Construction and exploration of particular curves and evolutes of theirs using GeoGebra dynamic software

Petra Surynková

surynkov@karlin.mff.cuni.cz
Department of Mathematics Education, Faculty of Mathematics and Physics, Charles University
Sokolovská 83, 186 75 Praha 8, Czech Republic

Abstract

The article deals with the synthetic construction of specific planar curves which are defined by geometric motions and with the modeling of these constructions in GeoGebra dynamic system. We focus on the constructions of trajectories, envelopes, and centers of curvature and osculating circles of specific curves. We present a particular construction of all centers of curvature of these curves, i.e. the construction of the evolute of a curve. Our aim is to investigate the synthetic constructions without the use of coordinates and formulate the proofs of them. We newly model selected types of curves in GeoGebra dynamic system which is an additional output to our theoretical conclusions and also a significant teaching aid at the same time. The constructions are meant to be dynamic and the curves are formed gradually as the traces of points or curves. All examples presented in this article are intended to be used in the undergraduate courses on kinematic geometry (mandatory courses for secondary pre-service mathematics teachers who study teaching mathematics and descriptive geometry). The synthetic constructions together with their proofs demonstrated in GeoGebra dynamic system bring a new light into this area. It allows students to imagine the main idea of the proofs of the constructions and to investigate the properties of the curves more easily based on the pure geometry and visual aspects. Dynamic constructions, i.e. the possibility of changing positions of points and curves play the significant role here. GeoGebra offers also algebraic expressions which represent another tool for students how they can study and manipulate with those curves.

Keywords: kinematic geometry, trajectory, envelope, centrode, synthetic construction, descriptive geometry, centers of curvature, osculating circles

1 Introduction

This article addresses the study of specific planar curves which are defined by geometric motions. The properties of these curves are well known and were studied through ages, in the 16th century and 17th century mainly. Let us mention several famous names which are inextricably linked to this field of mathematics - Girard Desargues (1593-1662), René Descartes (1596-1650), Pierre de Fermat (1601-1665), Johannes Kepler (1571-1630), Isaac Newton (1643-1727), Blaise Pascal (1623-1662), François Viète (1540-1603) and many more [5]. In this article we broaden the study of planar curves with another aspects - the visualizations of all constructions in GeoGebra dynamic system. All examples and constructions modeled in GeoGebra are used in the undergraduate courses on kinematic geometry (mandatory courses for pre-service mathematics teachers who study teaching of mathematics and descriptive geometry, i.e. the prospective secondary school teachers). GeoGebra software helps students to investigate the properties of and the relationships between points and curves through the observations and experiments based on basic constructions and characteristics of planar objects.

Information and communication technologies (ICT) became an inseparable part of school instructions and learning processes [18]. If we focus on software in mathematics education, dynamic software systems (DGS) and computer algebra systems (CAS) have the great potential to be used in education process [19]. GeoGebra dynamic system became widespread all over the world among teachers and students. Because of its popularity GeoGebra is even further modifying and extending [10]. GeoGebra is so popular because it is very easy to use even for the absolute beginner and it is open-source software.

Much of literature summarize that the integration of dynamic software into the education process has a positive effect on better understanding, motivation of students, and transformation of the school instruction [14]. Many researchers claim that GeoGebra and its using at secondary schools and colleges support students' discovering new mathematical facts, their experimentation, and develop their own explanations [10].

I have a long-term teaching experience at the Faculty of Mathematics and Physics (Charles University, Czech Republic) where I have been teaching geometric courses (Euclidean, descriptive, computational, or kinematic geometry) over 10 years. Moreover, I also started teaching mathematics instructions in the Grammar school in the Czech Republic two years ago. At the faculty I work with pre-service mathematics teachers - students who study the specialization of teaching mathematics and descriptive geometry, i.e. the prospective secondary school teachers. The Grammar school where I also teach provides upper secondary education. This perfectly complements one another. I prepare university students for their future career in the secondary schools and I am also a secondary school teacher. I am aware of students' needs at university and for what they should be prepared when they enter secondary schools and start teaching.

