Computational thinking in mathematical modelling: an investigative study

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Abstract:

It has been suggested that computational thinking, based on fundamental concepts of computing science, provides a useful approach to everyday problem solving. It has also been seen that computational approaches can play a major role in the work of professionals in various fields, and it is therefore pertinent that computational thinking be given some attention in schools to prepare students for the future workforce. One way to do so is to expose students to modelling challenges, and through these, provide opportunities for students to learn, practise and refine their skills and competencies in both computational thinking and practical problem solving. In this paper, we describe the interaction and interplay between computational thinking and mathematical modelling through students' experiences in an international mathematical modelling contest. Students' ability to apply computational thinking in the contest was inferred and investigated via case studies. Data sources in the form of report artefacts, videos, interviews and judges' comments formed the basis of the case studies. The investigation reveals that the constructs of computational thinking such as pattern recognition, abstraction, decomposition and algorithm creation play a critical role in the successful completion of the students' modelling tasks.

Introduction

The concept of computational thinking and its role in mathematics learning and teaching have been a topic of discussion among mathematics educators in recent times. Following the proposition from Wing that "computational thinking is a fundamental skill for everyone" [12], and with an increase in the use of digital technology in the classroom in the past decades, many researchers had carried out studies and debated on computational thinking, and how it can be introduced or taught in schools [4], [5], [11].

Although the idea of computational thinking seems to have been popularized by Wing in recent years, the term actually appeared earlier in Seymour Papert's book, *Mindstorms: Children, Computers and Powerful Ideas*, published in 1980. Papert envisioned a world where children would be using computers to learn and think, and we would be integrating computational thinking into everyday life [7].

Indeed, in more recent times, computational thinking is now thought of as an important 21st century skill that everyone should possess [6]. In fact, in many parts of the world, coding or computer programming is being aggressively promoted in schools and communities, in a bid to stay ahead of the curve [8], [9]. Yet, how these initiatives may actually bring about the development of computational thinking has not been extensively studied, and their impact on and role in problem solving remains not fully understood. In fact, there are researchers who have cautioned against "over-selling" computer science and "raising expectations that cannot be met" [10]. However, it seems possible to recognize and identify some aspects of computational thinking and how these are observed in problem solving.

In her paper, Wing did not define computational thinking in any precise way, and it was unclear how computational thinking could lead to improved problem solving, or how it can be taught. Nevertheless, what is clear about the idea of computational thinking is that it includes notions and constructs such as *abstraction*, *decomposition*, *pattern recognition* and *algorithm creation*. These four constructs have since formed what is sometimes called the "cornerstones" of computational thinking, as depicted in Figure 1.

