

Proceedings of the Twenty-Sixth Asian Technology Conference in Mathematics

A Rewarding Journey of Connecting Mathematics and Technology to Applications

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Published by Mathematics and Technology, LLC

ISSN 1940-4204 (Online version: <http://atcm.mathandtech.org/EP2021>)

December 13-15, 2021

Virtual Format Radford, Virginia

USA

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Counting Angle Bisection Theorems

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Abstract: *Some good geometry proof problems involve angles, angle bisectors, circle chords and isosceles triangles. We identify similarities between elements of such problems and present techniques for generating new problems from old.*

1. Introduction

In order to test symbolic geometry software, it is important to have access to a large number of geometry examples whose results are known. Over the course of the development of Geometry Expressions and its browser based sibling GXWeb, we have developed techniques for generating new geometry problems from old in order to broaden our suite of test cases. These techniques may be useful for educators who need to broaden their suite of test examples for students. It is with this in mind that we present these ideas. In particular, we consider angle theorems involving, in one disguise or another, the angle bisector.

The base of an isosceles triangle is perpendicular to the bisector of the angle at its apex, and hence has the same direction as the exterior angle bisector of that angle. We can, therefore, consider many theorems involving isosceles triangles to be angle bisector theorems. The chord of a circle makes an isosceles triangle with the radial lines at its ends, and hence many angle theorems involving circles can themselves be considered bisector theorems. The axis of a reflection is the external angle bisector of incident and reflected rays, and hence a reflection is simply an angle bisection viewed from a different point of view.

We examine ways of exploiting this perspective to generate new problems from old. We start with a very old problem, from Archimedes' Book of Lemmas [1], in figure 1.

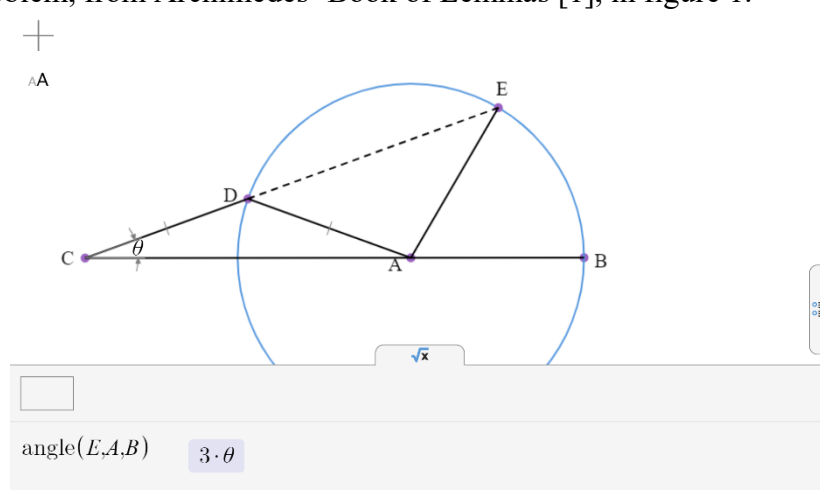


Figure 1: Problem from Archimedes' Book of Lemmas. Given that CD and AD are congruent, we are to show that angle EAB is three times angle DCA.

In the theorem, the center A of a circle with radius AB lies on the line AC . D lies on the circumference of the circle such that $CD=AD$. E is the second intersection between the circle and CD . The theorem states that angle EAB is 3 times angle DCA . (Archimedes uses this theorem in the opposite direction, as the basis for a method of trisecting the angle. The method involves a ‘neusis’ or sliding: try to work out how to draw the diagram with BAE given rather than BCE and you’ll see where this needs to occur. Try to draw the diagram in GXWeb with BCE specified and it will fail, as GXWeb does not do the neusis either.)

This theorem can be re-imagined in terms of angle bisectors (figure 2a) or even in a form with one angle bisector and one isosceles triangle (figure 2b). The thought process to use when doing this sort of re-imagination, is the topic of this paper. Once you have reimagined a theorem, you can use a dynamic geometry system with symbolic theorem proving [2] or a symbolic geometry system [3] to verify that it is correct. This technological safety net allows us to be reasonably loose in our approach, while maintaining confidence in the mathematical integrity of the end result.

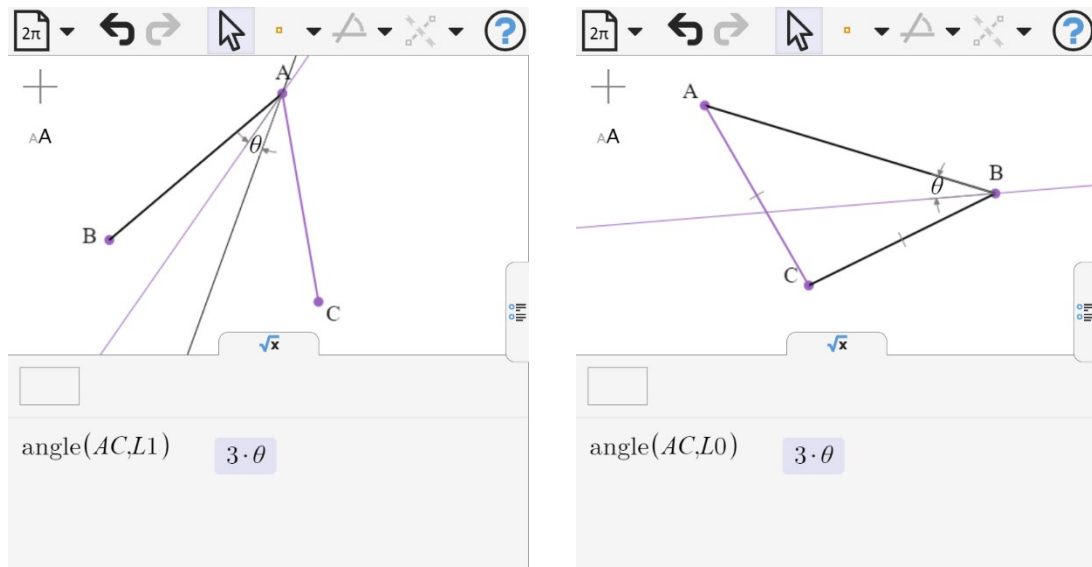


Figure 2: (a) $L0$ is the angle bisector of AB and AC , $L1$ is the angle bisector of $L0$ and AB . (b) $L0$ is the angle bisector of AB and BC , AC and BC are congruent

2. The Angle at the Center and the Angle at the Circumference

The familiar result that the angle subtended by a chord at the circumference of a circle is half that subtended at the center (fig. 3) is a useful place to start our exploration. In order to explore alternative castings of this theorem, we first make explicit the isosceles triangles inferred by the fact that lines BC and BD are chords of the circle centered at A . To do this, we add the line AB (dashed in figure 3a). A useful next step is to number the lines in the figure. With the numbering in figure 3a, line 4 is the base of an isosceles triangle between lines 1 and 2, while line 5 is the base of an isosceles triangle between lines 2 and 3.

The symmetry of an isosceles triangle implies that the perpendicular bisector of its base is the angle bisector of its apex. Hence the direction of the base is perpendicular to the direction of the angle bisector. It would thus seem reasonable that for any angle theorem involving isosceles triangles, there is a corresponding theorem involving angle bisectors. In figure 3b, we have started

with a triangle whose sides are labelled 1,2,3, then created the angle bisector of 1 and 2 (line 5) and the angle bisector of 2 and 3 (line 4). We have added an angle between lines 1 and 3, and measured the angle between lines 4 and 5. We note that, whereas in figure 3a the angle between lines 4 and 5 is $\frac{\theta}{2}$ in figure 3b the angle is $\frac{\pi+\theta}{2}$. Our goal in this endeavor is to find theorems, the fact that the angle's value is given by the inputs is what we claim. A dynamic geometry system (ideally with symbolic components) can be used to establish its actual value.

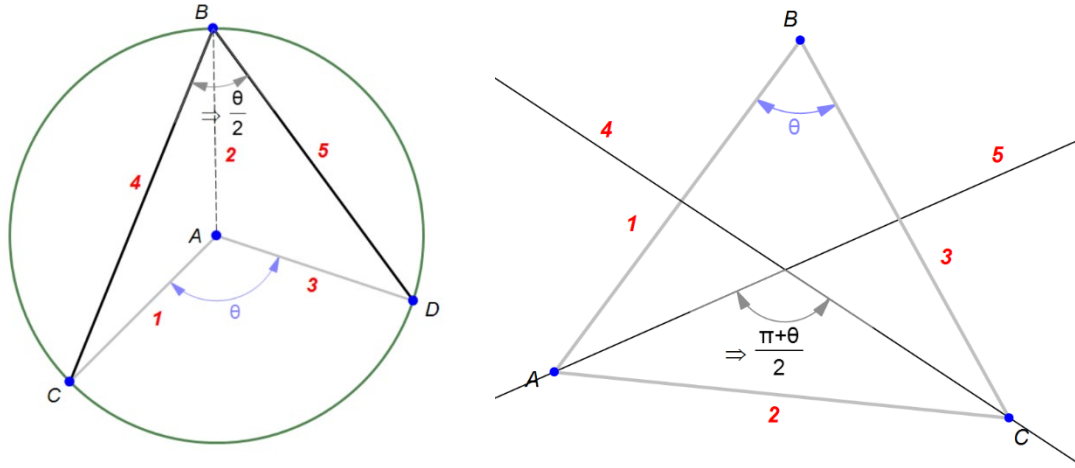


Figure 3: (a) An illustration of the theorem that the angle subtended at the center of a circle is twice that at the circumference. Lines are labelled for reference. (b) A theorem involving the angle at the vertex of a triangle and that between its angle bisectors.

We now have a theorem with two isosceles triangles and another theorem with two angle bisectors, it is natural to look for a theorem with one of each. In figure 4a, we start with triangle ABC whose sides are labelled 1, 2, 3. We use a circle centered at A to place D on BC such that ABD is isosceles and label AD as line 4. We then create the angle bisector of ACB, labelling it 5. We then specify the angle between lines 1 and 3, and measure the angle between lines 4 and 5.

If we have three lines L0, L1, L2 such that line L2 is the image of line L1 under reflection in line L0, then L0 is the angle bisector of L1 and L2. Hence, we could expect to be able to rephrase an angle bisector theorem as a reflection theorem. Figure 4(b) illustrates this in the context of the theorem of Figure 3(b). Lines 1, 2 and 3 show the path of a billiard ball which bounce in turn off lines 4 and 5. Whereas in the formulation of figure 3(b) it was natural for lines 1, 2 and 3 to be given and 4 and 5 derived, it is more natural here for lines 4, and 5 to be given (along with line 1) and line 2 and 3 to be derived. Therefore we have specified the angle between lines 4 and 5, and derived the angle between lines 1 and 3.

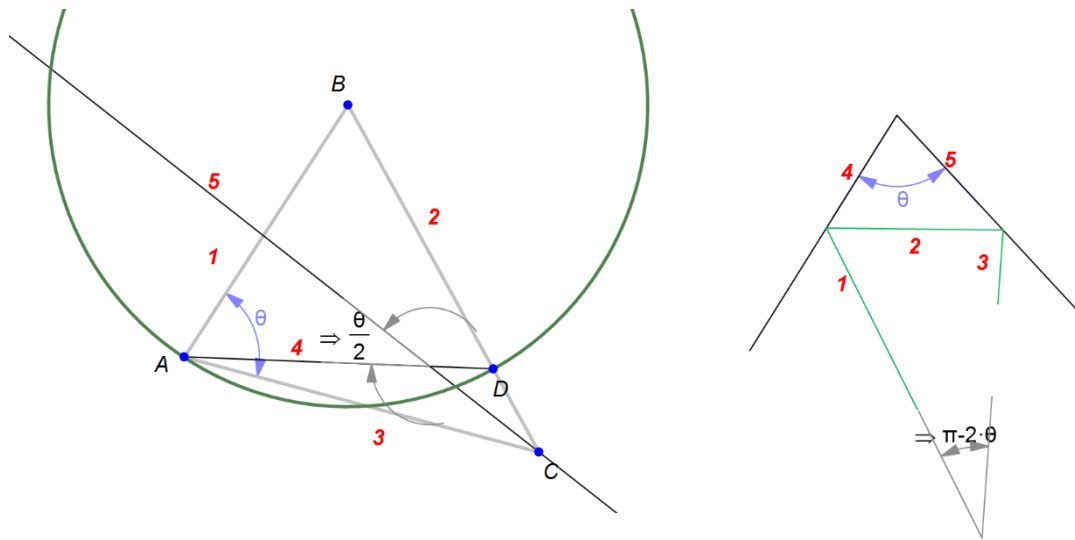


Figure 4: (a) A theorem relating a triangle angle with that between an angle bisector and the base of an isosceles triangle. (b) A theorem relating the angle of the initial and final paths of a billiard ball which bounces off two faces in succession.

3. Cyclic Quadrilaterals

Figure 5(a) shows a cyclic quadrilateral, with explicit radial lines numbered 1,2,3,4, and chords numbered 5,6,7,8. The theorem that the opposite angles are supplementary is illustrated as a relation between the angles between chords 5 and 6 and between chords 7 and 8.

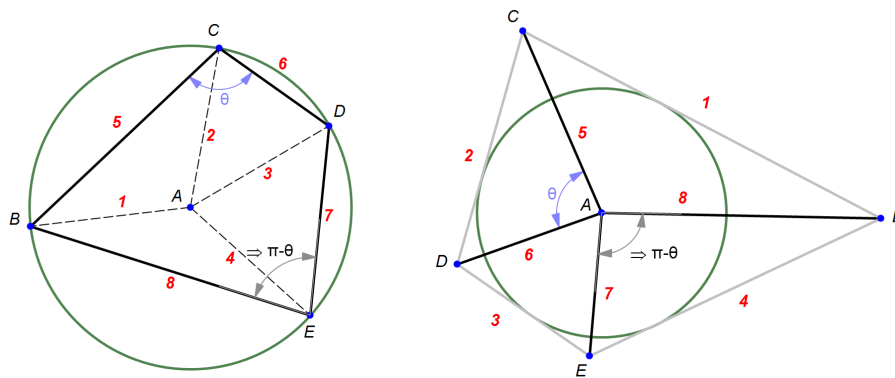


Figure 5: (a) A cyclic quadrilateral (b) A quadrilateral with inscribed circle..

Given arbitrary directions 1,2,3,4, the requirement that BCDE is cyclic imposes the conditions that line 5 is the base of an isosceles triangle with sides 1 and 2, that line 6 is the base of an isosceles triangle with sides 2 and 3, that line 7 is the base of an isosceles triangle with sides 3 and 4 and that line 8 is the base of an isosceles triangle with sides 4 and 1. We can't have just any directions for lines 5, 6, 7 and 8. In fact if the directions of 5, 6 and 7 are chosen, the direction of line 8 is determined, as the theorem attests.

To create a comparable diagram where angle bisector conditions are used rather than isosceles triangles, we can start with a quadrilateral which has an inscribed circle (figure 5b). In this diagram the sides of the quadrilateral are numbered 1,2,3,4. The angle bisectors meet at the center of the

inscribed circle. We number the angle bisectors 5,6,7,8. The angle between 5 and 6 is supplementary to the angle between 7 and 8.

4. Two Cyclic Quadrilaterals

A theorem in [4] extends two sides of a cyclic quadrilateral to form a second cyclic quadrilateral (fig 6a). The new cyclic quadrilateral shares three sides with the original. The theorem states that the fourth side of the new quadrilateral is parallel to the fourth side of the original quadrilateral. This theorem may be reimaged in a number of ways.

First, and most simply, we can phrase the theorem as follows: two cyclic quadrilaterals have three sides parallel in pairs, prove that the fourth sides are parallel.

If we cast one of the cyclic quadrilaterals as a quadrilateral with inscribed circle, we get the theorem of figure 6b. Whereas in figure 6a the fact that EFCB is cyclic is given and the parallelism of lines EF and AD is to be proved, in 6b the parallelism is given and the fact that AEDG is cyclic is to be proved.

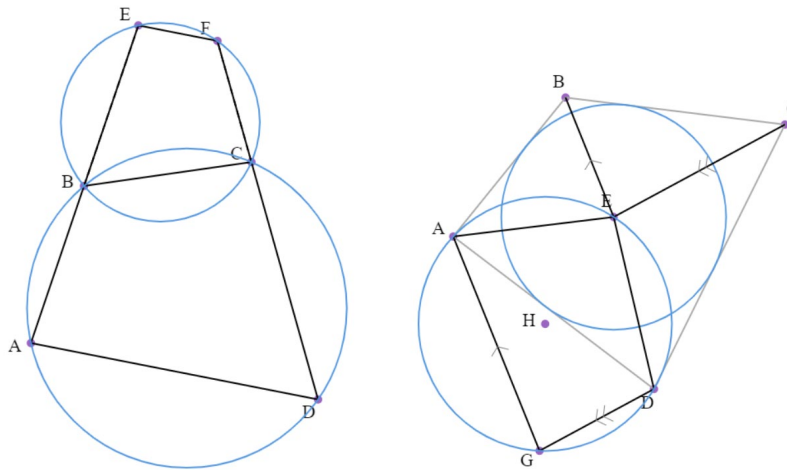


Figure 6: (a) ABCD is a cyclic quadrilateral. E lies on AB extended and F lies on DC extended such that EFCB is cyclic. EF is parallel to AD. (b) ABCD is a quadrilateral with inscribed circle center E. Let G be the intersection of the parallel to EB through A and the parallel to EC through D. AEDG is a cyclic quadrilateral.

Reimagining the theorem with angle bisectors for one quadrilateral (figure 7a), we have the succinct statement that the intersections of the angle bisectors of any quadrilateral form a cyclic quadrilateral.

If we take a cyclic quadrilateral and exscribe a quadrilateral such that it forms isosceles triangles with three sides, then it forms an isosceles triangle with the fourth (figure 7b)

If a billiard ball bounces off the four sides of a cyclic quadrilateral in turn (figure 7c) then its final path is parallel to its initial path.

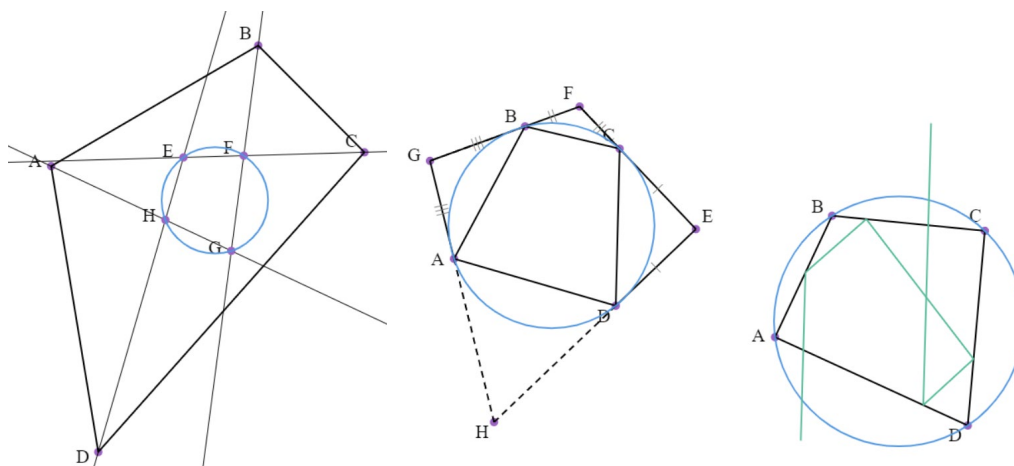


Figure 7: (a) The intersections of the angle bisectors of a quadrilateral are cyclic. (b) EFGH is an exscribed quadrilateral to cyclic quadrilateral ABCD such that DEC, BFC and BGA are isosceles, then ADH is also isosceles. (c) A billiard ball bounces off the four sides of a cyclic quadrilateral in turn. Its final path is parallel to its initial path.

5. Coalescing Lines

Another way of creating new theorems from old is by coalescing lines. For example, in figure 8(a), we have reproduced the diagram of figure 6(a) but drawn all the implicit radial lines, and numbered them (we see that our five lines and two circles have become thirteen lines!). Imagine what would happen if we allowed lines 1 and 8 in the diagram to get closer and closer to each other. Line 12 would remain parallel to line 13 and lines 1 and 8 would approach the perpendicular bisector of FG, which is perpendicular to line 13. Once they have coalesced, we have diagram 8(b) and this theorem: let F be the intersection of line AB and CD of cyclic quadrilateral ABCD; let G be the circumcenter of triangle BCF; then GF is perpendicular to AD.

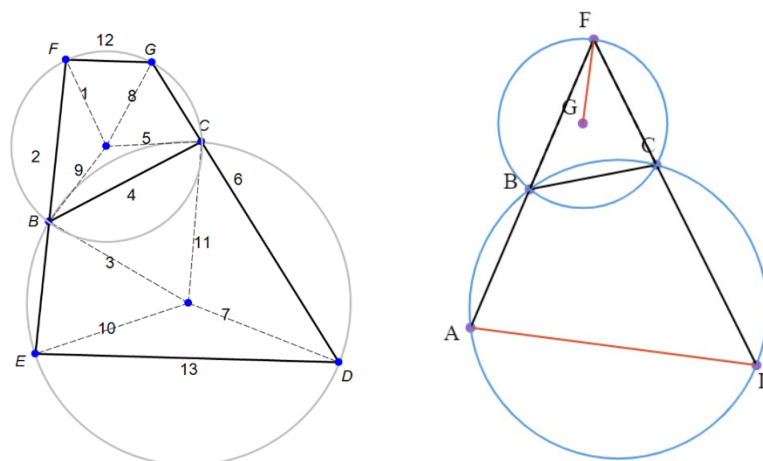


Figure 8 (a) The theorem of figure 6(a) with all the implied radial lines present and numbered. (b) Coalescing lines 1 and 8 leads to a new and distinct theorem.

A precise statement of this argument would involve calculus, however having stated the theorem, we can readily prove it geometrically. Our goal here is the discovery of theorems, the proof to be left to our geometric technology or to our geometry students. Hence the use of imprecise limits as part of the discovery process seems justified.

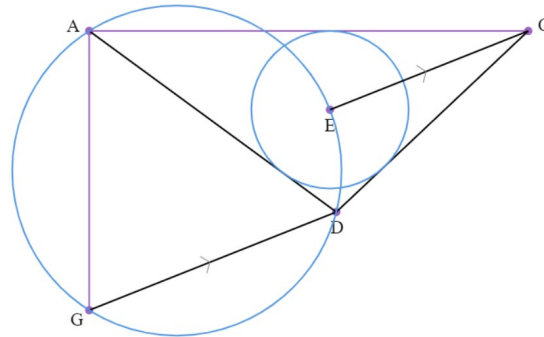


Figure 9. Coalescing the lines AB, BC in figure 6b yields this diagram and the theorem: Let E be the incenter of triangle ADC. Let G lie on the circumcircle of ADG such that DG is parallel to EC. Angle GAC is right.

A second example of this approach is given in figure 9. In this figure, we have coalesced the lines AB and BC of figure 6b. Line EC in figure 6b is the bisector of angle ABC and by definition parallel to line AG. In the limit the coalesced line ABC is perpendicular to the angle bisector. Hence in our new diagram (figure 9), we have AC perpendicular to AG.

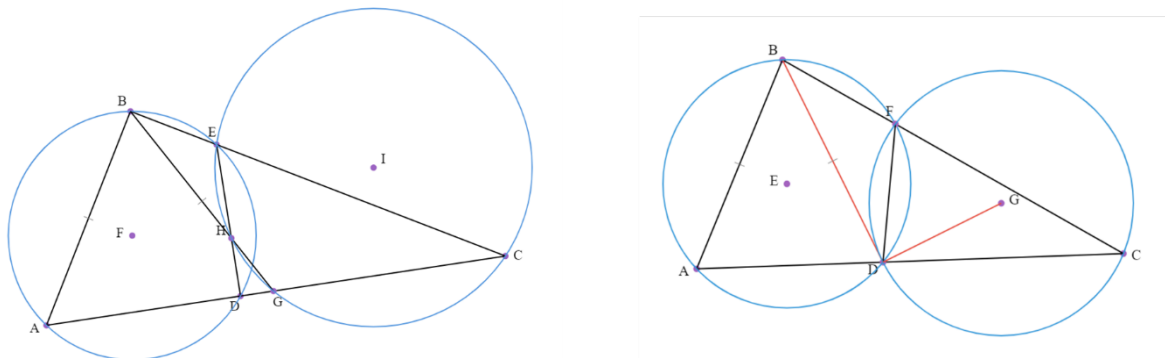


Figure 10: (a) D lies on side AC and E on side BC of triangle ABC such that ABED is cyclic. G lies on side AC such that ABG is isosceles. H is the intersection of ED and BG. EHGC is cyclic. (b) D lies on side AC of triangle ABC such that ABD is isosceles. F lies on side BC such that ABFD is cyclic. G is the circumcenter of CDF. GD is perpendicular to BD.