As it has been already pointed out, I will show the synthetic constructions of selected planar curves defined by geometric motions and the use of GeoGebra for studying their properties. The constructions modeled in GeoGebra were created with the intention to be used in my university courses on kinematic geometry.

2 Preliminaries and Terminology

Kinematic geometry in the plane is a field of Euclidean geometry describing the motions of points, curves, and systems of points and curves without considering the forces that cause them to move. Kinematic geometry studies the determination of motions, the principles of planar mechanism, or the properties of curves given or determined by the motion.

We assume that the curves that we study are determined as trajectories of moving points or as envelopes of moving curves in the plane. Usually, the tasks in kinematic geometry are to determine a trajectory of a point or an envelope of a curve from the given inputs, to find special points of curves such as cusps or inflection points, to construct centers of curvature and osculating circles of curves or to demonstrate and to study the properties and the underlying geometry of mechanisms such as gears, screw systems, robots [8], [11], [16], [17], [20], [22].

Let us briefly introduce the basic terms related to kinematic geometry which will be then used in the constructions.

A movable plane Σ slides over a fixed plane Π in an arbitrary direction [9]. The final position of the plane Σ is obtained from the initial position by a continuous motion. A movable plane Σ slides over a fixed plane Π and it is never "turned over". The plane Σ is considered to be rigid during the motion and each of its points describes a *trajectory* (also called a *path*) in the plane Π and each of its curves describes an *envelope* in the plane Π. A trajectory of a point A will be denoted by τ_A and an envelope of a curve a by (a). We also distinguish the individual discrete positions of the moving plane Σ and denote them by $\Sigma^1, \Sigma^2, \Sigma^3, \ldots$, analogically the positions of a moving point A are denoted by $A^1, A^2, A^3, \ldots$, and of a moving curve a by $a^1, a^2, a^3, \ldots$.

The simplest motions in the plane are the identity, translations (the points of the plane move in the same direction and of the same distance) and rotations (the points of the plane rotate by an angle about a fixed point). The trajectory of a point in translation is a line, in rotation is a circle.

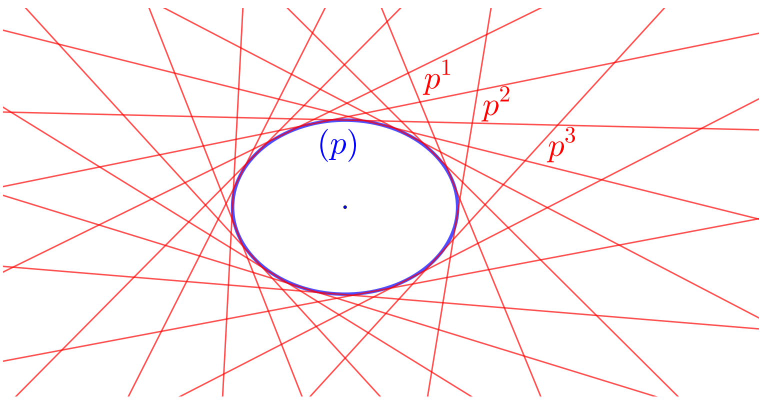

Figure 1: An ellipse (p) as an envelope of a moving line p.

To have a clear idea, an envelope of a moving curve a is a curve (a) which is tangent to the each curve a^1, a^2, a^3... at some point, and these points of tangency together form the whole envelope. Two examples of envelopes can be seen in Fig. 1 and 2.

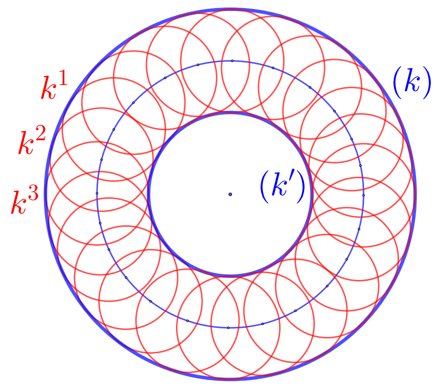

Figure 2: Two concentric circles (k) and (k') as an envelope with two branches of a moving circle k.

