Real Quantifier Elimination in the Classroom

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Abstract

We present the experimental command ***RealQuantifierElimination*** *in GeoGebra Discovery. The command provides quantifier elimination over the reals. We describe how this new command can be used in certain classroom situations. In our examples we focus on mathematical logic and elementary calculus (in particular, on definitions of basic notions and proving inequalities). Finally we conclude a potential impact of this new command in the educational world.*

1 Introduction

Quantifier elimination is a simplification method, well known in mathematical logic. Let us consider, for example, the statement S "for each x there exists y such that $x < y$" that contains two quantifiers, "$\forall$" and "$\exists$". Quantifier elimination requires us to find an equivalent statement S' that does not contain any more quantifiers. We can eliminate the quantifiers in certain cases, for example, if x and y range over integers, and – since the natural interpretation is "there is no greatest integer" in this case – the eliminated statement is "true".

Clearly, it is important where x and y come from. For example, if x and y are elements of a non-empty finite subset of $\mathbb{R}$, the quantifier elimination yields a false statement (since there is always a greatest element).

In the 20th century great efforts were made to learn when exactly quantifier elimination can always be made for an arbitrary quantified formula in a given theory. It turned out that certain theories are "well-behaved" and an automated way of quantifier elimination can always be done. These results go back to the exhaustive work of Alfred Tarski and others. Here we emphasize the Tarski-Seidenberg theorem [19] – it states that all quantified formulas that are interpreted on the reals and constructed from polynomial equations and inequalities by logical connectives and quantifiers can be equivalently expressed with another formula that has no quantifiers.

As a simple example we recall the classroom question: "When does the quadratic equation $ax^2 + bx + c = 0$ have two different solutions?" When using a formula, the question can be formulated as

$$S : a \neq 0 \ \wedge \ \exists x_1 \, \exists x_2 \, (ax_1^2 + bx_1 + c = 0 \ \wedge ax_2^2 + bx_2 + c = 0 \ \wedge x_1 \neq x_2).$$

An equivalent, quantifier-free formula is

$$S' : a \neq 0 \ \wedge \ b^2 - 4ac > 0.$$

As Tarski proved it in 1951, an equivalent quantifier-free formula can always be found (namely, in the first-order theory over the reals). But the original method, suggested by Tarski, was not computationally effective enough. The first effective way was published by Collins in 1975 [4] that was based on Łojasiewicz's concept [15] by using a cylindrical algebraic decomposition of the variable space of $\mathbb{R}^n$. Since then, several computer programs have been written that use either Collins' protocol, or some other similar methods. For more information we refer to [5, 7, 1, 13, 9, 18, 3].

Even if some of these implementations have been already existing for 30 years, until recently, however, no applications have been specifically designed for students, and there was no freely available web-based implementation either. This prevented many users, including secondary school teachers and students, from exploiting the benefits of real quantifier elimination.

This paper has two goals:

1. To illustrate how real quantifier elimination can be a fruitful process in stimulating the students' thinking in various mathematical topics.

2. To show how these topics can be supported with the freely available, effective implementation of real quantifier elimination in the software tool GeoGebra Discovery.

2 Implementation

We use the two introductory examples to show how GeoGebra Discovery [12], a free experimental version of GeoGebra [8], can handle inputs for real quantifier elimination, and how its outputs look like. See Figures 1 and 2 for the native Java version (available on Windows, Mac and Linux) and the web version, respectively.

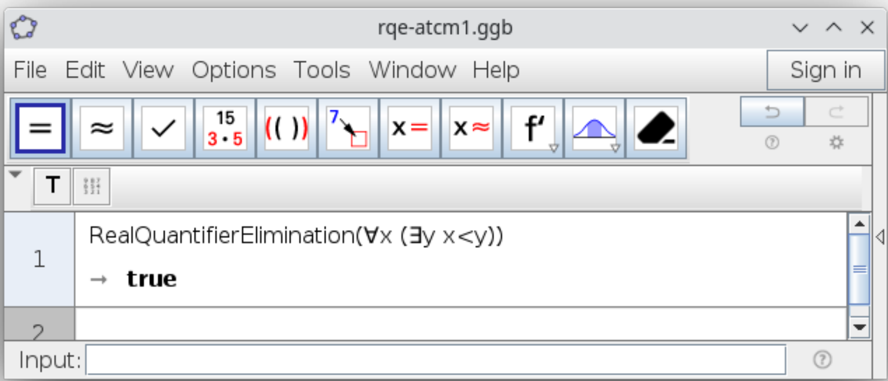

Figure 1: The Java version of GeoGebra Discovery (here, on platform Linux) checks if the statement "there is no greatest real number" is true.

