Sphere and Spherical Geometry: The Power of Visualization and Investigation Through Dynamic Geometry

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Abstract: *This article aims to provide a dynamic approach to the basic notions of spherical geometry. We will use the power of visualization of Cabri 3D to make more comprehensible the detail of the proofs establishing the formulas giving the area and the volume of a sphere. Experimenting with real spheres is a challenge already overcome with half hollow and transparent plastic balls on which you can stretch strings and write with markers. The use of Cabri 3D greatly enriches such teaching of spherical geometry in facilitating the 3D representation of the objects of this geometry as well as their manipulation. We will show how. All the results presented here are known, but it is the way in which they are approached and detailed that constitutes the originality of this article.*

1. Surface area and volume of a sphere

1.1. Surface area of a sphere

Visualization and manipulation of spherical coordinates: each point M of a sphere (yellow sphere of **Figure 1 left**, centered at O and passing through D, radius r) belongs to a half circle here SmN. This half circle can be dragged in dragging point m from D to D along the displayed equatorial circle of the given sphere; the position of m and by the way the position of the half circle is known in knowing angle $\widehat{DOm} = \theta$ with $0 \leq \theta < 2\pi$. All points of the half circle can be reached in dragging point M along it; the position of M is known in knowing angle $\widehat{mOM} = \varphi$ with $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$.

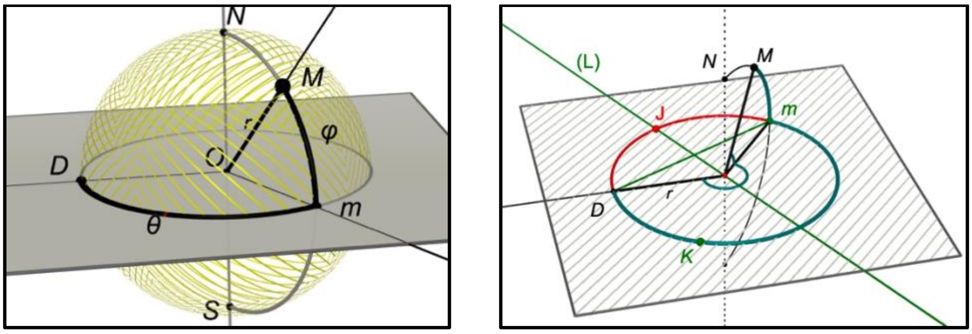

Figure 1: Spheric coordinates

A construction trick (**Figure 1 right**): An issue with the use of Cabri 3D is that arcs of more than 180° are not represented on a given circle. So, we need a tricky geometric construction of arc Dm in order to see it continuously during the whole displacement of point m along the horizontal circle. Here is this construction:

- Given point m on the circle centered at O passing through D.
- Construct segment $[Dm]$
- Construct line (L) perpendicular to this segment passing through O.

▪ J is the intersection point between the circle and this line belonging to the shortest arc defined by D and m.
▪ Measure the length of arc $DJm = L$.
▪ Calculate $l = L/2$.
▪ Define point K on the circle as the measurement transfer of l on this circle from point D.
▪ Arc DKm is the solution of our problem.

Proof mediated by Cabri 3D (Figure 2):
Highlighting the area element $dL.dl$ to evaluate the area of the sphere as a double integral: Figure 1 left.
Evaluation and calculation of dL, dl and $dL.dl$: Figure 2 right
$dL = r.d\varphi$, $dl = r.\cos(\varphi).d\theta$ and $dL.dl = r^2.\cos(\varphi).d\varphi.d\theta$
So, the area of the sphere is equal to $\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\theta=0}^{2\pi} r.\cos(\varphi).d\theta.r.d\varphi$
$= r^2.\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(\varphi).d\varphi.\int_{\theta=0}^{2\pi} d\theta = 4\pi r^2$
Finally: $\boxed{\textbf{Surface area of a sphere of radius } r = 4\pi r^2}$