Very special cases of trajectories and envelopes can occur during the motion. For example, curves can even degenerate into points, let us mention a *point envelope*. We can imagine a simple example when a line a is rotating around a point (a); then the point (a) is the point envelope of the line a. I refer the reader to an interesting literature on curves where other definitions are provided and special curves are studied [4], [7]. Curves of kinematic geometry can be studied also using the coordinate system and algebraic expressions of curves, see [2].

2.1 Determination of Motions in the Plane

Now let us focus on more general examples of motions than identity, translation, or rotation. It can be proved that the motion is completely determined by the form of two trajectories of two distinct points, each on its trajectory, see Fig. 3. Another possibility is the determination by the form of two envelopes of two distinct curves, each tangent to its envelope, see Fig. 4. We can even combine it, it means that the motion is completely determined by a trajectory of a point and by an envelope of a curve. The concrete examples can be found in [15].

As has been already mentioned, when a movable plane Σ slides over a fixed plane Π it remains of the same orientation (it does not "turn over"). Then an arbitrary position Σ^i of a movable plane Σ can be moved to another position Σ^j of a plane Σ using one single translation or one single rotation. The proof can be found in [9]. This means that every two positions of a moving object of a movable plane are congruent and of the same orientation.

We can consider translations as rotation by a zero angle about a point at infinity. If the positions Σ^i and Σ^{i+1} are infinitely close we can construct the center of rotation S_i about which the position Σ^i is rotated to the positions Σ^{i+1}. This center of rotation is called the *instantaneous center of the motion* at the instant i (for the translation it is a point at infinity) [23], [22].

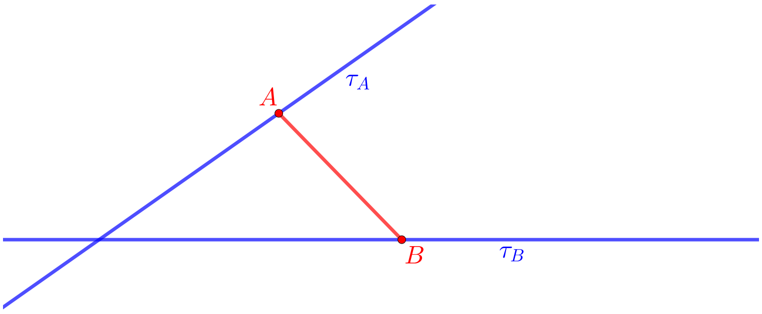

Figure 3: The determination of the motion by two trajectories τ_A and τ_B of two distinct points A and B.

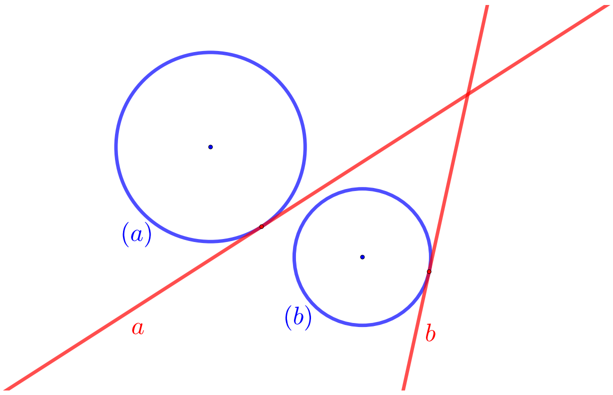

Figure 4: The determination of motions by two envelopes (a) and (b) of two curves a and b.

A locus of the instantaneous centers at every moment of the motion is a curve in the fixed plane Σ which is called the *fixed centrode* of the motion. The roles of the planes Σ and Π can be interchanged and the *inverse motion* is obtained. It means that what was moving in the original motion is fixed now and vice versa. A locus of the instantaneous centers at every moment of the inverse motion is a curve in the plane Σ which is called the *moving centrode* of the original motion. The fixed centrode of the original motion will be denoted by p and the moving centrode of the original motion by h, the i^{th} position of h will be denoted by h^i. The original and the inverse motion are given by each other. It can be proved that the fixed centrode of the original motion is the moving centrode of the inverse motion; analogically the moving centrode of the original motion is the fixed centrode of the inverse motion. It implies that with interchanging the roles of centrodes of the original motion we get the inverse motion. We defined the inverse motion so we could determine