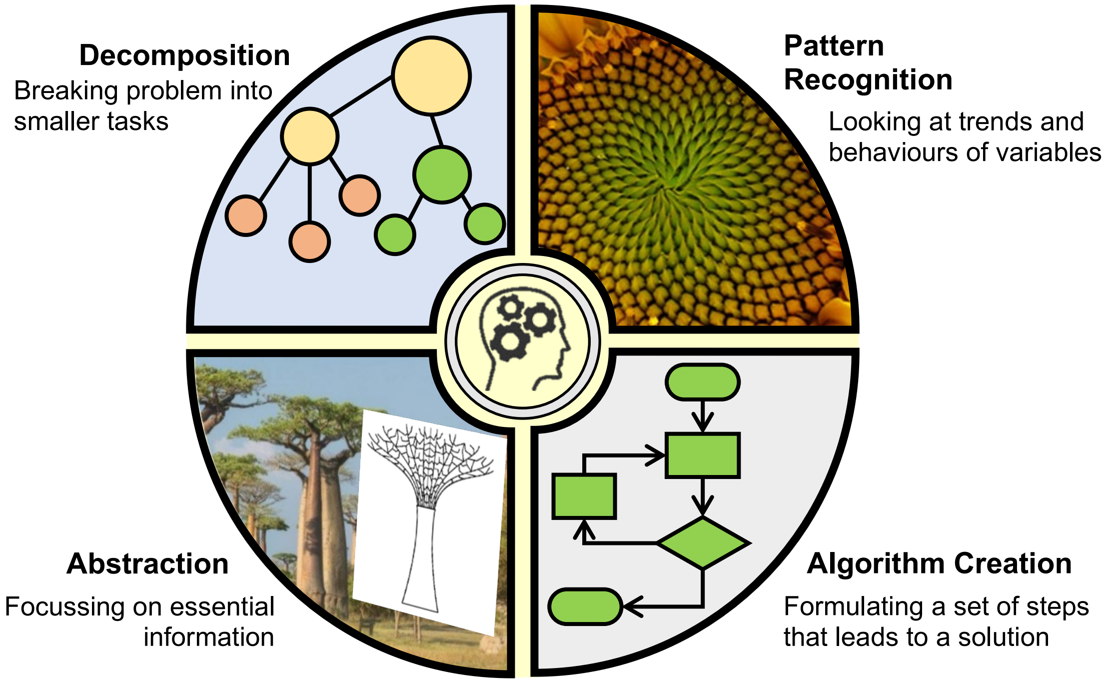

Figure 1: Aspects of Computational Thinking

For the purpose of our present discussion, we shall define computational thinking as a mental process of reasoning to approach, design and construct solutions to problems with a view to implementing and executing these solutions on the computer or using a computing tool. Therefore, the main goal is to study and examine problems in such a way that eventually yields a solution which can be implemented with a computing tool.

In general, this reasoning process will include the following.

- reducing a large, complex problem into simpler situations or smaller parts, and solving these first before building towards the complete problem solution (decomposition);
- examining the problem to detect possible trends or trajectories, and looking to see if some known or familiar solution approach can be employed (pattern recognition);
- collecting only the essential information in the problem and removing non-essential parts or components for solutioning (abstraction); and
- writing a set of step-by-step instructions to solve the problem (algorithm creation).

In fact, these mental processes are so important to problem solving that it has been suggested that these be developed in students as habits of mind, particularly in mathematical modelling tasks [2]. In other words, one hopes that when a student is confronted with a problem situation, the response that would first comes to the student's mind would be one or more of the above, thus making it a habit of mind to think computationally in problem solving. In the case of mathematical modelling, such habits can be developed quite effectively, simply because the nature of the task of mathematical modelling provides ample opportunities for these mental processes to be applied and practised.

Linking Mathematical Modelling and Computational Thinking

Mathematical modelling, its process and its place in the school teaching curriculum have been discussed extensively by many researchers in the past decades. Although there is no universal consensus on the actual process and there exist many versions of diagrams and pictorials depicting the process, there is some general agreement on the different stages of the modelling process. For the present discussion, the process shown in Figure 2 below is used.

