies, enabled a revival of the topic for a more general audience than the specialists [17] and at an early stage of education. There the focus was on families of plane curves. Then the interest went to the transition from 2D to 3D [20], and there to the difference of treatment for 1-parameter and 2-parameter families of surfaces [19].

Automated methods for the exploration of envelopes have been developed and implemented in software, for example in GeoGebra-Discovery, a beta version of new commands for automated exploration and proofs (see Section 2). They have been developed for envelopes of families of plane curves. We intend to explore here envelopes of parameterized families of surfaces in \mathbb{R}^3 , therefore we cannot use them, but we will use other features available in CAS (here Maple 2022) and DGS (Here GeoGebra). Of course, we will use commands such as **solve** to solve systems of equations (either polynomial or not); the pattern recognition of the specific software will make the decisions which algorithm to use. This may be transparent to the user, with the drawback of using the CAS as a blackbox. We will have profit both of the algebraic algorithms for exact computations, and of graphical features, plotting and animating the plots. Working together with a CAS and a DGS as in [21], offers opportunities to build bridges towards visual arts. Note that a CAS has strong plotting features, but not with the same dynamics as a DGS, and a DGS may contain a CAS component [26], but generally not as powerful as a CAS.

The present paper proposes new contributions to the exploration of envelopes of parameterized families of surfaces, related to the transitions mentioned above. Actually it reports on the opening of ways for the technology-assisted study of envelopes. In Section 3, we explore a family of spheres, its envelope and the singular locus of this envelope, comparing a similar situation in 2D. Another old-new object will be explored, namely pipe surfaces (also called tubular surfaces) and more general canal surfaces, which can be viewed as envelopes of families of spheres centred in a space curve (Section 5).

2 Envelopes of families of surfaces

The study of envelopes of 1-parameter families of plane curves, including working with technology, is well developed and documented, and important developments have been obtained during the last decade; see [9, 17, 8]. For 1-parameter families of plane curves, exploration with a Dynamic Geometry System (DGS) has revealed very fruitful and efficient. Tools have been developed for automated exploration and determination of envelopes; they are part of the package *GeoGebra Discovery*. A guide to the package is given in [32]. The commands **Envelope** and **Locus** of *GeoGebra-Discovery* are central tools for this automated exploration; see [27].

There exist several different and non-equivalent definitions of an envelope in the literature. For 1-parameter families of plane curves, they have been explained by [25, 9] and the relations between the different kinds of definitions have been settled. Plotting technologies help to visualize the differences between the definitions. One of these 4 definitions of an envelope, the only one used by Berger [5], and which has been used in [17], reads as follows:

Definition 1 *Let \mathcal{C}_k be a family of plane curves given by an equation $F(x, y, k) = 0$, where k is a real parameter. An envelope \mathcal{D} of the family is the curve determined by the solutions of the system of equations*

$$\begin{cases} F(x, y, k) = 0 \\ \frac{\partial}{\partial k} F(x, y, k) = 0 \end{cases} \quad (1)$$

The translation of Definition 1 to a 1-parameter family of surfaces in 3D-space reads as follows:

Definition 2 *Let \mathcal{S}_a be family of surfaces in the 3D-space given by an equation of the form $F(x, y, z, a) = 0$, where a is a real parameter. If it exists, an envelope of this family is the surface given by the solutions of the following system of equations:*

$$\begin{cases} F(x, y, z, u) = 0 \\ \frac{\partial F}{\partial u} F(x, y, z, u) = 0 \end{cases} \quad (2)$$

and for a 2-parameter family of surfaces, the corresponding definition reads as follows:

Definition 3 *Let $\mathcal{S}_{u,v}$ be family of surfaces in the 3D-space given by an equation of the form $F(x, y, z, u, v) = 0$, where u and v are real parameters. If it exists, an envelope of this family is the surface given by the solutions of the following system of equations:*

$$\begin{cases} F(x, y, z, u, v) = 0 \\ \frac{\partial F}{\partial u} F(x, y, z, u, v) = 0 \\ \frac{\partial F}{\partial v} F(x, y, z, u, v) = 0 \end{cases} \quad (3)$$

In this paper, we will mostly focus on 1-parameter families of surfaces.

Translation of two other definitions from [9] to 3D yields the following definitions. We consider a 1-parameter family of surfaces S_u , $u \in \mathbb{R}$.

Definition 4 *The envelope \mathcal{E}_1 is the limit of intersection of infinitesimally closed surfaces S_u and $S_{u+\varepsilon}$.*

Definition 5 *The envelope \mathcal{E}_2 is a surface tangent to all the surfaces S_u .*

It can be easily shown that $\mathcal{E}_1 \subset \mathcal{D}$ and $\mathcal{E}_2 \subset \mathcal{D}$. We will explore in Section 5 a case where the inclusion is not an equality, i.e. $\mathcal{E}_2 \neq \mathcal{D}$.

Families of planes have been studied in [17, ?]. In general, 1-parameter families of planes have an envelope, with the following properties:

- The envelope is a ruled surface.
- The envelope has singular points forming a cuspidal edge.
- The tangents to this cuspidal edge are generators of the envelope.

As an example consider the family of planes given by the polynomial $F(x, y, u) = x + uy + 2u^2z - 3u^3$, where u is a real parameter. The envelope obtained according to Definition 2 has parametric presentation

$$\begin{cases} x = 2u^2t - 6u^3 \\ y = -4ut + 9u^2 \\ z = t \end{cases}$$

and implicit equation

$$32xz^3 - 4y^2z^2 + 108xyz - 12y^3 + 243x^2 = 0.$$

It is shown in Figure 1:

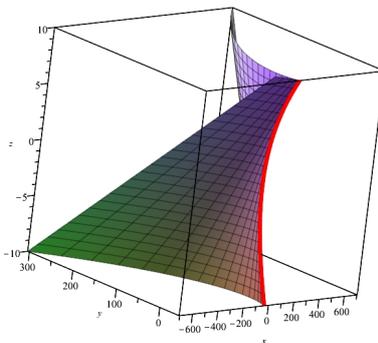


Figure 1: The envelope of the family of planes with equation $x + uy + 2u^2z - 3u^3 = 0$