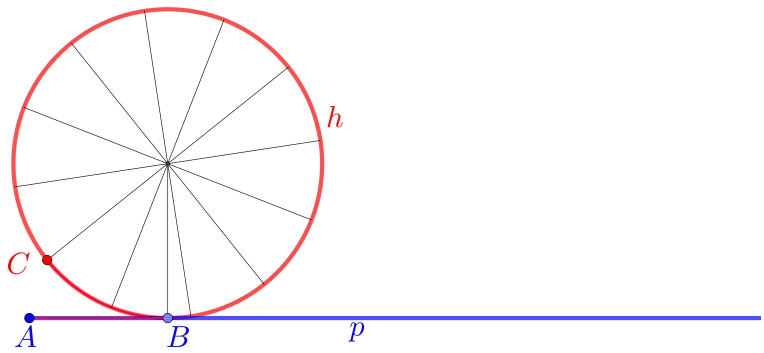

Figure 5: The determination of the motion by rolling the moving centrode h along the fixed centrode p.

the moving centrode. From this moment on, we consider only the original motion and we do not deal with the inverse motion.

Very important property of the instantaneous centers on the fixed centrode p is that these points are the intersections of the normals to the trajectories at their points in the particular position [12].

Considering the centrodes, we can add another possibility how to determine the motion. When a moving plane Σ slides over a fixed plane Π the moving centrode h in the moving plane Σ rolls (without sliding) along the fixed centrode p in the fixed plane Π. At each moment the centrodes are mutually tangent at the instantaneous center. Since the centrodes roll along each other without sliding it implies that the arc bounded by any two points on the fixed centrode p has the same length as the arc bounded by the corresponding points on the moving centrode h. See Fig. 5 where the motion is determined by the centrodes. For more detailed information regarding the basics of kinematic geometry see [3], [9], or [21].

3 Constructions using GeoGebra Dynamic Software

In this section we will show synthetic constructions of trajectories, envelopes, and centers of curvature and osculating circles of specific curves when some specific motion is given. The motions in the plane can be categorized according to the types of trajectories, envelopes or centrodes which determine them. The motions can be also sorted according to the types of curves which are created during the motion. We omit the categorization here and limit ourselves only on several examples. The categorization of motions and some other constructions are presented online and are accessible from [1].

All the constructions are newly modeled in GeoGebra dynamic software to demonstrate the motions in real time and to show the specific properties which can be hardly seen in the static pictures. I am using these constructions in my university courses on kinematic geometry.

We can even draw the constructions by hand and GeoGebra software use as the control aid. I am also practicing hand-drawn constructions with my students.

I also refer the reader to other literature which deals with the constructions of trajectories, envelopes, evolutes, and many other types of curves using dynamic software systems [6], [13].

3.1 Construction of a Trajectory

Let us consider the motion is determined by two straight line trajectories τ_A and τ_B of two distinct points A and B, see Fig. 6. We can construct a circle determined by three points A, B, and O (the intersection of the trajectories τ_A and τ_B). The points A and B are considered in an arbitrary position. It is obvious that the instantaneous center S constructed for this concrete position also lies on this circle. Let us denote this circle h and its radius r. We construct the circle p with a center O and the radius $2r$. It is easy to prove that the circle p is the fixed centrode and the circle h is the moving centrode which is depicted in one concrete position.