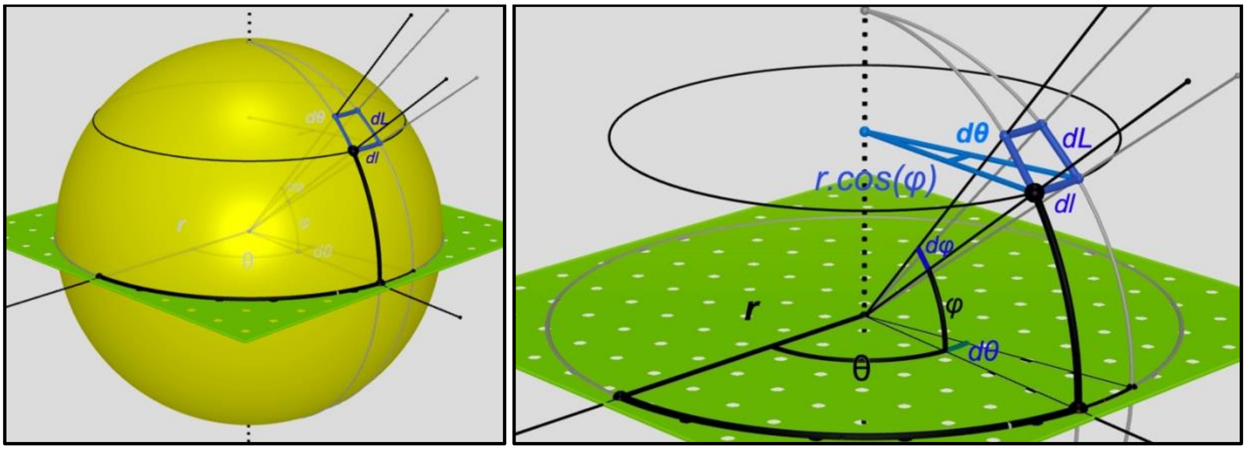

Figure 2: Surface area of a sphere

1.2. Volume of a sphere

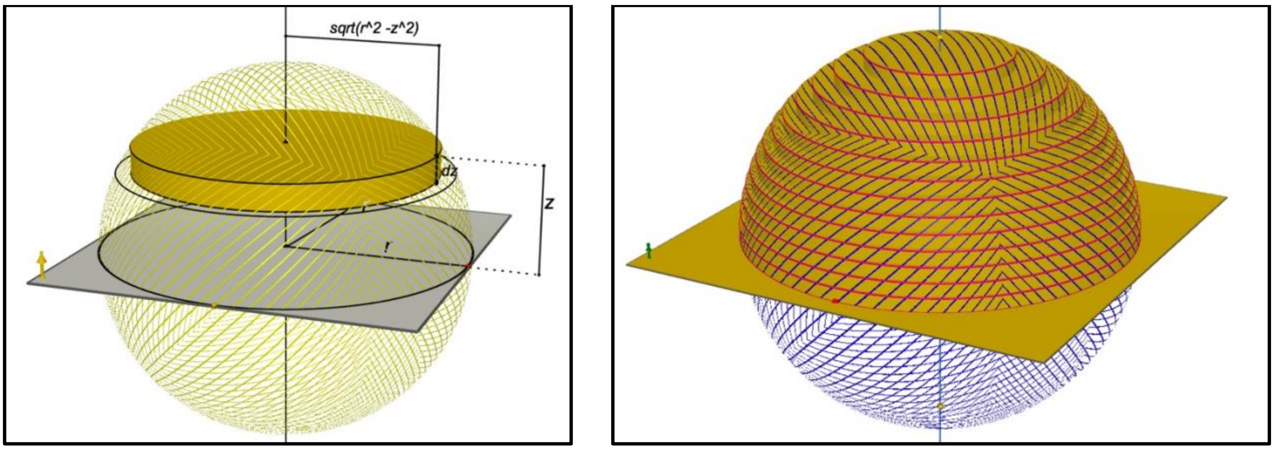

Figure 3: Volume of a sphere

For a sphere of radius r, we have represented one of the elementary cylinders we need to sum to obtain the volume of a half sphere as an integral (Figure 3 left). The radius of such a cylinder at height z is equal to $\sqrt{r^2 - z^2}$. The area of its base is $\pi.(\sqrt{r^2 - z^2})^2$. The sum of the volumes of these cylinders approaches the volume of the half-sphere (Figure 3 right). So, the volume of the upper half sphere is equal to:

$\int_0^R \pi.(\sqrt{r^2 - z^2})^2 . dz = \pi.\int_0^R (r^2 - z^2).dz = \frac{2}{3}\pi r^3$. Therefore:

$$\boxed{\textbf{Volume of a sphere of radius } \boldsymbol{r} = \frac{4}{3}\pi r^3}$$

2. Spherical geometry

2.1. Segments and lines on a sphere (geodesic lines) ([1'])

2.1.1. Experimenting before proving: we know that in planar geometry, a segment between two given points is the shortest path joining these points and the line defined by these points is obtained by extending as far as we want this segment (Euclidean definition). We will try to define segments and lines drawn on a sphere in the same way as we did for segments and lines of Euclidean plane geometry. As the simplest path joining two points on a sphere seems to be an arc, let us consider any arc defined by its two vertices A and M and passing through a third point of the sphere, point I (Figure 4 left). Let us measure the length of this arc and let us move point I in order to minimize the length of this arc I (Figure 4 middle). Very quickly we obtain a position of I, which seems to belong to the plane passing through the center of the sphere and the two points A and M (Figure 4 right). So, we can conjecture that the minimum is reached when the arc is supported by a great circle of the sphere and included entirely in the same half sphere.

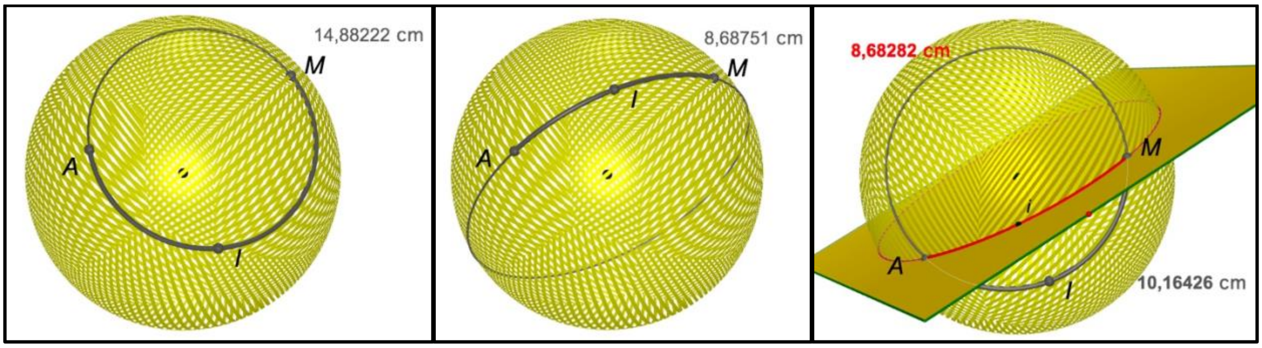

Figure 4: Experimenting in order to define a spherical segment

Now, we can hope that the definition of a spherical segment passing through two given points could be the shortest arc supported by the great circle containing these points of the sphere

2.1.2. Proof of the previous conjecture (Figure 5): let us start from an arc AM (called ***Arc1***) included in a great circle C_1 of a given sphere (Figure 5). This circle belongs to plane P. Let us consider another arc, ***Arc2***, in blue, with vertices also A and M. This arc belongs to a circle C_2 with radius less than the radius of C_1. We have to prove that the length of ***Arc2*** is always more than the length of ***Arc1***. To do so, it is sufficient to prove that the shortest arc included in C_2, with vertices are A and M is longer than ***Arc1***. C_2 is centered at i_2 which is the orthogonal projection of the center of the sphere on the plane containing C_2. We rotate circle C_2 and ***Arc2*** around line (AM), using the rotation transforming the plane of circle C_2 onto plane P. We obtain circle $r(C_2)$ centered at $r(i_2)$ and r(***Arc2***) included in plane P. So, the problem we have to solve is a problem of plane geometry described following Figure 5.