The inputs must be entered in the CAS View.

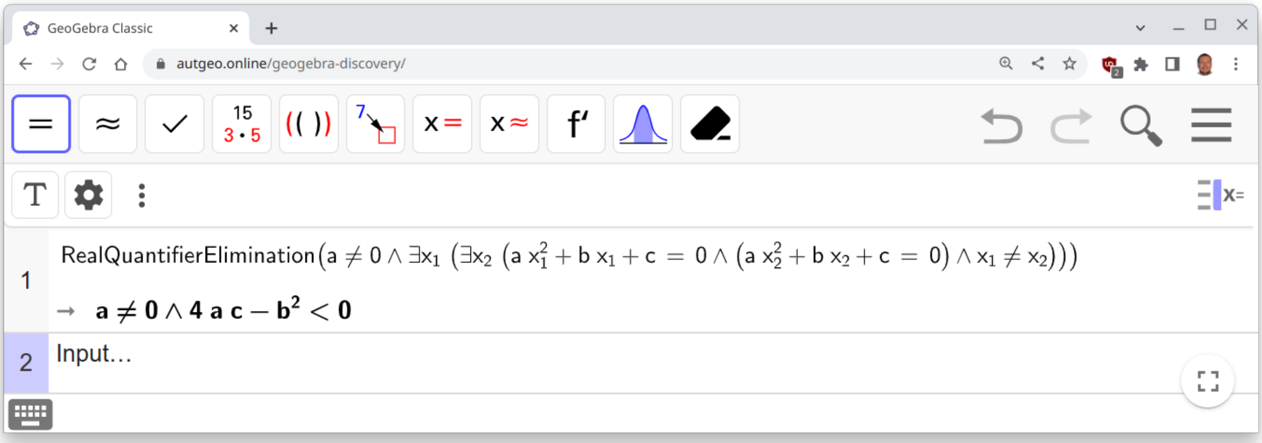

Figure 2: The web version of GeoGebra Discovery finds a quantifier-free formula for the statement "the quadratic equation $ax^2 + bx + c = 0$ has two different solutions".

The implementation is based on [20] and [2]. That is, an embedded version of the Tarski software system is used to ensure effective manipulations on quantified formulas and fast real quantifier elimination. At this point we need to warn the reader that *fast* here means a remarkable speed among competing systems, but not in absolute terms, because real quantifier elimination usually requires up to 2^{2^n} atomic steps where n is the number of variables (see [6]).

We do not go into the details how this effective implementation is programmed. The main idea (see [4]) is the cylindrical algebraic decomposition. In a nutshell, given the polynomials P in $\mathbb{R}^n$, a cylindrical algebraic decomposition is

1. a decomposition D of $\mathbb{R}^n$ into connected semialgebraic sets[1] called cells, on which each polynomial has constant sign (either $+$, $-$ or 0);

2. D must satisfy the following condition: If $1 \leq k < n$ and π is the projection from $\mathbb{R}^n$ onto $\mathbb{R}^{n-k}$ consisting in removing the last k coordinates, then for every pair of cells $c, d \in D$, one has either $\pi(c) = \pi(d)$ or $\pi(c) \cap \pi(d) = \emptyset$.

The main programming challenge is the implementation of an algorithm that creates a cylindrical algebraic decomposition of P by registering the cells in an exact symbolic way. This requires advanced computer algebra methods, including several operations on algebraic numbers of arbitrary degree.

GeoGebra Discovery is available for download on its GitHub page `https://github.com/kovzol/geogebra-discovery` and for direct web use at `https://autgeo.online` (in this paper we refer to version 2002Jul11).