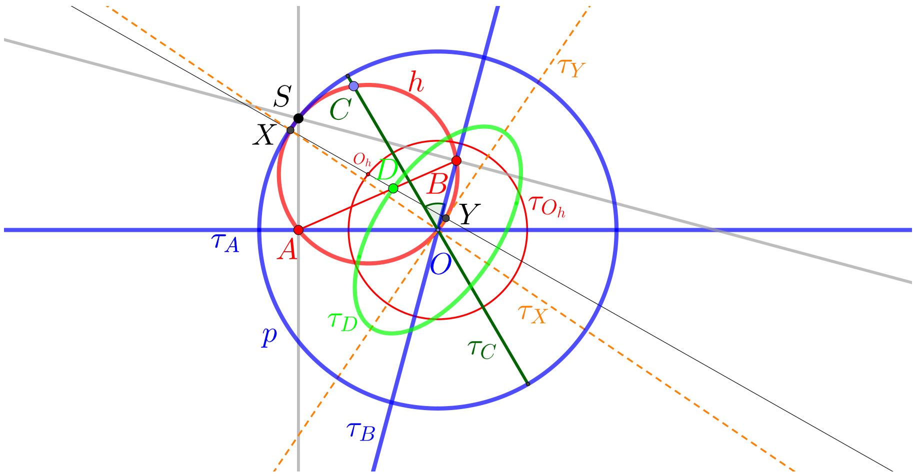

Figure 6: The trajectories of the elliptic motion.

The trajectories of points on the moving centrode h are segment lines - diameters of the fixed centrode p. It is clear that points A and B are moving on theirs trajectories τ_A and τ_B, if the point C is a point on the moving centrode h distinct from A and B then the length of an arc BC is the same in every position, thus the size on an angle $\angle BOC$ is constant. One ray τ_B of this angle is fixed, thus the second ray is fixed too. That is why the trajectory τ_C is the diameter of the fixed centrode p. The trajectory τ_{O_h} of the point O_h, the center of the moving centrode h, is the circle with the center O and the radius r. Finally, the trajectory τ_D of every other point D is an ellipse which is the reason for the name of this motion - the *elliptic motion*. A line DO_h intersects the moving centrode h in points X and Y which have straight line perpendicular trajectories τ_X and τ_Y. Thus, the trajectory τ_D is an ellipse with the axes τ_X and τ_Y and the semi-major axis of the length DX and the semi-minor axis of the length DY. This is a known construction of an ellipse called - the *paper strip method* or the *trammel method* [15].

From these observations implies that the elliptic motion can be determined by two circles as centrodes i.e. the moving circle h rolls on the inside of the fixed circle p with double the radius. That is the elliptic motion is a special case of the *hypocycloidal motion* [15].

We can also construct trajectories synthetically point by point using the observations that the moving plane Σ is rigid, i.e. the distances between points in the moving plane Σ are preserved during the motion. We obtain the trajectory as the locus of all the positions of a moving point.

3.2 Constructions of Centers of Curvature and Osculating Circles

A suitable aid how to precisely describe the shape of the curves is to construct the centers of curvature and osculating circles. The center of curvature of a curve can be defined as the intersection point of two infinitely close normal lines to the curves [15]. The *evolute* of the curve is the locus of all its centers of curvature. The original curve is called the *involute* [15]. We can also say that the evolute is the envelope of the normals to the curve. The evolute of the involute is the original curve.

Let us consider the *cycloidal motion* is given, i.e. the moving circle h with the center O_h and the radius r_h rolls on the fixed straight line p (we can consider the straight line p as a circle with the center at infinity). The moving circle h is given in the initial position. We shall determine the trajectory τ_C of a point C of the moving circle h, that is a *cycloid*, and the center of curvature and the osculating circle in each point of the trajectory τ_C, see Fig. 7. To find all centers of curvature of the trajectory τ_C actually means to find the evolute of this trajectory.

At each moment of the motion, the fixed centrode p is the tangent line to the moving centrode h at the instantaneous center S, i.e the i^{th} position of h and the fixed centrode p are mutually tangent at the instantaneous center S_i. As has been already pointed out, the instantaneous center on the fixed centrode p is the intersections of the normals to the trajectories at their points in the particular position. Thus, we can construct the normal n_C to the trajectory τ_C at the point C, i.e. $n_C = SC$. The tangent line t_C to the trajectory τ_C at the point C is perpendicular to the n_C. We construct the point C' on the circle h opposite to the point C. The trajectory $\tau_{C'}$ of the point C' is a cycloid in the fixed plane Π which is congruent to the cycloid τ_C just moved by $\pi \cdot r_h$ in a direction of the fixed centrode p. The tangent line $t_{C'}$ to the trajectory $\tau_{C'}$ at the point C' is perpendicular to the normal $n_{C'}$ where $n_{C'} = SC'$. It is obvious that $n_C \parallel t'_C$. A segment line TS where T is the intersection of t_C and h determines the direction and the distance of translation which moves the tangent line $t_{C'}$ to the normal n_C. It implies that an envelope of n_C (i.e.