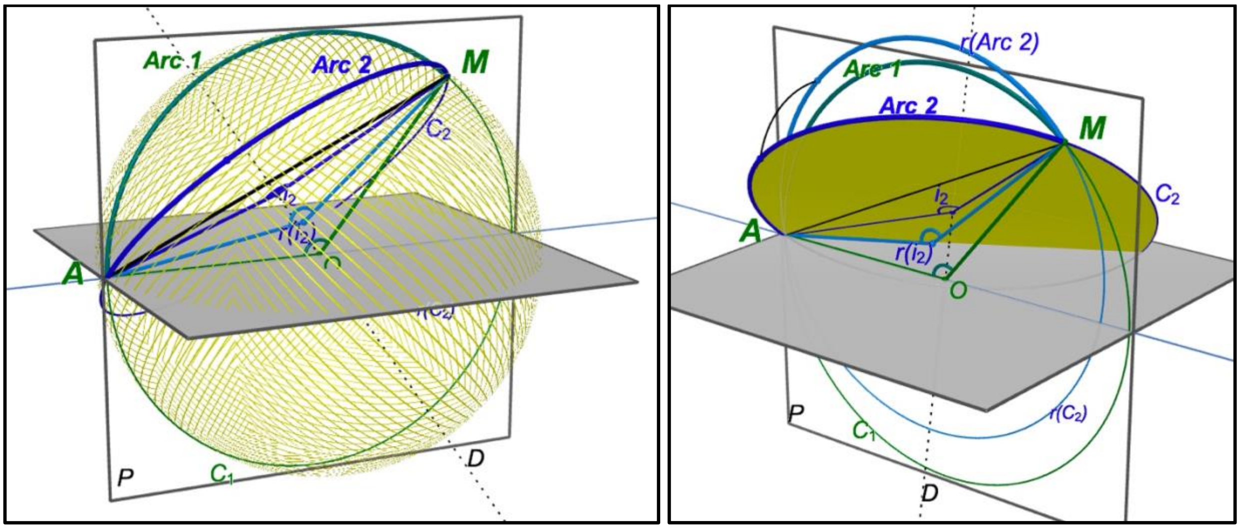

Figure 5: Proof of the minimization of the length of an arc

This planar problem is illustrated by Figure 6: C_1 is the given circle (corresponding to a great circle of a sphere). A and M belong to this circle; let us consider arc AjM which is the shortest arc of this circle defined by A and M (the longest is AiM). $[ij]$ is a diameter of circle C_1 perpendicular to segment $[MA]$. All the circles C_2 centered at O_2 where O_2 belongs to segment $[O_1H]$ are all the circles passing through A and M, with radii less thas the radius of C_1 and for which the arc supported by C_2 between A and M passing by l (not on the half plane defined by (AM) and O_1) is shorter than the other arc AkM. The other cases are obtained when O_2 belongs to the segment symmetric of $[O_1H]$ with respect to H. Note that, when O_2 moves from O_1 to H, circle C_2 moves from circle C_1 to the circle with diameter $[AM]$. We can notice, thanks to this interpretation that, the length of arcs AlM are greater than the length of arc AjM. As the length of arc AiM is greater than the one of AlM, all arcs defined by vertices A and M on a sphere are longer than arc AM included in a great circle, arc AM being the shortest of the two arcs defined by A and M on such great circle.

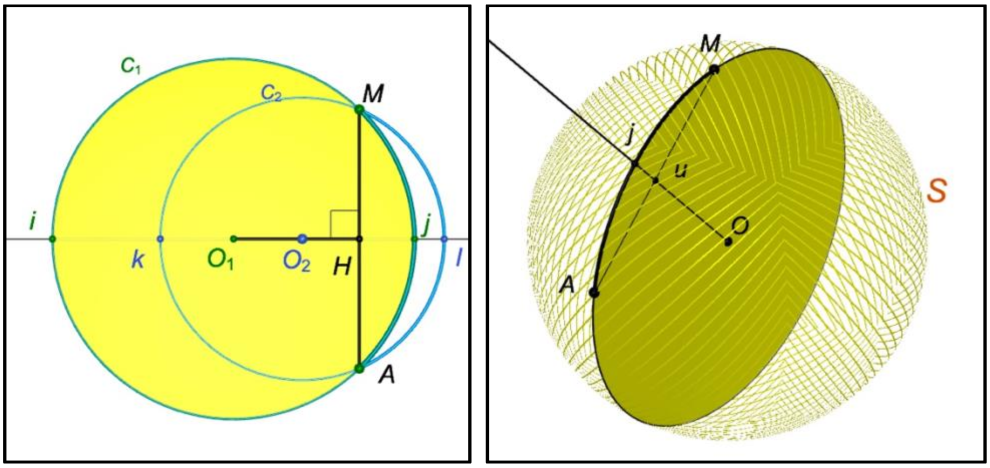

Figure 6: Plane proof of the minimization of the length of an arc

Finally, we can give the definition of a segment of a sphere which vertices are two given points of this sphere.

Definition 1: A segment between two points of a sphere is the shortest arc defined by these vertices on the great circle containing these points and the center of the sphere.

We can also deduce a construction of such an arc (Figure 6 right): if *A* and *M* are two given points of a given sphere *S* centered at *O*, the spherical segment *S(AM)* is the arc of the sphere defined by *AjM* where *j* is the intersection point between ray [*Ou*) and sphere *S* with *u* midpoint of segment [*AM*]

Definition 2: the spherical line defined by two distinct points of a sphere is the great circle containing these two points.
Notation: Spherical line (*AB*) noted *SL*(*AB*). Such a spherical line contains the spherical segment *S*(*AB*) and the other arc limited by *A* and *B*: let us call this arc, second spherical segment *SS*(*AB*). Let us remark that the extension of a spherical segment of the sphere is a great circle as the extension of a segment of the plane is the infinite Euclidean line.

2.2. Spherical triangles (Figure 7) ([2'])

2.2.1. Definition: if A, B and C are three points of a sphere, the spherical triangle is defined by the three spherical segments *S(AB), S(BC)* and *S(CA)*.

Remark: these spherical segments are constructed as detailed juste above and we know they are supported by three great circles (in red in Figure 7) of the sphere. These great circles are the three spherical lines *SL*(*AB*), *SL*(*BC*) and *SL*(*CA*). As three lines of a plane define one triangle, here the three spherical lines define eight spherical triangles whose vertices are three of the six points *A*, *B*, *C*, *A'*, *B'* and *C'* (where *A'*, *B'* and *C'* are the symmetric of points *A*, *B* and *C* with respect to *O*, center of the sphere): see Figure 7 middle and Figure 7 right.