Figure 10 illustrates a third application of this technique. Coalescing lines IG and IH of figure 10a leads to points H, G and D of the figure coalescing into the single point D of figure 10b and a right angle postulated at GDB. This can readily be confirmed geometrically.

6. Cyclic Polygons

In section 4 above, we were able to generate a number of different theorems from the theorem that two cyclic quadrilaterals with three parallel sides have a fourth parallel side. A natural question is this: do we have a similar theorem for cyclic pentagons? Do we have a similar theorem for cyclic hexagons? The answers to these questions are no and yes.

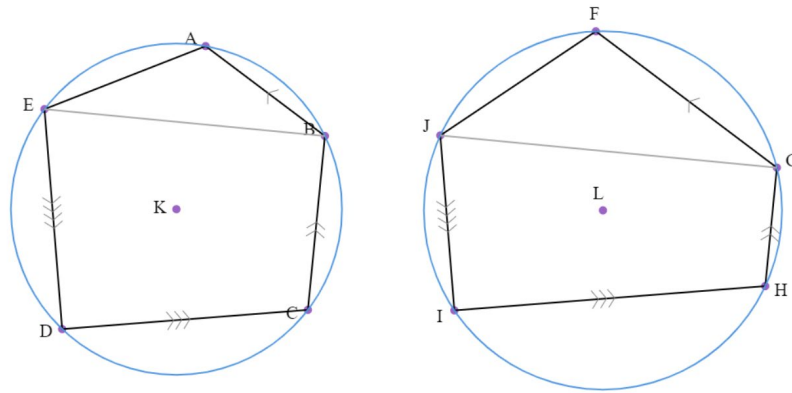


Figure 11: Cyclic pentagon $FGHIJ$ has sides FG, GH, HI, IJ parallel to sides AB, BC, CD, DE of cyclic pentagon $ABCDE$.

In Figure 11, two cyclic pentagons $ABCDE$ and $FGHIJ$ have parallel sides AB with FG , BC with GH , CD with HI and DE with IJ . The not-necessarily-parallel sides are EA and JF . As $BCDE$ and $GHIJ$ are cyclic quadrilaterals with 3 pairs of parallel sides, the fourth sides, EB and JG are also parallel. As FG and AB are also parallel, EA is only parallel to JF when angle EAB is equal to angle JFG . This is only true when the angle subtended at the center by EB is equal to the angle subtended at the center by GJ : that is when the two pentagons are similar.

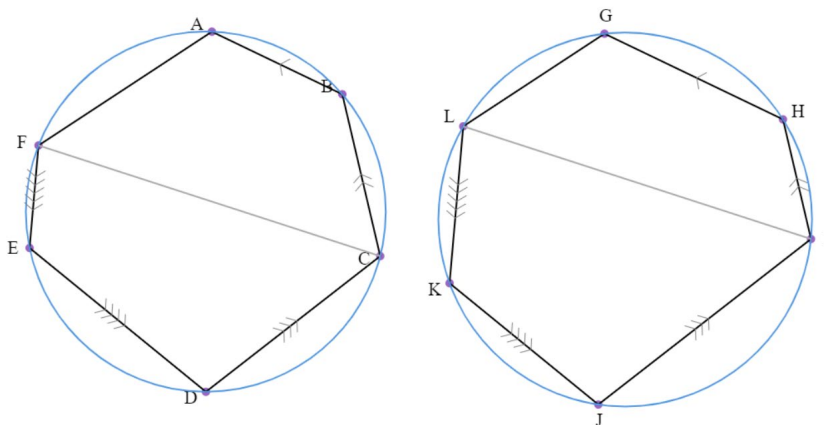


Figure 12: Cyclic hexagon $GHIJKL$ has sides GH, HI, IJ, JK, KL parallel to sides AB, BC, CD, DE, EF of cyclic hexagon $ABCDEF$.

The situation with cyclic hexagons is different. In figure 12, cyclic hexagons $ABCDEF$ and $GHIJKL$ have parallel sides GH with AB , HI with BC , IJ with CD , JK with DE and KL with EF .

As IJKL and CDEF are cyclic quadrilaterals, IL is parallel to CF. Now GHIL and ABCF are cyclic quadrilaterals with three pairs of parallel sides, hence AF is parallel to GL.

This argument can be adapted in a straightforward way to show that this property holds in general for cyclic polygons with an even number of sides, and does not hold for cyclic polygons with an odd number of sides.

7. Three Cyclic Quadrilaterals and Napoleon's Theorem

In section 4, we looked at theorems involving two cyclic quadrilaterals. In section 6, we generalized by looking at theorems with two cyclic polygons with more than 4 sides. In this section, we consider a different generalization axis and consider theorems involving more than two cyclic quadrilaterals.

In figure 13, the lines AD, BD, CD are not shown, however the cyclic quadrilaterals ADBH, BDCI and CDAJ play comparable roles to the two cyclic quadrilaterals in figure 6a. The fact that the sum of $\angle BDA$, $\angle ADC$ and $\angle BDC$ is 2π implies that the sum of angles $\angle AHB$, $\angle BIC$ and $\angle CJA$ is π and hence HA is parallel to AJ, from which we deduce that the line HJ passes through A.

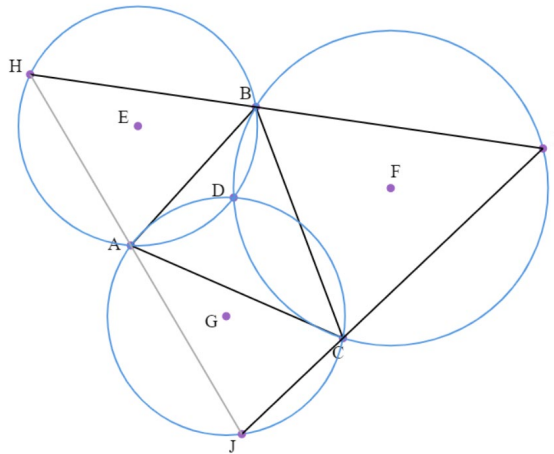


Figure 13: Given triangle ABC and point D, let H lie on the circumcircle of ADB, let I be the intersection of HB and the circumcircle of BDC and let J be the intersection of IC and the circumcircle of ADC. HJ passes through A

The key to this pattern is that each cyclic quadrilateral shares two edges with its neighbor: ADBH and BDCI share BD and HI, BDCI and ADCJ share DC and IJ, ADCJ and ADBH share AD and JH.

Figure 14 contains a diagram where this relationship is more obvious. ABCD is a cyclic quadrilateral. CGFE is another cyclic quadrilateral, with CE and CG the same lines as BC and DC. AKLJ is another cyclic quadrilateral with AK and AJ the same lines as DA and BA. Points L and F are put arbitrarily on the circumcircles of AKJ and GCE. The four lines KL, LJ, EF and FG, when extended form a cyclic quadrilateral.

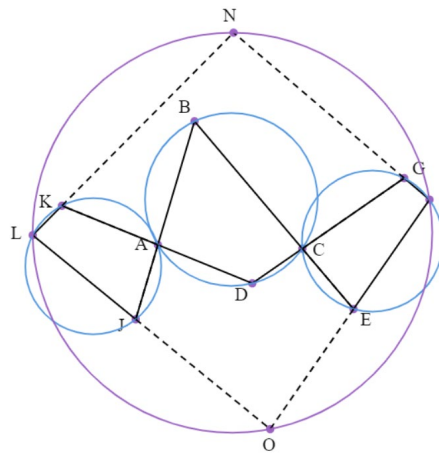


Figure 14: ABCD is a cyclic quadrilateral. G lies on DC extended, E lies on BC extended, J lies on BA extended and K lies on DA extended. CGFE and AKLJ are cyclic quadrilaterals. N is the intersection of LK and FG, O is the intersection of EF and JL. LNFO is a cyclic quadrilateral.

We leave the reader with the challenge of reimagining this pattern with different manifestations of the bisector and illustrate the power of the configuration with a proof of Napoleon’s Theorem.

Napoleon’s theorem states that the triangle formed by joining the centers of equilateral triangles constructed on the sides of a triangle is itself equilateral. A proof based on the above configuration takes the following form. Let D in figure 13 be the intersection of the circumcircles of equilateral triangles drawn on sides AB and BC of triangle ABC. Angles ADB and BDC (and hence CDA) are 120 degrees. Hence HIJ is equilateral. To complete the proof, we relate a particular HIJ to the Napoleon Triangle.

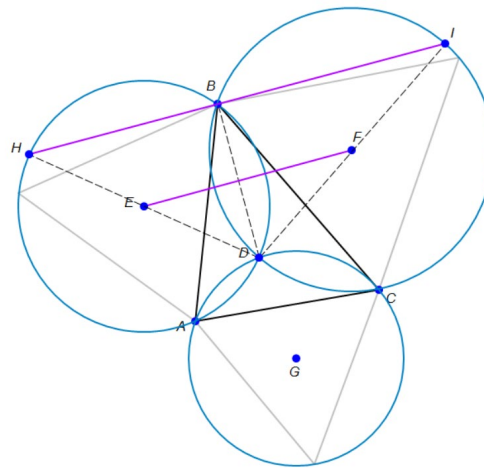


Figure 15: D is the Fermat-Toricelli point of triangle ABC, EF is one side of the Napoleon triangle, HI its dilation by a factor of 2 centered at D.

In Figure 15, E F and G are the centers of the circumcircles of equilateral triangles drawn on the sides of triangle ABC. D is the intersection of the circles. Napoleon’s triangle is EFG. If we dilate Napoleon’s triangle by a factor of 2 centered at D, its vertices lie on the three circumcircles and its edges pass through points ABC, and hence the dilated triangle is an instance of HIJ in figure 13.

The diagram of Figure 13 can be generalized so that instead of triangle ABC, we have a general polygon $A_1A_2 \cdots A_n$ and point D. In Figure 14, we could have a chain of n cyclic quadrilaterals rather than 3.

8. Conclusion

We encounter the angle bisector, in disguise, in theorems which feature reflections and in diagrams which contain isosceles triangles. As a circle chord and its two radial lines form an isosceles triangle, this latter category includes many diagrams containing circle chords. In this presentation we have described two ways of creating new angle bisector theorems from old. One method involves starting with a theorem which involves only circle chords and replacing some or all of these with angle bisectors. Another method involves coalescing lines of the original model and making an informal limit argument. The availability of software to check the validity of a theorem absolves us from the need to do calculus, rather we can use our limit argument to show us a candidate theorem, then check it using technology, or geometry.

Acknowledgements The author would like to thank the referees for suggesting a sensible restructuring of this presentation.

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Wavelet Neural Network Prediction of Stock Performance

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Abstract

Wavelet methods and artificial neural networks are incorporated to examine the forecasting performance of the daily closing price of the Microsoft stock, NASDAQ:MSFT. An experimental analysis is performed to demonstrate improved performance of the wavelet neural network. Results in this study suggest that for these neurowavelet models a long history with a short training is ideal for stock prediction. This model could be used by investors, financial managers, or others to enhance their ability to select desired stocks.

1 Introduction

Accurate stock market prediction is not only an important topic in the understanding of the economy but also proves to be a challenging task for investors. It is hard to forecast trends in the stock market. This unpredictability in market trends is pervasive for many reasons. Not only is the future inherently unpredictable, but preferences of society can exhibit high spontaneity and sudden fluctuations in demand. In this work we perform wavelet analysis using the closing price of a stock as a time series signal. Consider price action as a time series and hypothesize that through the use of wavelets, data science, and machine learning we may accurately project how future price moves. Most time series exhibit a high degree of temporal and spatial dynamics and are described by some nonlinearity and singularities. Financial analysts attempt to provide rational explanations to problems that arise in the stock market. Through the development of theoretical models and predictive algorithms, effective information can be obtained for use in financial planning, analysis and management. Traditional time series models are limited with the assumption that data are stationary and thus unable to accurately capture nonlinearities or singularities in the signal. Stock market prices are usually the outcome of interaction among different nonlinear phenomena, which fluctuate across different spatial and temporal scales producing a chaotic response curve that is difficult to predict. Our purpose is to develop an adequate model that characterizes high complexity, and nonstationarity of the stock price as well as provide accurate forecasting.

Various approaches have been tested for forecasting and prediction of financial time series. The last couple of decades have seen remarkable progress in the ability to develop accurate time series models [1, 2, 5, 6]. Artificial neural networks (ANN) shows great strength in efficiency of curve-fitting and simulation through the diverse network designs and available training algorithms. ANN have become a popular tool among stock market and elsewhere in the realm of time series analysis [4, 8, 9]. However, when working with chaotic signals such as a series of daily stock price, a major bottleneck in the capabilities of ANN has been the ability to generalize a well-trained simulation to the accurate prediction of extreme events or extended forecasting. By including multi-resolution information from a wavelet decomposition of the time series as input the ANN model is specialized to form the hybrid wavelet neural network (WNN) or *neurowavelet* model. Among many models developed over this period, WNN type models have shown to be among the most promising in simulating stock market. This is in part due to the provable ability of ANN to efficiently approximate highly nonlinear relationships [3]. The power of ANN is combined with the efficient multiscale representation granted by the wavelet transform to increase forecast accuracy. This is exemplified in the demonstrable potential for robust prediction of nonlinear time series by models incorporating a neurowavelet technique [5, 8, 11, 12].

Wavelet analysis is a useful and powerful tool in performing time-frequency or time-space analysis of a time series. The wavelet transform can be used as a decomposition of a time series in to precise resolution in both time and frequency scale planes. Wavelets with multiresolution (MRA) properties have become useful tools in many applications, which include sub-band coding data compression, characteristic points detection, and noise reduction and others.

In this study, we combine wavelet analysis and artificial neural networks (ANN) as a hybrid stock market forecasting model WNN. By ANN we are referring to the concept of using a machine learning approach to perform stock market forecasting. In order to control for the innate uncertainty produced by neural network we introduce two controls in the implementation of the neurowavelet system. As a first measure, a genetic algorithm is used to only select the best trained networks, this process is repeated to generate a set of networks. Each network is used to create a prediction, and then these results are averaged together to give the reported forecast horizon. We use a neuro-wavelet method to perform forecasting on the Microsoft stock.

There are two main experimental variables and two hypothesis which were tested in this analysis. In the implemented neurowavelet system there are two ways to control the amount data being used for prediction, and these are the variables of interest here. The first is the length of the time series to be analyzed by the wavelet transform. The second experimental variable is the length of the training period supplied to WNN, which we refer to as the *Lookback time*. The tested research questions are as follows: Does the addition of additional history data to the wavelet MRA improve stock market prediction? How much look-back time is needed to cleanly reproduce market trends?

This paper is organized as follows. In Section 2, we briefly describe wavelet method, neural networks, data sources and computer resources. We present our results in Section 3, followed by discussions in Section 4. We conclude with several comments and state our future plans in Section 5.

2 Methods

2.1 Wavelet Analysis

In what follows, we provide some background on wavelet analysis. This information includes a description of multiresolution analysis, scaling functions, wavelet functions, and the wavelet transform for both continuous and discrete signals.

A multiresolution analysis (MRA) [7, 10] consists of a sequence of successive approximation spaces $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ with the following properties:

- (i) $V_j \subset V_{j+1}$,
- (ii) $\lim_{j \rightarrow \infty} V_j = \bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,
- (iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,
- (iv) $f(x) \in V_j \iff f(2x) \in V_{j+1}$,
- (v) $f(x) \in V_j \iff f(x + 2^{-j}k) \in V_j, \forall k \in \mathbb{Z}$,
- (vi) There exists a function $\phi \in V_0$ so that $\{\phi(x - j)\}_{j \in \mathbb{Z}}$ is an orthonormal basis of V_0 .

ϕ is called a *scaling function* that generates a MRA with the above properties. Through translation and dilation of ϕ , a Riesz basis $\{\phi_{j,k}(x)\}_{k \in \mathbb{Z}}$ is obtained for the subspace $V_j \subset L^2(\mathbb{R})$ by the properties (iv)(v), where

$$\phi_{j,k}(x) = 2^{\frac{j}{2}} \phi(2^j x - k), \quad j, k \in \mathbb{Z}. \quad (1)$$

This family can be generally expressed as $\phi_{m,n}(x) = \frac{1}{a^{\frac{m}{2}}} \phi(\frac{x-nb}{a^m})$, for real numbers $a \neq 0$ and b .

Since $V_0 \subset V_1$, there is a set of coefficients $\{a_k\}_{k \in \mathbb{Z}}$, so that ϕ satisfies the two-scale equation or refinement equation

$$\phi(x) = \sum_k a_k \phi(2x - k). \quad (2)$$

For every $j \in \mathbb{Z}$, we define W_j to be the orthonormal complement of V_j in V_{j+1} , we then have

$$V_{j+1} = V_j \oplus W_j \quad (3)$$

and

$$W_j \perp W_{j'} \quad \text{if } j \neq j'. \quad (4)$$

It follows that, for $j > J$

$$V_j = V_J \oplus \left(\bigoplus_{k=0}^{J-j+1} W_{J-k} \right). \quad (5)$$

By virtue of (ii) and (iii) above, this implies

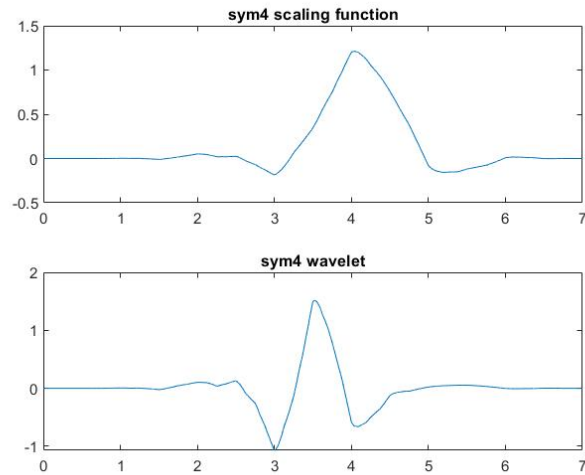


Figure 1: Symlet Scaling Function and Wavelet of Order 4

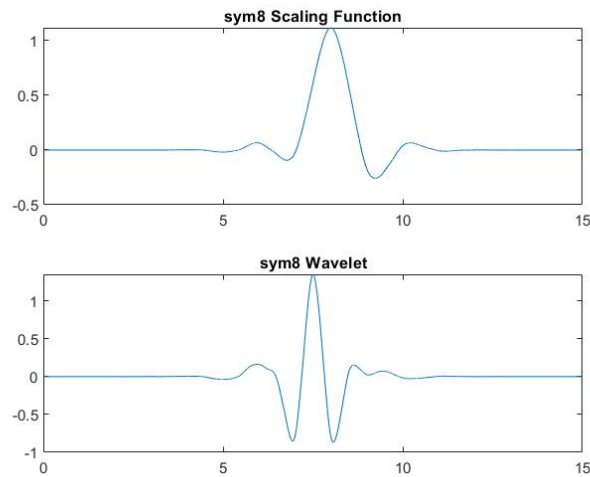


Figure 2: Symlet Scaling Function and Wavelet of Order 8

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j \tag{6}$$

which is a decomposition of $L^2(\mathbb{R})$ into mutually orthogonal subspaces. It turns out that a basis for W_0 can be obtained by dilating and translating a single function $\psi(x)$ called basic (mother) wavelet which is defined by (wavelet equation)

$$\psi(x) = \sum_k b_k \phi(2x - k) \tag{7}$$

where $b_k = (-1)^k a_{-k+1}$. In fact, $\{\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k)\}_{k \in \mathbb{Z}}$ forms an orthonormal basis for W_j . Examples of scaling functions and wavelets are presented in Figures 1-4.

Let P_j, Q_j denote the orthogonal projection $L^2 \rightarrow V_j, L^2 \rightarrow W_j$, respectively. Then

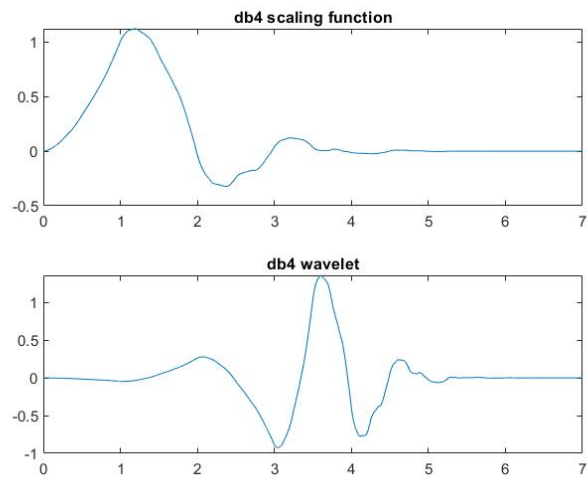


Figure 3: Daubechies Scaling Function and Wavelet of Order 4

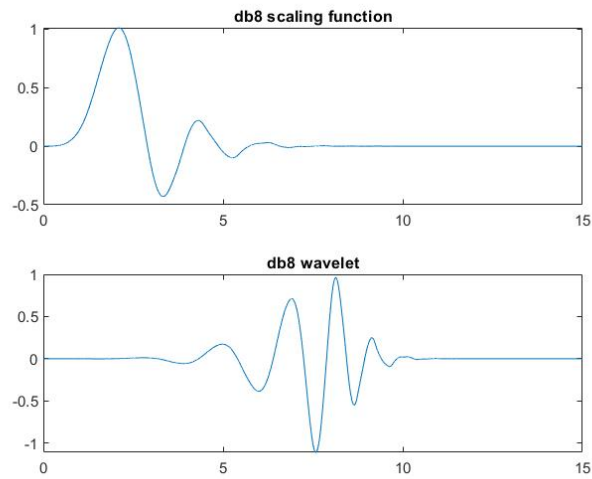


Figure 4: Scaling Function and Wavelet of Order 8

$$P_j f(x) = \sum_k \alpha_{j,k} \phi_{j,k}(x), \quad (8)$$

$$Q_j f(x) = \sum_k \beta_{j,k} \psi_{j,k}(x), \quad (9)$$

where the coefficients $\alpha_{j,k}$, $\beta_{j,k}$ are given by the following inner products respectively:

$$\alpha_{j,k} = \langle f, \phi_{j,k} \rangle = \int_{-\infty}^{\infty} f(x) \phi_{j,k}(x) dx, \quad (10)$$

$$\beta_{j,k} = \langle f, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx. \quad (11)$$

$P_j f$ converges to f in the L^2 norm which is the best approximation of f in V_j .

More precisely, the above coefficients can be obtained by applying wavelet transforms which are defined as follows.

The continuous wavelet transform is defined as:

$$[w_\psi x(t)](a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t) \psi^* \left(\frac{t-b}{a} \right) dt \quad a > 0, b \in R, \quad (12)$$

where the symbol $*$ represents the complex conjugate, $x(t)$ is the given signal and ψ is a wavelet. The discrete wavelet transform is defined as:

$$[Dw_\psi x(n)](a, b) = \sum_{n \in Z} x(n) g_{j,k}(n), \quad a = 2^j, b = k2^j, j \in N, k \in Z, \quad (13)$$

where $g_{j,k}$ are the coefficients of the wavelet equation associated with ψ .

To provide input data, we calculate wavelet coefficients by choosing appropriate wavelets and the level of MRA resolution.

2.2 Neural Networks

Artificial neural network (ANN) is one of the most promising methods in artificial intelligence. It is a mathematical method for data analysis based on learning and analyzing abilities. The idea is to simulate the human brain in the knowledge acquisition process to solve problems such as clustering, classification and prediction. ANNs set up components that possess essential properties of neurons and these are connected by specific weights. An ANN has input and output layers connected by a hidden layer. These successive layers receive the input information and propagate it towards the output layer. More precisely, the structure of the information processing system consists of a large number of interconnected processing elements (neurons). The networks learn the systems by adjusting to the synaptic connections that exist between neurons. The nonlinear autoregressive neural network is one of the basic models of ANN appropriate for estimation of future values of the input variable. It performs multistep neural network prediction which is for multi-step ahead prediction. In this context, dynamic networks with feedback can be transformed between open-loop and closed-loop modes. Closed-loop networks continue to predict when external feedback is missing, by using internal feedback.

There are many training functions used to train an ANN. The idea is to forecast future values of a time series, based on its historical (previous) values, utilizing additional external time series with some time delay parameters. The network training is performed, by some back propagation algorithm, and uses steepest descent method [8] to obtain least error between the real data and the predicted values.

2.3 Wavelet Neural Network

The idea of wavelet neural network (WNN) is to combine wavelet analysis and ANN as a hybrid time series forecasting model. By ANN we are referring to the concept of using a machine learning approach to perform forecasting. We first use wavelet tools to obtain wavelet coefficients of input data, we then use them to process through ANN which is one of the most powerful and useful methods in artificial intelligence. ANN is a typical mathematical method for data analysis based on learning and analyzing abilities so that we simulate the data through the knowledge acquisition process to solve problems as we described in the previous sub-session. The incorporation of wavelet analysis and ANN gives rise to more accurate, efficient and effective ways of classification and forecasting. The robustness and flexibility of our WNN system have been proven through additional experiments on other data. In addition, our model has been extended to incorporate multifarious extraneous measured input signals, without loss of algorithmic efficiency.