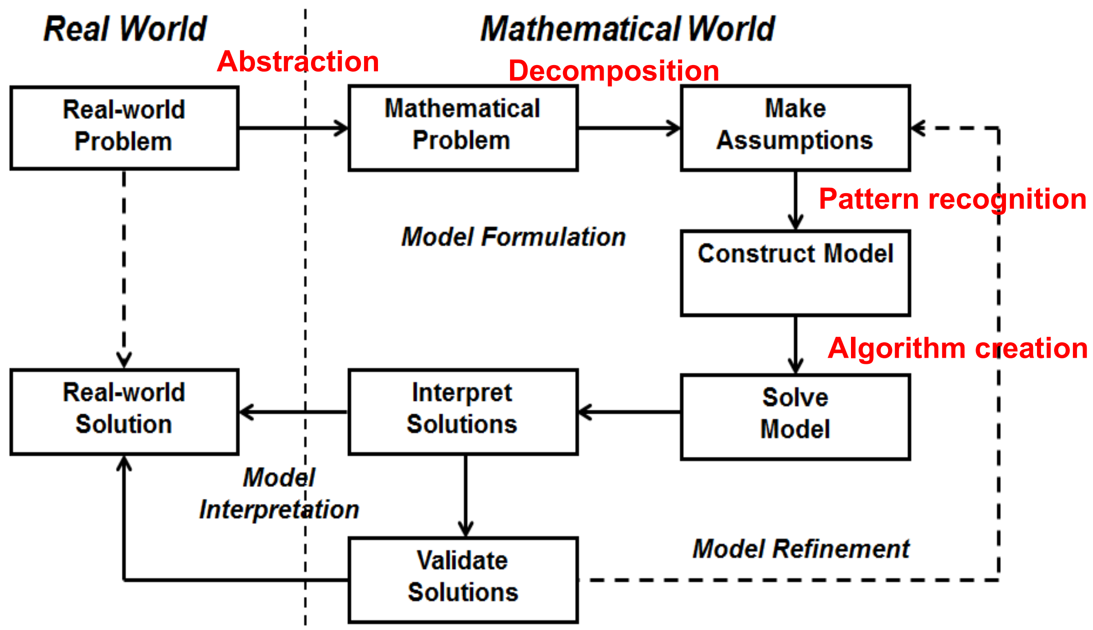

Figure 2: The modelling process (adapted from Ang, 2019, p.3)

In any modelling process, we begin with a real-world problem, and our goal is to reach a real-world solution. However, this direct path is often not trivial and sometimes impossible. The idea then is to cast the real-world problem into a mathematical problem. The process of moving from the real world to the mathematical world would often involve a fair amount of abstraction. Non-essential parts of the problem are removed, leaving only factors or variables that matter.

Even with abstraction, if the problem remains too large, the next modelling step would be to simplify the problem with some assumptions, or to break it down into more manageable bits. This process of reducing and scoping results in a decomposition of a large problem into smaller parts. This is one important and common strategy in mathematical modelling.

The next stage in the modelling process would be to attempt to construct a model based on the assumptions. By now, one would have identified the factors or variables that should be considered or included in the model. A common practice in this stage of the modelling process is to study the variables and their behaviour, examine available data, identify trends or patterns, and see if some existing, known model or solution methods can be used and applied. This requires, to a large extent, the ability to recognize patterns.

To implement the solution, especially if it is based on some form of simulation or iterations of steps, would often require a systematic set of procedure. This is where the ability to think in terms of an algorithm becomes very relevant and useful.

The above discussion essentially links the important stages of the modelling process to the important aspects or "cornerstones" of computational thinking. As shown in Figure 2, these aspects fit very naturally and appropriately in the different stages. Consequently, it would be reasonable to suggest that the skills associated with computational thinking are directly applicable and relevant to the competencies required for mathematical modelling.

Over the years, by working with students who have participated in modelling contests, it is observed that students who have some basic knowledge or experience in solving problems computationally – that is, they are able to apply their computational minds – are often more successful in the modelling task. At the same time, as students engage in more modelling exercises and activities, they tend to be more able to design and construct computational solutions to the modelling problem. In other words, there is a certain inter-dependency and interaction between computational thinking and mathematical modelling [3].

In summary, development of competencies in mathematical modelling can be supported by computational thinking. At the same time, the different aspects of computational thinking are evident in process of mathematical modelling. It is this interplay between these two pieces of the puzzle, as depicted in Figure 3 below, that we wish to investigate in the present study.