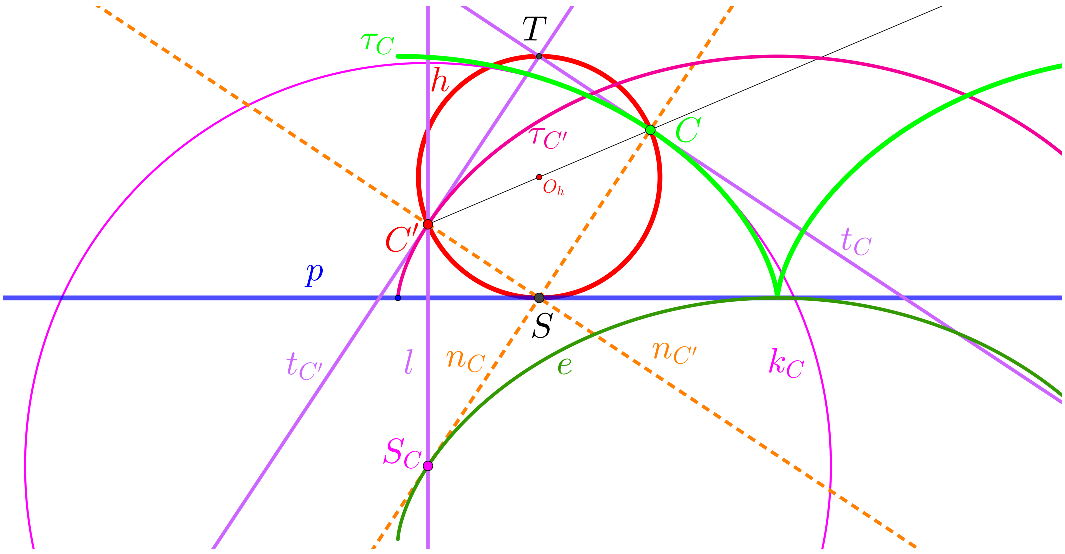

Figure 7: The center of curvature S_C of the trajectory τ_C at the point C of the moving circle h together with the osculating circle k_C.

an evolute of the τ_C) is the cycloid e congruent to $\tau_{C'}$. The points of tangency on $\tau_{C'}$ and e correspond in the same translation. It means that the intersection S_C of a straight line $l, l \parallel TS, C' \in l$ and n_C is the center of curvature of τ_C at C. The circle k_C with the center S_C and the radius $|S_C C|$ is the *osculating circle*.

Analogically, we could describe the construction for the epicyclodial or hypocycloidal motion, i.e. the motion where the moving circle h rolls on the outside of the fixed circle p or where the moving circle h rolls on the inside of the fixed circle p.

4 Conclusion and Future Work

We showed several synthetic constructions of trajectories, envelopes, and centers of curvature and osculating circles of specific curves when the geometric motion in the plane is given. We modeled all examples of constructions in GeoGebra dynamic software to demonstrate the properties of curves. The categorization of motions and some other constructions are accessible online [1].

I am using GeoGebra applets in my kinematic geometry lessons at university. The pre-service mathematics teachers who attend my courses on kinematic geometry will not teach kinematic geometry in secondary schools. My aim is to show them that even such a complex issue as kinematic geometry can be taught in an exploratory and experimental way. They can apply this approach in mathematical topics which are included in secondary school mathematics. This will be demonstrated during the presentation with examples on the inscribed and central angles that subtend the same arc on the circle (the inscribed angle theorem). The special case when the inscribed angle is a right angle will be considered too.

Regarding the future work I plan to extend the database of examples and study also another types of curves and their synthetic constructions. My aim is to describe the constructions of special points of trajectories and envelopes such as cusps or inflection points.

5 Acknowledgments

The paper was supported by the project PROGRES Q17 Teacher preparation and teaching profession in the context of science and research and by Charles University Research Centre No. UNCE/HUM/024.

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