[1] A semialgebraic set in $\mathbb{R}^n$ is a finite union of sets defined by a finite number of polynomial equations and inequalities, i.e., by a finite number of statements of the form $p(x_1, \ldots, x_n) = 0$ or $q(x_1, \ldots, x_n) > 0$ for polynomials p and q.

3 Real quantifier elimination for students

One important consequence of the Tarski-Seidenberg theorem is that real quantifier elimination is always possible to do, at least in theory. That is, each question S that can be formulated with polynomial equations and inequalities by logical connectives and quantifiers can be answered either with a yes/no answer, or a condition S' can be given when S is true. This opens up an extremely wide range of applications. One such application is in real analysis.

In this paper we will mostly focus on real analysis. But there exist other important fields, for example, Euclidean planar geometry with algebraic translation. This is, however, out of scope in our paper.

3.1 Formulation

A very important discipline characteristic of mathematics is to express mathematical content in an exact way. At a certain level in schools there is already a need to formulate mathematical content precisely. For example, in Austria, 14-years-old students of secondary level usually begin their first math classes with the topic of logic and sets. Textbooks usually demonstrate precise formulations of simple statements. For example, the schoolbook [16] starts with statements, logical operations (negation, conjunction and disjunction) on them, and implications and equivalences between statements.

One of the first exercises, 1.03, asks the student to decide if some statements are equivalences or implications of each other. For example, the student must decide if

$$x \neq 0 \iff x < 0 \vee x > 0.$$

This simple question can already be challenging, but it can be well supported by technical means. We show how this exercise can be approached with our implementation in GeoGebra Discovery.

First, the student needs to open the CAS View. A direct way of learning if the first part of the equivalence, namely,

$$x \neq 0 \Rightarrow x < 0 \vee x > 0$$

is correct, by typing `RealQuantifierElimination(`$x \neq 0 \to x < 0 \vee x > 0$`)` which gives true. The logical operators can be typed via some keyboard shortcuts (Alt-H for $\neq$, Alt-J for $\vee$), or the virtual keyboard can be used to select them accordingly.

Manipulating parts of the implication, namely, to store the first or the last parts of the implication, and then perform further operations, is also possible in our implementation. However, some technical difficulties (and mathematical issues) may arise. In Figure 3 we assign the statement $x \neq 0$ to the variable A, this statement will be silently rewritten to $(0 > x) \vee (x > 0)$ by the program, but we ignore this output now. We create statements $x < 0$ and $x > 0$ as lines \$2 and \$3 and refer to them on the 4th line. (Here, for example, \$2 is shorthand for the formulae found in line number 2.) Note that GeoGebra Discovery automatically added a couple of parentheses to ensure the correct evaluation order. Refraining a logical operation between \$2 and \$3 is also supported on line 5. In addition, line 6 gives a fully accurate output on when exactly the implication $x \neq 0 \Rightarrow x > 0$ occurs. Finally, line 7 shows that in general $x \neq 0 \Rightarrow x > 0$ is false.

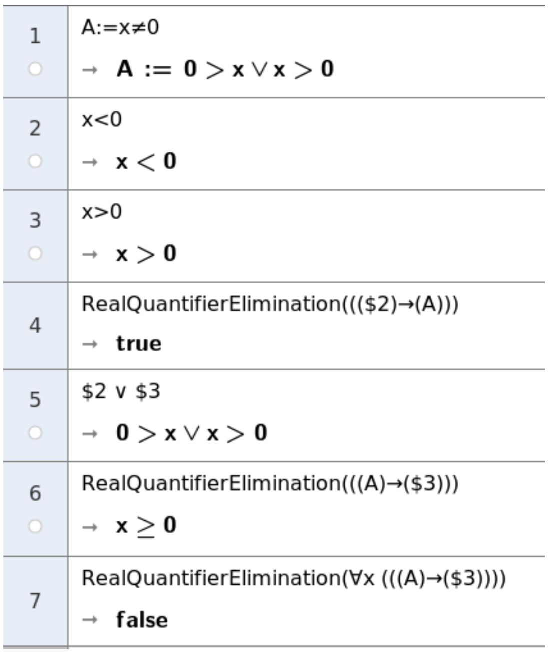

Figure 3: Some introductory examples on handling logical formulas in GeoGebra Discovery.