The algebraic abilities of a Computer Algebra System (CAS) are identical for the case of plane curves and for the two cases of surfaces. We mean that the same commands, such as **solve**, can be utilized. The software uses pattern recognition and the eventual options entered by the user to decide which algorithm to use. The same remarks apply to visualization, with the **plot3d** command instead of **plot**, together with the suitable options. Note that the same **plot3d** command is useful to plot surfaces and curves in \mathbb{R}^3 , but for the last a **spacecurve** command is also available. Note also that visualization is more problematic in the 3D case than

in the plane [18, 19], and this makes the transition non-trivial. In particular, the representation on the screen (a 2-dimensional surface) of objects in 3D space requests the choice of a suitable mesh. The standard choice is rectangular, and yields often strange plots, as explained in [16]. Choosing an accurate mesh is a must, and generally a parametric presentation of the surface provides an accurate plot. Figure 1 shows that quite well. Anyway, a consequence is that the respective role importance of a DGS and of a CAS are switched: in the 3D case, the CAS is dominant.

In this paper, we work with 2 definitions only among those given by [9]: definition 2 (in Section 3) and the definition of offsets (see Section 5).

3 Envelope of a 2-parameter family of spheres

We consider the ellipsoid \mathcal{E} in \mathbb{R}^3 whose respective half-axes are denoted by a, b, c respectively; these are positive parameters. A parametric presentation for the ellipsoid \mathcal{E} is as follows:

$$\begin{cases} x = a \cos u \sin v \\ y = b \sin u \sin v \\ z = c \cos v \end{cases} \quad (4)$$

For $a = b = c = 1$, \mathcal{E} is the unit sphere. We leave to the reader the study of this special case; this can be done by substitution into the equations appearing later.

A family of spheres \mathcal{S}_k , centered on \mathcal{E} and with radius $1/k$, where $k > 0$ is determined by the implicit equation

$$(x - a \cdot \cos u \sin v)^2 + (y - b \cdot \sin u \sin v)^2 + (z - c \cdot \cos v)^2 - \frac{1}{k^2} = 0. \quad (5)$$

We denote by $G_k(x, y, z, u, v)$ the polynomial in the left-hand side of Equation (5). If the family $\{\mathcal{S}_k\}$ has an envelope, it is determined by the solutions of Equations (3), substituting the polynomial G . With Maple's **solve** command, we solve Equations (3); as the output involves the *RootOf* placeholder, we need to use the **allvalues** command. Actually, in order to explore the envelopes for different values of the radius, we write a procedure in Maple²:

```
with(plots):
srfce := plot3d([2*cos(u)*sin(v), sin(u)*sin(v), cos(v)], u = 0 .. 2*Pi, v = 0 .. Pi,
  transparency = 0.5, color = red, scaling = constrained) # the given ellipsoid
s := proc(k, a, b, c)
local G, derGu, derGv, envlp, envlp1, envlp2;
G := (x - a*cos(u)*sin(v))^2 + (y - b*sin(u)*sin(v))^2 + (z - c*cos(v))^2 - 1/k^2;
derGu := diff(G, u); derGv := diff(G, v);
envlp := allvalues(solve({G = 0, derGu = 0, derGv = 0}, {x, y, z}));
envlp1 := plot3d([rhs(envlp[1][1]), rhs(envlp[1][2]), rhs(envlp[1][3])],
  u = 0 .. 2*Pi, v = 0 .. Pi, color = yellow, transparency = 0.5);
envlp2 := plot3d([rhs(envlp[2][1]), rhs(envlp[2][2]), rhs(envlp[2][3])],
  u = 0 .. 2*Pi, v = 0 .. Pi, color = blue, transparency = 0.5);
display(sph, envlp2, envlp1);
end proc
```

²It is often possible to modify the code, to make it shorter, sometime to use loops where we do not; what is proposed can work with any version of the software.

Figure 2 shows the output of the procedure for $(a, b, c) = (4, 1, 2)$ and radius $r = 1/k$ equal to 2, 1, 1/2, 1/4.

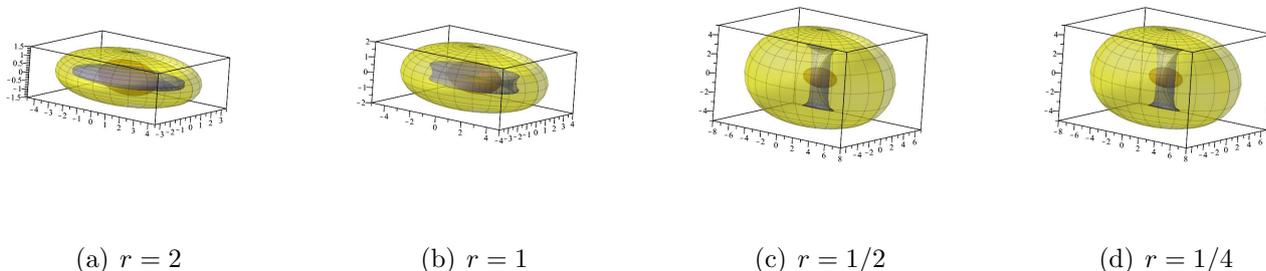


Figure 2: Envelopes of spheres centred on an ellipsoid

We should mention that some experiments have been made writing them separately; only then a general pattern appeared and enabled to write the code above, i.e. a loop. The various experiments reveal that the envelope has always two components³, an external one seeming smooth, and an internal one which may have non isolated singularities. The study of these singularities requires heavy computational machinery. Two possibilities exist:

1. If an implicit equation of the envelope has been obtained, eventual candidates for singularity are given by the vanishing of the gradient. For this, commands are available in various versions such as **Gradient** or **Nabla** (belonging to Maple's *VectorCalculus* package). A non trivial drawback is that it is generally impossible to distinguish the two components by algebraic means. We mean that the equation is given by an irreducible polynomial, whose vanishing locus is the union two distinct components. Another difficulty is that it may be impossible (at least in a reasonable time), to derive an implicit equation for the envelope. This happened with an example described later. Sometimes, when the envelope has a polynomial equation, the singular locus can be obtained in an automated way, using the *algecurv* package and its command **singularities**.
2. Working with the parametric presentation of the envelope, the candidates for singularity are given by the vanishing of the first derivative vector, which may be easier to find using teh CAS, but here too the computations may be unilluminating. The eventual singular points seem to be non isolated. Moreover, as shown in Figure 2, the singular locus may be the union of two components.