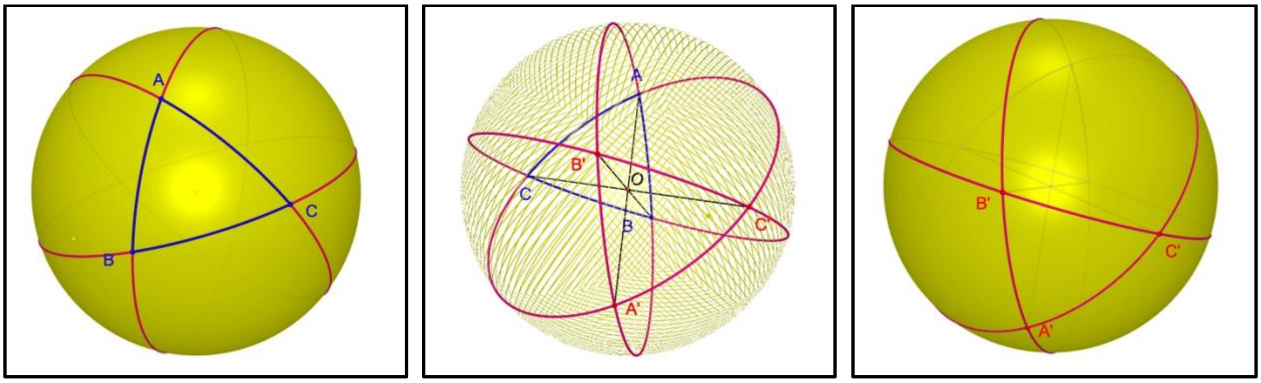

Figure 7: Spherical triangles

In the half plane defined by plane *BCC'B'* and the point *A*, we have four spherical triangles: triangles *ABC*, A*B'C'*, *ABC'* and *ACB*'. In the other half plane limited by the same plane, we have four other spherical triangles which are the symmetric of the previous ones with respect to the center of the sphere.

2.2.2. Notion of angle between two segments

If *A*, *B* anc *C* are three points of a sphere, let us define angle $\widehat{BAC}$. Figure 8 left: we must consider T_B and T_C, respectively tangent lines to the great circles containing arc *AB* and arc *AC* (or spherical segments *AB* and *AC*). *I* and *K* are the intersection points respectively between T_B and ray [*Oi*) and T_C and ray [*Ok*) (*O* center of the sphere, *i* midpoint of [*AB*] and *k* midpoint of [*AC*]. T_B is defined as the perpendicular line to [*OA*] in the plane containing *O*, *A* and *B*. It is also the tangent line at *A* to the great circle containing *A* and *B*. Same construction for T_C. Eventually, the spherical angle $S(\widehat{BAC})$ is defined as the plane angle $\widehat{IAK}$. The two other spherical angles at *B* and *C* to the spherical triangle *ABC* are defined similarly (see Figure 8 middle). These angles are measured and their measurements displayed. Their sum is also displayed and we can notice that this sum is more than180°. In Figure 8 right we have redefined points A, B and C at the vertices of a quarter of half sphere: we have obtained a spherical triangle with three angles measuring each 90° (sum: 270°). See [4'].

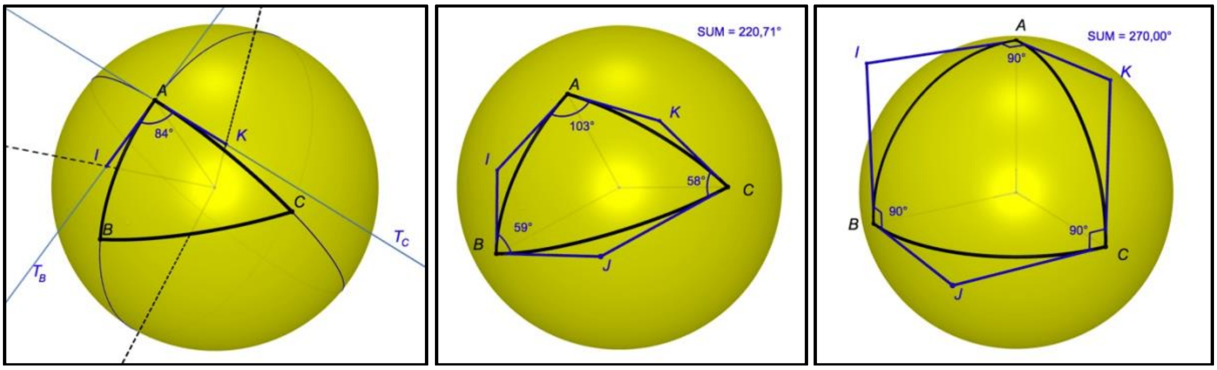

Figure 8: Angle of two spherical segments

2.2.3. Maximum sum of the angles of a spherical triangle

Let us enlarge triangle *ABC* until positions of *A*, *B* and *C* are closer and closer to a great circle but in the same half plane defined by this circle. We can notice that each angle of the triangle approaches 180°, so the sum approaches 540°. Let us show how to experiment to reach this result:

In Figure 9 left, we construct a sphere centered at a point belonging to the great circle containing A and B, the circle intersection between the big sphere and the small one (the radius of the small sphere is given by a number created with the calculator of Cabri 3D and displayed: here 0.8 that can be changed at any moment) and the arc, part of this circle included in the same half plane as *C* (in bold red). Then, we redefine point *C* on this arc (Figure 9 middle). Eventually we move point *C* on this arc until it reaches the great circle (two positions): we can observe (Figure 9 right) that the sum displayed is 540° as each angle of the spherical triangle *ABC* is evaluated as 180°.

