7-Circle Waltz

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What is a 7-Circle Waltz?

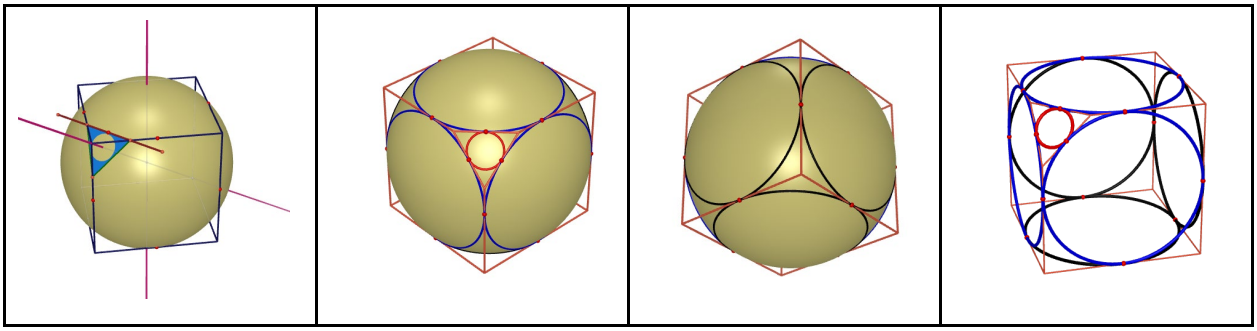
A polyhedron is said to be circumsriptible [1] if there exists a single point P (the circumcenter) equidistant r from all edges (not produced). This paper deals with the seven incircles of a circumscriptible heptahedron having a pair of parallel faces. We are thankful to the capabilities of taking inverse of the sphere as a built-in feature of the dynamic software Cabri 3D, the visual phenomena, full of geometric invariant properties, can be created and experimented. We call such experiments “7-Circle Waltz”.

How to build a circumscriptible heptahedron?

Method A: Truncation

Example 1: (5,5,5,4,4,4,3)

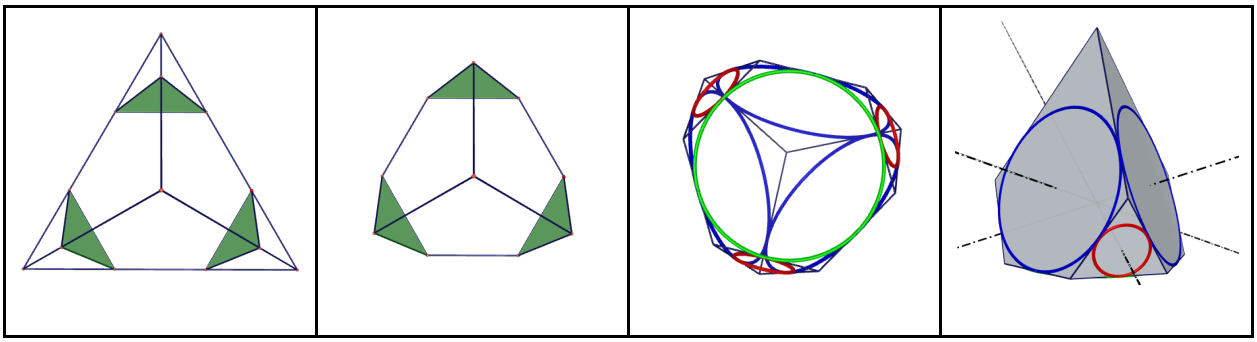
Select one corner and make the truncation there.The resulting heptahedron has three pentagons, three squares and one triangle as faces. Following [5] the list (5,5,5,4,4,4,3) is assigned to indicate its topological type.



Animated GIF: Forming circumscriptible heptahedron (5,5,5,4,4,4,3) by truncating a corned of the cube)

Example 2: (6,5,5,5,3,3,3)

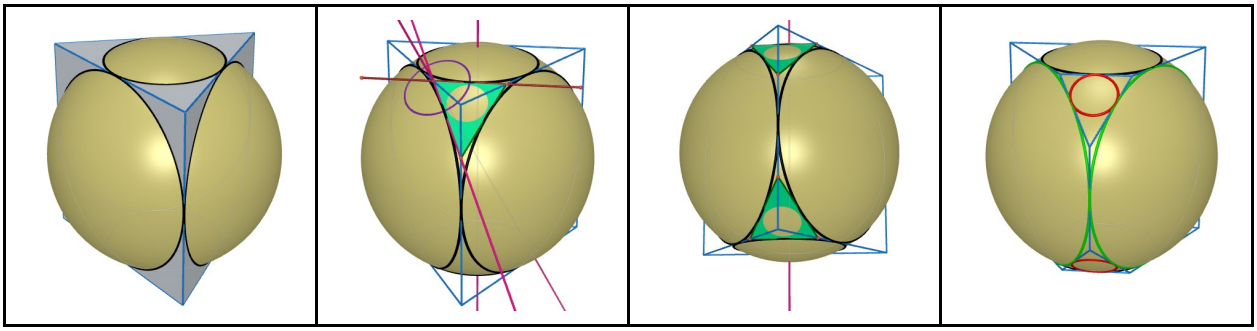
This heptahedron is constructed by truncating 3 vertices of a regular tetrahedron at one third of the original edge length. Note that this polyhedron has 3 pairs of parallel (5,3) faces.