2.4 Data Sources

Microsoft (MSFT) price can be found at <https://www.macrotrends.net/> which includes 35 years stock prices and other financial information.

2.5 Computer Resources

All computational experiments were performed using the High Performance Computing Center (HPCC) at Michigan State University's Institute for Cyber Enabled Research (ICER). The calculations were run on a single machine equipped with a AMD EPYC 7H12 64-Core Processor @ 2.6GHz and 996 GB of DDR4 ECC RAM. By way of this configuration a homogeneous parallel cluster was generated using 101 logical cores with 4 GB of memory allocated to each core and used for each experimental trial.

The predictive neurowavelet system was implemented as a MATLAB function. Computations were performed using MATLAB/2021a and the following toolboxes: Deep Learning, Parallel Computing, and Wavelet.

3 Results

The data was divided as 52% for training, 15% for validation and the final 33% as testing data.

After extensive testing with 100,500,2500, and 7500 days of lookback superior results were observed with 100 days of training. Based on this it was determined that a more accurate stock prediction is determined using less days WNN training. This is an unexpected result and stands in contrast of previous experiments applying the same system to a more natural signal

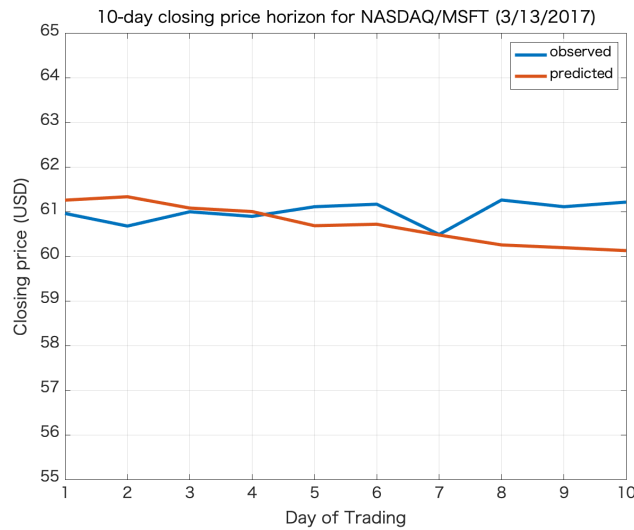


Figure 5: 10 day forecast horizon (based on the data since 1986)

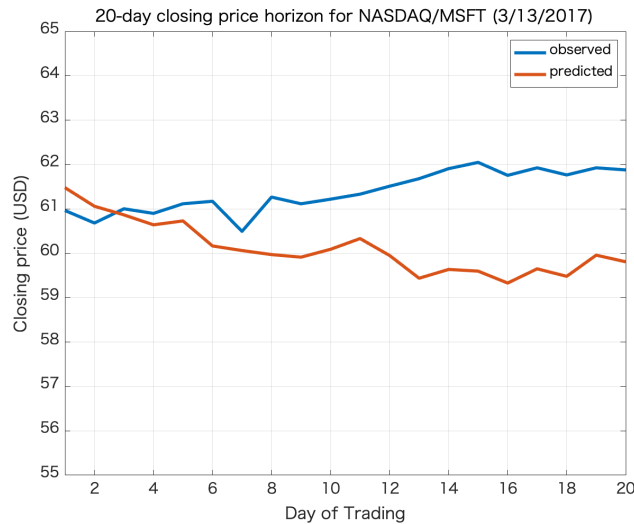


Figure 6: 20 day forecast horizon (based on the data since 1986)

[1]. Based on further detailed tuning, it was determined that 66 days of lookback should be used as the optimal length of history for the target signal.

To perform the WNN process, we use 6 delays, one hidden layer of 13 neurons, level 13 DWT with sym3 wavelet. In order to get reasonably good results, we run 100 trials and took the average. Based on the data since 1986, we obtained 10-day, 20-day, and 30-day closing price forecasting for NASDAQ/MSFT as shown in Figures ??, ??, and 7. In a similar way, we based on the data since 1995, we obtained and presented the results in Figures 8, 9, and 10. The model performance is measured by Mean Squared Error (MSE), Root Mean Square Error(RMSE), RMS Relative Error (RMSRE), and Mean Absolute Percent Error as defined in Table 3. We calculated the performance of the above methods and present the results in Table 3 and Table 4. The steps of the algorithm are presented in the appendix.

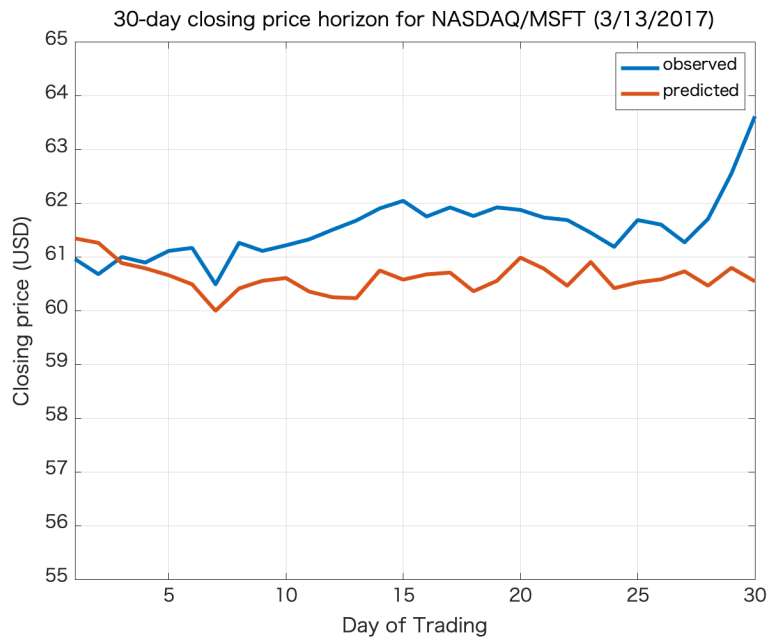


Figure 7: 30 day forecast horizon (based on the data since 1986)

Table 1: Performance Statistics

MSE	$\frac{1}{n} \sum_{t=1}^n e_t^2$
RMSE	$\sqrt{\frac{1}{n} \sum_{t=1}^n e_t^2}$
RMSRE	$\sqrt{\frac{1}{n} \sum_{t=1}^n \left(\frac{e_t}{y_t}\right)^2}$
MAPE	$\frac{1}{n} \sum_{t=1}^n \left \frac{e_t}{y_t}\right $

Table 2: Model Configurations (Hyperparameters)

Model	Hidden Layers	Wavelet Family	Resolution	Buffer	Lookback	History
WNN86	(13)	<i>Sym3</i>	13	6	66	7814
WNN95	(13)	<i>Sym3</i>	13	6	66	5539
MLP[4]	(70,28,14,7)	-	-	1	-	2750
UA [4]	(70,70)	-	-	-	-	2750

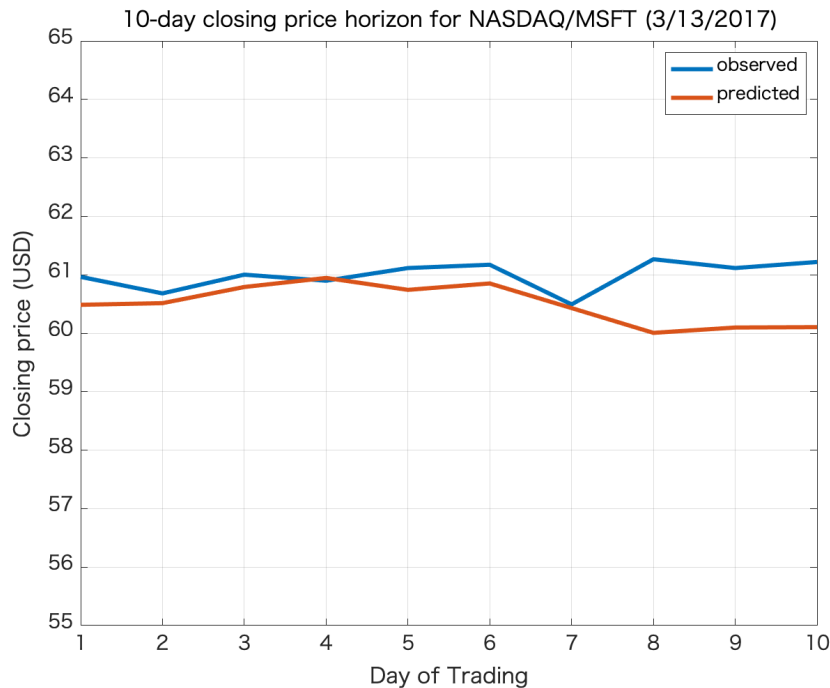


Figure 8: 10 day forecast horizon (based on the data since 1995)

4 Discussions

The performance with longer history data shows better than less history data. Remarkably, this means forecasting performs better if we are able to supply a longer history to the WNN during the closed-loop training and testing segments of model prediction. However, in calibrating the length of lookback it was determined that a shorter training data produces the best result. Together, this seems to imply that a long history with a short training is ideal for stock prediction.

Our WNN method shows better results than some neural network method [4, 12]. Figure 11 presents predictive results of a final response showing an improvement by the WNN presented here over the neurowavelet designs developed in [12]. The details of the hyperparameters representing some comparable models from [4] are shown in Table 3. Table 5 provides an explicit comparison of the measured statistics further demonstrating the superiority of the WNN method.

The performance gain seen with WNN method is likely due to the higher level wavelet decomposition incorporating more information of the time series. Also, WNN method can do better jobs on non-stationary time series and allows choosing different levels of resolution.

5 Conclusions

We use wavelet method together with ANN in predicting stock price and see the effects of our method by comparing various errors. Wavelet analysis has the ability to improve forecasting by capturing useful information on various resolution levels of a signal. ANN is very useful in

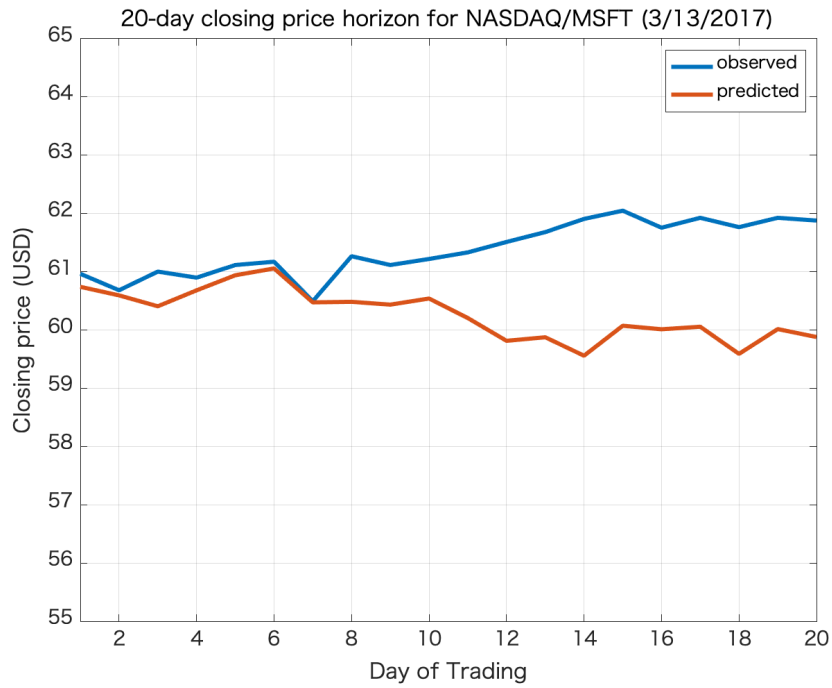


Figure 9: 20 day forecast horizon (based on the data since 1995)

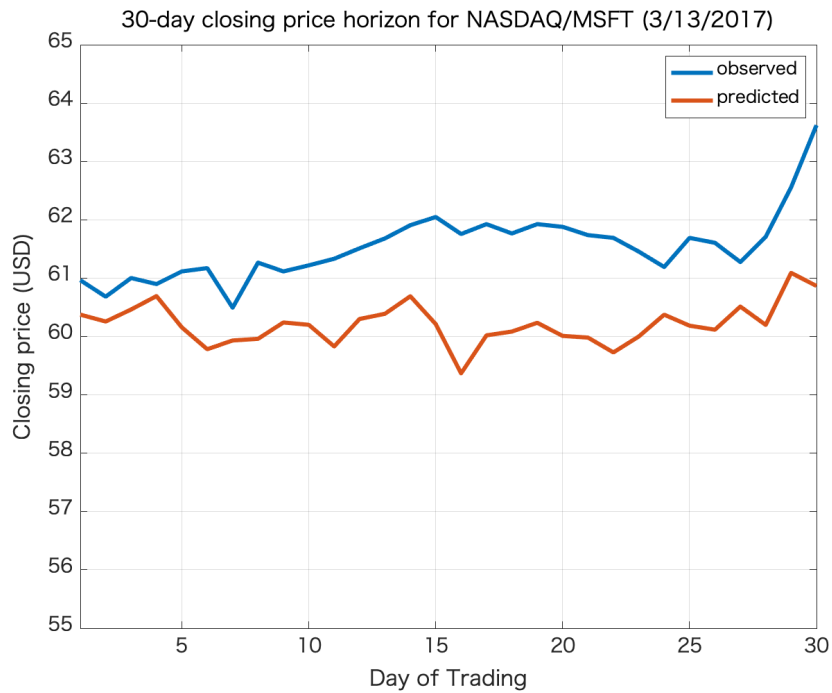


Figure 10: 30 day forecast horizon (based on the data since 1995)

modeling and forecasting time series. Combining both techniques, we obtain better results than other methods. Overall, WNN model using longer training produces better results than other methods. In fact, it turns out that short-term traders are usually better served by waiting for

Table 3: Forecast Performance (history: 3/13/1986-3/10/2017)

starting 3/13/2017		
<i>horizon (days)</i>	<i>MSE</i>	<i>RMSRE</i>
10	0.40	0.00045
20	2.52	0.0010
30	1.27	0.00050

Table 4: Forecast Performance (history: 3/13/1995-3/10/2017)

starting 3/13/2017		
<i>horizon (days)</i>	<i>MSE</i>	<i>RMSRE</i>
10	0.44	0.00081
20	1.89	0.00091
30	2.10	0.00072

Table 5: Comparison of Method Performance

Model	RMSE	MAPE
WNN86-10	0.6296	0.00045
WNN95-10	0.6642	0.00082
MLP-SP500[4]	44.5137	0.0118
UA-SP500[4]	25.4851	0.0067

confirmation of an output at hand, rather than trying to predict what an output will be in the long run. From what is learned of the outcome, traders establish significant stages to buy or sell that should be based on what price is actually doing, rather than what we expect it to do.

We will explore various tasks related to the method we presented in this paper, such as the level of wavelet resolution, genetic evolution parameters, number of neurons, size of delay buffer, and the structure of the hidden layer. We will also perform forecasting for other stocks. We envision our study will have impact on understanding and predicting the trends crucial in maintaining a stable marketplace environment with capacity for safe, effective, and profitable

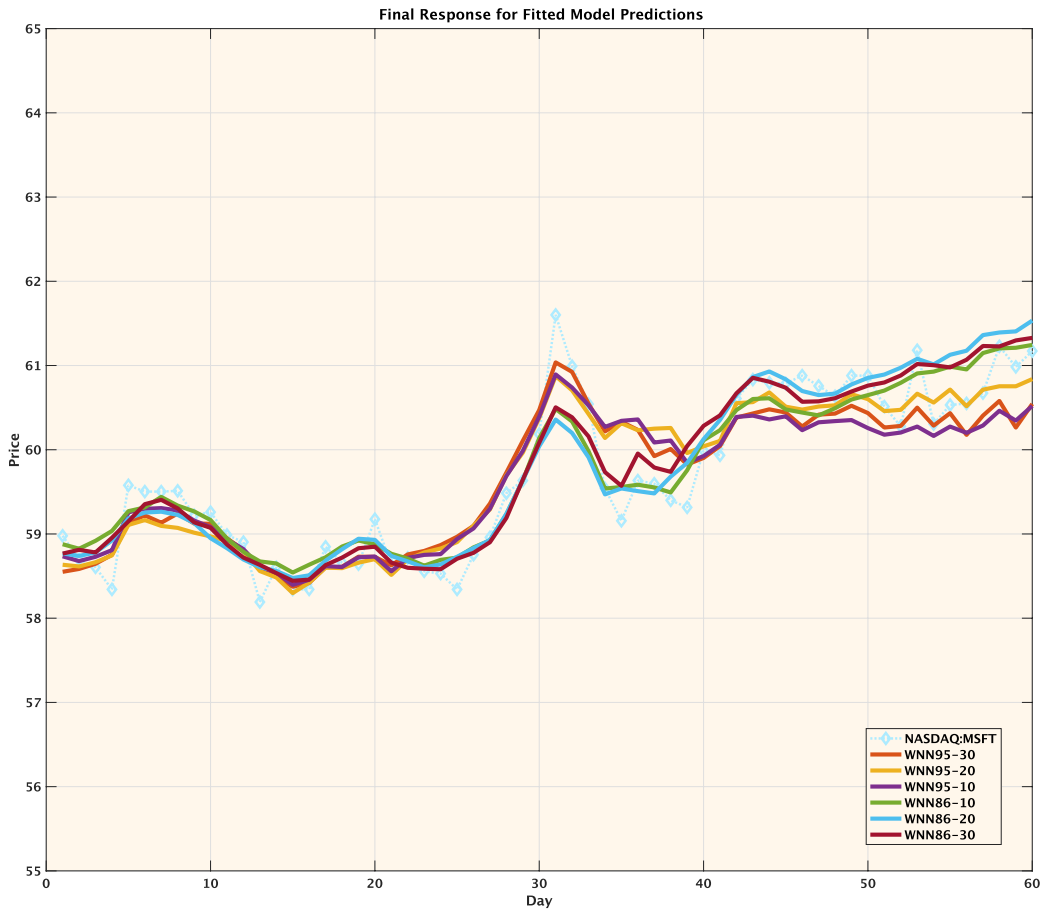


Figure 11: Comparison of prediction to fitted testing data for each WNN method.

stock trading. In this way we hope to produce work that helps improve the international economy and the well-being of our society.

Appendix

The complexity and power of ANN is achieved by the interaction of several neurons through the nonlinear process. The algorithm is based on the following Scaled Conjugate Gradient (SCG) training algorithm which is briefly described as follows. The idea is basically to minimize the error function $E(w)$ of the weight vector w with the following steps.

- Step 1. Select the initial weight vector w_1 and let $k=1$.
- Step 2. Determine a search direction p_k and a step size h_k so that $E(w_k + h_k p_k) < E(w_k)$.
- Step 3. Update vector $w_{k+1} = w_k + h_k p_k$.
- Step 4. If $E'(w_k) \neq 0$ then set $k = k + 1$ and go to Step 2 else return w_{k+1} as the desired minimum.

Acknowledgment

The authors wish to thank anonymous referees' comments which improve the manuscript.

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Understanding Geometric Pattern and its Geometry

Part 6 – Using Geometer’s Sketchpad for designing sizeable geometric projects

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Abstract: *This document aims to explain a few aspects of dealing with geometric objects in Geometer’s Sketchpad that are not entirely related to geometry. We discuss how one can reduce the amount of data needed to create bulky geometric constructions without losing the quality and accuracy of the final design. We use a real complex geometric pattern to demonstrate our conclusions.*

Introduction

Designing geometric patterns with GSP (Geometer’s Sketchpad) or any other geometry software¹ means two things – creating geometric objects and dealing with the software’s non-geometric features. We know a lot about geometry, but we often get into trouble with its non-geometric aspects.

While drawing complex geometric patterns using GSP, I observed that sometimes my computer slows down, and it takes a longer time to redraw the image on the screen. Some of my students, using older machines, reported that their computers crashed while producing such designs. In this paper, we deal with this issue. We will show how one can design even very bulky geometric patterns without slowing the performance of their computers and still obtaining good quality designs.

While working with the program, GSP keeps its sketch entirely memory- resident. No part of the drawing, selecting, or dragging objects require access to the file whatsoever. The file on the disk is completely independent and is used for long-term storage and data transfer. Thus, in reality, we have two documents – one the computer file and another one the in-memory sketch.

Generally, GSP files aim to be small. They contain a mathematical description of objects used in the sketch. The in-memory sketches aim to be as fast as possible. While loading a file from the disk, GSP will allocate, compute, and store the equations and parameters of each object most recently sampled to rapidly redraw the picture without having to perform any arithmetic every time it needs to redraw even a small portion of the window. Thus, we have a reasonably small file on the computer disk, while the in-memory sketch is a maximal object.

¹ Most of the information in this paper is valid also for GeoGebra.

How much information is needed for an object?

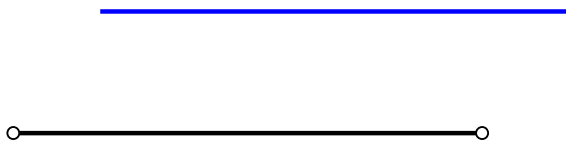
Let us examine some of the objects we deal with in Geometer's Sketchpad.

A point from the geometric point of view is a small dot. A point means much more in GSP (and any other geometry software) – its coordinates, color, size, and flag for object's features – hidden or visible. All this information is stored in the in-memory sketch. There is also a visual representation of a point – two small circles, one black and another smaller using a specific color.

Thus, if we have a few thousand points in our design, we already have a considerable amount of hidden data. But points are not the bulkiest objects.

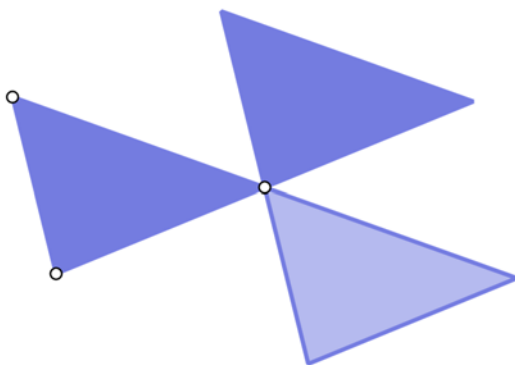
A segment is a collection of two points used to create it, the equation of the line passing through these points, the color of the segment, its thickness, and its style. This means that for every segment, we have about 3 times more information than for a point. But there is something hidden that we didn't notice. The information about the endpoints of a segment is doubled. Here is an example:

The black segment below was created by connecting the two points shown here. The blue segment is a translation of the black one. I changed color to distinguish them. The top segment does not have visible endpoints, but they still exist. This means that if we draw a segment, information about its ends is stored twice – (1) in coordinates of points used to create a segment and (2) in the description of the segment (look at the blue part).



All this means again a lot of hidden, sometimes unnecessary information is stored in the in-memory sketch.

A polygon should be understood as a collection of many objects: vertices, sometimes (!) segments joining these vertices, and filling of the polygon area. This is a lot. Here are also a few things that we should consider. While creating a polygon, we use points for its vertices, but while creating a polygon, we also create coordinates of its vertices. The next illustration shows what we really create.



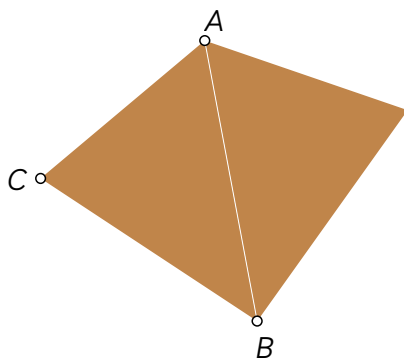
Anatomy of a polygon

In this drawing, the left triangle was created using the three points shown here. The top-right polygon is its translation, and the drawing right-down is another translation of the original polygon. For the last one, we used 50% opacity to show its real structure.

These two right images show that any polygon itself contains information about its vertices even if they are not visible, and (this is important) the polygon has in its definition also edges (let us call them **frames**). They are clearly seen on the right-bottom triangle.