The user interface uses the implication sign "$\to$". This may differ from the notation "$\Rightarrow$" that is used in certain textbooks. On the other hand, entering logical operations and quantifiers is relatively simple by using the keyboard shortcuts Alt-Y ("$\to$"), Alt-Z ("$\neg$"), Alt-K ("$\wedge$"), Alt-X ("$\exists$") and Alt-V ("$\forall$"), or the virtual keyboard.

Correct syntax of the input is mandatory. In most cases equations and quantified expressions must be entered in parentheses. We think that the expected syntax is, however, very simple, and there are just a couple of software systems that provide easier syntax (see GNU Aris, `https://www.gnu.org/software/aris/`, for an example). Easy syntax can be very important for students to make them possible to focus on the mathematical content – but, on the other hand, indications towards mathematical rigor are also unavoidable.

3.2 Textbook examples

We already mentioned a typical question on quadratic equations in the Introduction. Here we repeat a similar exercise (task 3.49 in [16]): *Find all $k \in \mathbb{R}$ such that the equation $x^2 - kx + 1$ has at least one solution.* Here GeoGebra Discovery can solve this question as shown in Figure 4. We use GeoGebra's variable definition feature for the polynomial $p(x)$ on line 1. Line 2 rewrites the input via the Tarski subsystem and line 3 performs the real quantifier elimination; these steps can be combined as a single step on line 4.

We think that this kind of communication between user and machine should be acceptable for students.

Another example from [16] is related to basic notions of calculus like monotonicity and extrema. Exercise 6.20 raises the question: *Given the function $f : [-3, 1] \to \mathbb{R} \mid x \mapsto x^2 + 2x$. Find its monotonicity constraints, its extrema and its zeros.* The exercise shows a helpful plot of the graph to help the students at this level. In fact, the notions given in this example are

1	p(x):=x^2-k x+1 → $p(x) := x^2 - k\,x + 1$
2	∃x p(x)=0 → $\exists x\ (-1\ (x\,k - x^2 - 1) = 0)$
3	RealQuantifierElimination($2) → $k - 2 \geq 0 \vee k + 2 \leq 0$
4	RealQuantifierElimination(∃x p(x)=0) → $k - 2 \geq 0 \vee k + 2 \leq 0$

Figure 4: A solution for task 3.49 in [16].

not yet exactly introduced, just later in the next school year (see [17, p. 42]), but for that second time they appear in a precisely defined form. Nevertheless, it is possible to use real quantifier elimination to express the most precise mathematical content and get the solution in a not completely trivial form – see Figure 5 for finding the monotonicity constraints. (In the provided solution we check *strict* monotonicity.)

1	f(x):=x^2+2x → $f(x) := x^2 + 2\,x$
2	x_1>=a ∧ x_1<=b → $x_1 \geq a \wedge b \geq x_1$
3	x_2>=a ∧ x_2<=b → $x_2 \geq a \wedge b \geq x_2$
4	x_1 < x_2 → $x_2 > x_1$
5	(($2 ∧ $3 ∧ $4)→(f(x_1) < f(x_2))) → $a > x_1 \vee x_1 > b \vee a > x_2 \vee x_2 > b \vee x_1 \geq x_2 \vee x_2^2 + 2\,x_2 > x_1^2 + 2\,x_1$
6	RealQuantifierElimination(∀x_1 (∀x_2 ($5))) → $a + 1 \geq 0 \vee -(b - a) \geq 0$

Figure 5: A solution for task 6.20 in [16].

We remove the restriction of working with the interval $[-3, 1]$ since it is not really helpful in the automated solution. Instead, we set up an interval $[a, b]$ to search for monotonicity on it. Now $6 yields the sought solution, namely, that $a \geq -1$, and the second part of the disjunction is just an extra condition (the negation of $a < b$) that clarifies that the input should indeed be an interval.