Animated GIF: Forming (6,5,5,5,3,3,3) by truncating three corners of a tetrahedron

Example 3: (6,6,4,4,4,3,3)

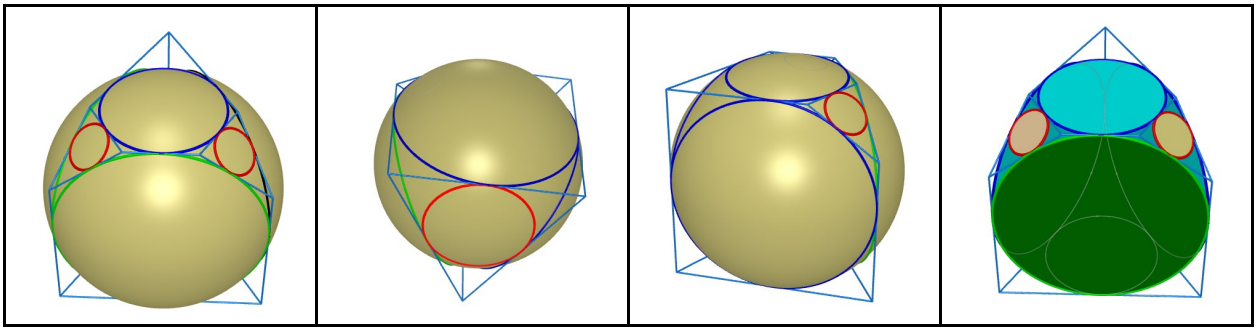
Starting from a prism, select two corners of the same vertical edge. After the truncation of the two corners, the remaining polyhedron appears as:



Animated GIF: Forming (6,6,4,4,4,3,3) by truncating two corners of (4,4,4,3,3)

Example 4: (5,5,5,5,3,3,3)

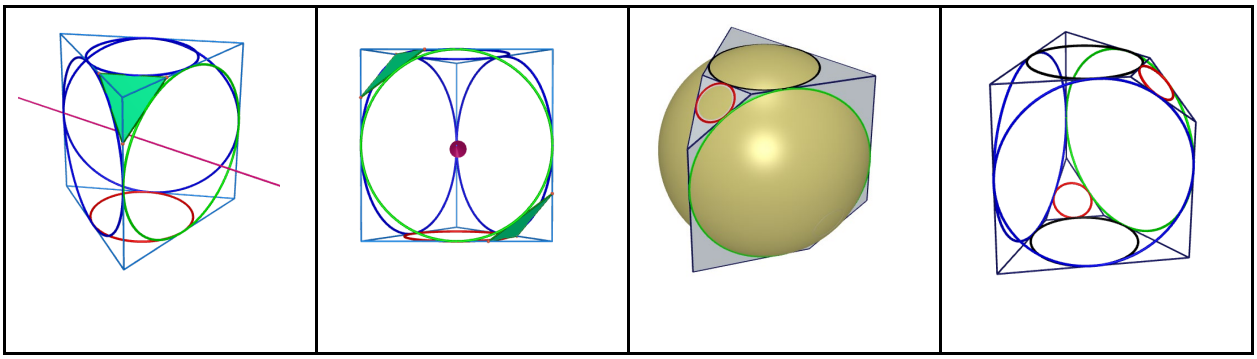
Starting from a prism two corners of the same height are truncated.



Animated GIF: Forming (6,5,5,5,3,3,3) by truncating two corners of (4,4,4,3,3)

Example 5: (6,5,5,4,4,3,3)

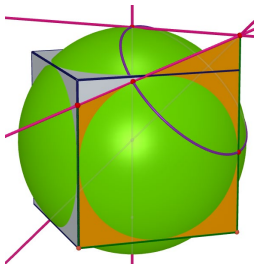
Starting from a prism two diagonal corners of a square are truncated.