4 Envelope of the family of unit spheres centered on an ellipsoid

The envelopes of families of circles centered on an ellipse have been en explored in [17]. In certain cases they revealed isolated singular points, generally cusps. A first transition should

³Actually, each component is *an* envelope of the given family. The discussion of the non-uniqueness of an envelope is beyond the scope of the present paper.

be to study spheres centered on a given curve, here an ellipse. We chose to postpone this topic to Section 5. We consider here the family of unit spheres centered on an ellipsoid. The experimental work, based on visualization, leads to conjecture that the internal component of an envelope has non-isolated singularities. On the one hand, this is not a surprise, we encountered such a situation in Section 2 with the cuspidal edge of an envelope of a family of planes. On the other hand, the actual existence of these singularities has to be proven. For this, we use the algebraic affordances of the software.

A canonical ellipsoid E , i.e. an ellipsoid having the coordinate axes as symmetry axes, is given by the following polynomial equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where a, b, c are positive real parameters, or equivalently, by the following parametrization:

$$\begin{cases} x &= a \cos u \cos v \\ y &= b \sin u \cos v \\ z &= c \sin v \end{cases} \quad (6)$$

Actually, E has also a rational parametrization:

$$\begin{cases} x &= \frac{2au}{1+u^2+v^2} \\ y &= \frac{2bv}{1+u^2+v^2} \\ z &= \frac{c(1-u^2-v^2)}{1+u^2+v^2} \end{cases} \quad (7)$$

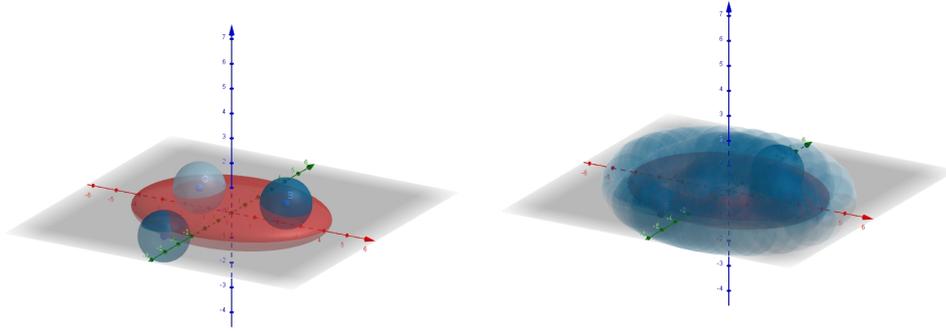
The proof is similar to the proof for a circle in the plane (see [15]), we leave it to the reader. Once again, a trigonometric parametrization provides a more accurate plot than a rational one (see [38, 14]), and we use it for graphics. For the algebraic computations, a rational parametrization such as given in Equation (7) is preferred: it leads to polynomial equations, enabling the usage of Gröbner bases packages, elimination and resultants; see [10, 14, 21].

We begin the exploration using GeoGebra, as the software allows to drag points directly on the plot using the mouse. Plot the ellipsoid, and for a general point on it, plot a unit sphere centered at this point. Figure 3(a) show three of the possible unit spheres. Using the **Move point** feature with **Trace On**, we obtain the output displayed in Figure 3(b). The interaction man-and-machine provides a "cloudy" picture of what can be the envelope. Nevertheless, previous experience in 2D (for envelopes of unit circles centered on an ellipse [17]), we conjecture that two components exist: one external, visible on the screen, and one internal, quite invisible here. Even changing the transparency of the plots does not improve visibility of what happens inside the ellipsoid. Therefore, switching to precise algebraic computations is a must, more important than in 2D.

With the trigonometric parametrization of the ellipsoid \mathcal{E} in Eq. (6), we obtain a 2-parameter implicit equation for the family of unit spheres centered on \mathcal{E} :

$$(x - a \cos u \cos v)^2 + (y - b \sin u \cos v)^2 + (z - c \sin v)^2 - 1 = 0. \quad (8)$$

Denote the left hand side of Equation (8) by $F(x, y, z, u, v)$. According to Definition 3, an



(a) 3 unit spheres

(b) experimenting the envelope

Figure 3: Dynamic exploration with GeoGebra

envelope of the family of spheres, if it exists, is determined by the following system of equations:

$$\begin{cases} (x - a \cos u \cos v)^2 + (y - b \sin u \cos v)^2 + (z - c \sin v)^2 - 1 = 0 \\ (a^2 - b^2) \sin 2u \cos^2 v + 2by \cos u \cos v - 2ax \cos v \sin u = 0 \\ (a^2 - b^2) \cos^2 u \sin 2v - 2ax \cos u \sin v + (b^2 - c^2) \sin 2v \\ \quad - 2by \sin u \sin v + 2cz \cos v = 0 \end{cases} \quad (9)$$