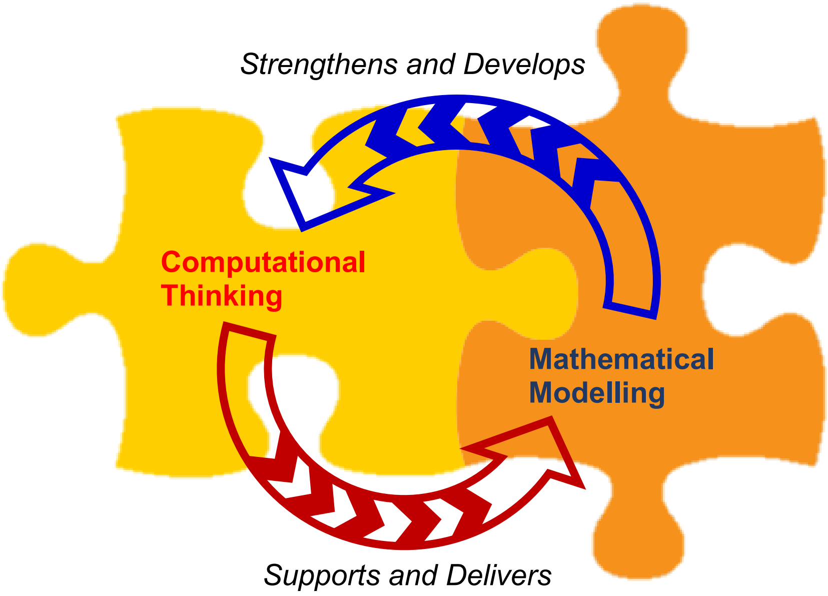

Figure 3: Interaction between computational thinking and mathematical modelling

Mathematical Modelling Contests

One of the ways of promoting the learning and encouraging the practice of mathematical modelling is through modelling contests. The International Mathematical Modelling Challenge (IMMC) is one such international contests, and it aims at promoting the teaching and learning of mathematical modelling at all educational levels for all students (https://www.immchallenge.org). Established in 2015, the IMMC is based on the belief that by tackling and solving real world problems outside of mathematics in a realistic context, students and teachers will get to experience the power of mathematics and its relevance to real world problem solving.

Each year, each country or region may send up to two teams to participate in the IMMC. Each team may consist of up to four students from secondary (including pre-University) students, and one faculty advisor. While the challenge may run for two months, each team may only work on the problem for five consecutive days and submit their solutions within these five days. In other words, the team may choose any five-day block within the two months during which the challenge is open to work on the modelling task. Teams may use any *inanimate* source of data, material, computers, software, references, books, websites and so on. Specifically, they may not consult or seek help from another person (including the faculty advisor) during the challenge.

The role of the faculty advisor is to assist the team in their preparation for the challenge, including any training or teaching before the start of the challenge, if necessary, and is responsible for the various administrative processes involved. Once the challenge begins – that is, once the team obtains the problem statement and the five-day duration commences – the faculty advisor's role is only administrative and will not be allowed to assist the team in the modelling task.

Over the years since the inaugural IMMC in 2015, it is noted that the IMMC problems are generally quite complex, and while the goals are usually well defined, the subgoals or subtasks are not. Moreover, developing a model or designing a solution to the problem would usually require knowledge from different disciplines, with factors and variables that may not have been explicitly mentioned and that can be inter-dependent. It is also noted that that the modelling task would demand creative use of various problem-solving strategies, including but not limited to computational approaches or methods.

The solutions submitted will be judged by an international panel of experts, and solutions will be recognized as "Successful participant", "Honorable Mention", "Meritorious" and "Outstanding", in order of increasing prestige, with the "Outstanding" award being the top award. Solution papers of outstanding teams are published on the IMMC website.

Given such complex settings, and, for most students, with limited experience and practice in mathematical modelling, how do IMMC participants cope with the challenge? What kind of strategies or approaches do they use to successfully complete the modelling challenge, and what lessons have they learnt? What role does computational thinking play in the participants' modelling practice and process?

In the next section, we will discuss our observations through case studies to investigate the interaction between computational thinking and mathematical modelling, and how one influences, supports and develops the other.