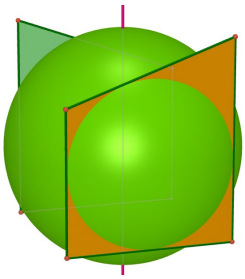
Animated GIF: Forming (6,5,5,4,4,3,3) by truncating two corners of (4,4,4,3,3)

Method B: Convex Hull

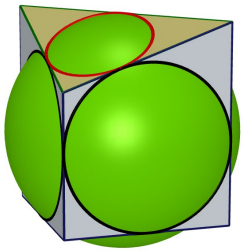
Example: (4,4,4,4,4,3,3)



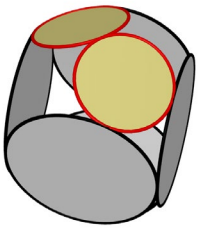
Modify a square face to a trapezoid circumscribing the sphere.



Construct the second trapezoid symmetric to the vertical diameter.



The required heptahedron is obtained by taking the convex hull of the pair.

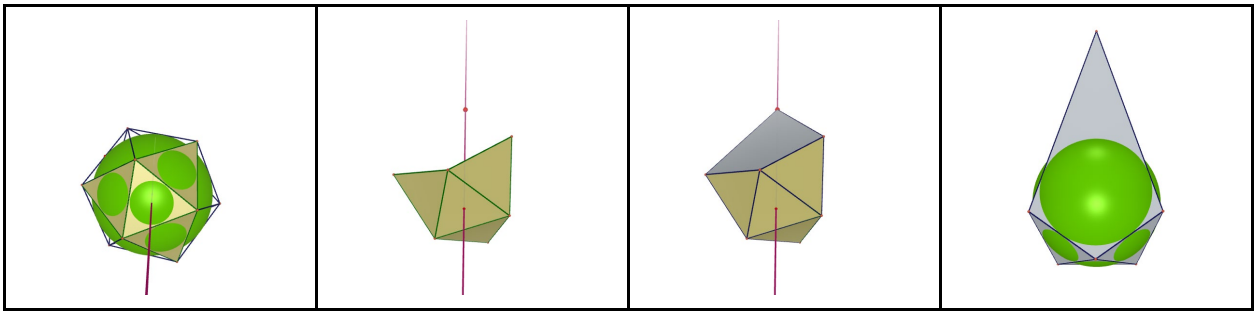


Masterpiece made from recycled vinyl disks.

Animated GIF: Forming (4,4,4,4,4,3,3) by taking the convex hull of a pair of trapezoids

Method C: Build from the Icosahedron

Example:(4,4,4,3,3,3,3;1) constructed from Icosahedron



Animated GIF: (4,4,4,3,3,3,3;1) constructed from Icosahedron 28 frames

Systematic Method

There is a systematic way to construct the circumscriptible heptahedron following a three-step process

1) Step 1: perform a plane geometric construction of an RSF
2) Step 2: taking an inversion of RFS from Step 1
3) Step 3: construct the 7 faces of the required heptahedron one-by-one

Example: Concrete construction of the heptahedron (4,3,3,3,3,3,3;1)

1) Step 1: Construction of RSF(4,3,3,3,3,3,3;1) 5 frames.gif
2) Step 2: RSF(4,3,3,3,3,3,3;1) inverted 6 frames.gif
3) Step 3: 7 faces of RSF(4,3,3,3,3,3,3;1) constructed one-by-one 12 frames.gif

Important inversion

Basic inversion taking any pair of disjoint circles on a sphere to circles having the same axis

Google Drive links of files associated with 27 animations

Part A. Heptahedra assigned with the list (6,x,x,x,x,x,x). (Visually, each polyhedron has 1 or 2 hexagonal faces.)

7-Circle Waltz(6,6,4,4,4,3,3) 8 frames.gif	RSF 6 frames	mov 1 min 29 sec 81.4 MB
7-Circle Waltz(6,5,5,5,3,3,3) 8 frames.gif	RSF 3 frames	mov 3 min 34 sec 61 MB
7-Circle Waltz(6,5,5,4,4,3,3) 8 frames.gif	RSF 6 frames	mov 2 min 42 sec 144 MB
7-Circle Waltz(6,5,4,4,3,3,3;2) 8 frames.gif	RSF 4 frames	mov 2 min 35 sec 102.4 MB
7-Circle Waltz(6,5,4,4,3,3,3;1) 8 frames.gif	RSF 4 frames	mov 2 min 2 sec 163.7 MB
7-Circle Waltz(6,4,4,4,4,3,3) 8 frames.gif	RSF 4 frames	mov 3 min 46 sec 218.9 MB
7-Circle Waltz(6,4,4,3,3,3,3;2) 9 frames.gif	RSF 4 frames	mov 3 min 26 sec 218.7 MB
7-Circle Waltz(6,4,4,3,3,3,3;1) 8 frames.gif	RSF 5 frames	mov 2 min 16 sec 122.5 MB