Note that we *proved* something automatically. Even if the used atomic steps (in the cylindrical algebraic decomposition of $\mathbb{R}^4$) for the users are completely hidden, the resulted output can be considered as the artifact of a mechanical proving process.

Proofs and proving are often part and parcel of secondary school mathematics – in fact, several exercises ask for explaining certain properties or statements. For example, the tasks 3.02ab in [17] ask for explaining why strict monotonicity implies monotonicity. Also, tasks 3.02cd ask for the converse and request either an explanation or a counterexample.

Some concepts with exact definitions may appear just later, eventually at undergraduate level. *Boundedness* or *convergence* are defined mostly at university level with a completely exact description. However, [17, p. 134] (written for 15-years-old students) gives the following definition:

> The number a is the limit of sequence $(a_n \mid n \in \mathbb{N}^*)$, written as $a = \lim_{n\to\infty} a_n$ if following holds: For each $\varepsilon \in \mathbb{R}^+$ there exists an index $n_0 \in \mathbb{N}^*$ such that $|a_n - a| < \varepsilon$ for all $n \geq n_0$. (Here $\mathbb{N}^*$ denotes the positive integers.)

There is no doubt that this definition can be extremely challenging to understand at this level for most students. In many schools in Austria, therefore, such definitions are skipped (but kept for the interested students and for future reference). Luckily, real quantifier elimination can be helpful to enlighten the meaning of such complicated mathematical constructions.

In fact, studying the convergence of a sequence is not supported in real quantifier elimination, but we can use the definition of the limit of a *function* instead. Since we need to use polynomials, an entire translation of concepts would lead to study the behavior of a polynomial at infinity, namely, for example $\lim_{x\to\infty} x^2$ which is ∞, that is, instead of convergence we have divergence. Maybe a better start is to study

$$\lim_{x\to 3} x^2$$

which is 9, but here we lose a major part of infinity in the problem setting. Another option is to use rational functions like $f(x) = \dfrac{1}{x}$ that are divisions of polynomials, but in this case we must be careful and take care of denying the division by zero.

By using GeoGebra Discovery's CAS View, we can issue the following input commands to formalize and prove several statements of introductory calculus:

- `RealQuantifierElimination(`$\forall M\ (\exists N\ (\forall x\ (x > N \to x^2 > M)))$`)` yields true and it proves that the function $y = x^2$ diverges to infinity.
- `RealQuantifierElimination(`$\forall\varepsilon\ \varepsilon > 0 \to (\exists\delta\ \delta > 0 \land (\forall x\ |x-3| < \delta \to |x^2 - 9| < \varepsilon))$`)` yields true and this proves that the function $y = x^2$ is continuous at point $x = 3$ and it converges to 9.
- `RealQuantifierElimination(`$\forall x_0\ (\forall\varepsilon\ \varepsilon > 0 \to (\exists\delta\ \delta > 0 \land (\forall x\ |x-x_0| < \delta \to |x^2-x_0^2| < \varepsilon)))$`)` yields true and it proves that the function $y = x^2$ is continuous at point x_0 and it converges to x_0^2.

The correct formulation of such complex statements may be far from trivial. Still, it seems fruitful to build up mathematics with the support of precise logic, including clear syntax and immediate response from the computer if something is entered incorrectly.

To avoid constructing very difficult statements in the beginning, teachers may consider experimenting with simpler definitions which may also be part of the curriculum. The simpler exercises provide scaffolding for students to become familiar with the syntax and some non-trivial features of quantified formulas, including negations of implications. Some ideas to experiment with:

- Solving (in)equations or systems or (in)equations. For example, the inequalities $y > x^3 - 1$ and $1 - x^3$ can be graphically easy to solve, but many computer algebra systems have difficulties with this. By contrast, GeoGebra Discovery can find the solution after issuing `RealQuantifierElimination(`$\exists y\ (y > x^3 - 1 \wedge y < 1 - x^3)$`)` (it is $x < 1$).
- Generalizing or specializing the examples from the Introduction. For example,

  $$\texttt{RealQuantifierElimination(}\exists x\ (x^3 + a\,x^2 + b\,x + c = 0)\texttt{)}$$

  returns true because each cubic function has a zero; or

  $$\texttt{RealQuantifierElimination(}\forall x\ (\exists y\ (y < 1000 \wedge x < y))\texttt{)}$$

  returns false (since a maximizing constraint for y prevents fulfilling the statement for big x values).
- Simple properties of a real function like *periodicity*, *evenness* or *oddity* are surprisingly easy to formulate, and to (dis)prove. For example,

  $$\texttt{RealQuantifierElimination(}\forall x\ (x^2 = (x + p)^2)\texttt{)}$$

  yields $p = 0$, this means that the function $y = x^2$ is not periodic.

The collection [10] shows several additional examples that aim mainly at undergraduate students, but there the Mathematica frontend is used. Many examples of that list can be tried in GeoGebra Discovery as well with success, however, some reorganizing of the input may be required. (For example, divisions must be eliminated first, because the undelying system Tarski does not support division by polynomials.) Here we show two examples of [10] after preparing the input for use with GeoGebra Discovery and Tarski:

12. `RealQuantifierElimination(`$xy(x^2 - y^2) > 0 \wedge (x^2 - 1)(x^2 - y^2) < 0$`)` yields

    $$\begin{aligned} x \neq 0 \wedge x + 1 \neq 0 \wedge y \neq 0 \wedge x - 1 \neq 0 \wedge y - x \neq 0 \wedge y + x \neq 0 \wedge \\ (x + 1 < 0 \wedge y + x > 0\ \vee\ x + 1 > 0 \wedge y < 0 \wedge y - x > 0 \vee \\ y > 0 \wedge x - 1 < 0 \wedge y - x < 0\ \vee\ x - 1 > 0 \wedge y + x < 0). \end{aligned}$$

    This output is however somewhat complicated as given by Mathematica.

22. `RealQuantifierElimination(`$(a\,d - b\,c = 1) \to a^2 + b^2 + c^2 + d^2 + a\,c + b\,d > 1$`)` proves the expected implication.

Such examples are not very typical in classroom situations, but may be of interest for contest challenges. As a final example we show the mechanical solution for a recent problem of the Austrian Mathematical Olympiad:

> Show that for all numbers $x > -1$, $y > -1$ and $x + y = 1$ the inequality
>
> $$\frac{x}{y+1} + \frac{y}{x+1} \geq \frac{2}{3}$$
>
> holds. When does equality occur?[2]

In Figure 6 we can learn that the statement is true. By changing the inequation sign to

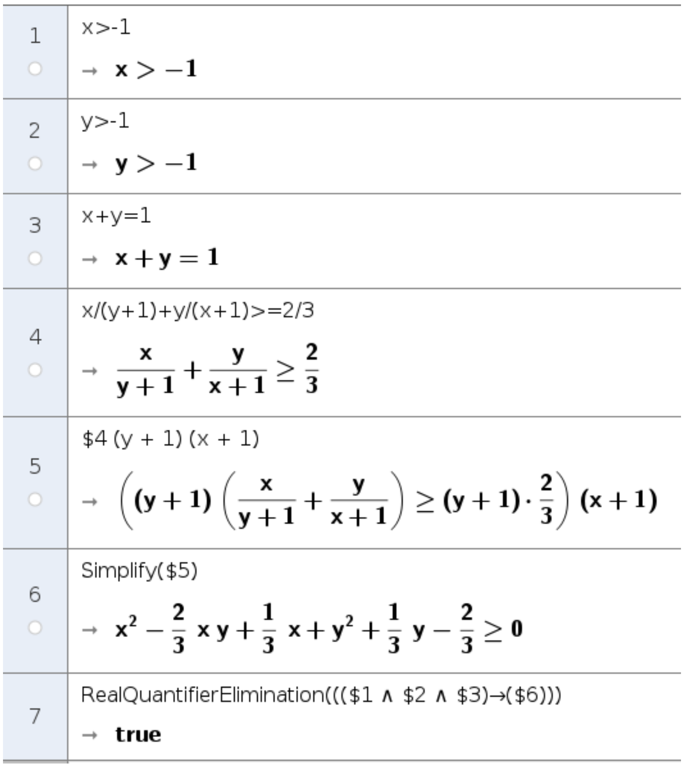

Figure 6: A solution for JRW-2022-1 of the Austrian Mathematical Olympiad (ÖMO).