The above analysis concludes that a simple triangle like the one on the drawing contains a large amount of data outside of our awareness. Thus the question is: how can we reduce all this information to absolutely necessary elements? This way, we could reduce the number of objects and data that, in reality, we do not need and thus make in-memory sketches smaller and faster displaying their content. Here are some conclusions:

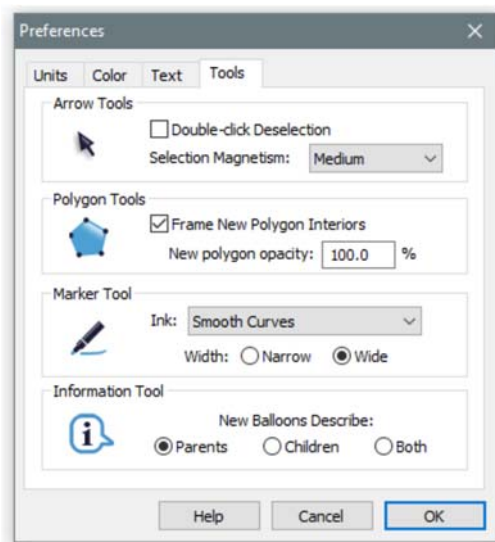
1. We could reduce the size of our constructions and consequently in-memory sketches by not drawing edges of polygons. They are already there. Unfortunately, they have the same color as the whole polygon, which cannot be changed.
2. If we want to reduce the number of points, we can consider translating or rotating polygons and segments without their endpoints.
3. We can also remove the frames of polygons like the one seen in the drawing below. But this possibility has some drawbacks.



Frames of polygons

In this illustration, the left triangle was created by drawing a polygon with vertices ABC. The right one is its reflection about a line passing through points A and B. In both cases, we removed the frames of each polygon. The drawback is this tiny gap between them.

This means that we can remove these frames without any problem in some cases, and in some others, we will get unpleasant artifacts in the form of a gap between polygons. How do we fix this issue? There are two ways.

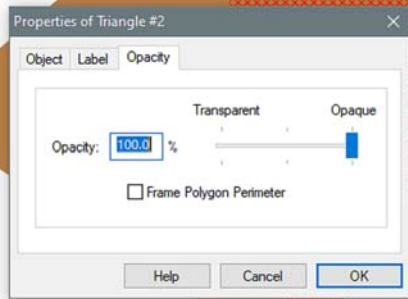


The preferences panel

In GSP's Edit menu, at the very bottom, there is the [Preferences] option. The illustration to the left shows its appearance in the Windows version. Here we can change several parameters but the most important for us, at the very moment, are polygon tools. Here we can decide if we want polygon frames created for us and no white gaps between polygons, and here we also determine if we want to have 100% opaque (non-transparent polygons).

This way, we decide the features of our polygons globally. But of course, later, we can select any polygon and change its own specific features.

Important – there is a bug in GSP. Suppose that we created an unframed polygon with the Frame New Polygon option checked. Then each copy of it obtained by translations, rotations, or reflections will get a frame.



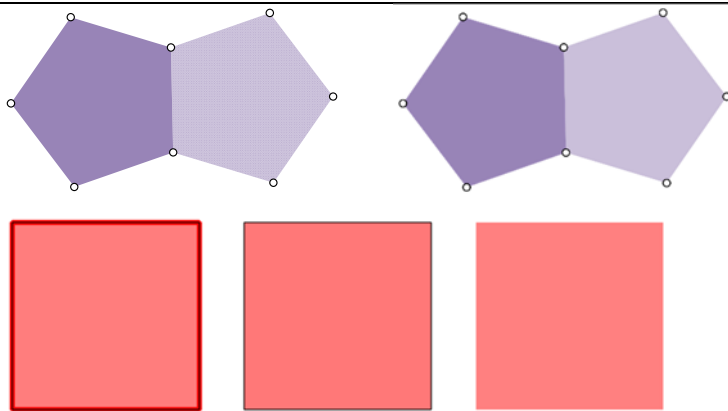
The properties of a polygon

By the right-mouse click on any polygon, we get several options, and one of them is the Properties panel. Here we also can apply opacity and framing but only for the selected polygons. All other polygons will not be changed at all.

All this discussion brings us to the two questions: should polygons be created with frames or without, transparent or not? You can decide on your own what you prefer and see how this fits into your needs.

In all my works, when I deal with tangent polygons, I use polygons with frames. If I do not have tangent polygons or a few tangent only, I prefer to have polygons without frames, and these few I change to framed by hand. I rarely use polygons with lower than 100% opacity.

The opacity of polygons brings us to a few unexpected problems. Transparent polygons at, say, 50% opacity are great at showing an image behind the polygon. But they are awful when we take an image from GSP, paste it into MS Word, and then print the Word file. In such cases, I make a right-click on this image in Word, and I ask to save it as a PNG file. I then take this PNG file and copy and paste it into the Word document, replacing the original image.



Left – original drawing from GSP (obtained by cut and paste), right is its copy created by MS Word (save as Picture). We do not see any difference between these two images in an MS Word document, but we see the difference after printing it.

In this image, we show three 50% transparent squares (from the left): a polygon with thick frames and edges drawn by us; a polygon without frames but with edges drawn by us; a polygon with no frames and no extra edges. The left one uses maximum data, the right one uses the minimum of necessary data.

A few more conclusions – we can avoid framed polygons if we use contrasting colors, one dark and another light. This way, the gaps between polygons with two contrasting colors will still exist, but they will be less visible. Tangent polygons with the same dark color should always be framed. If we use contrasting colors, we do need to draw extra edges of polygons at all. A good solution is to choose one color as a background for the whole design and draw it in the form of a large polygon, usually a rectangle.

Custom color palettes

The original set of colors in GSP may not be enough for a more sophisticated design. Thus we can create our own color scheme/palette and save it as a GSP tool. Here I show step by step how one can do it.



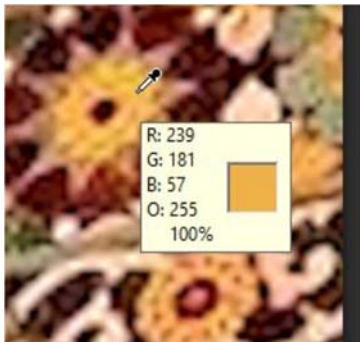
A sample photo

The photo shown here has a very bad quality but an excellent color scheme. We will try to develop a color scheme based on this photo.

For this reason, we will need any graphics program for photo editing. It can be the Paint program from Windows, or Photoshop, or anything else. In most of these programs, we have a Color Picker tool that looks like the one shown in the next image. Positioning it on a particular area will display the color parameters of the given pixel.

Here we see that the point has color parameters R:239, G:191, B:57, and O:255. This is the amount of each color component for the given point in the image: red, green, blue, and opacity. Ignore the last one.

Now, we have to choose color parameters for a few different and characteristic for this image places. Write them down on a piece of paper and go with them to GSP. In Sketchpad, we can create, for example, a rectangle divided into smaller cells, and each of them fill with a polygon. To each of these polygons, we apply color parameters taken from our photograph. Such an object can be saved as a new tool. The next drawing shows the color palette that I created for this image. I selected each polygon and through the menu



Display > Color > Other

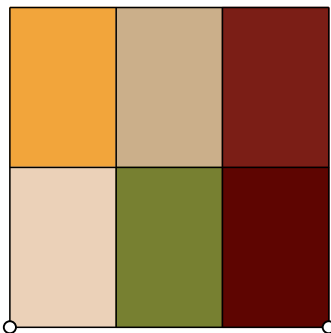
I was able to change the color of each polygon.

In this color palette, the following colors were used:

[R:242, G:165, B:59], [R:203, G:175, B:138], [R: 123, G:30, B:22]

[R:235, G:209, B:184], [R:119, G:128, B:49], [R:94, G:5, B:1]

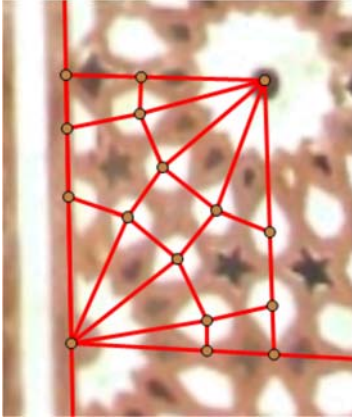
If we want to use this color palette in our design, we have to put it on the screen and select an object in our design, then with [shift] down, select the desired color from the palette and again through the menu:



Display > Color > Other

apply the new color. We do not need to type color parameters again. They are already there. This way, we can change the colors of any number of objects in our design. Important – while changing colors, the undo operation does not work. We cannot get back old colors by using the ctrl-z key.

A real example using the user-created color palette

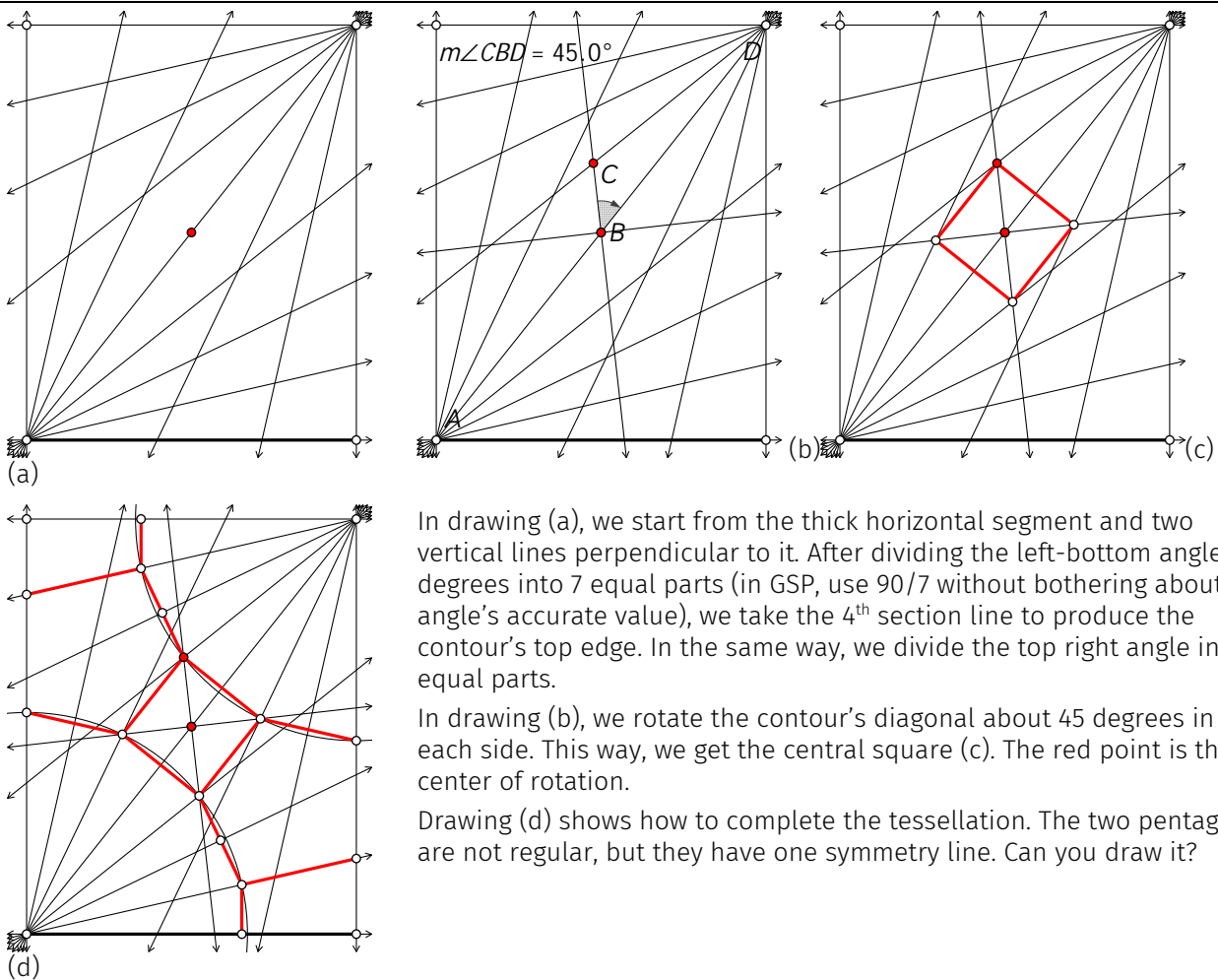


Pattern from Fatehpur Sikri

In this example, we will use an existing pattern from Mughal architecture, from Fatehpur Sikri. I got it from Richa Raut, an Indian architect, already suggesting a tessellation for it. This is our starting point. We will develop this pattern using the color palette created a while ago.

The pattern itself is very interesting. It contains regular stars with 14 vertices and pentagonal stars with one symmetry line only. A significant feature of this pattern is the middle star with four arms. It is an object with four-fold symmetry. Thus we may assume that it can be wrapped in a square as it is shown here.

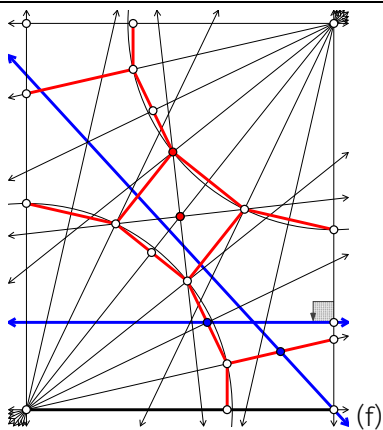
The next drawings show the step-by-step construction of the contour, tessellation, and pattern.



In drawing (a), we start from the thick horizontal segment and two vertical lines perpendicular to it. After dividing the left-bottom angle 90 degrees into 7 equal parts (in GSP, use $90/7$ without bothering about the angle's accurate value), we take the 4th section line to produce the contour's top edge. In the same way, we divide the top right angle into 7 equal parts.

In drawing (b), we rotate the contour's diagonal about 45 degrees in each side. This way, we get the central square (c). The red point is the center of rotation.

Drawing (d) shows how to complete the tessellation. The two pentagons are not regular, but they have one symmetry line. Can you draw it?



Here we show how one can approach the construction of the pattern on this tessellation. The slant blue line passes through the midpoint of the pentagon's edge (blue point) and the contour's corner. This part is evident from the photograph.

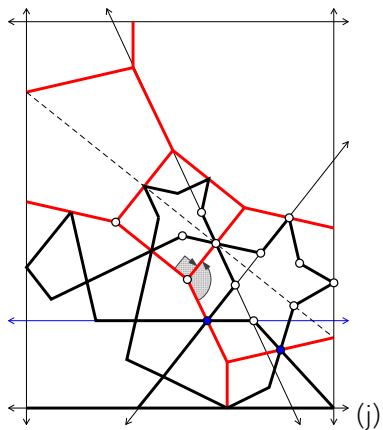
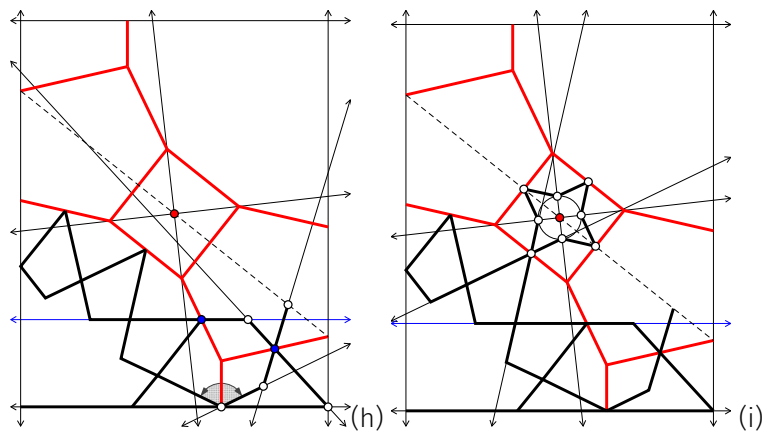
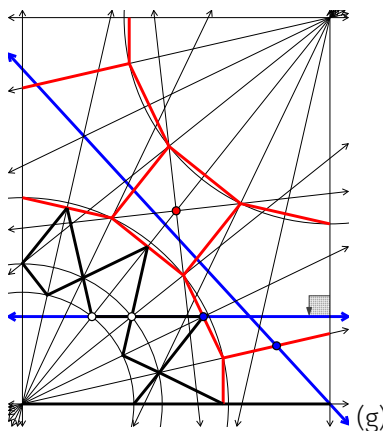
The second blue line can be drawn in a few different ways. Here I made it horizontal, but it can be slightly slanted in both ways. The photograph does not show the pattern accurately.

The selection of both blue lines determines the construction of the pattern.

Drawing (g) shows how we can create the quarter of the large star.

In (h), we see how we could develop the pattern in half of the trapezium (bottom-right).

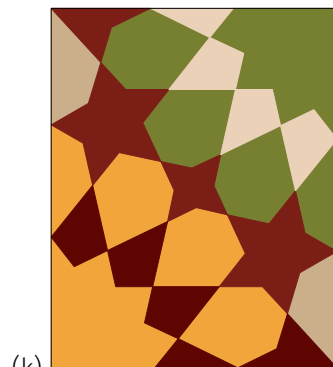
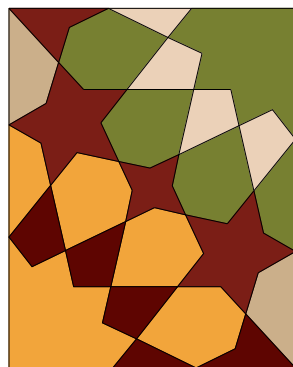
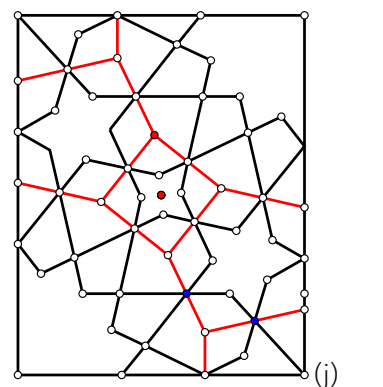
In (i), we see how we can draw the pattern inside the central square.

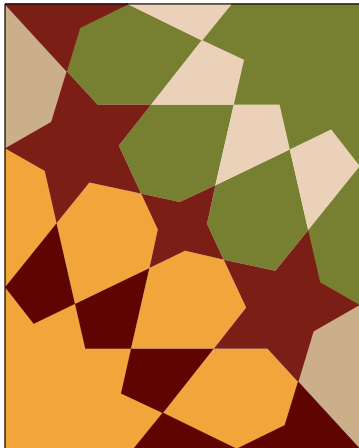


In this drawing (j), we see how we can develop the pentagonal star. It will never be regular. The reason for this fact is that the two angles shown here are different.

Below – the three drawings show the complete pattern with tessellation; the pattern using black segments for edges of polygons; the pattern using only polygons with contrasting colors.

It is essential to notice that the number of segments in drawing (k) is not very large, but we may get a large set of edges when we make a large pattern out of it. Thus using the version with framed polygons only (drawing (l)) we can produce a design with a significantly smaller number of elements – no points, no edges, frames, and polygons only.





An image with an absolute minimum of data

This image uses tangent polygons with contrasting colors. There are no borders created by the user.

Polygons inside the contour are without frames. Polygons tangent to the contour edges have frames. There are 39 objects in this creation. In the above images, the central pattern (k) has 116 objects, and the right one (l) has 54 objects.

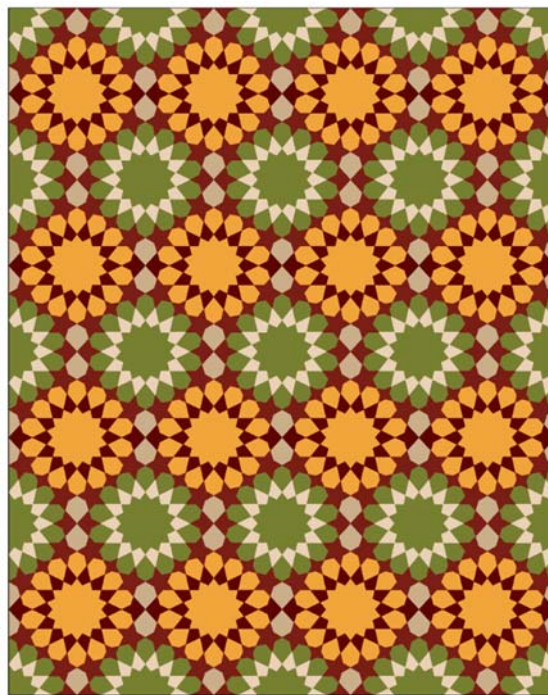
Thus if we create a pattern using 8x8 copies of a template, we will produce objects with

$$8 \times 8 \times (\text{number of objects in the template}) - 7 \times 7$$

This gives us numbers 575, 815, and 1807. The 7x7 is for removing edges of the contour and each it's copy.

Here we have the final version of a pattern using framed polygons only. This is important to notice that after creating a large design from the template using any type of polygons, we do not replicate anything that is not visible.

Another important fact – a geometric pattern with a large number of elements copied into an MS Word document will slow it down. The best solution is to paste it into MS Word, ask MS Word to save it as a picture. We will get a bitmap – not vector graphics. Then replace the image in the MS Word document with the bitmap file. In the MS Word file and then in the PDF file, we will not see a difference between these two images.



Size of graphic files

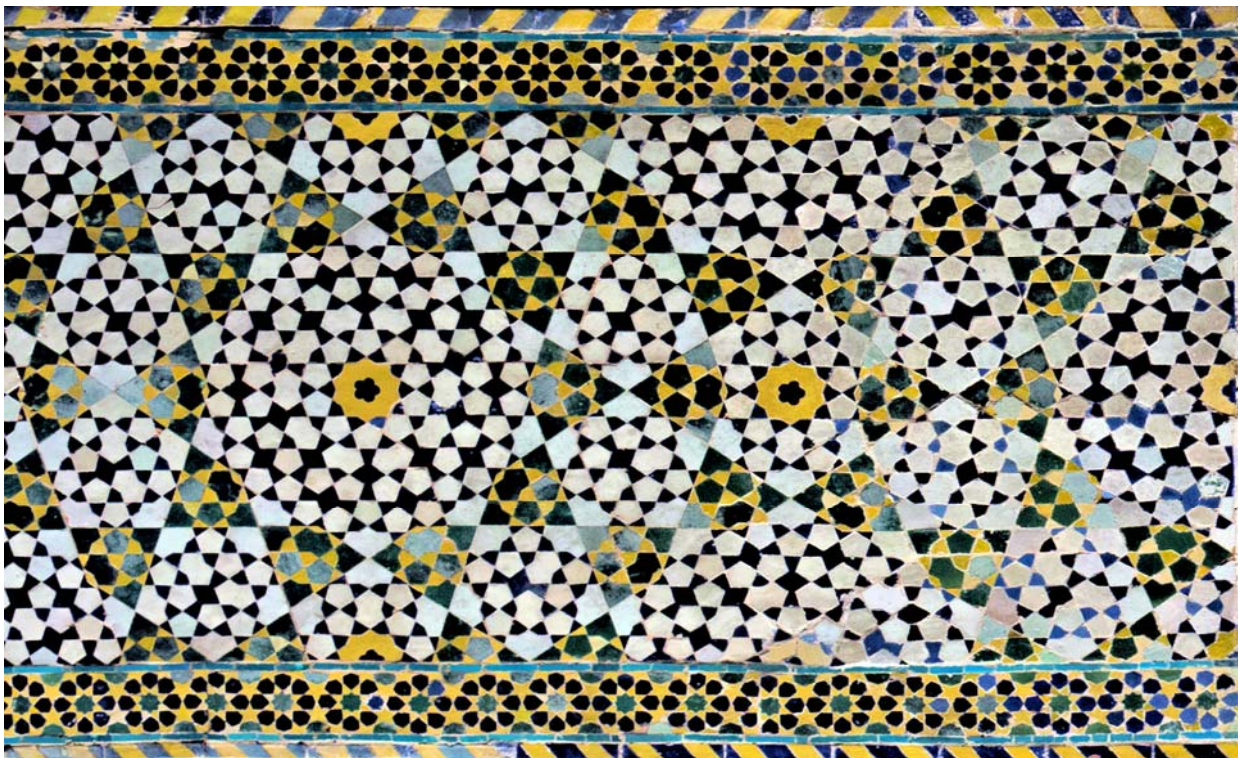
GSP graphics are always in vector format. This means the size of graphics copied from the GSP screen is dependent on the number of elements used in the construction and their parameters. In designs with a lot of details, this can be a huge number. While saving an image from MS Word as a picture, we produce a bitmap image. The size of such an image depends on the number of pixels in it. The bitmap file can be much smaller than the original GSP vector file for the constructions with a vast number of elements and complex parameters.

For smaller designs, it can be much better to use images copied directly from the GSP notebook. They are smaller than their bitmap versions.

GSP for Windows can save a construction in the EMF format. This is also a vector graphics file. Lines are very crisp, but they can be bulky for large designs.

A bulky example of Persian origin

In this example, we will show how one can create a very complex geometric pattern following the principles discussed in this paper.



A decagonal pattern with two color schemes and a large number of elements

The photo shows a typical Persian design using two different color schemes. We have dark greenish and yellow ceramic tiles and areas with black and very light blue slightly pinkish ceramic tiles.

In western literature, design from the photo is described as a mysterious pattern with some symbolic features and as a self-similar fractal-like object. A person with some basic knowledge of fractals will notice that there is nothing fractal-like in it. In the next few pages, we will reconstruct this design using a new custom-made color palette. We will show that there is nothing mysterious in it, only pure decagonal geometry. Our main goal will be to produce an optimal design with possibly the lowest number of elements. Thus we will simplify everything possible. We will cut most of the unnecessary details.