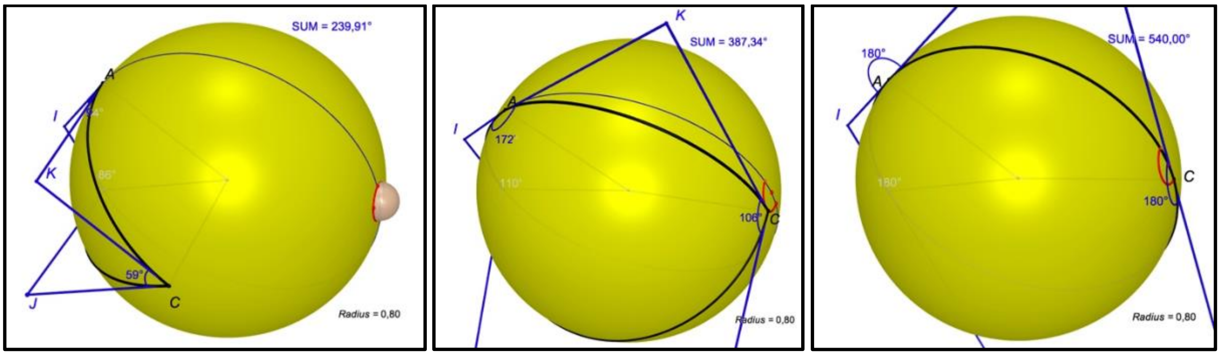

Figure 9: Maximum sum of the angles of a spherical triangle

2.2.4. Area of a spherical triangle (Girard theorem)

2.2.4.1. Area of a slice: Figure 10 left displayed such a slice between two half planes P_1 and P_2 intersecting in line (*NS*) which is the line joining a point *N* of the sphere to *S* its symmetric point with respect to the center of the sphere. The angle of this slice is the angle of P_1 and P_2 or the angle $\widehat{N_1NN_2}$ where (NN_1) is perpendicular to (*NS*) in P_1 and (NN_2) is perpenpendicular also to (*NS*) in P_2. The area of this slice is proportional to its angle knowing that when this angle is 2π, the area is the surface area of the sphere quoted in 1.1. which is $\boldsymbol{4\pi r^2}$. So, the area of a slice of angle θ (between 0 and 2π) in a sphere of radius *r* is equal to $\frac{\theta}{2\pi}\boldsymbol{.}\ 4\pi r^2$. Finally, the formula simplifies to: $\boldsymbol{2r^2\theta}$.

2.2.4.2. Girard theorem: *ABC* is a spherical triangle of a sphere of radius *r*. Measurements in radians of angles $\widehat{BAC}$, $\widehat{ABC}$ and $\widehat{BCA}$ are respectively called *a*, *b* and *c* (Figure 10 right). The great circles containing each side of the triangle allows us to create six slices of this sphere. In reality three pairs of two equal slices, two green equal slices bordered by the great circles of *S*(*AB*) and *S*(*AC*) (area of each slice: $2r^2a$), two blue equal slices bordered by the great circles of S(*BA*) and S(*BC*) (area of each slice: $2r^2b$) and two purple equal slices bordered by the great circles of S(*CB*) and S(*CA*) (area of each slice: $2r^2c$). If *s* is the area of the spherical triangle, we evaluate, in two ways, the area of the sphere:
Way 1 using the known formula: $\boldsymbol{4\pi r^2}$.
Way 2 in adding areas of slices:

$\boldsymbol{2.2r^2a}$	+	$\boldsymbol{2.2r^2b-2s}$	+	$\boldsymbol{2.2r^2c-2s}$
2 green slices		2 blue slices − 2 areas of ABC		2 purple slices − 2 areas of ABC

From the equality, $\boldsymbol{2.2r^2a}$ + $\boldsymbol{2.2r^2b-2s}$ + $\boldsymbol{2.2r^2c-2s}$ = $\boldsymbol{4\pi r^2}$, we can deduce the area *s* of the spherical triangle:
Area of a spherical triangle of angles *a*, *b* and *c* (in a sphere of radius *r*): $\boxed{\boldsymbol{s=(a+b+c-\pi).r^2}}$

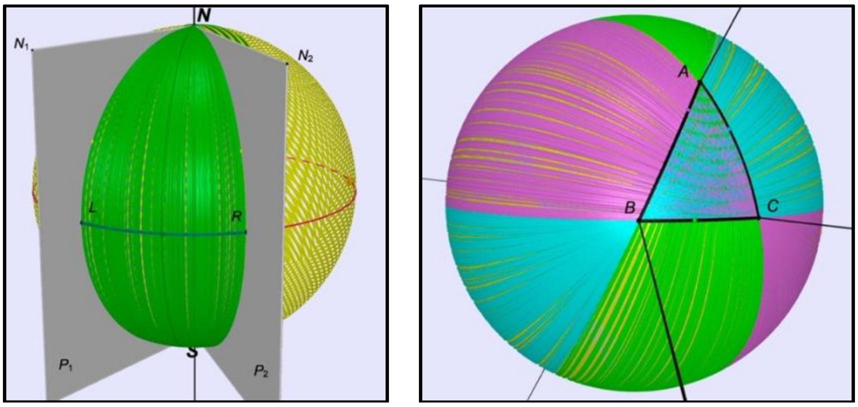

Figure 10: Area of a spherical triangle

2.3. Particular lines and points of a spherical triangle (investigations)
The following investigations aims to answer the question: can we expect that the known particular points of the geometry of the triangle in the plane exist in spherical geometry? That is to say: can we expect that the medians of a spherical triangle meet at the same point? The same question applies for the perpendicular bisectors, the angle bisectors and the altitudes. If yes can we expect that these points are collinear as they are in plane geometry (*Euler's line*). The first problem to solve is: "the constructions of these lines in spherical geometry".

2.3.1. Medians and centroid
As displayed in Figure 11 left, we can state that the three medians intersect at the same point even if we move points *A*, *B* or *C* to different positions. The justification is easy (Figure 11 middle): points *I*, *J* and *K* are the intersections between rays [*Oi*), [*Oj*) and [*Ok*) with the great circle containing arcs *BC*, *CA* an *AB* (where *i*, *j* and *k* are the midpoints of segments [*BC*], [*CA*] and [*BA*]. So, *I*, *J* and *K* are the midpoints of spherical segments S(*BC*), S(*CA*) and S(*AB*). As we know, in triangle *ABC*, the medians intersect at the centroid *g*, necessarily, the spherical medians intersect at a same point *G* which is the intersection between ray [*Og*) and the sphere. The reason is because *g* is also located on the intersection of the three planes *OAi*, *OBj* and *OCk* which is (*Og*); these planes are the same as the one containing the three spherical segments S(*BJ*), S(*CK*) and S(*AI*). But, knowing that *g* is at the

two thirds point of segment [*Ai*], angle $\widehat{gOA}$ is not necessarily the two thirds of angle $\widehat{gO\imath}$ and so *G* is not necessarily at the two thirds of spherical segment S(*AI*). The same remark applies for the location of *G* on the other spherical segments S(*BJ*) and S(*CK*). An investigation displayed on Figure 11 right confirms this remark: as the ratio between *gA* and *gI* is evaluated by 2, the ratio between angles $\widehat{gOA}$ and $\widehat{gO\imath}$ is evaluated by 1,36.