Case Studies

In the present discussion, the case study approach was used for a detailed examination of students' practice of Computational Thinking in developing mathematical models. The cases comprise two teams of students, Team A and Team B, who represented Singapore at IMMC 2021. Team A was accorded the Honorable Mention Award and Team B was accorded the top-tier Outstanding Award. Multiple data types in the form of report artefacts, presentation videos, judges' commentaries and interviews were collected to examine the participating students' use and practice of Computational Thinking in developing the mathematical models.

The IMMC 2021 problem statement is stated as follows.

We read all the time in the sports pages about an athlete being called the G.O.A.T. - the Greatest Of All Time. What does that really mean and how can that truly be determined?

(1) Develop a mathematical model for determining the **greatest woman tennis player in 2018** on the basis of Grand Slam tournament results (data provided)

(2) Choose one example of an individual sport and develop a mathematical model from any factors and data you find significant, measurable, and obtainable for **determining the G.O.A.T. in that sport**

(3) Discuss **any changes** your G.O.A.T. models from #2 would **require to determine the G.O.A.T. of a team sport**

At first glance, the tasks outline above do look daunting, and it was indeed true that many teams did not know where or how to begin. Teams must first study the data provided, and then work out an approach towards arriving at a solution or model that could help answer the question and identify the G.O.A.T. based on sound reasons.

As it turned out, participants from the two case study teams, Teams A and B, appeared to have been actively engaged in the practice of the Computational Thinking process, as can be seen from the approach that they had adopted. Aspects of the Computational Thinking process that were demonstrated through their development of the mathematical models include pattern recognition, decomposition, abstraction and algorithm creation.

It has to be pointed out that students from Team A have had some experience in computational thinking from their work in other mathematics projects assigned by their school. Students from Team B, on the other hand, had gained experience in Computational Thinking through their participation in solving programming problems at a competitive event known as the *National Olympiad in Informatics Singapore*.

a) Pattern Recognition

It is likely that Team B's engagement in pattern recognition have led to their creative mathematical modelling approach to determine the greatest woman tennis player in 2018. Team B had developed a model based on a weighted directed graph to represent the different tennis matches played by different players, who are represented as nodes in the graph. The directed edge is drawn from the winning player to the losing player, and the thickness of the edge is an indicator of edge weight or the winning margins. See Figure 4 for the network representation of Team B's model of the directed graph, which suggests that Simona Halep (represented by the red node) is consistently outperforming many of the most skilled players (represented by the green nodes) in singles women tennis by significant margins.

By observing the directed graph path patterns, Team B noticed the important fact that not all tennis players played each other and creatively used the Floyd-Warshall or All pairs shortest Path algorithm to predict the results of every possible game. This enabled them to find the relative ability of players and calculate the odds ratio of winning margins for each tennis player.

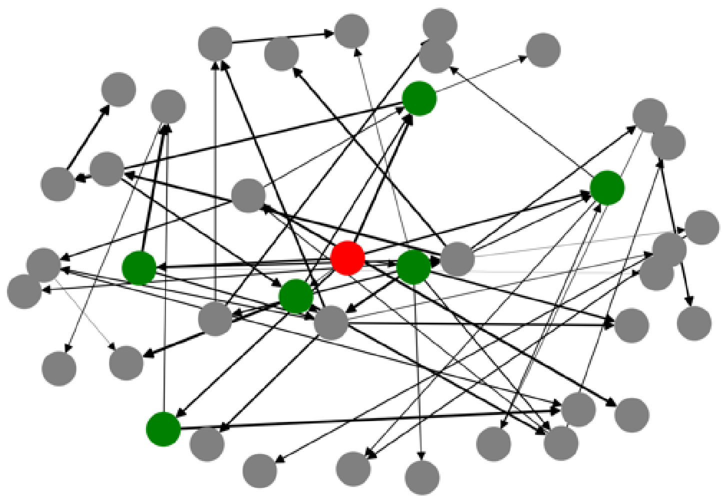

Figure 4: Weighted directed graph of athletes competing in women's singles tennis in 2018

To further confirm and ascertain that Team B's ability to recognise patterns in a problem situation had led to their creative mathematical modelling approach, an interview was carried out.