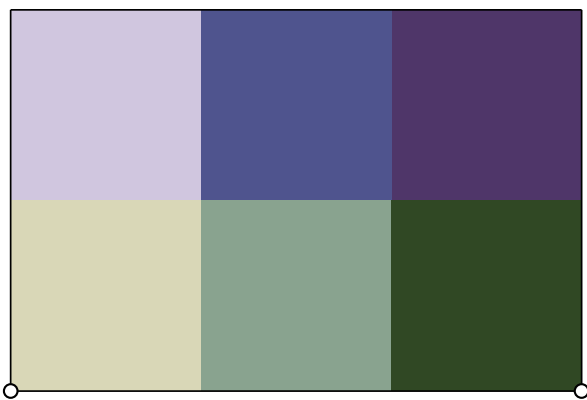
Pattern analysis

By analyzing the colors used in this design, we can notice that we have four significant areas: kites (dark green and yellow parts), regular pentagons, halves of regular pentagons, and large stars (black and bluish

tiles). These figures form the so-called upper-level pattern. These figures are filled with Persian style patterns using elements that we created already in our earlier works (see, for example, [2] or [3]). This is the lower-level pattern. Here are the following steps in this project:

1. Create two distinct color palettes using contrasting colors.
2. Construct the overall structure of the template – contour, tessellation, and upper-level pattern.
3. Construct the lower-level pattern for each mentioned shape – kites, pentagons, etc. There are only four of them.
4. Create a large top-level pattern and fill it with the lower-level pattern.

Color palettes



For this project, we will use two color palettes – one with a blue component with colors:

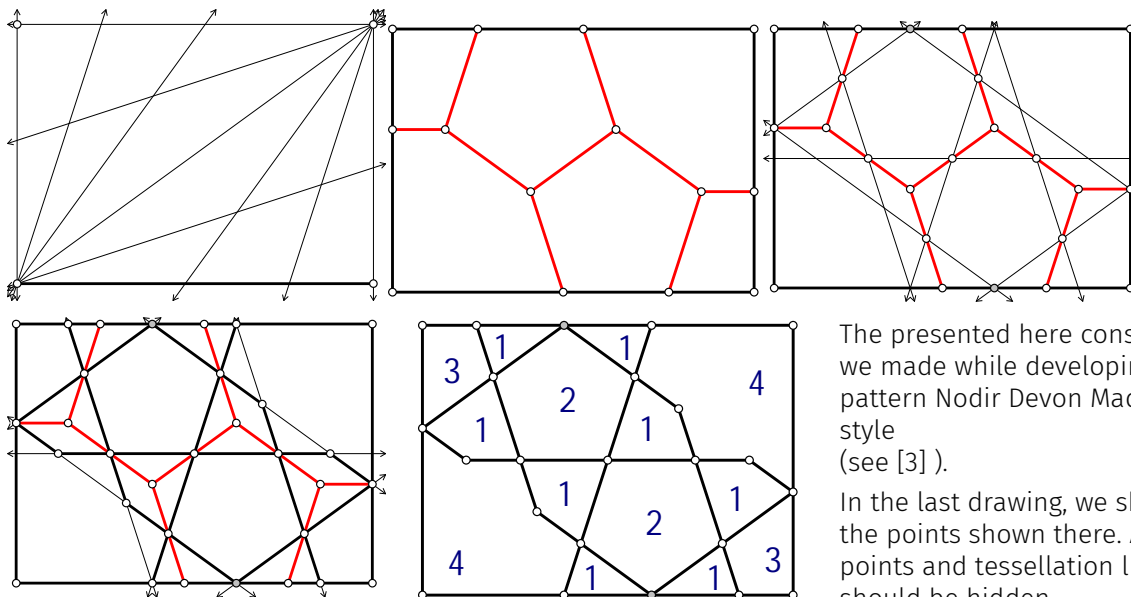
[R:208, G:198, B:223],
 [R:79, G:84, B:142],
 [R:79, G:54, B:105],

and another one with green component:

[R:217, G:215, B:183],
 [R:137, G:163, B:143],
 [R:48, G:72, B:36].

Template for the upper-level pattern

This part is easy. We did it already a few times on various occasions ([3]).



The presented here construction we made while developing the pattern Nodir Devon Madrasah style (see [3]).

In the last drawing, we should keep the points shown there. All other points and tessellation lines should be hidden.

We have here only 4 different shapes numbered from 1 to 4.

Multilevel patterns in Persian art

A particular feature of Persian art are designs with two or more patterns overlapping each other. In Iran, we often see designs where a large pattern forms a framework for another more detailed pattern. Usually, there are two levels. We will call them upper-level and lower-level. However, it is possible creating patterns with three levels. These designs will be dense-packed with a pattern that may be very difficult to render in any type of material.

The example discussed in this text contains two levels. The upper-level pattern is shown in the above drawing. The lower level will be created on the next few pages.

It is essential to notice that nodes of the upper-level pattern usually are centers of stars or rosettes of the lower-level pattern.

Tools for this design

In this paper, we will make intensive use of two kinds of GSP custom tools. There will be basic tools representing copies of elements of the lower-level pattern and tools for pattern filling large blocks of the upper-level pattern.

A tool to draw tangent decagons

This design will need a tool to draw parts of two tangent decagons provided that their centers are given.

$m\angle CBD = 18^\circ$ $m\angle DBD' = 36^\circ$

Start with line AB and create the midpoint C. Rotate the line 18 degrees up and down about point B. Create the line perpendicular through point C. This way, we will obtain points D and E. All other points in this drawing are rotations of the point D around A and B at an angle of 36 degrees.

NOTE – usually, in similar situations, I construct two tangent decagons. In this example, we do not need complete decagons. We will need only parts of them. It is also essential to notice that, shown here, polygons are required only for a brief moment, and we can delete them. External points D''' and D'' can be deleted (deleted, not hidden) if we do not need them.

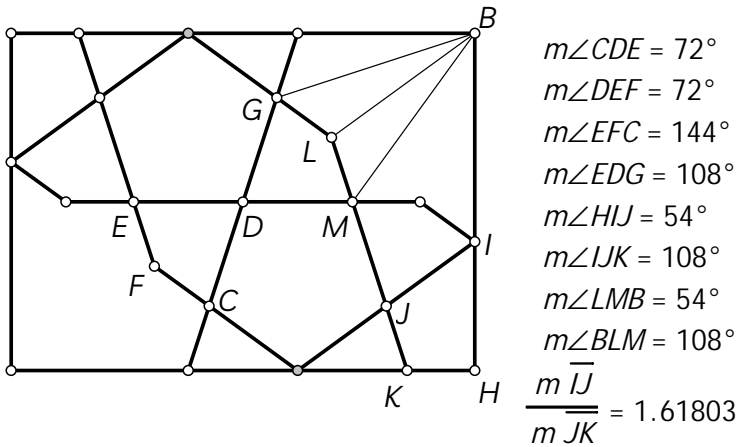
Hide or delete

In Sketchpad and in any other geometry software, we are often tempted to delete unwanted objects. Sometimes after deleting an item, the whole construction is still not affected, but in some other cases deleting a single point or a segment may destroy the entire construction. All geometry programs use the parent-child model. Every element that was used to create a new object is a parent for this object. For example, a line B created as a parallel to a given line A and passing through the given point a is a child of A and a. These two objects are parents of line B.

Elements that are parents of other objects cannot be deleted by deleting these objects. But we can always hide them, and this will not affect the rest of the construction.

Angles and lengths

The drawing presented below displays all necessary angles and lengths needed in further steps of this design. I assume that the reader knows how to construct the golden section of a segment.



Tools for basic elements

There are only two basic elements that we need for creating lower-level. These are the black shapes in the light areas and yellow kites in the yellow/green areas. Thus we have two shapes: a small kite and a flower vase-like shape. The first one is the fill for a long triangle tile or a tall trapezium. The second one is the fill for the long hexagon. All other elements can be considered as a background.

In the table, we show steps in constructing each of them.

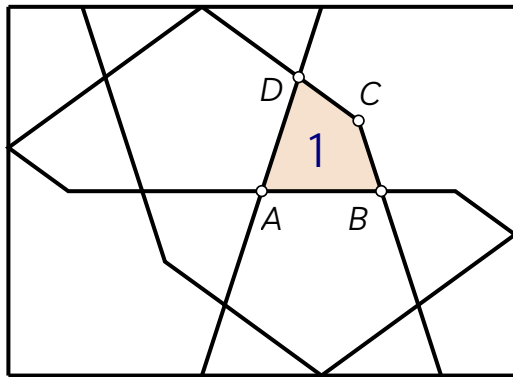
Construction	Framed shapes using multiple polygons	Unframed shapes using single polygon	
			<p>The vase-like shape was created on a long hexagon with all edges equal and lines forming 54 degrees angle.</p> <p>We need two tools: a tool with four identical parts (middle). The symmetry lines separate them. Each polygon in this tool is framed, with thin frames, to avoid gaps between them; a tool with one unframed polygon (right).</p>
			<p>We create four tools for kites. Kites in the middle are split into two tangent framed triangles (thin lines)</p> <p>Kites in the right are created in the form of one unframed polygon.</p>

The shapes in the middle column have a slightly different color than those in the right column. This way, we can easier show where we use them. Later we can adjust their colors if we want this.

Now we can start developing patterns for the shapes in the upper-level design.

A tool for the pattern inside the kite (number 1)

Our first step is to produce a pattern for the quadrilateral ABCD and other identical shapes.



$$m\angle CBA = 72^\circ \quad m\angle BCD = 144^\circ$$

$$m\angle DAB = 72^\circ$$

Left – the upper-level pattern with angles for the kite ABCD.

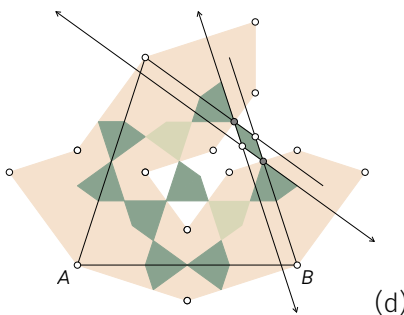
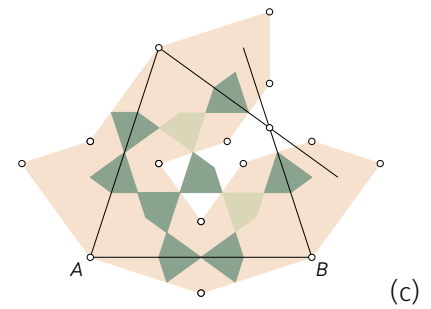
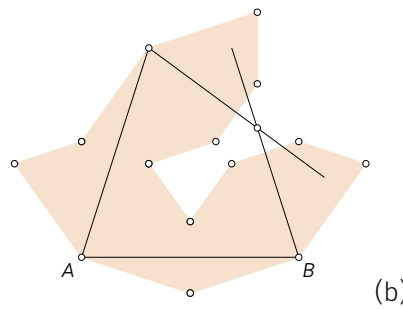
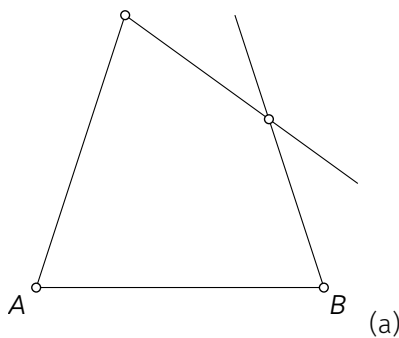
Below – construction of the kite and the pattern for it.

Start with segment AB and rotate it about points A and B according to the angles shown here (a). Use twice the tool to draw tangent decagons (b)

Fill the beige polygons with two versions of the small kite. These crossed by the kite's edge should be made out of small kites with two framed polygons. These inside the large kite should be one-piece polygons unframed (c).

The drawing (d) shows how to fill the gap in the wide corner of the large kite.

All polygons outside the large kite should be deleted. Some but not all points outside can also be deleted.

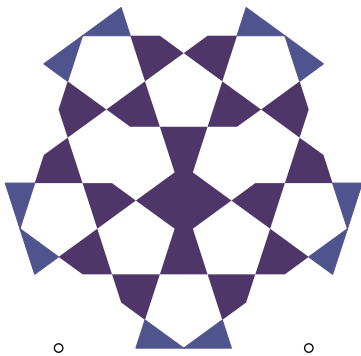
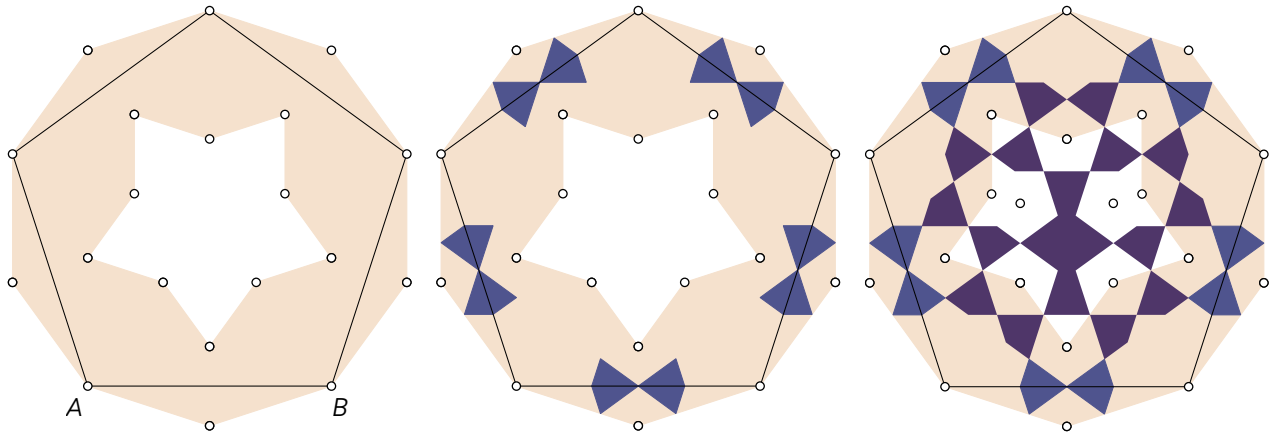


Left – the final pattern for the large kite. This is important – along the kite symmetry line, we also have small kites created out of two tangent triangles. Why we did this?

Select the two points below the pattern (e), and then all polygons and create a new tool 'kite pattern.'

The pattern for large kites is the simplest one. Now we can design the pattern for pentagons. This is a slightly more complex task.

A tool for the pattern inside the large pentagon (number 2)



Construction of pattern for the large pentagon

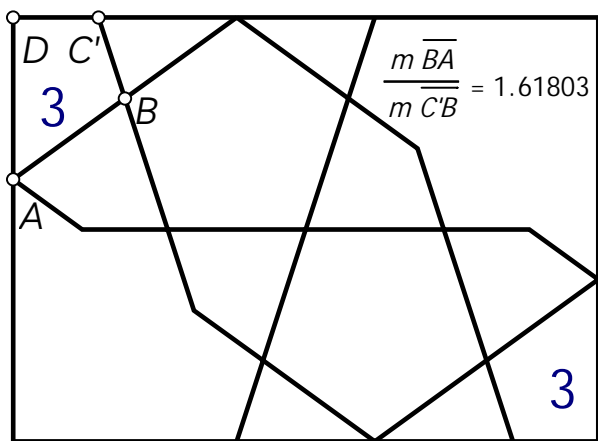
Start from segment AB and by rotating it 108 degrees about its endpoints, create the pentagon. For each edge of the pentagon, use the tool to draw tangent decagons. This is what is shown in the first image. Then fill it with kites and vase-like shapes. Start filling the central white area by inserting the bottle shape (one framed polygon).

Delete all polygons that are outside of the pentagon. Hide all points leaving only points AB. We need them to create the tool for filling with the pattern for the pentagon.

Select the two points shown here and then select the pattern and create a new tool, 'pentagon pattern.'

Next comes the pattern for shape in the top-left and bottom-right corners of the upper-level pattern – the figure with number 3.

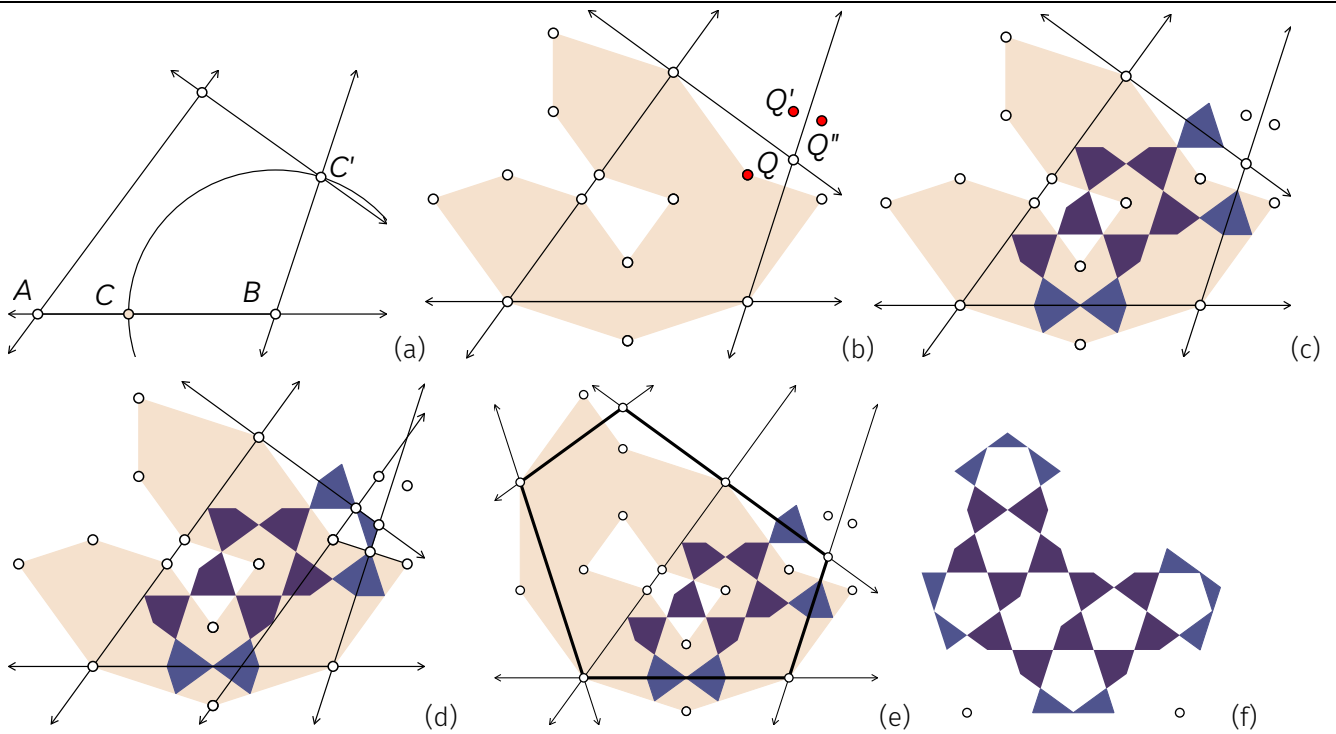
A tool for pattern filling corners (number 3)



$m\angle DAB = 54^\circ \quad m\angle ABC' = 108^\circ$

The corner polygon is half of a figure that can be easily obtained by cutting a regular pentagon. However, the lower-level pattern filling it will be different than for the pentagon.

The drawing shows all necessary angles and the ratio of the two edges DE and EF. We can recognize here the golden ratio. Thus we have an easy way to construct the shape and the pattern for it.



A brief description

In the drawing (a), we start with the two points A and B. Point C is the point marking the golden section of AB.

In (b) point Q' is a mirror reflection of the point Q, and point Q'' is a mirror reflection of Q'.

Drawing (d) shows how to fill the space in the top-right corner.

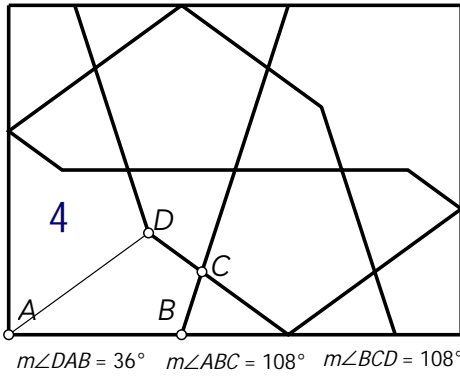
Drawing (e) shows the complete shape. We developed a pattern for only half of it. In the same way, we develop a pattern for the other half.

Drawing (f) is the final pattern for the filling shape number 3 and its reflected part. All kites around the edge are framed. All other kites are single polygons, unframed.

Select the two points shown here and then all polygons and create a new tool, 'corner pattern'.

A tool for a pattern for the big star (shape number 4)

The last part of this long example is constructing a pattern for the big star (shape number 4). This design can be developed in a few different ways. We can split the big star into 10 segments and create a pattern for one of them. We can also develop a complete design for a quarter of the star.

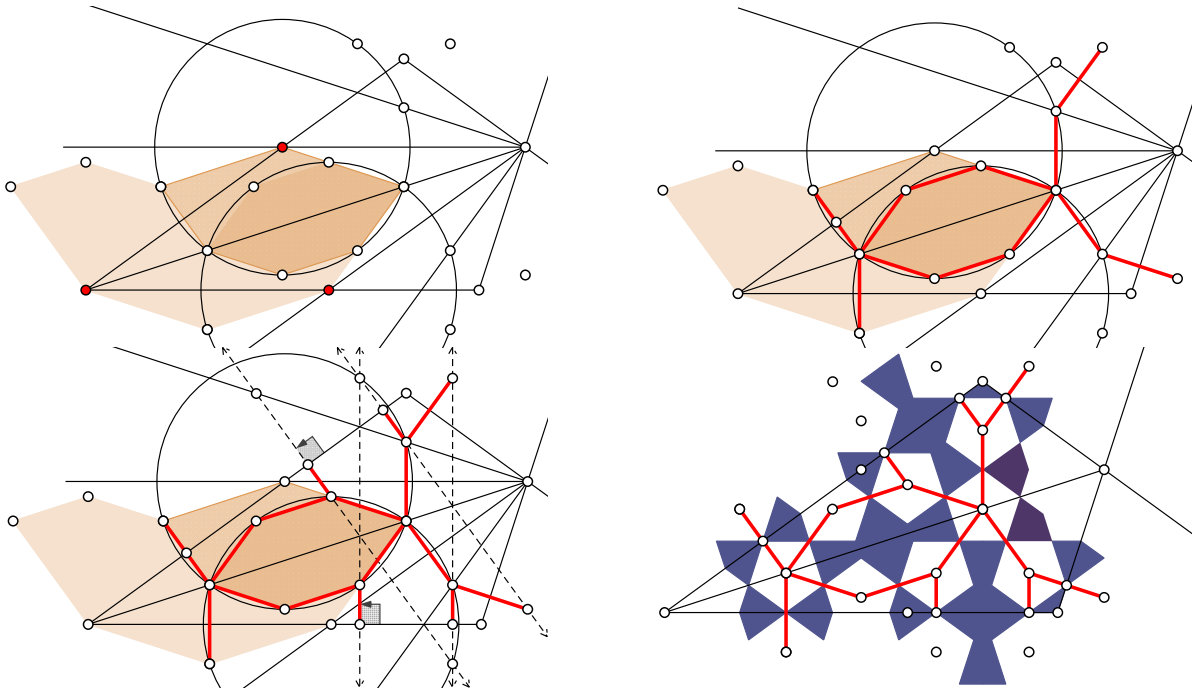


Angles and properties of the big star

If we split the star into 10 identical kites, we will get a polygon that can be easier to fill with a pattern. It can be convenient to start from segment AB and then obtain the remaining edges by rotations 36 and 108 degrees.

There will be a few shapes cut along their symmetry lines. Thus we will have to use for them two or four framed polygons.

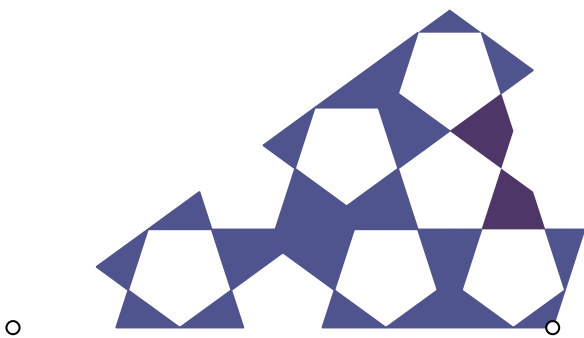
Below – construction of the long kite and constructing a tessellation using twice the tool for creating tangent decagons.



Above - left finishing tessellation, right the tessellation and the pattern. See how we created the pattern for the corners.

Left – complete pattern for the long kite. Each of the light blue figures uses two or four framed polygons. Why? There are only two figures made out of a single unframed polygon.

Select the two points with all polygons and create a new tool, 'long kite pattern'.