Using the **solve** command, we obtain parametric presentations for two components. Denote

$$D = (((a^2 - b^2)c^2 \cos^2 u + a^2(b^2 - c^2)) \cos^2 v - a^2b^2).$$

Then a parametric presentation for one component of the envelope is:

$$\begin{cases} x = \frac{1}{aD} [\cos u (a^2c^2((a^2 - b^2) \cos^2 u + a^2(b^2 - c^2)) \cos^2 v - a^4b^2 \\ \quad + \sqrt{-a^2(((a^2 - b^2)c^2 \cos^2 u + (b^2 - c^2)a^2) \cos^2 v - a^2b^2)b^2c}) \cos v] \\ y = \frac{1}{bD} [\cos v (b^2((a^2 - b^2)c^2 \cos^2 u + a^2(b^2 - c^2))) \cos^2 v - a^2b^4 \\ \quad + c \sin u \sqrt{-b^2a^2(((a^2 - b^2)c^2 \cos^2 u + (b^2 - c^2)a^2) \cos^2 v - a^2b^2)}] \\ z = \frac{1}{D} [\sqrt{-b^2a^2(((a^2 - b^2)c^2 \cos^2 u + (b^2 - c^2)a^2) \cos^2 v - a^2b^2)} + c((c^2(a^2 - b^2) \cos^2 u \\ \quad + a^2(b^2 - c^2) \cos^2 v - a^2b^2)) \sin v \end{cases} \quad (10)$$

and for the second component

$$\begin{cases} x = -\frac{1}{aD} [-\cos u \cos(v) (((-a^2 + b^2)a^2c^2 \cos^2 u + (-b^2 + c^2)a^4) \cos^2 v + a^4b^2 \\ \quad + \sqrt{-a^2(((a^2 - b^2)c^2 \cos^2 u + (b^2 - c^2)a^2) \cos^2 v - a^2b^2)b^2c}) \cos v] \\ y = \frac{1}{bD} [-\sin u (-((a^2 - b^2)c^2 \cos^2 u + a^2b^2 - a^2c^2)b^2 \cos^2 v + a^2b^4 \\ \quad + \sqrt{-a^2(((a^2 - b^2)c^2 \cos^2 u + (b^2 - c^2)a^2) \cos^2 v - a^2b^2)b^2c}) \cos v] \\ z = \frac{1}{D} [\sin v (-\sqrt{-a^2(((a^2 - b^2)c^2 \cos^2 u + a^2(b^2 - c^2)) \cos^2 v - a^2b^2)b^2} \\ \quad + ((c^2(a^2 - b^2) \cos^2 u + a^2(b^2 - c^2) \cos^2 v - a^2b^2) c)] \end{cases} \quad (11)$$

The algebraic differences are very small (a minus sign instead of a plus sign, from time to time, which explains why, by successive squaring, both parametrizations provide the same polynomials).

Figure 4 shows the case $(a, b, c) = (4, 3, 1)$. and Figure 5 shows the case $(a, b, c) = (4, 4, 2)$

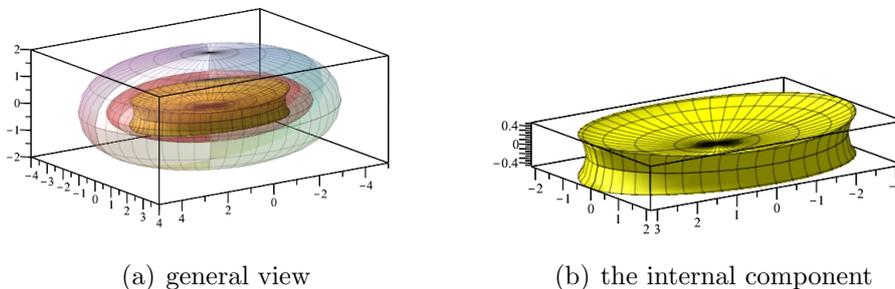


Figure 4: Envelope of unit spheres centered on an ellipsoid - $(a, b, c) = (4, 3, 1)$

In both cases, the obtained envelope is the union of two components. The external one seems

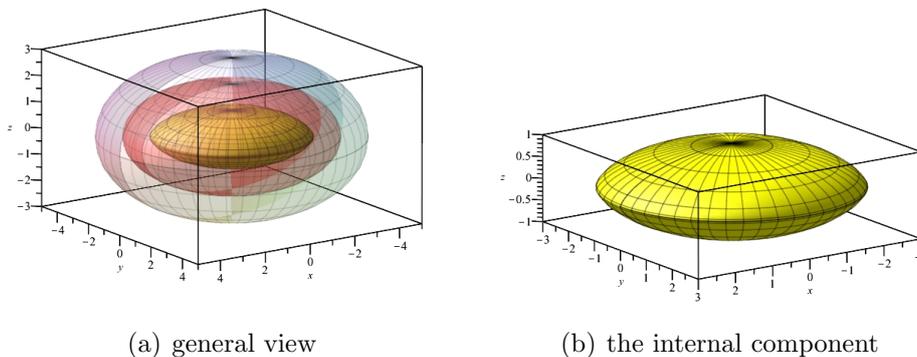


Figure 5: Envelope of unit spheres center on an ellipsoid - $(a, b, c) = (4, 4, 2)$

to be smooth, but the internal one seems to have singular points. It appears that, in the two examples, the sets of singular points have different structures, whence the need for a fine study.

In order to study the singularities, we use the method described in [4], chap. 5.2. Equations (10) determine a mapping $F := \mathbb{R}^2 \rightarrow \mathbb{R}^3$. We denote $F(u, v) = (F_1(u, v), F_2(u, v), F_3(u, v))$, where $F_k(u, v)$ is the right hand side of equation $n^\circ k$ in Equations (??). The jacobian matrix of F is

$$JF := \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} \end{pmatrix} \quad (12)$$

Let us study the case where $(a, b, c) = (4, 3, 1)$. Using the *LinearAlgebra* package and the command **Rank**, we find that generically, the rank of this matrix is equal to 2. This mapping is regular at points where its Jacobian matrix has maximal rank. The singular points are the vanishing points of the 3 minors of the matrix. The Jacobian is computed by the command **Jacobian** from Maple's *VectorCalculus* package. Then, in order to use the **Minor** command from the *LinearAlgebra* package, it is necessary to add a column on the right of the Jacobian matrix. This, and the subsequent computations, can be unilluminating, and we did not obtain a result in a "reasonable" time. Therefore we went in another direction.