Below is an excerpt of the interview during which the team participants were asked if it was the first time they had employed the Floyd-Warshall shortest path algorithm at IMMC.

Interviewer: *Is this the first time at IMMC that you have used the Floyd-Warshall shortest path algorithm to compute the relative ability of tennis players that did not play against each other?*

Team B Member A: *No, some of us are involved in competitive programming and we have often used the Floyd-Warshall shortest path algorithm in solving the computational problems. Hence we found it to be relevant in helping us solve the IMMC problem.*

In this interview excerpt, we see how Team B had used the Floyd-Warshall shortest path algorithm frequently in their competitive programming situations. This enabled them to recognise a similar problem structure in the directed graph (see Figure 4) that not all pairs of nodes are connected to an edge, but the shortest paths for these pairs of nodes will need to be computed in order to calculate the odds ratio of winning margins for each tennis player. This had led them to employ the Floyd-Warshall shortest path algorithm for this purpose.

b) Decomposition

For each of the three tasks outlined in the IMMC 2021 problem, Team A and Team B had decomposed it to its respective subtasks. Specifically, Team A had decomposed the first task of developing a mathematical model for determining the greatest woman tennis player in 2018 into the following subtasks:

- Subtask 1: Determine and analyze the major factors in finding the greatest women tennis player
- Subtask 2: Build a mathematical model that predicts the greatness of women tennis players in 2018 Grand Slam tournament
- Subtask 3: Use our mathematical model to determine the greatest woman tennis player in 2018
- Subtask 4: Check against 2018 women tennis players' rankings.

The second task of developing a mathematical model for determining the G.O.A.T. of a chosen individual sport was decomposed as:

- Subtask 1: Develop a mathematical model to find the G.O.A.T. of man's badminton singles event
- Subtask 2: Study and comment on the results obtained by the model
- Subtask 3: Adapt our model to determine the greatest of all time for any individual sport
- Subtask 4: Explain how the models are different for different sport

Team A had also decomposed the third task of modifying the mathematical model for determining the G.O.A.T. of a chosen individual sport to find the G.O.A.T. of a team sport as:

- Subtask 1: Adapt the model used earlier to construct a new model which can determine the G.O.A.T of a team sport
- Subtask 2: Explain how the model is different for individual and team sports

Team A noted that their general approach to solving a complex problem such as the IMMC problem is "*to identify the achievable subtasks*", explaining that "*solving the subtasks one after another can help us to move closer to solving the problem*".

c) Abstraction

In formulating their mathematical models for the IMMC problem, Team A and Team B had abstracted the important variables of interest for determining the G.O.A.T. of their chosen individual sport. The identification of important variables was validated by the performance of sensitivity analysis on their respective models.

As shown in Table 1, Team A had formulated the Gross Greatness Index ($GGI_{badminton}$) for determining the G.O.A.T. of man's badminton individual event based on the abstracted variables in the Winning Consistency ($W_{consistency,\ badminton}$) and Performance Index ($PI_{badminton}$) of a player. The Performance Index ($PI_{badminton}$) of a player is in turn dependent on his Achievement ($A_{badminton}$) and dominance ($D_{\mu,\ badminton}$).

$$GGI_{badminton} = 100\% \cdot \frac{90 \cdot \sum PI_{badminton} + 10 \cdot \left(1 - W_{consistency,\ badminton}\right)}{T_{max} \cdot 90 \cdot 100 + 10 \cdot \left(1 - \frac{0}{100}\right)}$$

$$= 100\% \cdot \frac{9 \cdot \sum PI_{badminton} + \left(1 - W_{consistency,\ badminton}\right)}{900 T_{max} + 1},$$

where $PI_{badminton} = 0.6 \cdot A_{badminton} + 0.4 \cdot D_{\mu,\ badminton}$.