"=" on line \$4, we obtain that $y + 1 \leq 0$ or $y + x - 1 \neq 0$ or $y - 2 \geq 0$ or $2y - 1 = 0$. The first two parts of this disjunction can be excluded (they are negations of \$2 and \$3), so the expected answer is $y \geq 2$ or $y = \frac{1}{2}$. Accordingly, $x \leq -1$ or $x = \frac{1}{2}$, but the first assumption is contradictory to \$1. So the only solution is $x = y = \frac{1}{2}$.

4 Didactical comments

Understanding many notions and definitions of basic calculus is far from being straightforward for most students. Both for students and teachers, it remains a challenge to explain why the

[2] Obtained from `https://www.math.aau.at/OeMO/Downloads/datei/818`.

actual precise definitions are necessary, and what outcome can be reached if proper notions are used.

For example, definition of convergence and the related notions (divergence, continuity) requires a very sophisticated and long description. Why do we require such a complicated definition? Can we leave some parts of the definition out to get the same or some useful meaning? Such questions could be answered with experimenting. If these experiments are performed with the help of a computer, that is, the logical syntax of the input can be verified in a simple way by technological means, it seems something beneficial. Also, if the truth of a formula can be immediately evaluated by a software tool, the student can immediately check if his/her experiments harmonize with the mathematical theory or not.

In Figure 7 a short experiment is shown that tries out some inputs related to the definition of convergence of the sequence $1/n$. The first input contains an error – this is a technical

1	∀ε (∃N (n>N → 1/n<ε)) → Sorry, something went wrong. Please check your input
2	∀ε (∃N (n>N → 1<n ε)) → $\forall \varepsilon\ (\exists \mathbf{N}\ (\mathbf{n} - \mathbf{N} \leq \mathbf{0} \vee \varepsilon\ \mathbf{n} - \mathbf{1} > \mathbf{0}))$
3	RealQuantifierElimination($2) → **true**
4	¬$2 → $\exists \varepsilon\ (\forall \mathbf{N}\ (\mathbf{n} - \mathbf{N} > \mathbf{0} \wedge \varepsilon\ \mathbf{n} - \mathbf{1} \leq \mathbf{0}))$
5	RealQuantifierElimination($4) → **false**

Figure 7: An experiment with the definition of convergence.

problem since divisions need to be rewritten to use polynomials only. The second line contains a formula that is already correct, at least, syntactically. By checking its logical truth on the third line we can be confirmed that the constructed formula *seems* to describe something very similar to the definition of convergence – but be warned, there was no assertion made on the expected positivity of ε and no absolute value of some of the formulas was introduced. Also, we did not take care of occasional change of the direction of the relation. So we should have a deeper look why these assertions can be omitted in this special case.

Some authors emphasize the importance of *negation* of logical formulas to deepen understanding of their meaning. For example [14], a textbook widely used in Hungarian teacher training for many decades, gives a detailed explanation on how to deny the definition of the limit of a function at a given point, when using Cauchy's definition (similarly to line 2 in Figure 7). Here, by interpreting the outputs of $2 and $4, a student can discover that the implication in $2 was substituted by a disjunction and its negation in $4 became a conjunction (by using the well-known formula by De Morgan). As expected, the evaluation of $4 as line 5 yields false, and this can also be a comforting result.

The notation in Figure 7 follows the usual notation for the limit of *sequence* intentionally. This can be misleading for some students: they may think that we are working over the integers when the variables N and n are introduced. But, of course, we are still doing all computations over the reals! Thus we (and our students) need to double check if the original problem setting still fits the changed mathematical model of the experiment. The reader may say that our experiments may differ *a lot* from the original problem setting, hence the practical use of these tools are questionable. But let us insist: Solving just a *similar* problem can also be beneficial. At the end of the day, our students should understand the notions and solve the problems on *their* own. Utilizing a computer is just an aid on the road of learning.