Part B. Heptahedra assigned with the list (5,x,x,x,x,x,x) (Visually, no hexagonal faces and at least one pentagonal face)

7-Circle Waltz(5,5,5,4,4,4,3) 8 frames.gif	RSF 4 frames	mov 2 min 16 sec 120.2 MB
7-Circle Waltz(5,5,5,4,3,3,3;2) 9 frames.gif	RSF 4 frames	mov 1 min 13 sec 50.8 MB
7-Circle Waltz(5,5,5,4,3,3,3;1) 10 frames.gif	RSF 11 frames	mov 1 min 40 sec 113.4 MB
7-Circle Waltz(5,5,4,4,4,4,4) 10 frames.gif	RSF 9 frames	mov 2 min 43 sec 91.9 MB
7-Circle Waltz(5,5,4,4,4,3,3;2) 10 frames.gif	RSF 5 frames	mov 3 min 25 sec 201.9 MB
7-Circle Waltz(5,5,4,4,4,3,3;1) 9 frames.gif	RSF 8 frames	mov 2 min 17 sec 92.5 MB
7-Circle Waltz(5,5,4,3,3,3,3;2) 10 frames.gif	RSF 4 frames	mov 2 min 26 sec 153.2 MB
7-Circle Waltz(5,5,4,3,3,3,3;1) 14 frames.gif	RSF 5 frames	mov 1 min 49 sec 109.9 MB

7-Circle Waltz(5,5,4,3,3,3,3;1) 8 frames.gif	RSF 4 frames	mov 1 min 59 sec 128 MB
7-Circle Waltz(5,4,4,4,4,4,3) 11 frames.gif	RDF 6 frames	mov 2 min 59 sec 128.9 MB
7-Circle Waltz(5,4,4,4,3,3,3;3) 42 frames.gif	RSF 6 frames	mov 2 min 19 sec 93.3 MB
7-Circle Waltz(5,4,3,3,3,3,3;1) 16 frames.gif	RSF 5 frames	mov 1 min 53 sec 78.4 MB

Part C. Circumscriptible heptahedra assigned with the list (4,x,x,x,x,x,x) (Visually, only quadrilateral and triangle faces)

7-Circle Waltz(4,4,4,4,4,3,3) 10 frames.gif	RSF 3 frames	mov 35 sec 60.7 MB
7-Circle Waltz(4,4,4,3,3,3,3;5) 8 frames.gif	RSF 2 frames	mov 2 min 9 sec 122.6 MB
7-Circle Waltz(4,4,4,3,3,3,3;4) 8 frames.gif	RSF 5 frames	mov 2 min 2 sec 124.6 MB
7-Circle Waltz(4,4,4,3,3,3,3;2) 15 frames.gif	RSF 7 frames	mov 2 min 7 sec 141.4 MB
7-Circle Waltz(4,4,4,3,3,3,3;1) 16 frames.gif	RSF 3 frames	mov 59 sec 177.3 MB
7-Circle Waltz(4,3,3,3,3,3,3;2) 11 frames.gif	RSF 6 frames	mov 1 min 38.3 MB
7-Circle Waltz(4,3,3,3,3,3,3;1) 11 frames.gif	RSF 4 frames	mov 1 min 1 sec 77.3 MB

Byproduct:

Links to ideas motivated by the Kinetic Art

Stereographic images containing a pair of concentric circles

References

[1] Altshiller-Court, Nathan (1935), Modern pure solid geometry, Macmillan.

[2] Chasles, Michel Floréal (1860), Les Trois Livres de Porismes d'Euclide, Paris: Mallet-Bachelier.

[3] Evelyn, C. J. A.; Money-Coutts, G. B.; Tyrrell, J. A. (1974). The Seven Circles Theorem and Other New Theorems. London: Stacey International. ISBN 978-0-9503304-0-2.

[4] Wells, David(1991). The Penguin Dictionary of Curious and Interesting Geometry. New York: Penguin Books. pp. 227–228. ISBN 0-14-011813-6.

[5] https://en.wikipedia.org/wiki/Heptahedron.

[6] The Association for the Development of Education, Sofia, Bulgaria; Sangaku Journal of Mathematics, an open-access electronic journal devoted to geometry problems in the Wasan tradition. http://www.sangaku-journal.eu/

[7] http://www.wasan.jp/english/

[8] All animations in this work can be found in https://drive.google.com/drive/folders/1aElN1rrvSMEm0KK7YwCzOphsiD-9mzBl?usp=sharing