Table 1: Team A's abstracted variables for formulating the Gross Greatness Index to determine the G.O.A.T. of man's badminton individual event

Variable	**Definition**
$D_{\mu,\ badminton}$	Average dominance of a badminton player to his opponents across the competition.
$W_{consistency,\ badminton}$	Consistency of a particular badminton player's performance index across his career
T_{max}	Maximum number of tournaments we are accounting for.
$A_{badminton}$	The total achievement received by a particular badminton player.
$GGI_{badminton}$	The gross greatness index for a particular badminton player.
$PI_{badminton}$	The performance index of a particular badminton player in a single year.

In their directed graph model, Team B had taken into account the Average winning margin between player u and player v $(h(u,v))$ and its related variables in formulating the degree ratio, which is used to determine the greatest woman tennis player in 2018 (see Table 2).

Table 2: Team B's abstracted variables for formulating the Degree Ratio to determine the greatest woman tennis player in 2018

Variable	Definition
u_i, v_i	The two players involved in match i
$m_{u,v}$	Total number of matches played between player u and player v
s_{u_i}	Score of player u for match i
$g(u,v)$	Difference in the total score of player u and player v for match i i.e. $\sum s_{u_i} - \sum s_{v_i}$; if $g(u_i, v_i)$ is positive, this represents a directed edge from u to v and vice versa
$h(u,v)$	Average winning margin between player u and player v based on number of matches
d_u	Degree ratio $\frac{\sum_{i=1}^{n} h(u,i)}{\sum_{j=1}^{m} h(j,u)}$ i.e. the ratio of the sum of all the weights of outward edges to the sum of all the weights of inward edges

In performing sensitivity analysis to validate the significance of their abstracted variables, Team A and Team B made systematic changes to the input quantity of the parameters used in calculating their respective G.O.A.T. score. Consistency in their G.O.A.T. results was maintained, hence indicating the robustness of their model. This implies that the dynamic interplay of relationships for determining the G.O.A.T. score in their respective models can be explained sufficiently by the variables that were abstracted.

d) Algorithm creation

To achieve the efficient computation of the G.O.A.T. score from big data sets, Team A and Team B had each generated algorithms for their respective model solutions. Team A's algorithm included the computation steps to add the input of the variables to separate lists organized by the match type (e.g. qf: Quarter-finals, sf: Semi-finals, f: Finals). Lists for matches won and matches played with game points appended were then made to determine the achievement score of players. The Performance Index were then assigned to the respective player before generating the Gross Greatness Index as an output file.

Besides the algorithm to compute the outward winning and inward losing edges in their directed graph models, members from Team B members had systematically crafted and created a flowchart as an overall model to address the problem. The flowchart (see Figure 5), which clearly and evidently demonstrated the team's experience and ability to engage in algorithmic thinking, summarizes how their G.O.A.T. models were adapted for the different types of sports under consideration.