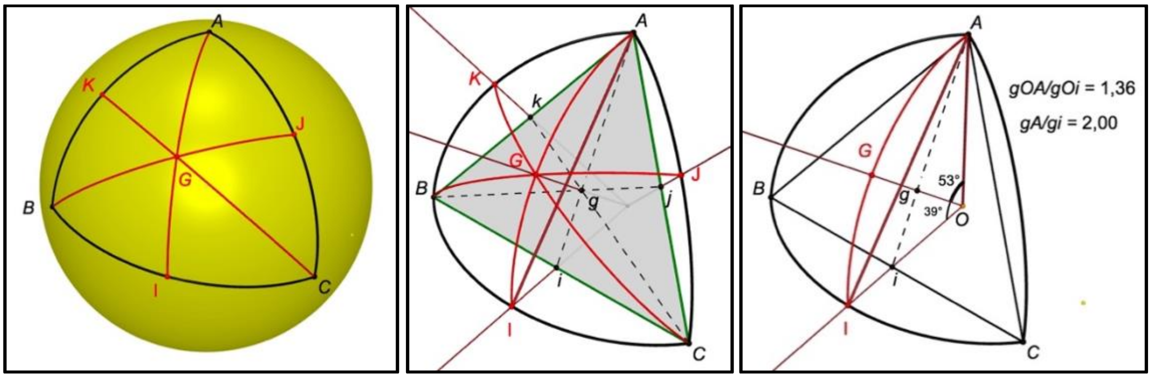

Figure 11: Centroid of a spherical triangle

2.3.2. Perpendicular bisectors and circumscribed circle

In Figure 12 left, we have constructed the three spherical perpendicular bisectors of the three sides of the spherical triangle *ABC* (three great circles). We can state (in fact, conjecture!) that these three spherical lines intersect at the same point *c* and at its symmetric point with respect to the center of the sphere. We can check (Figure 12 middle) that these two points are the centers of two spheres whose intersection is a circle containing *A*, *B* and *C*. This circle is the circumscribed spherical circle of the spherical triangle *ABC* because the three spherical arcs c*A*, c*B* and c*C* are equal. In Figure 12 right, we show how to construct the spherical perpendicular bisector of spherical segment S(*BC*) as the **circle** intersection between the sphere and the perpendicular bisector plane of segment [*BC*]. As the perpendicular bisector line of [*BC*] in triangle *ABC* belongs also to the plane of this circle, this proves the previous conjecture.

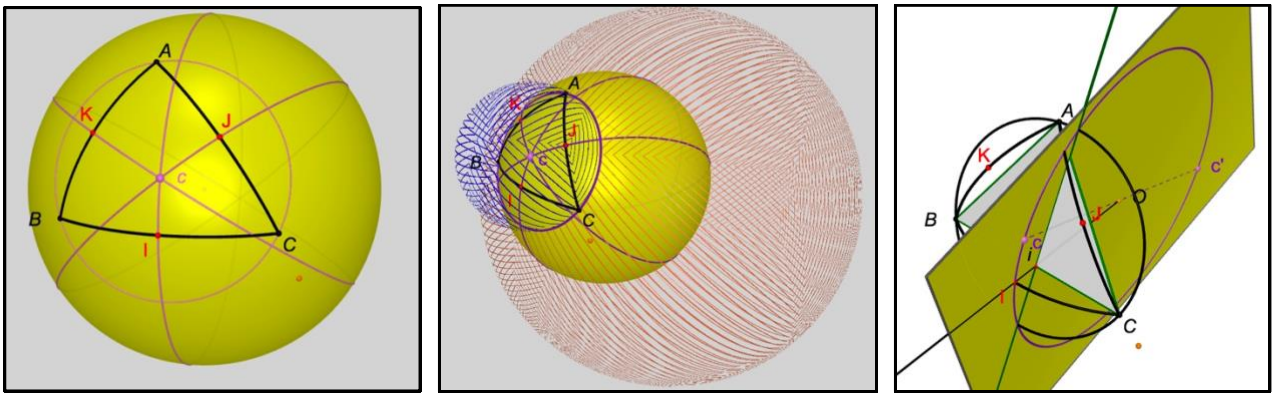

Figure 12: Perpendicular bisectors and circumscribed circle

2.3.3. Angle bisectors and inscribed circle

In Figure 13 left, we have conducted a construction inspired by the planar construction and we state that we are led to the same result in spherical geometry, that is to say: the three spherical bisector arcs of spherical triangle *ABC* (in green) intersect at the same point *s* which is the center of a circle tangent to each side of this spherical triangle. The radius of this circle is defined by knowing the position of *h*, intersection between arc *BC* and the great circle perpendicular to arc *BC* from *s*.

Constructions of the great circle containing *A* (spherical bisector of spheric angle $\widehat{BAC}$): see Figure 13 middle. Construct first, if not already constructed, the great circles supporting arcs *AB* and *AC* (using *A*, *b* and *B* for the first one and *A*, *c* and *C* for the second one). Construct the plane perpendicular to (*OA*) passing through *O* and its circle intersection with the sphere. This circle intersects the circles supporting arcs *AB* and *AC* at *B'* and *C'*. The perpendicular bisector plane of *B'* and *C'* intersects the sphere at the great circle containing *A* (in green). The same constructions produce three great circles of Figure 13 left. For the construction of point *h* (Figure 13 right), construct plane containing *B*, *C* and *O* and its perpendicular at *O*: line (*NS*) (*N* and *S* on the sphere). Construct the circle containing arc *BC*. *h* is the intersection between this circle and the circle containing *N*, *s* and *S*. *s* is the center of a sphere passing through *h* cutting the previous circle in the inscribed circle.