At this point, we wish to recall a study from [17]: the envelope of a family of circles centered on an ellipse shows generally two components. The external one is smooth, and the internal one has four cusps. An example is displayed in Figure 6. The case $r = 1$ seems to have 2

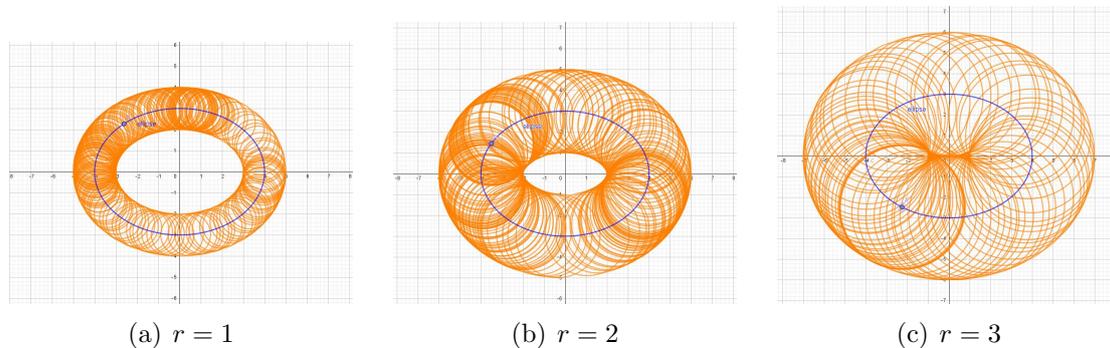


Figure 6: Families of circles centred on the same ellipse with radii 1,2,3.

smooth components; this can be easily proven. For $r = 2$ the plot suggests that there may be 2 singular points. If $r = 3$, we identify an internal component with 4 cusps. In all these cases, the algebraic computations are similar, the actual way depending on whether the curve is given by a parametrization or by an implicit equation.

For our problem here in 3D, the parametric presentation of the envelope is quite heavy. we may intersect the ellipsoid and the two components by a plane parallel to a coordinate plane, as shown in Figure 7. What remains to do is to prove that the plane curve obtained as the

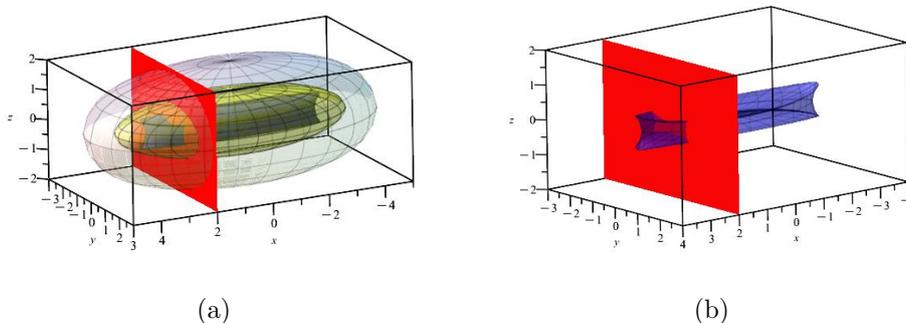


Figure 7: The intersection of a plane with the surfaces - cusps appear

intersection of the plane with the internal component of the envelope has generally 4 cusps. This is standard work in a plane.

We said earlier that the usage of a trigonometric parametrization yields a nice output; see [38]. This is the reason why we began as above. Nevertheless, the study of singularities should be easier with an implicit equation. For this purpose, it is more efficient to use a rational parametrization of the surface (if possible), leading to polynomial expressions and enabling further use of the Gröbner package. Of course, in theory, this may be possible only with rational curves.

We start from the rational parametrization of the ellipsoid in Equation (7). We consider now the family of unit spheres centered on the ellipsoid E . Their general equation is:

$$\left(x - \frac{2au}{1 + u^2 + v^2}\right)^2 + \left(y - \frac{2bv}{1 + u^2 + v^2}\right)^2 + \left(z - c \frac{1 - u^2 - v^2}{1 + u^2 + v^2}\right)^2 - 1 = 0. \quad (13)$$

Denote the left-hand side by $F(x, y, z, a, b)$. We solve now the corresponding system of equations (3). The desired envelope has 2 components. As the output is quite heavy, we do not present it here; we provide Maple code in a companion file. Transforming the parametric equations into polynomials using **expand**, **sort**, and commands from the Gröbner package with the *grlex* option, we obtain 3 polynomials in 5 variables:

$$\begin{aligned} P_1(x, y, z, u, v) = & (9216 + (576v^2 + 36)x^2)u^6 + (18432v^2 - 16164 + (864v^4 + 136v^2 - 216)x^2)u^4 \\ & + (9216v^4 - 14408v^2 + 9144 + (576v^6 + 164v^4 - 376v^2 + 36)x^2)u^2 \\ & + 144(v^2 + 1)^2 \left(-\frac{1}{4} + \left(v^4 - \frac{14}{9}v^2 + 1\right)x^2\right) - 2304u^7x \\ & + (-6912v^2 + 1728)u^5x + (-6912v^4 + 3008v^2 + 1728)u^3x \\ & - 2304(v^2 + 1) \left(v^4 - \frac{14}{9}v^2 + 1\right)ux + 144u^8x^2 \end{aligned}$$

$$\begin{aligned} P_2(x, y, z, u, v) = & (11520 + (1296u^2 + 144)y^2)v^6 + (23040u^2 - 18432 + (1944u^4 + 369u^2 - 360)y^2)v^4 \\ & + (11520u^4 - 20700u^2 + 11520 + (1296u^6 + 306u^4 - 846u^2 + 144)y^2)v^2 - 3888v^7y \\ & + (-11664u^2 + 2160)v^5y + (-11664u^4 + 5076u^2 + 2160)v^3y \\ & - 3888(u^2 + 1) \left(u^4 - \frac{7}{4}u^2 + 1\right)vy + 324v^8y^2 + 324(u^2 + 1)^2 \left(u^4 - \frac{7}{4}u^2 + 1\right)y^2 \end{aligned}$$

$$\begin{aligned} P_3(x, y, z, u, v) = & -135 \left(u^2 + \frac{128}{135}v^2\right)(u^2 + v^2 - 1)^2 \\ & + 72(u^2 + v^2 + 1) \left(u^4 + \left(2v^2 - \frac{7}{4}\right)u^2 + v^4 - \frac{14}{9}v^2 + 1\right)(u^2 + v^2 - 1)z \\ & + 36(u^2 + v^2 + 1)^2 \left(u^4 + \left(2v^2 - \frac{7}{4}\right)u^2 + v^4 - \frac{14}{9}v^2 + 1\right)z^2 \end{aligned}$$

What remains to be done is to define an ideal in $\mathbb{R}[x, y, z, u, v]$ generated by these 3 polynomials and to eliminate the variables u, v using the **EliminationIdeal** command. At the time we write, we were unable to obtain an answer with the CAS in a reasonable time. Indeed, implicitization is a crucial issue, in particular for engineers. Therefore approximate methods have been developed, for example in [34].