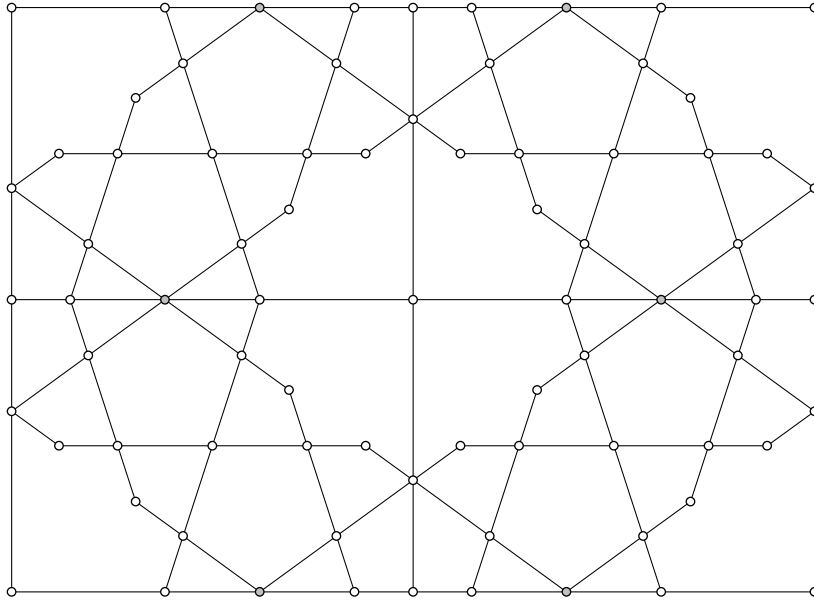


Assembling the complete design

Assembling the complete design is the very last and the least complicated task. We start from a large template created out of four copies of the upper-level pattern template. This is our master template. Now we have to fill it with all the elements that we made.

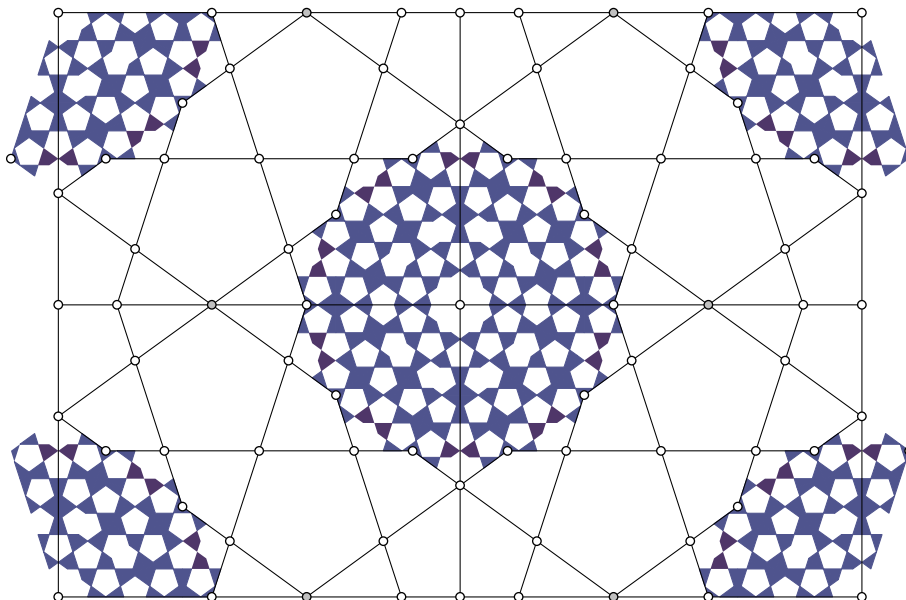
STEP 1 – complete the upper-level pattern

We have here four copies of the upper-level pattern template created at the beginning of this project.



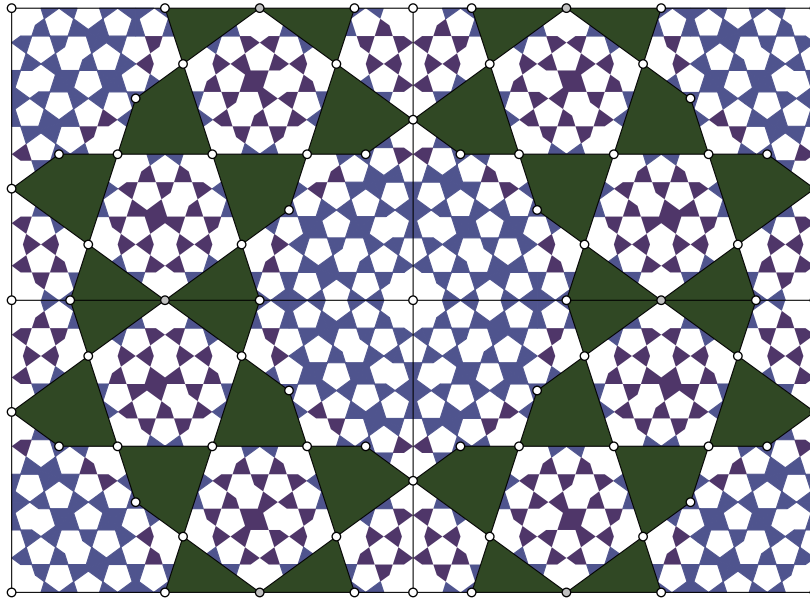
STEP 2 – add fills for the big star

We use the 'long kite pattern tool' to fill the center and all four corners.

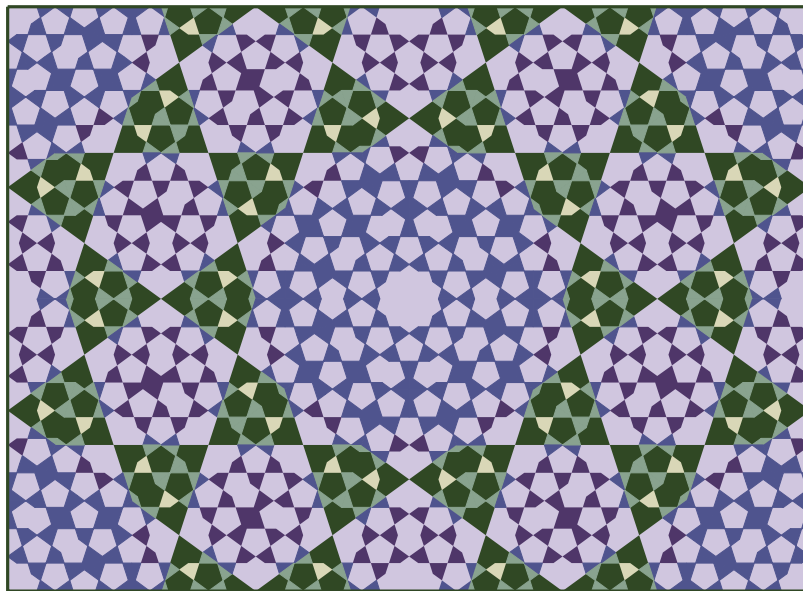


STEP 3 – fill pentagons with the pattern

Before filling kites with the pattern, create there the dark green background from the second color palette. Use unframed polygons. Now, use the ‘pentagon pattern’ tool to fill each pentagon shown on the drawing. Note how the pattern for pentagons was inserted. Its direction is different than in the original design.



Step 4 – fill remaining shapes with the pattern, hide the upper-level pattern, hide all points



The design we created contains 1784 elements. If we use all framed polygons, then this number will be at least 4 times higher. The most significant saving we get through unframed polygons and one global background for the whole design. The light blue color is one rectangle – not a few hundreds of pentagons, etc. Each small kite also has one background (see step 3).

In this design, we use polygons on a few layers. The top layer contains small shapes. The middle layer form the dark green kites. The bottom layer is one large light blue rectangle.

Final comments

In this text, we discussed methods for optimizing large geometric patterns. The final result looks exactly like the pattern created without using the techniques presented here. However, it contains much fewer elements. The example discussed in this paper is relatively simple. The number of elements may grow to 50 thousand or more elements – segments and polygons in more complex examples. Such a quantity of objects is still acceptable for Geometer's Sketchpad. But it will slow down its work. However, if we copy a GSP design to any other vector graphics program, we may wait a long time to render our graphics.

In this text, we did not discuss the differences between vector graphics files and bitmap files. Such a discussion can be found in any introductory textbook of graphics design or an appropriate Wikipedia page.

References⁽²⁾

- [1] Majewski. M. (2020). *Practical Geometric Pattern Design: Geometric Patterns from Islamic Art*. Kindle Direct, Independently published (February 10, 2020).
- [2] Majewski. M. (2020). *Understanding Geometric Pattern and its Geometry (Part 1)*, eJMT, vol. 14, Nr 2, pages 87-106.
- [3] Majewski. M. (2020). *Understanding Geometric Pattern and its Geometry (Part 2) – Decagonal Diversity*, eJMT, vol. 14, Nr 3, pages 147-161.
- [4] Majewski. M. (2021). *Practical Geometric Pattern Design: Decagonal Patterns in Persian Traditional Art*. Kindle Direct, Independently published (March 29, 2021)

² Papers [2] and [3] can be downloaded from <https://www.researchgate.net/profile/Mirosław-Majewski>

Locus of Antipodal Projection When Fixed Point is Outside a Curve or Surface

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Abstract

We continue the investigations regarding the locus problems for certain curves and quadric surfaces, which are discussed in [6] and [7] in 2D and 3D cases. We explore the intersecting curves when the fixed point A is outside the surface and when A is at an infinity. The discoveries from explorations with the help of technological tools in this paper will assist learners to conduct further research in the area of projective geometry and beyond.

1 Introduction

In [7], we considered the following:

Original problem: *We are given a fixed point A and a generic point C on a specified curve or surface Σ such that the line l passes through A and C and intersects a well-defined D on Σ , we want to determine the locus curve or locus surface Δ generated by the point E , lying on CD , which satisfies $\overrightarrow{ED} = s\overrightarrow{CD}$, where s is a real number parameter.*

We remark that the original locus problem leads to many interesting projects thanks to several parameters that need to be taken care of. The location of a fixed point A certainly determines the locus once the original surface Σ is chosen. The parameter s determines size of the locus surface Δ . As we discussed in [7], the 3D surface Σ we consider in this paper is either an ellipsoid or a hyperboloid with two sheets. In this paper, we discuss how the locus surface Δ will behave when point A is outside the specified surface Σ . Furthermore, since the location of the fixed point A will determine the locus surface, we shall distinguish the case when A is either at an infinity or not. We recall that we calculated the exact expression for the antipodal point D corresponding to the point C in [7]. In Section 2, we remind readers how we apply the Vieta formula to find the locus surface. In Section 3, we discuss the scenario, when A is outside a specified surface Σ but not at an infinity, how we can find the intersecting curves between Σ and its locus surface Δ . It is interesting to note that once the fixed point A is chosen, the intersecting points or curve stays fixed regardless of the parameter s . In Section 4, we explore

the locus surfaces when A is at an infinity. In such case, the point D_{inf} can be viewed as the “limit” of point D when A goes radially to infinity, or as the projection of C along the line l through the point A “fixed at infinity”. The explorations lead to linear transformations involve further discussions in [8].

2 Generic methodology to find locus surface

If Σ is the quadric surface $F(x, y, z) = 0$ we recall from [7] how we find the locus surface when the fixed point $A = (x_0, y_0, z_0)$ does not go to infinity. We represent a generic point on Σ as

$$C = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \quad (1)$$

In order to calculate the coordinates of point $D = (x, y, z)$ (which is different from C), as the intersection between the quadric Σ and the line l passing through A and C , we make use of the parametric equation of line l as follows:

$$\begin{aligned} x - x_0 &= \lambda(\hat{x} - x_0), \\ y - y_0 &= \lambda(\hat{y} - y_0), \\ z - z_0 &= \lambda(\hat{z} - z_0). \end{aligned}$$

Hence, we obtain

$$\frac{y - y_0}{x - x_0} = \frac{\hat{y} - y_0}{\hat{x} - x_0}, \quad (2)$$

$$\frac{z - z_0}{x - x_0} = \frac{\hat{z} - z_0}{\hat{x} - x_0}. \quad (3)$$

Then we define two auxiliary functions, namely

$$k \doteq k(\hat{x}, \hat{y}) = \frac{\hat{y} - y_0}{\hat{x} - x_0} \quad (4)$$

$$m \doteq m(\hat{x}, \hat{y}) = \frac{\hat{z} - z_0}{\hat{x} - x_0} \quad (5)$$

Since both intersection points, C and D , satisfy the implicit equation of Σ , we can use (4) and (5) to get the x -coordinate of D , say x_1 , by calculating the roots of the polynomial

$$p(x) = a_2x^2 + a_1x + a_0,$$

It follows from $p(\hat{x}) = 0$ and the Vieta's formulas that

$$x_1 = -\frac{a_1}{a_2} - \hat{x}.$$

It follows from (2) and (3) that

$$y_1 = y_0 + k(x_1 - x_0) \quad \text{and} \quad z_1 = z_0 + m(x_1 - x_0).$$

For a given s , the locus surface generated by point $E = sC + (1 - s)D$ is defined as

$$\Delta(x_0, y_0, z_0) = \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} s\hat{x} + (1 - s)x_1 \\ s\hat{y} + (1 - s)y_1 \\ s\hat{z} + (1 - s)z_1 \end{bmatrix}.$$

We remark that once the fixed point A is chosen, since A and C together determine the point E , the locus surface is thus fixed too.

3 Motivations

In this section, we shall explore how various factors such as, the original surface Σ , the fixed point A , and the scalar s will affect a locus surface $\Delta(\Sigma, A, s)$. For example, we may investigate the following scenarios:

1. How will the radius ρ of the fixed point $A = (\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$ affect a locus surface?
2. How will the angle (u_0, v_0) of the fixed point $A = (\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$ affect a locus surface?
3. How will the parameter $s > 1$ affect a locus surface?

We first use the following two dimensional ellipse to motivate our findings:

3.1 The ellipse case when the fixed point A is not at infinity

Consider the ellipse c

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{6}$$

and let $A = (x_0, y_0)$ be a fixed point “outside” c . In [6], the locus curve $\gamma : [0, 2\pi] \rightarrow \mathbf{R}^2$ was determined in parametric form as

$$\gamma(t) = \begin{bmatrix} s a \cos(t) + (1 - s) \frac{a^3 y_0^2 \cos(t) - 2a^2 b x_0 y_0 \sin(t) - a b^2 x_0^2 \cos(t) + 2a^2 b^2 x_0 - a^3 b^2 \cos(t)}{a^2 y_0^2 - 2a^2 b y_0 \sin(t) + b^2 x_0^2 - 2a b^2 x_0 \cos(t) + a^2 b^2} \\ s b \sin(t) + (1 - s) \frac{a^2 b y_0^2 \sin(t) + (2a b^2 x_0 \cos(t) - 2a^2 b^2) y_0 - b^3 x_0^2 \sin(t) + a^2 b^3 \sin(t)}{a^2 y_0^2 - 2a^2 b y_0 \sin(t) + b^2 x_0^2 - 2a b^2 x_0 \cos(t) + a^2 b^2} \end{bmatrix}$$

The Figure 1 shows the locus curves (orange) and the original ellipse (blue) for $a = 8$, $b = 6$, $A = (10, 10)$ and $s = 0.75, 1.5$, and 2.0 respectively.

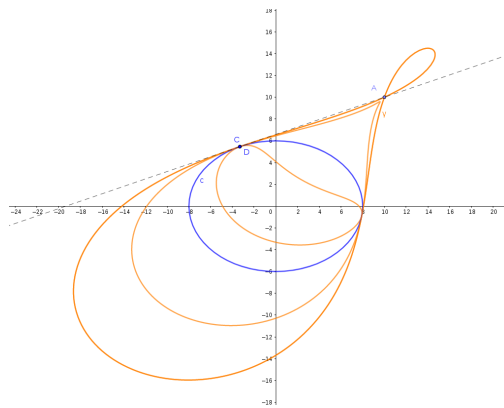


Figure 1. Locus curves when $s = 0.75, 1.5$ and 2.0 respectively.

With the help of [1] we see that, for any $s > 1$, the corresponding locus curve intersects tangentially the ellipse at the same points, say P_1 and P_2 . Consider now the following experiment: construct a point C on the ellipse and draw the corresponding tangent line, using for example the command `Tangent(Point, Conic)`. Turns out that the tangent lines to the ellipse at points P_1 and P_2 will contain the point A (this is what is expected from the geometric construction of the locus curve generated by point $E = sC + (1 - s)D$, because when $D = C$, we see $E = C$).

The intersecting points does not depend on the parameter s ; the idea is that intersecting points are those that are equal to their “antipodal” points and are equal to the points of tangency of the lines from A to the ellipse. The Figure 2 below shows the construction when we drag C on P_1 .

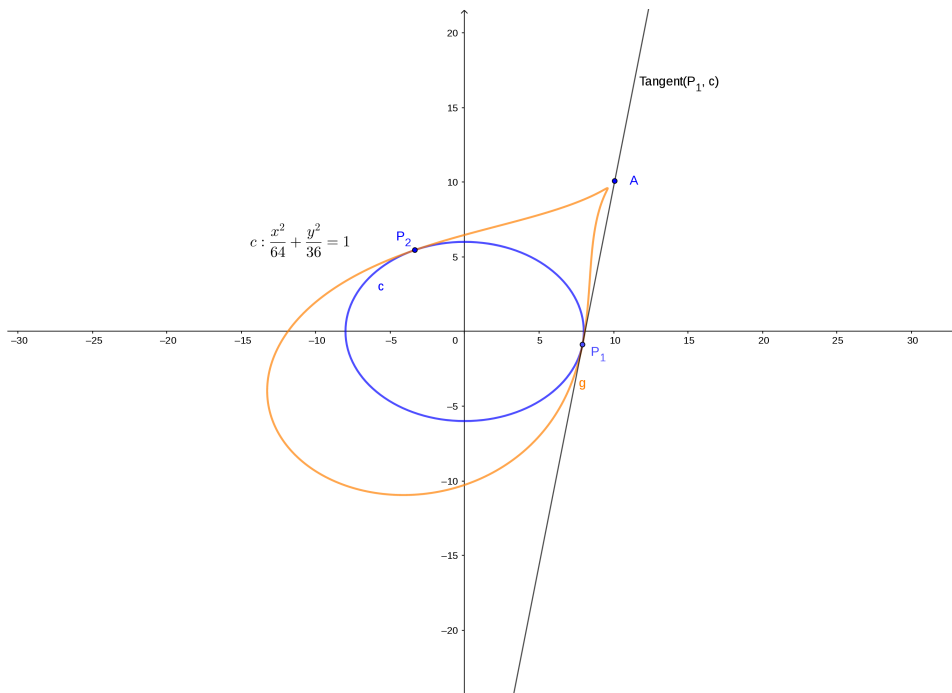


Figure 2. Tangent line for the ellipse at intersecting point P_1 .

Therefore, when the fixed point A is given, the problem of finding the tangency points of the ellipse with any locus curve reduces to the simple problem to find the tangent lines passing through point A . We describe the procedure as follows:

1. We set the Eq1 as the equation the ellipse c :

$$F(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad (7)$$

2. We calculate the gradient of $F(x, y)$,

$$\nabla F(x, y) = \left(\frac{2x}{a^2}, \frac{2y}{b^2} \right) \quad (8)$$

3. For generic point $P(x, y)$, we consider the vector $\overrightarrow{PA} = (x - x_0, y - y_0)$.

4. The condition of $\nabla F(x, y) \perp \overrightarrow{PA}$ yields to the following:

$$2x \frac{x - x_0}{a^2} + 2y \frac{y - y_0}{b^2} = 0. \quad (9)$$

5. We solve, using [1] (see [S1]), Eq1 and Eq2 for x and y to get the intersecting points,

$$P_1 = \left(\frac{a^2 y_0 \sqrt{a^2 y_0^2 + b^2 x_0^2 - a^2 b^2} + a^2 b^2 x_0}{a^2 y_0^2 + b^2 x_0^2}, -\frac{b^2 x_0 \sqrt{a^2 y_0^2 + b^2 x_0^2 - a^2 b^2} - a^2 b^2 y_0}{a^2 y_0^2 + b^2 x_0^2} \right)$$

and

$$P_2 = \left(-\frac{a^2 y_0 \sqrt{a^2 y_0^2 + b^2 x_0^2 - a^2 b^2} - a^2 b^2 x_0}{a^2 y_0^2 + b^2 x_0^2}, \frac{b^2 x_0 \sqrt{a^2 y_0^2 + b^2 x_0^2 - a^2 b^2} + a^2 b^2 y_0}{a^2 y_0^2 + b^2 x_0^2} \right)$$

3.2 The ellipsoid case when fixed point A is not at infinity

We want to find the tangent plane T at a point P on the ellipsoid $C = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$ such that

T is passing through the fixed point $A = (x_0, y_0, z_0)$. If the ellipsoid is the level surface of $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. Then the gradient at a point of the ellipsoid is $\nabla F(x, y, z) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right)$, then we see the tangent plane as follows:

$$T(x, y, z) = \nabla F(x, y, z) \cdot (x - x_0, y - y_0, z - z_0) = 0, \quad (10)$$

We thus solve $F(x, y, z) = 0$ and $T(x, y, z) = 0$ for the variables x, y and with the help of [7], we obtain the followings:

$$x = -\frac{a^2 \left(\text{RootOf} \left(\begin{array}{l} (a^2 y_0^2 + b^2 x_0^2) _Z^2 + a^2 c^4 - 2 a^2 c^2 z z_0 \\ + a^2 z^2 z_0^2 - c^4 x_0^2 + c^2 x_0^2 z^2 \\ + (-2 a^2 c^2 y_0 + 2 a^2 y_0 z z_0) _Z \end{array} \right) y_0 - c^2 + z z_0 \right)}{c^2 x_0}, \quad (11)$$

$$y = \frac{\text{RootOf} \left(\begin{array}{l} (a^2 y_0^2 + b^2 x_0^2) _Z^2 + a^2 c^4 - 2 a^2 c^2 z z_0 \\ + a^2 z^2 z_0^2 - c^4 x_0^2 + c^2 x_0^2 z^2 \\ + (-2 a^2 c^2 y_0 + 2 a^2 y_0 z z_0) _Z \end{array} \right) b^2}{c^2} \quad (12)$$

Consequently, we have two branches for each of the respective variables x and y . We list them accordingly below:

$$\begin{aligned}
 x_1 &= \frac{a^2}{c^2 x_0 (a^2 y_0^2 + b^2 x_0^2)} \left(- \sqrt{-c^2 x_0^2 \left(\begin{array}{l} ((a^2 y_0^2 + b^2 x_0^2) c^2 + a^2 b^2 z_0^2) \\ (\cos(t))^2 - 2 a^2 c z_0 \cos(t) b^2 \\ + c^2 ((b^2 - y_0^2) a^2 - b^2 x_0^2) \end{array} \right)} y_0 \right. \\
 &\quad \left. + b^2 c x_0^2 (-\cos(t) z_0 + c) \right) \\
 x_2 &= \frac{a^2}{c^2 x_0 (a^2 y_0^2 + b^2 x_0^2)} \left(\sqrt{-c^2 x_0^2 \left(\begin{array}{l} ((a^2 y_0^2 + b^2 x_0^2) c^2 + a^2 b^2 z_0^2) \\ (\cos(t))^2 - 2 a^2 c z_0 \cos(t) b^2 \\ + c^2 ((b^2 - y_0^2) a^2 - b^2 x_0^2) \end{array} \right)} y_0 \right. \\
 &\quad \left. + b^2 c x_0^2 (-\cos(t) z_0 + c) \right) \\
 y_1 &= \frac{b^2}{(a^2 y_0^2 + b^2 x_0^2) c^2} \left(\sqrt{-c^2 x_0^2 \left(\begin{array}{l} ((a^2 y_0^2 + b^2 x_0^2) c^2 + a^2 b^2 z_0^2) \\ (\cos(t))^2 - 2 a^2 c z_0 \cos(t) b^2 \\ + c^2 ((b^2 - y_0^2) a^2 - b^2 x_0^2) \end{array} \right)} \right. \\
 &\quad \left. + a^2 c y_0 (-\cos(t) z_0 + c) \right) \\
 y_2 &= \frac{b^2}{(a^2 y_0^2 + b^2 x_0^2) c^2} \left(- \sqrt{-c^2 x_0^2 \left(\begin{array}{l} ((a^2 y_0^2 + b^2 x_0^2) c^2 + a^2 b^2 z_0^2) \\ (\cos(t))^2 - 2 a^2 c z_0 \cos(t) b^2 \\ + c^2 ((b^2 - y_0^2) a^2 - b^2 x_0^2) \end{array} \right)} \right. \\
 &\quad \left. + a^2 c y_0 (-\cos(t) z_0 + c) \right)
 \end{aligned}$$

Accordingly, we have two branches of the space curves, which we describe them, respectively, as follows:

1. $r_1(t) = (x_1^*(t), y_1^*(t), z^*(t))$, where $z^*(t) = c \cos t$, and $x_1^*(t), y_1^*(t)$ are shown below respectively. First, we let

$$\delta = \sqrt{-c^2 x_0^2 \left(\begin{array}{l} (c^2 (a^2 y_0^2 + b^2 x_0^2) + a^2 b^2 z_0^2) (\cos(t))^2 \\ - 2 b^2 c a^2 \cos(t) z_0 + c^2 ((b^2 - y_0^2) a^2 - b^2 x_0^2) \end{array} \right)} \quad (13)$$

$$x_1^*(t) = \frac{-a^2}{c^2 x_0 (a^2 y_0^2 + b^2 x_0^2)} (\delta y_0 + b^2 c x_0^2 (-\cos(t) z_0 + c)), \quad (14)$$

$$y_1^*(t) = \frac{b^2}{c^2 (a^2 y_0^2 + b^2 x_0^2)} (\delta + a^2 c y_0 (-\cos(t) z_0 + c)). \quad (15)$$

2. $r_2(t) = (x_2^*(t), y_2^*(t), z^*(t))$, where $z^*(t) = c \cos t$, and $x_2^*(t), y_2^*(t)$ can be shown below respectively:

$$x_2^*(t) = \frac{a^2}{c^2 x_0 (a^2 y_0^2 + b^2 x_0^2)} (\delta y_0 + b^2 c x_0^2 (-\cos(t) z_0 + c)), \quad (16)$$

$$y_2^*(t) = \frac{-b^2}{c^2 (a^2 y_0^2 + b^2 x_0^2)} (\delta + a^2 c y_0 (-\cos(t) z_0 + c)). \quad (17)$$

We use $a = 5, b = 4, c = 3, x_0 = 7, y_0 = 8, z_0 = 9$ and plot the intersecting curve together with the ellipsoid and its locus surface when $s = 3$ in Figure 3. We remark that the intersecting curve in red will not vary when s varies once the fixed point $A = (x_0, y_0, z_0)$ is fixed. We use [S2] for exploration for this observation.