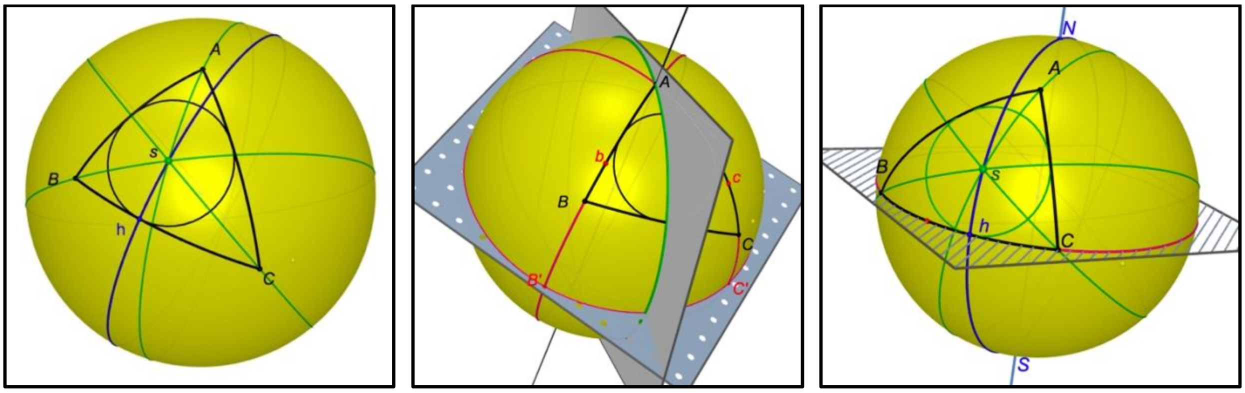

Figure 13: Angle bisectors and inscribed circle

2.3.4. Altitudes and orthocenter

In Figure 14 left we have constructed the three spherical altitudes of the spherical triangle *ABC*. The technique has been used in the previous paragraph to construct point *h* (Figure 13 right). We can state (conjecture) that these three great circles have two common points, *H* and *H'* (symmetric point of *H* with respect to the center of the sphere), called orthocenters of *ABC*. Figure 14 middle allows to state that this conjecture is true in all cases of the figure.

Proof: let us construct the two altitudes (*Bb*) and (*Cc*) intersecting at *H*. (Figure14 right). We know that (*Bb*) is the intersection between the sphere and the plane P_B containing O and perpendicular to the plane (*OAC*). We know also that (*Cc*) is the intersection between the sphere and the plane P_C perpendicular to the plane (*OAB*). The intersection line between the two planes P_B and P_C is line (*OH*). We know ([1]) that the third plane P_A containing *O* and perpendicular to the plane (*OBC*) contains the line (*OH*). Therefore, the intersection between P_A and the sphere is the third altitude of spherical triangle (*ABC*) which contains necessarily point *H*. That completes the proof.

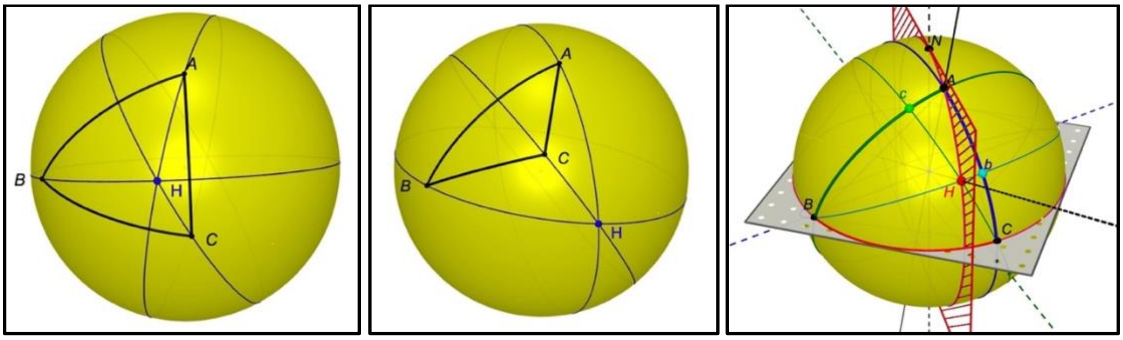

Figure 14: Altitudes and orthocenter

2.3.5. Euler’s line

We know that the four previous particular points are collinear in Euclidean plane geometry. The line defined by these points is known as *Euler’s line*. We construct in Figure 15 these four points in spherical geometry, in using the previous techniques, and we can state very quickly that these points are not collinear. In this figure we have also constructed the spherical lines (great circles) passing through any two of these four points. None of them contain one or two other points.

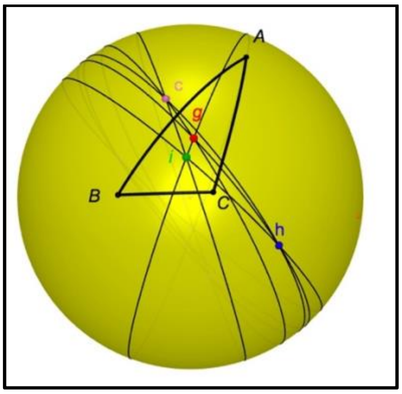

Figure 15: Investigations for a spherical *Euler’s line*

3. Stereographic projection and inversion

3.1. Stereographic projection (definition, investigations)

Definition: Let’s consider a sphere (S) and two points U and D such that [UD] is a diameter of the sphere (Figure 16 left). Let’s consider plane (P) tangent to this sphere at D. If M is a point of (S) different from U, its stereographic projection with repect to (S) is the point M' of (P), intersection of ray [UM) and (P).

Investigation: Let’s consider now a circle (C) of this sphere, defined as the intersection between (S) and another sphere (T) centered at a point t of (S) (be careful! the radius of the sphere (T) is not the radius of (C)). See Figure 16 middle and [3’]. To construct the image of circle (C) by the previous stereographic projection, we construct first the cone defined by point U and circle (C) and then we display the intersection between this cone and plane (P). We can state (Figure 16 right) that this image (C’) seems to be a circle. Cabri 3D says that this intersection curve is a “circular ellipse”. To corroborate this conjecture, we construct a circle passing through three points of curve (C') and note that this circle seems to be superimposed to (C') even if we change the radius of (C) or the position of the center t of sphere (T): you can notice in Figure 16 right that two sliders to command these possible modifications have been created. See [5’].