5 Offsets, canal surfaces and pipes

Definition 6 *Let \mathcal{C} be a curve in the 3-dimensional space given by a parametrization $M(t)$ and r a function which associates a positive real number $r(M)$ to every point on \mathcal{C} . For a point M on \mathcal{C} , we denote by \mathcal{S}_M the sphere centered at M with radius $r(M)$. If it exists, the envelope of the family of spheres \mathcal{S}_M is called a canal surface. The curve \mathcal{C} is the spinal curve of the canal surface.*

If the function r is constant, the envelope is called a pipe surface or a tubular surface[6].

We can describe a pipe surface as the offset at distance r of the curve \mathcal{C} .

Figure 8 shows two examples of pipe surface created interactively with GeoGebra. A spinal curve has been created with the **Curve** command, then a point A has been attached to the curve and a unit sphere centered at A has been created. Finally, the point A has been moved along the curve; the **Trace** command created the pipe surface. On the left, the pipe surface generated by unit spheres on the helix given by $(x, y, z) = (2 \cos t, 2 \sin t, t)$. On the right, the pipe surface generated by unit spheres on the loop given by $(x, y, z) = (2 \cos t, 2 \sin t, 3 \sin(2t))$. The first one seems more smooth, as the mouse has been moved slowly which provides more spheres in the family. For the second one, the moves have been made sometimes quicker, and its has effect that the intervals between two successive values of the parameter are bigger, and the pipe surface is somehow incomplete.

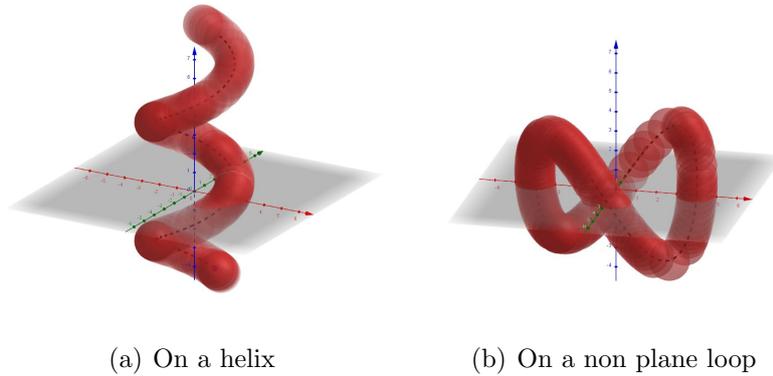


Figure 8: Two pipe surfaces studied interactively with GeoGebra

If the functions M and r are smooth, then the canal surface is smooth and non-degenerate under the condition

$$|M'(t)|^2 > r'(t)^2 \quad (14)$$

A proof is to be found in [31]. An example of a non smooth canal surface is given in subsection 5.4.

Moreover, a condition for the canal surface to be non self-intersecting is derived in [37], using a parametrization of the spinal curve by arc length.

Canal surfaces are important in Computer Assisted Geometric Design (CAGD) and numerous industrial applications, such as planning robot paths and safety zones, shapes of pipes, etc. An interesting survey of some applications in CAGD is given in [36]. It has been observed (op. cit.) that if the spinal curve \mathcal{C} and the radius function r are rational, then the canal surface has a rational parametrization. The task is now, in ever example, to find such rational parametrization. In our framework, the algebraic computations may be heavy and this is where a CAS is useful.

5.1 Pipe surfaces - the spinal curve is the unit circle in the x, y -plane

Consider the following rational parametrization of the unit circle \mathcal{C} in the x, y -plane:

$$\begin{cases} x = \frac{1-u^2}{1+u^2} \\ y = \frac{2u}{1+u^2} \end{cases}, \quad u \in \mathbb{R} \quad (15)$$

. Unit spheres centered on the circle \mathcal{C} have implicit equation

$$\left(x - \frac{1 - u^2}{1 + u^2}\right)^2 + \left(y - \frac{2u}{1 + u^2}\right)^2 + z^2 = 1 \quad (16)$$

According to Definition 2, the family of spheres has an envelope. The output of the `solve` command gives two parametrizations. Both cases lead to the same implicitization, namely:

$$x^4 + 2x^2y^2 + 2x^2z^2 + y^4 + 2y^2z^2 + z^4 - 4x^2 - 4y^2 = 0 \quad (17)$$

Actually, the computer output has an extra factor y^2 , which is somehow irrelevant to our question (It means that the x - z plane is tangent to all the spheres in the family). This is a consequence of the fact that actually, the algebraic computations provide a Zariski closure, and there may appear irrelevant components. This issue has been discussed in [7] and is beyond the scope of the present paper. What is true here is that the z -axis is tangent to all the spheres. Spheres and the pipe surface, namely a singular torus, are showed in the given animation.

5.2 Pipe surfaces - the spinal curve is a circle of radius 2 in the x, y -plane

Consider now spheres with radius $1/2$. The obtained pipe surface has equation

$$6x^4 + 32x^2y^2 + 32x^2z^2 + 16y^4 + 32y^2z^2 + 16z^4 - 104x^2 - 104y^2 - 40z^2 + 25 = 0 \quad (18)$$

Here too, the software produces an extra factor, irrelevant to the question. The animation shows a regular torus, enhancing the name *pipe surface*. This example, together with the example of the previous subsection, are displayed in Figure 9.