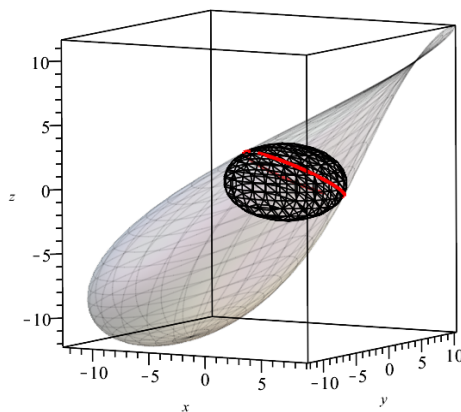


Figure 3. Intersecting curve between an ellipsoid and its locus surface.

3.3 Locus surface for a sphere

We let Σ be the sphere $x^2 + y^2 + z^2 = r^2$, and let the fixed point $A = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \rho \cos v_0)$ be on $S_1 : x^2 + y^2 + z^2 = \rho^2$ with $\rho \neq r$ and $\rho < \infty$. Because Σ is symmetric with respect to the origin and in view of preceding exploration with the locus for ellipsoid, it is natural to expect the shape of the locus for Σ stays unchanged and is coordinate free. Specifically, if we move the fixed points $A_1, A_2, \dots, A_n \in S_1$ sequentially:

$$A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n \quad (18)$$

with $A_n = A$. Then Δ_i , the locus surface of Σ with respect to A_i , for $i = 1, 2, \dots, n$, moves sequentially

$$\Delta_1 \rightarrow \Delta_2 \rightarrow \dots \rightarrow \Delta_n, \quad (19)$$

and we would expect that $\Delta_n = \Delta$. However, we shall explore that even if $\Delta_n = \Delta$, these two surfaces may be of different structures from differential geometry points of view. To begin with, we remark that it is an easy exercise to note that the antipodal points D of C for quadric surfaces by applying the Vieta's formula discussed in [7] is a **non-linear transformation** when $\rho < \infty$ and the fixed point A is not on the x, y or z axis. In the following, we describe how we can establish a sequence of non-linear transformations from a sphere to its locus surface Δ^* with the property of $\Delta = \Delta^*$.

Theorem 1 For $s > 0$ given, let Σ be the sphere $x^2 + y^2 + z^2 = r^2$, $A_1 = (0, 0, \rho)$ and $A = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \rho \cos v_0)$. We denote Δ_1 to be the locus surface of Σ with respect to A_1 and Δ to be the locus surface of Σ with respect to A . If $R_y(v_0)$ represents the rotation by v_0 radians around y -axis, and $R_z(u_0)$ represents the rotation by u_0 radians around z -axis, then $R_z(u_0) \circ R_y(v_0)(\Delta_1) = \Delta$.

Proof. Method 1. ($R_z(u_0) \circ R_y(v_0)(\Delta_1)$ is congruent to Δ .)

Let us consider the transformations $T = R_z(u_0) \circ R_y(v_0)$ and $T^{-1} = R_y(-v_0) \circ R_z(-u_0)$. For arbitrary $E \in R_z(u_0) \circ R_y(v_0)(\Delta_1)$, there exists $E_1 \in \Delta_1$ such that $E = T(E_1)$ where

$$E_1 = sC_1 + (1-s)D_1 \quad \text{with} \quad C_1, D_1 \in \Sigma \cap \overleftrightarrow{A_1C_1}.$$

A direct calculation shows that $T(A_1) = A$ and, since T is a rigid transformation, for $C \doteq T(C_1)$ and $D \doteq T(D_1)$, we have that

$$E = sC + (1-s)D \quad \text{with} \quad C, D \in \Sigma \cap \overleftrightarrow{AC}.$$

This shows that $R_z(u_0) \circ R_y(v_0)(\Delta_1) \subset \Delta$. The contention of $\Delta \subset R_z(u_0) \circ R_y(v_0)(\Delta_1)$ follows from a similar argument by using T^{-1} .

Method 2. (Expressing $R_z(u_0) \circ R_y(v_0)(\Delta_1)$ via a non-linear transformation) We let Δ_1 be the locus surface for Σ when $u_0 = v_0 = 0$, and let $C_1 = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \in \Sigma$ and D_1 be the antipodal point C_1 . By the definition of D_1 , we may write

$$D_1 = \begin{bmatrix} x_1^1 \\ y_1^1 \\ z_1^1 \end{bmatrix} = \frac{1}{\rho^2 + r^2 - 2\rho r \cos v} \begin{pmatrix} \rho^2 - r^2 & 0 & 0 \\ 0 & \rho^2 - r^2 & 0 \\ 0 & 0 & \frac{-(\rho^2 + r^2) + 2r^2\rho}{\cos v} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}. \quad (20)$$

Consequently, we define $T_1 : \Sigma \rightarrow \Delta_1$ by

$$\begin{aligned} T_1 \left(\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \right) &= s \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} + (1-s) \begin{bmatrix} x_1^1 \\ y_1^1 \\ z_1^1 \end{bmatrix}, \\ &= s \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} + \frac{1-s}{\rho^2 + r^2 - 2\rho r \cos v} \begin{pmatrix} \rho^2 - r^2 & 0 & 0 \\ 0 & \rho^2 - r^2 & 0 \\ 0 & 0 & \frac{-(\rho^2 + r^2) \cos v + 2r^2\rho}{\cos v} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \\ &= \begin{pmatrix} s + \frac{(\rho^2 - r^2)(1-s)}{\rho^2 + r^2 - 2\rho r \cos v} & 0 & 0 \\ 0 & s + \frac{(\rho^2 - r^2)(1-s)}{\rho^2 + r^2 - 2\rho r \cos v} & 0 \\ 0 & 0 & s - \frac{(1-s)((\rho^2 + r^2) \cos v - 2r\rho)}{(\rho^2 + r^2 - 2\rho r \cos v) \cos v} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \\ &= \begin{pmatrix} sr \cos(u) \sin(v) - \frac{(1-s) \cos(u) (r^2 - \rho^2) \sin(v) r}{-2\rho \cos(v) r + r^2 + \rho^2} \\ sr \sin(u) \sin(v) - \frac{(1-s) r \sin(u) \sin(v) (r^2 - \rho^2)}{-2\rho \cos(v) r + r^2 + \rho^2} \\ sr \cos(v) - \frac{(1-s) r ((r^2 + \rho^2) \cos(v) - 2r\rho)}{-2\rho \cos(v) r + r^2 + \rho^2} \end{pmatrix} \\ &= \Delta_1 \end{aligned} \quad (21)$$

We remark that matrix

$$M = \begin{pmatrix} s + \frac{(\rho^2 - r^2)(1-s)}{\rho^2 + r^2 - 2\rho r \cos v} & 0 & 0 \\ 0 & s + \frac{(\rho^2 - r^2)(1-s)}{\rho^2 + r^2 - 2\rho r \cos v} & 0 \\ 0 & 0 & s - \frac{(1-s)((\rho^2 + r^2) \cos v - 2r\rho)}{(\rho^2 + r^2 - 2\rho r \cos v) \cos v} \end{pmatrix}$$

is a non-linear transformation and Δ_1 is an image of a non-linear transformation. If we let $R_y(v_0)$ and $R_z(u_0)$ represent the rotation matrix around y and z axes respectively as follows:

$$R_y(v_0) = \begin{pmatrix} \cos v_0 & 0 & \sin v_0 \\ 0 & 1 & 0 \\ -\sin v_0 & 0 & \cos v_0 \end{pmatrix}, R_z(u_0) = \begin{pmatrix} \cos u_0 & -\sin u_0 & 0 \\ \sin u_0 & \cos u_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (22)$$

We leave it to a CAS to prove that $R_z(u_0) \circ R_y(v_0)(\Delta_1)$ produces the same surface as Δ .

Remarks:

1. In the preceding theorem, $R_z(u_0) \circ R_y(v_0)(\Delta_1)$ is congruent to Δ . However, these two locus surfaces have different characteristics in differential geometry sense. For example, the cross sections are different for $R_z(u_0) \circ R_y(v_0)(\Delta_1)$ and Δ respectively, when u or v is being kept as constant, and yet they have different characteristics in differential geometry sense. See the Example below.
2. In view of the preceding Theorem, if the fixed point A is on the x, y or z axis, the transformation from Σ to the corresponding locus surface is a non-linear transformation. The following result is trivial, which we omit the proof.

Corollary. If Σ is the sphere $x^2 + y^2 + z^2 = r^2$, and we let $A = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \rho \cos v_0)$ be on $S_1 : x^2 + y^2 + z^2 = \rho^2$ with $\rho \neq r$, and A is not on the x, y or z axis. Then there exists a transformation T^* on Σ such that $T^*(\Sigma)$ and Δ are same surface but with different cross sections, where Δ is the locus surface of Σ with respect to the fixed point A .

We use the following Example to demonstrate the effect of the preceding theorem.

Example 2 Consider the sphere S_0 of $x^2 + y^2 + z^2 = 25$, and the fixed point of

$$A = \left(\left(7 \sin \frac{\pi}{4} \right) \left(\cos \frac{\pi}{4} \right), \left(7 \sin \frac{\pi}{4} \right) \left(\sin \frac{\pi}{4} \right), 7 \cos \frac{\pi}{4} \right). \quad (23)$$

We shall show that to obtain the locus surface Δ for A when $s = 3$. We may pursue in the following ways. First, we pick $u_0 = v_0 = 0$ for the fixed point

$$A_1 = ((7 \sin 0) (\cos 0), (7 \sin 0) (\sin 0), 7 \cos 0) = (0, 0, 7), \quad (24)$$

and let the locus surface Δ_1 be the one with respect to A_1 . We first depict the surface Δ for A when $s = 3$ together with S_0 in Figure 4 below.

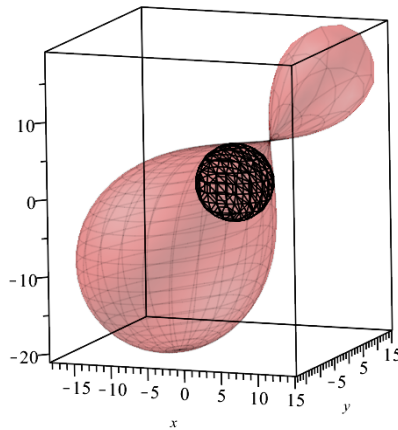


Figure 4. Locus surface Δ together with S_0 .

We notice that Δ is tangent to S_0 at an intersecting curve as we have discussed in the preceding section. Furthermore, we depict the locus Δ in Figure 5(a). Next we compute $R_z(u_0) \circ R_y(v_0) \circ \Delta_1$, with $u_0 = v_0 = \frac{\pi}{4}$, and the plot can be seen as in Figure 5(b).

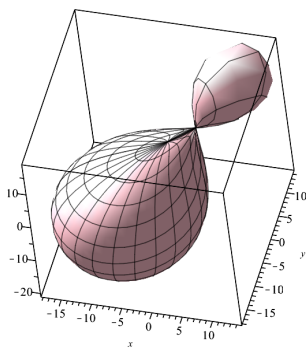


Figure 5(a). The locus for Δ .

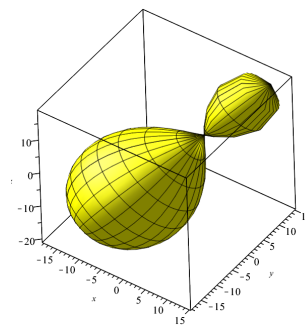


Figure 5(b). The locus surface $R_z(u_0) \circ R_y(v_0) \circ \Delta_1$.

The traces for Δ and $R_z(u_0) \circ R_y(v_0) \circ \Delta_1$ when $u = \frac{\pi}{2}$ can be seen in Figures 5(c) and 5(d) respectively.

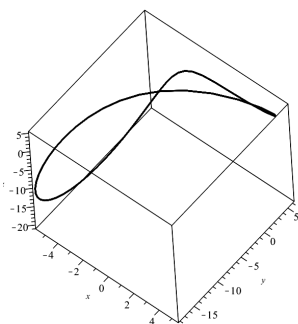


Figure 5(c) The traces for Δ when $u = \frac{\pi}{2}$.

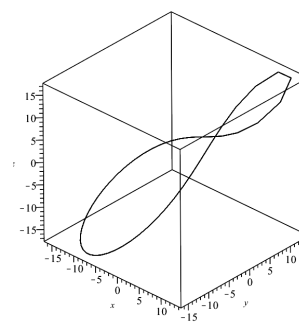


Figure 5(d). The traces for $R_z(u_0) \circ R_y(v_0) \circ \Delta_1$ when $u = \frac{\pi}{2}$.

It is a good exercise, which we leave it to readers to explore with a CAS or DGS that the the trace for Δ when $u = \frac{\pi}{2}$ (Figure 5(c)) does not lie on the same plane but the trace for $R_z(u_0) \circ R_y(v_0) \circ \Delta_1$ when $u = \frac{\pi}{2}$ does lie on the same plane (Figure 5(d)). See [S3] for explorations.

In the next section, we show that when the fixed point A is at infinity, the mapping which sends a quadric surface to its locus surface is a linear transformation.

4 When the fixed point is at an infinity

We first describe two approaches how we may obtain the locus surfaces when the fixed point A , written in its spherical coordinate, $(\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$, is at an infinity. We remark that the following Method 2 is essentially identical to the Method 1 after letting $\rho \rightarrow \infty$.

Method 1. We let the radius of spherical coordinate for $A = (\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$, ρ , go to infinity

We describe the locus surface in the following steps:

1. If $A = (\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$. Let us note that 4 and 5 become,

$$k = \frac{\hat{y} - \rho \sin u_0 \sin v_0}{\hat{x} - \rho \cos u_0 \sin v_0} \quad (25)$$

$$m = \frac{\hat{z} - \rho \cos v_0}{\hat{x} - \rho \cos u_0 \sin v_0} \quad (26)$$

2. We follow the usual procedure to find the intersection between the line AC and the quadric surface at $D = (x_1, y_1, z_1)$ respectively by adopting the Vieta's formula.
3. Next we let $\rho \rightarrow \infty$ to obtain the corresponding intersection point $D_{\text{inf}} = (x_{1 \text{ inf}}, y_{1 \text{ inf}}, z_{1 \text{ inf}})$
4. The corresponding locus surface, is defined as $E_{\text{inf}} = (x_{e \text{ inf}}, y_{e \text{ inf}}, z_{e \text{ inf}})$ where

$$x_{e \text{ inf}} = s\hat{x} + (1 - s)(x_{1 \text{ inf}})$$

$$y_{e \text{ inf}} = s\hat{y} + (1 - s)(y_{1 \text{ inf}})$$

$$z_{e \text{ inf}} = s\hat{z} + (1 - s)(z_{1 \text{ inf}}).$$

Method 2. We take k (4) and m (5) to be fixed angles after letting $\rho \rightarrow \infty$.

1. We fix the angles $u_0 \in (0, 2\pi) - \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ and $v_0 \in (0, \pi)$, and let the point A going to infinity in the direction $(\sin v_0 \cos u_0, \sin v_0 \sin u_0, \cos v_0)$. Taking the limit of 25 and 26 when $\rho \rightarrow \infty$ we get,

$$k_0 \doteq k(u_0, v_0) = \frac{\sin v_0 \sin u_0}{\sin v_0 \cos u_0} = \tan u_0, \quad (27)$$

and

$$m_0 \doteq m(u_0, v_0) = \frac{\cos v_0}{\sin v_0 \cos u_0} = \cot v_0 \sec u_0. \quad (28)$$

2. By using the followings and substitute into the implicit equation of the quadric, $F(x, y, z) = 0$,

$$\begin{aligned} y &= \hat{y} + k_0(x - \hat{x}), \\ z &= \hat{z} + m_0(x - \hat{x}), \end{aligned}$$

we follow the Vieta's formula to find the x -coordinate of the the antipodal point D'_{inf} , say $x'_{1\text{inf}}$, by calculating the roots of the polynomial

$$p(x) = a_2x^2 + a_1x + a_0.$$

3. For a given s , the locus surface generated by point $E'_{\text{inf}} = sC + (1 - s)D'_{\text{inf}}$ is defined as

$$\Delta'_{\text{inf}}(s, u_0, v_0) = \begin{bmatrix} x'_{e\text{inf}} \\ y'_{e\text{inf}} \\ z'_{e\text{inf}} \end{bmatrix} = \begin{bmatrix} s\hat{x} + (1 - s)x'_{1\text{inf}} \\ s\hat{y} + (1 - s)y'_{1\text{inf}} \\ s\hat{z} + (1 - s)z'_{1\text{inf}} \end{bmatrix}.$$

Calculations in Exploration [S4] shows that $D_{\text{inf}} = D'_{\text{inf}}$, and therefore $E_{\text{inf}} = E'_{\text{inf}}$, so the locus surfaces $\Delta_{\text{inf}}(s, u_0, v_0)$ and $\Delta'_{\text{inf}}(s, u_0, v_0)$ produced by Method 1 and Method 2, respectively, are identical.

We explore the locus surfaces for ellipsoids and hyperbolic with two sheets when A is at an infinity in the following subsections.

4.1 Ellipsoid

Let Σ be the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Following the methodology set out in the previous section, we calculate the roots of the polynomial

$$p(x) = a_2x^2 + a_1x + a_0,$$

where

$$a_2 = \frac{a^2b^2m^2 + a^2c^2k^2 + b^2c^2}{a^2b^2c^2} \tag{29}$$

$$a_1 = \frac{2(z_0b^2m + y_0c^2k - x_0(b^2m^2 + c^2k^2))}{b^2c^2} \tag{30}$$

$$a_0 = \frac{x_0^2(b^2m^2 + c^2k^2) - 2x_0(z_0b^2m + y_0c^2k) + y_0^2c^2 + z_0^2b^2 - b^2c^2}{b^2c^2}. \tag{31}$$

The explicit expressions for D_{inf} , E_{inf} and $\Delta_{\text{inf}}(s, u_0, v_0)$ are calculated in Exploration [S5]. See [S6] for dynamic explorations.

We depict the locus surface (blue) when $s = 2, a = 5, b = 4, c = 3$, with $u_0 = \frac{\pi}{3}, v_0 = \frac{\pi}{4}$ and $\rho \rightarrow \infty$ together with the original ellipsoid (yellow) in Figure 6.

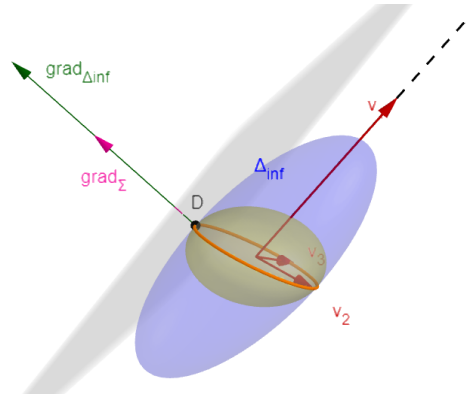


Figure 6. Locus ellipsoid when $s = 2$.

4.2 Hyperboloid with two sheets

Let Σ be the hyperboloid with two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$. Following the methodology set out in the previous section, we calculate the roots of the polynomial

$$p(x) = a_2x^2 + a_1x + a_0,$$

where

$$a_2 = \frac{a^2c^2k^2 - a^2b^2m^2 + b^2c^2}{a^2b^2c^2} \quad (32)$$

$$a_1 = \frac{2y_0c^2k - 2x_0(c^2k^2 - b^2m^2) - 2z_0b^2m}{b^2c^2} \quad (33)$$

$$a_0 = \frac{2x_0z_0b^2m - 2x_0y_0c^2k - x_0^2(b^2m^2 - c^2k^2) + y_0^2c^2 - z_0^2b^2 + b^2c^2}{b^2c^2}. \quad (34)$$

The explicit expressions for D_{inf} , E_{inf} and $\Delta_{\text{inf}}(s, u_0, v_0)$ are calculated in Exploration [S7]. See [S8] for dynamic explorations.

We depict the locus surface (blue) when $s = 0.8, a = 5, b = 4, c = 3$, with $u_0 = \frac{\pi}{6}, v_0 = \frac{\pi}{3}$ and $\rho \rightarrow \infty$ together with the original hyperboloid (yellow) in Figure 7.

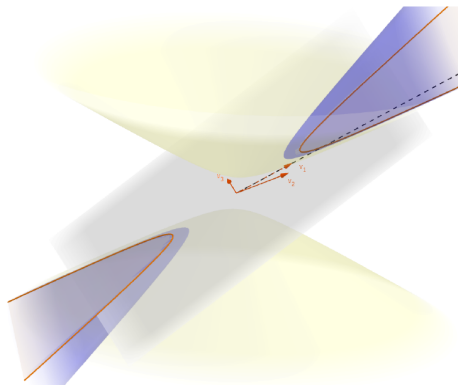


Figure 7. Locus hyperboloid when $s = 0.8$

5 Special Cases

We show that both methods coincide when the point $A = (x_0, y_0, z_0)$ is at infinity on x -axis, y -axis or z -axis respectively. In other words, when $x_0 \rightarrow \pm\infty, y_0 \rightarrow \pm\infty$ or $z_0 \rightarrow \pm\infty$ respectively, both methods produce the same locus surfaces.