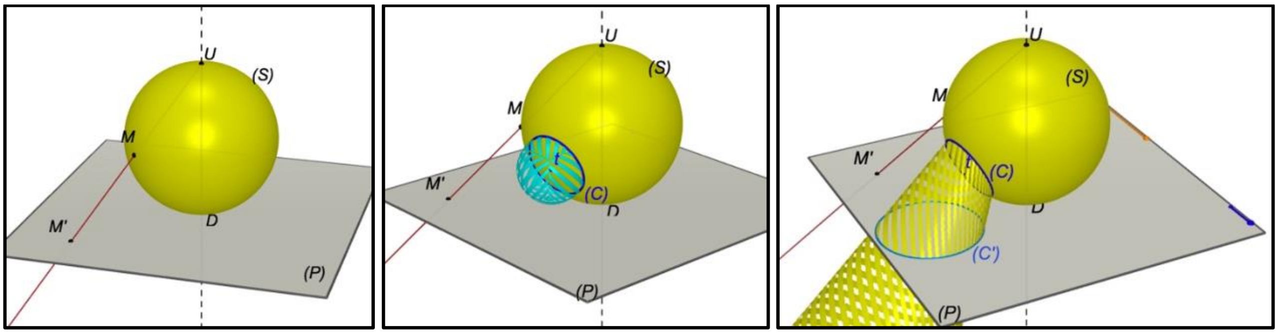

Figure 16: Stereographic projection of a circle

3.2. Image of a circle by a stereographic projection (proof)

In order to prove easily that the image of a circle by a stereographic projection is another circle, we will prove first that any sterographic projection is nothing else than an inversion. As shown in Figure 17 left, we conduct our reasoning in plane (*Q*) (the plane containing (*UD*) and *M* and by the way *M'*). The red circle is the one with diameter [*UD*] and passing through *M*. The two triangles *DUM'* and *MUD* are two similar right-angle triangles, therefore: $\frac{UD}{UM'} = \frac{UM}{UD}$ and finally $UM.UM' = UD^2$.

This last equality means that *M*' is the image of *M* by the inversion centered at *U* with scale of UD^2 or, which is the same thing, *M*' is the image of *M* by the inversion of the sphere centered at U of radius *UD*. As we know that the image of a circle (which does not contain the center of the inversion) is a circle by any inversion, we have established the result previously conjectured. See [6'].

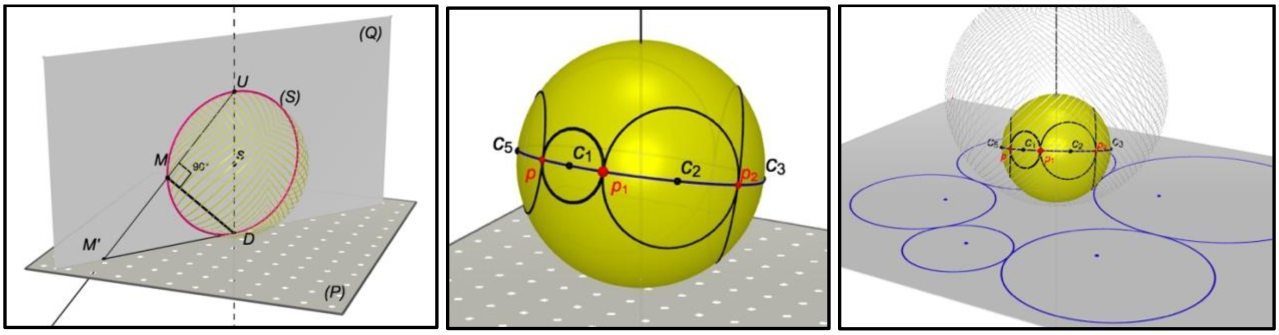

Figure 17: Stereographic projection and inversion

3.3. A nice dynamic consequence (inspired by Professor Chuan)

In Figure 17 middle, we construct a first circle with a little sphere centere at c_1. On this circle we create a point p_1 that can be animated. Then, we create the equatorial circle containing c_1 and p_1. With the same technique we create a second circle (with a sphere centered at c_2 on the equatorial circle) passing through p_1 and by the way tangent to the first circle. We iterate the process to create three other circles with the fifth one tangent to both the fourth one and the first one. We need a trick to find the position of c_5. c_5 is the intersection point between the ray [*si*) and the equatorial circle (*s* is the center of the big initial sphere and *i* the midpoint of segment [p_5 p_1]). When p_1 is animated along the first circle, the four other circles are animated on the initial sphere. If we transform these five circles by the stereographic projection of the given sphere with respect to the north pole or with the inversion of the sphere centered at the north pole and passing through the south pole, we obtain five tangent circles of the horizontal plane (Figure 17 right). You can observe that the images of points c_1, c_2, c_3, c_4 and c_5 are not the centers of the images of the five circles of the sphere. But we use these images to create flat cones that can be coloured as shown in Figure 18. See [7'].

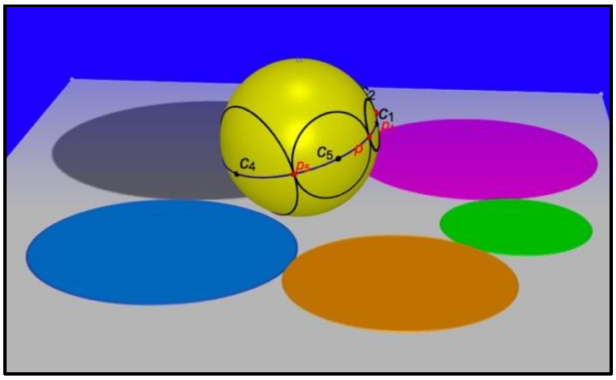

Figure 18: Five animated tangent circles

4. Conclusion

The aim of this paper was not at all to present new results about spherical geometry which is a subject that has given rise to a large number of very technical productions, especially in spherical trigonometry. Instead, the aim was to show an approach of spherical geometry that can convince teachers of the power of dynamic geometry to understand notions deemed difficult to teach. The last part related to stereographic projection aimed to show the powerful link with the notion of inversion and the possibilities of creating animations in the plane thanks to simple spherical constructions.

References

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YouTube videos links (videos in French)

[1'] Video of YouTube channel « jjdahan »: Droite et segment en géométrie sphérique https://www.youtube.com/watch?v=m0ejFi9NfUM

[2'] Video of YouTube channel « jjdahan »: Triangle en géométrie sphérique https://www.youtube.com/watch?v=TMoOKFzzu8k

[3'] Video of YouTube channel « jjdahan »: Cercle en géométrie sphérique https://www.youtube.com/watch?v=mWLdYXmnbM4

[4'] Video of YouTube channel « jjdahan »: Triangle rectangle en géométrie sphérique https://www.youtube.com/watch?v=ix0d_d7ow7Y

[5'] Video of YouTube channel « jjdahan »: Projection stéréographique d'un cercle https://www.youtube.com/watch?v=kE6aC51LMv0

[6'] Video of YouTube channel « jjdahan »: Lien projection stéréographique inversion https://www.youtube.com/watch?v=O0RM7C6kb_A

[7'] Video of YouTube channel « jjdahan »: Cercles tangents et projection stéréographique https://www.youtube.com/watch?v=f-vqmbFl3gM

Software

Cabri 3D by Cabrilog at http://www.cabri.com