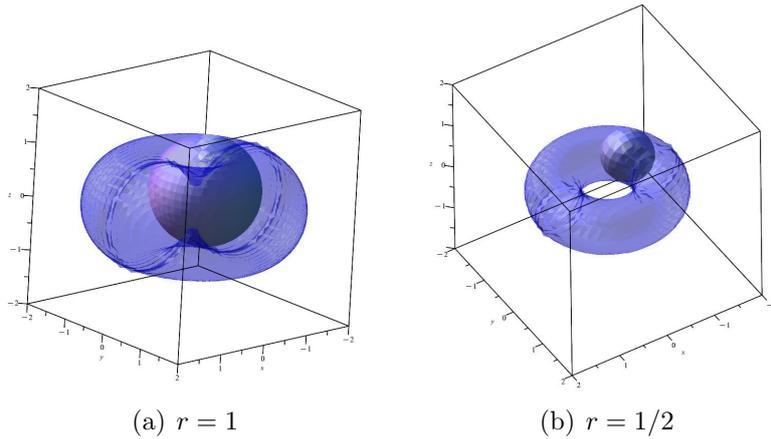


Figure 9: Two pipe surfaces

5.3 A pipe whose spinal curve is a parabola

Other easy examples are provided by curves which can be parameterized. For example, consider the curve \mathcal{C} defined by the parameterizations $(x, y, z) = (t, t^2, t)$, $t \in \mathbb{R}$. The offset at distance

1 of \mathcal{C} is the envelope of spheres whose equation is $(x-t)^2 + (y-t^2)^2 + (z-t)^2 - 1 = 0$. Denote by $F(x, y, z)$ the right-hand side of this equation. The offset is then determined by the solutions of the following system of equations:

$$\begin{cases} (x-t)^2 + (y-t^2)^2 + (z-t)^2 - 1 = 0 \\ -2x + 4t - 4(-t^2 + y)t - 2z = 0 \end{cases} \quad (19)$$

The solutions are given by the following parametrizations:

$$\begin{cases} x = t^3 - tv + t - \frac{1}{2}\sqrt{-4t^6 + 8t^4v - 2t^4 - 4t^2v^2 + 4t^2v - 2v^2 + 2} \\ y = v \\ z = t^3 - tv + t + \frac{1}{2}\sqrt{-4t^6 + 8t^4v - 2t^4 - 4t^2v^2 + 4t^2v - 2v^2 + 2} \end{cases} \quad (20)$$

and

$$\begin{cases} x = t^3 - tv + t + \frac{1}{2}\sqrt{-4t^6 + 8t^4v - 2t^4 - 4t^2v^2 + 4t^2v - 2v^2 + 2} \\ y = v \\ z = t^3 - tv + t - \frac{1}{2}\sqrt{-4t^6 + 8t^4v - 2t^4 - 4t^2v^2 + 4t^2v - 2v^2 + 2} \end{cases} \quad (21)$$

(actually here t is the original parameter). The pipe surface is shown in Figure 10.

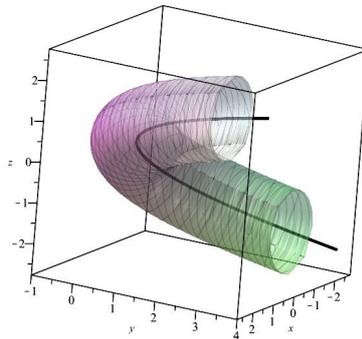


Figure 10: An example of a pipe surface

An animated GIF of a Maple animation shows a unit sphere moving along the "bone", here a parabola, within the envelope. The Maple code for creating the animation is the following:

```
restart: with(plots); setoptions(scaling = constrained):
spinal := spacecurve([t, t^2, t, t = -2 .. 2], numpoints = 7000, thickness = 4):
spheres := plots[animate](plot3d, [[t + cos(u) cos(v), t^2 + sin(u) cos(v),
t + sin(v)], u = -Pi .. Pi, v = -Pi/2 .. Pi/2, labels = [x, y, z],
color = black], t = -2 .. 2);
s := (x - t)^2 + (-t^2 + y)^2 + (z - t)^2 - 1;
ders := diff(s, t);
pipe := solve({ders = 0, s = 0}, {x, y, z}); #output contains RootOf
xpipe := allvalues(pipe[1]);
```

```

ypipe := allvalues(pipe[2]);
zpipe := allvalues(pipe[3]);
pipeplot := plot3d({[rhs(xpipe[1]), rhs(ypipe), rhs(zpipe[1])],
                    [rhs(xpipe[2]), rhs(ypipe), rhs(zpipe[2])]}},
                  y = -1 .. 3, t = -2 .. 2, numpoints = 15000,
                  labels = [x, y, z], transparency = 0.5)
display(spinal, spheres, pipeplot)

```

The envelope has been obtained as the union of two "half-pipes". Anyway, implicitization of both parametrizations yield the same implicit equation for the envelope⁴. Note the apparent inhomogeneity of the notation in the definition of *pipeplot*. It is due to the specific output of the **allvalues** command. This enhances the need to be in control of what the CAS does. Other examples may allow more automatic work.

```

spheres := plots[animate](plot3d, [[cos(t) + cos(u) cos(v),
                                   sin(t) + sin(u) cos(v), t + sin(v)],
                                   u = -Pi .. Pi, v = -Pi/2 .. Pi/2, labels = [x, y, z],
                                   scaling = constrained, color = black], t = -2 Pi .. 5 Pi)
CartEqSpheres := (x - cos(t))^2 + (y - sin(t))^2 + (z - t)^2 - 1;
DerCartEqSpheres := diff(CartEqSpheres, t);
env := solve({CartEqSpheres = 0, DerCartEqSpheres = 0}, {x, y, z});
comp12 := allvalues(env[1]);
comp34 := allvalues(env[2]); %c'est ce qu'on va employer
comp[1] := simplify(comp12[1], trig);
comp[2] := simplify(comp12[2], trig);
comp[3] := simplify(comp34[1], trig);
comp[4] := simplify(comp34[2], trig);
tube[3] := plot3d([rhs(comp[3][1]), rhs(comp[3][2]), rhs(comp[3][3])],
                 t = -2 Pi .. 4 Pi, y = -3 .. 3, scaling = constrained);
tube[4] := plot3d([rhs(comp[4][1]), rhs(comp[4][2]), rhs(comp[4][3])],
                 t = -2 Pi .. 4 *Pi, y = -3 .. 3, scaling = constrained);
display(spheres, tube[3], tube[4])

```

Consider now spheres with a variable radius centered on the same parabola as above. In Figure 11, one of the sphere appears within the canal. Here the radius is equal to t . An animated GIF of a Maple animation shows a unit sphere moving along the "bone", here a parabola, within the envelope.

montrer un exemple de construction selon Peternell et Potmann cite pp page 2

5.4 A canal surface with periodic radius

We consider spheres centered on the x -axis with radius equal to the absolute value of the x -coordinate of the center. The canal surface described by these spheres is shown on Figure 12. An animated GIF shows how the canal surface is generated. The computations have been performed in the same way as in the previous examples. We should only mentioned that they provide a parametric presentation of the canal surface, but no polynomial presentation should be looked for.