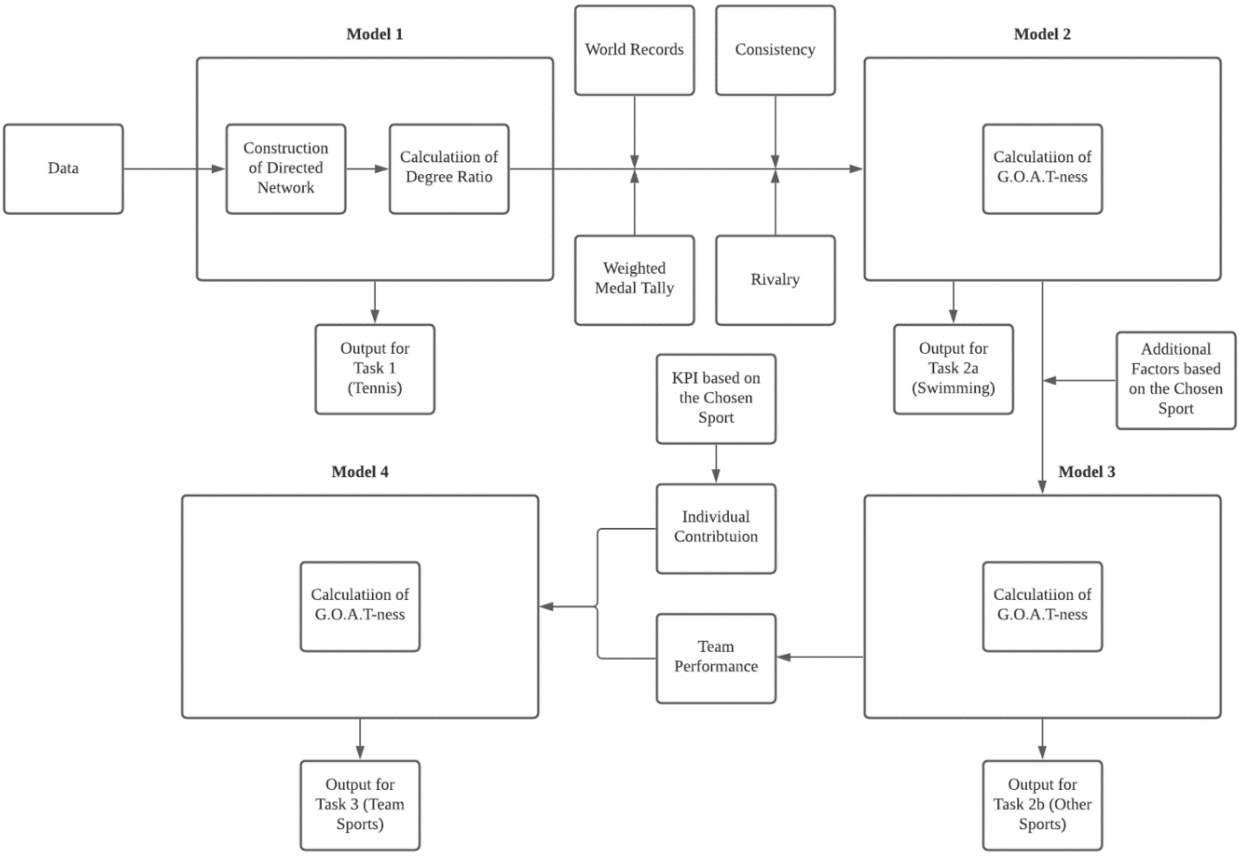

Figure 5: Summary flowchart produced by Team B.

For details and a more complete explanation of the models that Team B had produced in IMMC 2021, readers may refer to the actual solutions which are available at the IMMC website (https://immchallenge.org/Contests/2021/Solutions.html).

Concluding Remarks

In this investigative study, case studies involving two participating teams of the IMMC 2021, an international modelling contest, were carried out to demonstrate students' engagement in the various aspects of computational thinking when tackling modelling tasks. By examining artefacts, including solution reports, as well as through interviews, it was found that student members of the two teams had consciously and intentionally applied a computational mindset when approaching the modelling tasks.

It was revealed that the students' past experiences in computational thinking had come into play when they were confronted with the present modelling problem. In addition, it is clear that because of the complexity and the closeness to a real-world situation of the problems, mathematical modelling in general and modelling contests in particular, provide excellent opportunities for one to develop one's sense of thinking in a computational way.

Although the link between computational thinking and mathematical modelling may not be particularly obvious, this study has established that there is indeed a close relationship between these two notions. Future work and more research in this area will be necessary to provide deeper insights into the inter-dependency of computational thinking and mathematical modelling.

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