5 Conclusion

We demonstrated some novel ways to teach logical formulas and some concepts of elementary calculus with the help of GeoGebra Discovery, an experimental version of GeoGebra. The command **RealQuantifierElimination** can be a powerful tool in many classroom situations. It is based on an embedded version of the free software system Tarski. By using free software components, teaching mathematical logic and function calculus may be supported by technology even better than before.

We admit that our presentation is just a beginning of the plans of a longer research. We have not yet received enough feedback to draw conclusions of the usefulness of our developments yet. Also, the implementation is not completely matured yet in the sense of internal representation of quantified formulas (currently only prenex formulas are fully supported). We also need to improve the cooperation between Giac (GeoGebra's main embedded computer algebra system [11]) and Tarski in the future.

As mentioned above, real quantifier elimination is theoretically always successful, but practical use has its limits. Above a certain number of variables the computations may be very slow, too heavy for classroom use, or even infeasible. Geometry problems can be on the border of computational complexity, however, some simple theorems can be well illustrated via this method. (See Figure 8 for a proof and an illustration of the well-known theorem "the altitudes of a triangle are concurrent".) Thus, technical improvements (concerning speed as first priority) remain an important item for future work.

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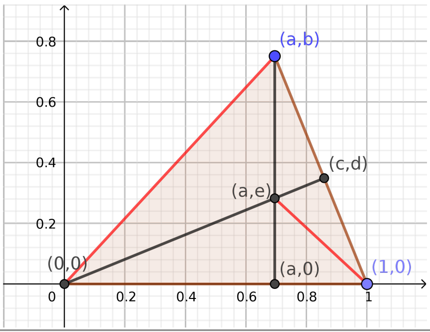

1	H1:=c^2-c+d^2=0 → $\mathbf{H1 : c^2 + d^2 - c = 0}$
2	H2:=a d+b-b c-d=0 → $\mathbf{H2 : a\,d - b\,c + b - d = 0}$
3	H3:=a d-e c=0 → $\mathbf{H3 : a\,d - c\,e = 0}$
4	T:=a^2-a+b e=0 → $\mathbf{T : a^2 + b\,e - a = 0}$
5	RealQuantifierElimination(∀a (∀b (((H1 ∧ H2 ∧ H3)→T)))) → $\mathbf{d^2 + c^2 - c \neq 0 \vee d \neq 0 \vee e < 0 \wedge (c - 1 = 0) \vee e > 0 \wedge (c - 1 = 0)}$
6	RealQuantifierElimination(∀a (∀b (((H1 ∧ H2 ∧ H3 ∧ b > 0 ∧ e > 0)→T)))) → **true**
7	RealQuantifierElimination(∀a (∀b (((H1 ∧ H2 ∧ H3 ∧ b ≠ 0 ∧ e > 0)→T)))) → **true**
8	RealQuantifierElimination(∀a (∀b (((H1 ∧ H2 ∧ H3 ∧ b ≠ 0 ∧ e ≠ 0)→T)))) → **true**
9	RealQuantifierElimination(∀a (∀b (((H1 ∧ H2 ∧ H3 ∧ c≠1)→T)))) → $\mathbf{d^2 + c^2 - c > 0 \vee c > 0}$

Figure 8: A mechanical proof of the concurrence of altitudes of a triangle. Without loss of generality we can assume that the vertices of the triangle are $(0,0)$, $(1,0)$ and (a,b). Now hypothesis $H1$ assumes that line $(0,0)$–(c,d) is perpendicular to (a,b)–$(1,0)$, $H2$ assumes that (a,b), (c,d) and $(1,0)$ are collinear, and $H3$ assumes that $(0,0)$, (a,e) and (c,d) are collinear. The expected conclusion T claims that the line $(0,0)$–(a,b) is perpendicular to (a,e)–$(1,0)$. But this conclusion cannot be drawn directly, only if some extra conditions (namely, the non-degeneracy of the triangle) are additionally assumed. Luckily, \$5 gives us very useful hints on how the extra conditions should look like.