⁴This is a consequence of the fact that in reality the envelope which is computed is a Zariski closure. This discussion is beyond the scope of the present paper.

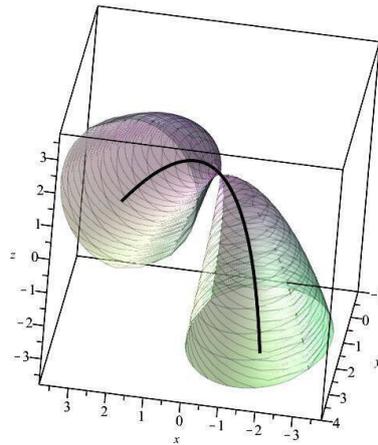


Figure 11: An example of a canal surface

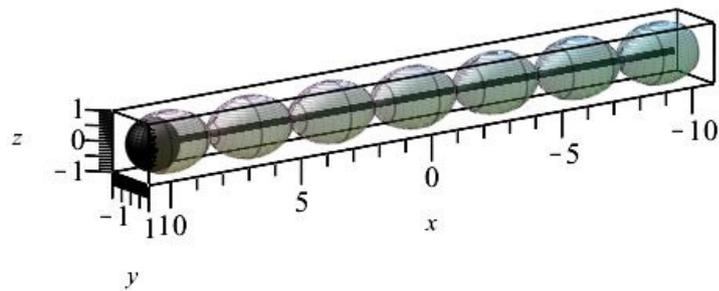


Figure 12: Variable spheres centered on a line

Now we choose a radius twice the radius of the previous example. The canal surface is shown in Figure 13. Here too, an animated GIF shows how the canal surface is generated. Note the intersection of the generating bulbs. The canal surface is only the external part of the variety on display.

6 Discussion

The Institut Henri Poincaré in Paris has an large collection of models of surfaces in 3-dimensional space. These models have been built in 19th century, and not many people know about their existence. Such collections exist at other institutions around the world, but this one is really impressive, It has been also photographed, at least partially, by Man Ray. A recent description has been written by Apéry [1], accompanied by photos.

Constructions with wood or other materials are for sure a wonderful tool for teaching. It seems that today it is not fully known how the Parisian models have been calculated and constructed. Nowadays, other techniques exist, such as 3D printing. This modern technology requests the description of the surface by mathematical means and their implementation into mathematical software and from there into the language understood by the 3D printer. Whence

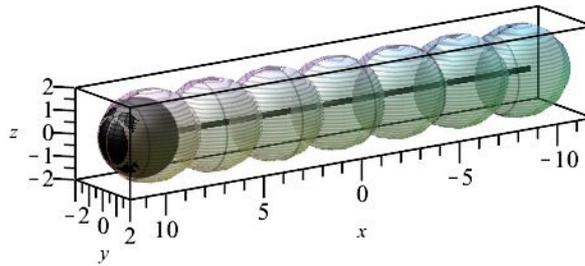


Figure 13: Variable spheres centered on a line - 2nd example

the importance to acquire the mathematical skills needed, accompanied by the requested technological discourse, according to Artigue [2]. It is common sense that the usage of a Computer Algebra System has here two main components, algebraic and graphical. The first one requests mathematical knowledge, such as what is described in Section 2 to solve systems of equations and algebraic knowledge on polynomials, elimination, etc.; see [10]. The graphical part requests both mathematical understanding in order to choose the most suitable mesh for plotting. We mentioned already that plotting using a parametric presentation provides a better plot, because of the choice of the mesh made by the software, which is generally more accurate than the standard rectangular mesh used by the `implicitplot3d` command. Nevertheless, this does not a definitive answer to the plotting issue. 3D printing avoids the problem of visualisation on a screen, i.e. a copy of a plane, of an object in 3D such as a space curve or a surface. Many of the educational benefits of this technology are described in [28]. In Section 3, we could derive a parametric representation of the envelope under study, but were unable to obtain an implicit equation, probably because the computations were too heavy for our current system. At a certain extent, the usage of multiple technologies may help to fill the gap, at least for the exploratory part of the work. After all, in education, open problems are more appealing than final answers.

Working out examples of rational curves and radii given by rational function provides a setting where polynomials can be derived and the algebraic packages of the CAS can be applied. Nevertheless, the meaning of the parameter has an influence. For example, we refer to the difference between a rational and a trigonometric parametrization of a circle. The second case gives a more uniform behaviour of the animations, and the plot is also more readable; see [17]. For more advanced examples, in particular in 3D, other choices of the parameter may be useful. In [?] canal surfaces are described (and analyzed for their curvature and other properties) using the arc length on the spinal curve as parameter. The quality of the obtained plots is high, as the mesh totally fits the properties of the surface; see also [38]. We did not use such a parametrization for the sake of the reader who may not be familiar with the usage of arc length.

In any case, visualization is the name of the game. It requests its own technological skills. The transition from 2D to 3D made dragging points less useful for exploration⁵. The mouse is still the good tool to rotate the plot in order to have numerous viewpoints, but it is difficult to

⁵The interest in dragging points in the plane is described in [33].

identify a point on a given surface on the screen.

Finally, we wish to emphasize that we chose to work mostly with Definition 2. Further work is needed for 2-parameter families. We must mention that the different definitions of an envelope correspond to different problems, safety zones, robotic paths, design of pipeline networks, etc.

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