1. If the fixed point $A = (x_0, y_0, z_0)$ is on the x -axis and we let $x_0 \rightarrow \pm\infty, D = (x_1, y_1, z_1) = (-\hat{x}, \hat{y}, \hat{z})$ is simply a reflection of C along the x -axis. Consequently, the locus surfaces is

$$\begin{bmatrix} s\hat{x} + (1-s)(-\hat{x}) \\ s\hat{y} + (1-s)\hat{y} \\ s\hat{z} + (1-s)\hat{z} \end{bmatrix} = \begin{bmatrix} (2s-1)\hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}. \quad (35)$$

2. If the fixed point $A = (x_0, y_0, z_0)$ is on the y -axis and we let $y_0 \rightarrow \pm\infty, D = (x_1, y_1, z_1) = (\hat{x}, -\hat{y}, \hat{z})$ is simply a reflection of C along the y -axis. Consequently, the locus surfaces is

$$\begin{bmatrix} s\hat{x} + (1-s)\hat{x} \\ s\hat{y} + (1-s)(-\hat{y}) \\ s\hat{z} + (1-s)\hat{z} \end{bmatrix} = \begin{bmatrix} \hat{x} \\ (2s-1)\hat{y} \\ \hat{z} \end{bmatrix}. \quad (36)$$

3. If the fixed point $A = (x_0, y_0, z_0)$ is on the z -axis and we let $z_0 \rightarrow \infty, D = (x_1, y_1, z_1) = (\hat{x}, \hat{y}, -\hat{z})$ is simply a reflection of C along the z -axis. Consequently, the locus surfaces is

$$\begin{bmatrix} s\hat{x} + (1-s)\hat{x} \\ s\hat{y} + (1-s)\hat{y} \\ s\hat{z} + (1-s)(-\hat{z}) \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ (2s-1)\hat{z} \end{bmatrix}. \quad (37)$$

5.1 Remarks for the ellipsoid case

When the point $A = (x_0, y_0, z_0)$ is at infinity on x -axis, say $x_0 \rightarrow +\infty$, the above calculations show the followings:

- 1 The locus surface $\Delta_{\text{inf}}(s, u_0 = 0, v_0 = \pi/2)$ is the image of Σ under the linear transformation given by the matrix

$$L = \begin{bmatrix} 2s-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let us note that this locus surface is an ellipsoid, say:

$$\Delta_{x\text{inf}} \doteq \Delta_{\text{inf}}(s, u_0 = 0, v_0 = \pi/2) = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{((2s-1)a)^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$$

- 2 If $s \in \mathbb{R}^+ \setminus \{1\}$ then Σ and $\Delta_{x\text{inf}}$ intersect tangentially just at an elliptical curve, say:

$$\gamma_{x=0} = \left\{ (x, y, z) \in \mathbb{R}^3 : x = 0, \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}.$$

Proof. Clearly, $\gamma_{x=0} \subseteq \Sigma \cap \Delta_{x \text{ inf}}$. Now, $(\bar{x}, \bar{y}, \bar{z}) \in \Sigma \cap \Delta_{x \text{ inf}}$ implies that

$$1 = \frac{\bar{x}^2}{a^2} + \frac{\bar{y}^2}{b^2} + \frac{\bar{z}^2}{c^2} = \frac{\bar{x}^2}{((2s-1)a)^2} + \frac{\bar{y}^2}{b^2} + \frac{\bar{z}^2}{c^2}$$

so, $(2s-1)^2 \bar{x}^2 = \bar{x}^2$. Since $s \neq 1$, we conclude that $\bar{x} = 0$, that is, $(\bar{x}, \bar{y}, \bar{z}) \in \gamma_{x=0}$, and therefore $\Sigma \cap \Delta_{x \text{ inf}} \subseteq \gamma_{x=0}$.

Remark 3 1 Let us denote by

$$\underline{\Sigma} = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

the solid ellipsoid which boundary is Σ . Then we see $\Delta_{x \text{ inf}} \subsetneq \underline{\Sigma}$ for $0 < s < 1$.

Proof.

Given $(x, y, z) \in \Delta_{x \text{ inf}}$, choose $(\hat{x}, \hat{y}, \hat{z}) \in \Sigma$ such that $x = (2s-1)\hat{x}$, $y = \hat{y}$ and $z = \hat{z}$. Then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{(2s-1)^2 \hat{x}^2}{a^2} + \frac{\hat{y}^2}{b^2} + \frac{\hat{z}^2}{c^2} < \frac{\hat{x}^2}{a^2} + \frac{\hat{y}^2}{b^2} + \frac{\hat{z}^2}{c^2} \leq 1.$$

2 Let us denote by

$$\underline{\Delta_{x \text{ inf}}} = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{((2s-1)a)^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

the solid ellipsoid which boundary is $\Delta_{x \text{ inf}}$. Then, $s > 1$ implies $\Sigma \subsetneq \underline{\Delta_{x \text{ inf}}}$.

Proof.

The proof is similar to (1). Similar statements hold when $x_0 \rightarrow -\infty, y_0 \rightarrow \pm\infty$ or $z_0 \rightarrow \pm\infty$ respectively.

In view of 1, we prove the result for A being at an infinity as follows:

Theorem 4 For $s > 0$ given, let Σ be the sphere $x^2 + y^2 + z^2 = r^2$, A_1 be at the infinity on the z axis, and $A = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \rho \cos v_0)$ when $\rho \rightarrow \infty$. We denote Δ_1 to be the locus surface of Σ with respect to A_1 and Δ to be the locus surface of Σ with respect to A . If $R_y(v_0)$ represents the rotation by v_0 radians around y -axis, and $R_z(u_0)$ represents the rotation by u_0 radians around z -axis, then $R_z(u_0) \circ R_y(v_0)(\Delta_1) = \Delta$.

Proof. We let $\rho \rightarrow \infty$ for $A_1 = (0, 0, \rho)$, and compute the matrices in method 2 in 1.

It turns out, as expected, that the antipodal point D_1 is $\begin{pmatrix} r \cos u \sin v \\ r \sin u \sin v \\ -r \cos v \end{pmatrix}$ and the locus Δ_1 is

$$\begin{pmatrix} sr \cos u \sin v + (1-s)r \cos u \sin v \\ sr \sin u \sin v + (1-s)r \sin u \sin v \\ sr \cos v - (1-s)r \cos v \end{pmatrix}. \quad (38)$$

We proceed and compute $R_z(u_0) \circ R_y(v_0)(\Delta_1)$ and let a CAS to verify that $R_z(u_0) \circ R_y(v_0)(\Delta_1) = \Delta$.

5.2 Remarks for the hyperboloid with two sheets

When the point $A = (x_0, y_0, z_0)$ is at infinity on x -axis, say $x_0 \rightarrow +\infty$, the above calculations show the followings:

- 1 The locus surface $\Delta_{\text{inf}}(s, u_0 = 0, v_0 = \pi/2)$ is the image of Σ under the linear transformation given by the matrix:

$$L = \begin{bmatrix} 2s-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let us note that this locus surface is an hyperboloid with two sheets, say:

$$\Delta_{x\text{inf}} \doteq \Delta_{\text{inf}}(s, u_0 = 0, v_0 = \pi/2) = \left\{ (x, z, z) \in \mathbb{R}^3 : \frac{x^2}{((2s-1)a)^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \right\}$$

- 2 If $s \in \mathbb{R}^+ \setminus \{1\}$ then Σ and $\Delta_{x\text{inf}}$ intersect tangentially just at an hyperbolic curve, say:

$$\gamma_{x=0} = \left\{ (x, y, z) \in \mathbb{R}^3 : x = 0, \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \right\}.$$

Proof. Clearly, $\gamma_{x=0} \subseteq \Sigma \cap \Delta_{x\text{inf}}$. Now, $(\bar{x}, \bar{y}, \bar{z}) \in \Sigma \cap \Delta_{x\text{inf}}$ implies that,

$$-1 = \frac{\bar{x}^2}{a^2} + \frac{\bar{y}^2}{b^2} - \frac{\bar{z}^2}{c^2} = \frac{\bar{x}^2}{((2s-1)a)^2} + \frac{\bar{y}^2}{b^2} - \frac{\bar{z}^2}{c^2}$$

so, $(2s-1)^2 \bar{x}^2 = \bar{x}^2$. Since $s \neq 1$, we conclude that $\bar{x} = 0$, that is, $(\bar{x}, \bar{y}, \bar{z}) \in \gamma_{x=0}$, and therefore $\Sigma \cap \Delta_{x\text{inf}} \subseteq \gamma_{x=0}$.

- 3 Let us denote by

$$\underline{\Sigma} = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} \leq -1 \right\}$$

the solid hyperboloid with two sheets whose boundary is Σ . Then we have $\Delta_{x\text{inf}} \subsetneq \underline{\Sigma}$ for $0 < s < 1$.

Proof. Given $(x, y, z) \in \Delta_{x\text{inf}}$, choose $(\hat{x}, \hat{y}, \hat{z}) \in \Sigma$ such that $x = (2s-1)\hat{x}$, $y = \hat{y}$ and $z = \hat{z}$. Then,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{(2s-1)^2 \hat{x}^2}{a^2} + \frac{\hat{y}^2}{b^2} - \frac{\hat{z}^2}{c^2} < \frac{\hat{x}^2}{a^2} + \frac{\hat{y}^2}{b^2} - \frac{\hat{z}^2}{c^2} \leq -1.$$

- 4 Let us denote by

$$\underline{\Delta}_{x\text{inf}} = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{((2s-1)a)^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} \leq -1 \right\}$$

the solid hyperboloid with two sheets whose boundary is $\Delta_{x\text{inf}}$. Then we have $\Sigma \subsetneq \underline{\Delta}_{x\text{inf}}$ for $s > 1$.

Proof. The proof is similar to (3). Similar statements hold when $x_0 \rightarrow -\infty, y_0 \rightarrow \pm\infty$ or $z_0 \rightarrow \pm\infty$ respectively.

6 Main results

This section is to say the locus problem is a linear transformation between the specified surface Σ and the locus surface Δ when A is at an infinity. However, due to the length requirement for the paper, we simply state the following results and their respective proofs can be found in [8].

Theorem 5 *Let Σ be a quadric surface, and let $A_{\text{inf}}(u_0, v_0)$ be the fixed point at infinity in the direction of $(\cos u_0 \sin v_0, \sin u_0 \sin v_0, \cos v_0)$, $C \in \Sigma$ and D_{inf} be the “antipodal” point of C corresponding to $A_{\text{inf}}(u_0, v_0)$ as described in previous sections. Then there exists an affine transformation $\mathcal{A}_D : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\mathcal{A}_D(C) = D_{\text{inf}}$.*

Corollary. Given $s > 0$, consider same hypothesis as in Theorem 5 and let $E_{\text{inf}} = sC + (1 - s)D_{\text{inf}}$. Then the affine transformation

$$\mathcal{A}_E = sI + (1 - s)\mathcal{A}_D$$

is such that $\mathcal{A}_E(C) = E_{\text{inf}}$, where I is the identity mapping from R^3 to R^3 .

Proposition 6 *In Theorem 5, if Σ is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then there exists a matrix $L_D^e = [l_{ij}^e]_{3 \times 3}$ such that $L_D^e C = D_{\text{inf}}$.*

Corollary. Given $s > 0$, consider same hypothesis as in Proposition 6 and let $E_{\text{inf}} = sC + (1 - s)D_{\text{inf}}$. Then the matrix

$$L_E^e = sI + (1 - s)L_D^e$$

is such that $L_E^e C = E_{\text{inf}}$, and therefore, the locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ is the image of Σ under the linear transformation given by the matrix $L_E^e = [l_{ij}^e]_{3 \times 3}$.

Proposition 7 *For $s \in \mathbb{R} \setminus \{1\}$, the ellipsoid Σ and locus ellipsoid $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect themselves tangentially at an elliptical curve.*

Exploration [S6] contains an animation to exemplify the previous result.

Proposition 8 *For $s \in \mathbb{R}^+ \setminus \{1\}$, if the hyperboloid with two sheets Σ and corresponding locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect themselves, they do it tangentially at an hyperbolic curve.*

Finally, we can verify that the gradient of Σ and $\Delta_{\text{inf}}(s, u_0, v_0)$ are colinear when evaluated at any point on γ when Σ is a hyperboloids with two sheets, see Figure 8 and exploration [S8].

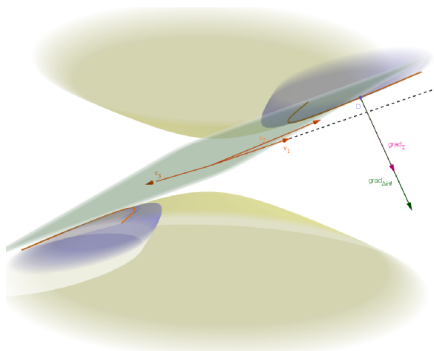


Figure 8. Intersection of the hyperboloid Σ and its corresponding locus.

In [8], we shall further discuss how the eigenvectors of a linear transformation on a quadric surface, when the fixed point A is at an infinity, will affect the shapes of locus surfaces when s gets larger and larger and when $s \rightarrow \infty$ respectively.

7 Conclusions

In this paper, we have explored the locus problems when the fixed point A is outside a specified curve or surface. When A is not at an infinity, although the projection $T : \Sigma \rightarrow \Delta$ is not a linear map, the result is interesting because the intersecting points in 2D or intersecting curve in 3D remains fixed regardless of the parameter s . When the fixed point A is at an infinity, the projection $T : \Sigma \rightarrow \Delta$ becomes a linear transformation. We shall further discuss this linear transformation in [8]. It is delighted to see a simple college entrance exam problem originated from China [6] has led to many interesting discoveries in projective geometry, differential geometry (see [6]), and possibly other areas. The explorations, discussed all papers that are related to this locus problem, are very accessible to undergraduate or graduate students, we believe that the concepts involved can be comprehended to future math teachers. Only when the math contents are enriched for our math teachers, can we increase our success in teaching math for our future generations.

It is clear that technological tools provide us with many crucial intuitions before we attempt more rigorous analytical solutions, and lead to many unexpected discoveries. Here we have gained geometric intuitions while using a DGS. In the meantime, we use a CAS for verifying that our analytical solutions are consistent with our initial intuitions. Incorporating a DGS and CAS into exploring a problem definitely has made mathematics fun and accessible on one hand, but they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging. We do not expect that exam-oriented curricula will change in many parts of the world. However, encouraging a greater interest in mathematics for students, and in particular, providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil and paper, is an important step for cultivating creativity and innovation.

8 Supplementary Electronic Materials

- [S1] GeoGebra worksheet for ellipse case in Section 3.1.
- [S2] GeoGebra worksheet for ellipsoid case in Section 3.2.
- [S3] Maple worksheet for Sections 3.2, 3.3 and Example 2.
- [S4] wxMaxima worksheet for methods 1 and 2 in Section 4.1.
- [S5] wxMaxima for ellipsoid case in Sections 4.1 and 6.
- [S6] GeoGebra worksheet for ellipsoid case in Sections 4.1 and 6.
- [S7] wxMaxima worksheet for hyperboloid case in Sections 4.2 and 6.

[S8] GeoGebra worksheet for hyperboloid case in Sections 4.2 and 6.

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Mathematical Patterns formed by the Spheres

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INTRODUCTION

In Graph Theory, the contact graph of a collection of spheres is a graph whose nodes are represented by the spheres, and each edge corresponds to two externally tangent spheres. In this work, we focus on the construction and the pattern design of models, each having a Platonic solid, an Archimedean solid, or a Catalan solid as the contact graph. The (monochrome) spheres in each model are then assigned with k colors obeying the rule that the spheres receiving two different colors are congruent in the Euclidean sense that there exists a Euclidean motion (a translation, a point reflection, a line reflection, a plane reflection, a rotation w.r.t. a line together with their compositions) taking all spheres of one color onto spheres of another. Whenever such a coloring of the model is possible, the pattern is said to have k -color symmetry.

The models/patterns are organized according to the hierarchy: (1) The number of spheres. The possible numbers are 4, 6, 8, 12, 14, 20, 24, 26, 32, 48, 60, 62, 120, as tabulated from the number of vertices in all the polyhedra of interest. (2) The contact graphs. Two models having the same number of spheres may have 1, 2, or 3 different contact graphs. (3) Color symmetry. Two k -color patterns are considered distinct if they have non-congruent parts. Distinct patterns are labeled pattern 1, pattern 2, etc.

There is no preference for particular colors so long as the arrangement causes no mathematical ambiguity.

The completed work of the construction and the design in the native *.cg3 and the screen output *.mov files produced by Cabri 3D can be downloaded for further examination. [7]

This paper is meant to be a survey of a large number of possibilities in designing distinct patterns. We do not attempt (1) to explain in full the motivation for each design; (2) to automate the creative process of designing; (3) to make an encyclopedia of sphere model design. By way of illustration, we are to focus on the model having 120 spheres only.

COMPUTING/GEOMETRIC CONSTRUCTION ENVIRONMENT

All geometric constructions faithfully follow the tradition established by Euclid. This work is made possible under the Interactive Geometry environment created by the software Cabri 3D.

This work is completed under the environment provided by Google Workspace for Education Fundamentals (formerly G Suite for Education).

For the benefit of academic discussions, links to the *.cg3 files saved from Cabri 3D, the animations files *.mov save from video capture of Cabi 3D, and the explanation notes *.pdf can be found at [7].

APPLICATIONS

Designing patterns on the sphere model may serve as a challenging exercise in Concrete Geometry [1,2,3,4,6]. The pre-production and the production of the models as video segments lead the Cabri 3D users to appreciate the visual art of counting when the iPad is seen almost everywhere. At the same time, few people use it for mathematical thinking. Technical terms used for the polyhedra have their origin in the study of Mineralogy. Many models studied here are of current interest in Chemistry [5].

RESULTS

CONSTRUCTION OF THE 120-SPHERE MODEL

Only the Great Rhombicosidodecahedron [4] has 120 vertices among the Platonic, Archimedean, and Catalan solids. Therefore, this is the only possible contact graph for any 120-sphere models for a Platonic solid or an Archimedean-Catalan solid. Since all faces of the Great Rhombicosidodecahedron are regular polygons, all edges have the same length. One by one, the required spheres can be constructed by taking a vertex as the center and one-half length an edge as radius. In practice, it suffices to construct (Fig. 1) only one sphere and then construct the other 119 by taking successive planar reflections, line reflections, and the point reflections compatible with the vast symmetries enjoyed by the polyhedron.

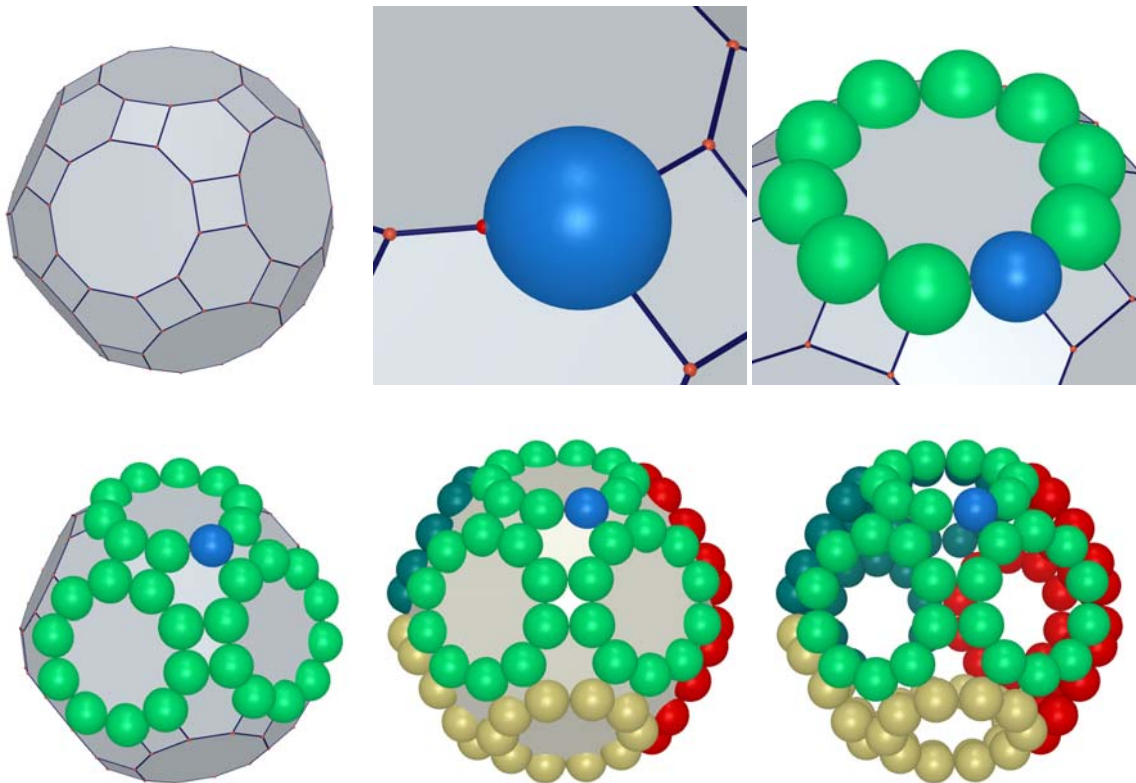


Fig. 1 Construction of the 120-Sphere Model from the Great Rhombicosidodecahedron

PATTERNS ON THE 120-SPHERE MODEL

2-COLOR SYMMETRY

Pattern 1 - Each color consisting of six 10-cycles

The design (Fig. 2) follows the dissection of a baseball into two parts marked by the stitches. The two parts are not symmetric with respect to the center. They are line-symmetric. The appearance suggests the name “The Yin-Yang Pattern.”

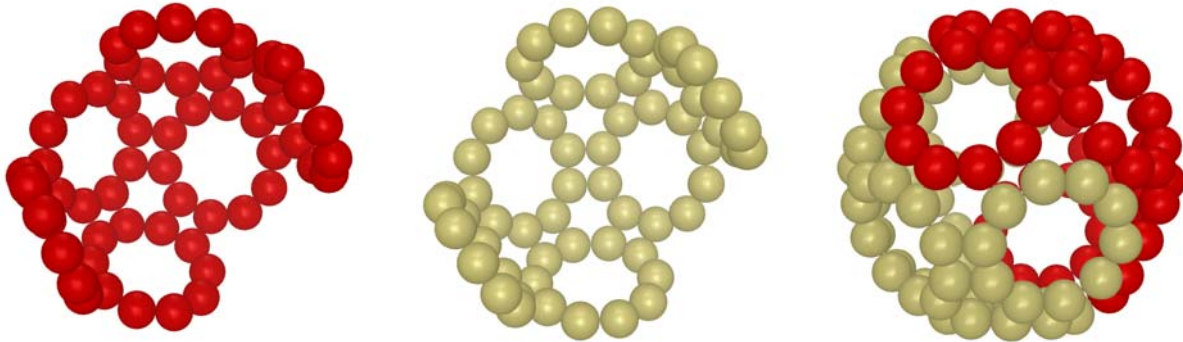


Fig. 2 The Yin-Yang Pattern

Pattern 2 - Each color consisting of 60 connected spheres formed by a single 10-cycle with 5 arms attached

Here is a pattern (Fig. 3, left) having 2-color symmetry, with each color consisting of five connected components formed by 12 spheres. By swapping colors for 10 spheres, we have created (Fig. 3, right) a pattern having 2-color symmetry with each color formed by a single 10-cycle with 5 arms attached (Fig. 4).

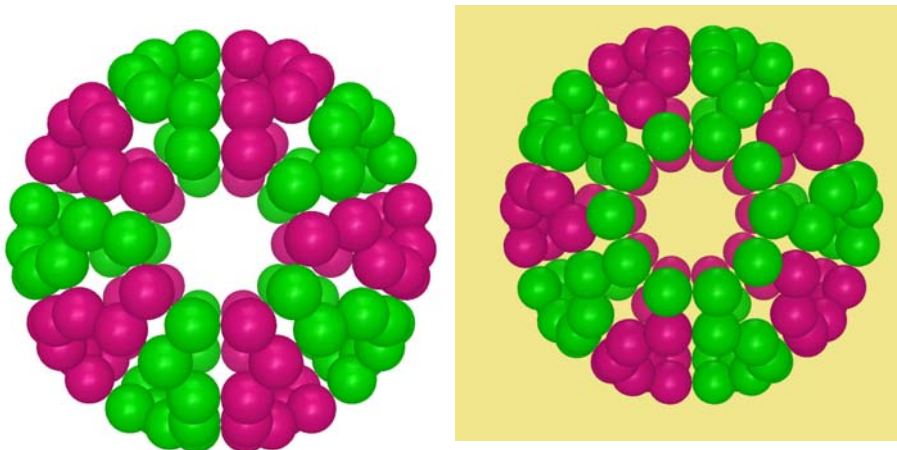


Fig. 3 Swapping colors to create monochrome 10-cycles

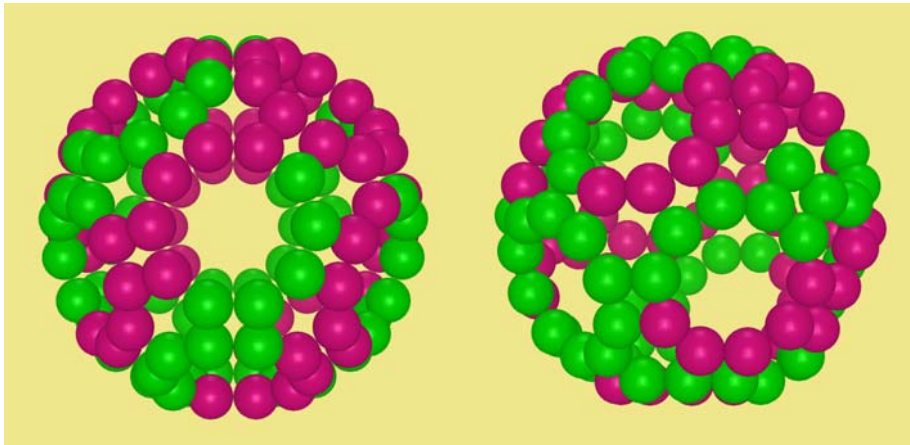


Fig. 4 Each color formed by a single 10-cycle with 5 arms attached

Pattern 3 - Each color consisting of 60 connected spheres with no cycles

Pattern 2 almost depicts a spider, except for the hole at the center. After a slight modification yields a pattern having 2-color symmetry that has no cycles. This pattern suggests the name “the Yin-Yang Spiders” (Fig. 5).

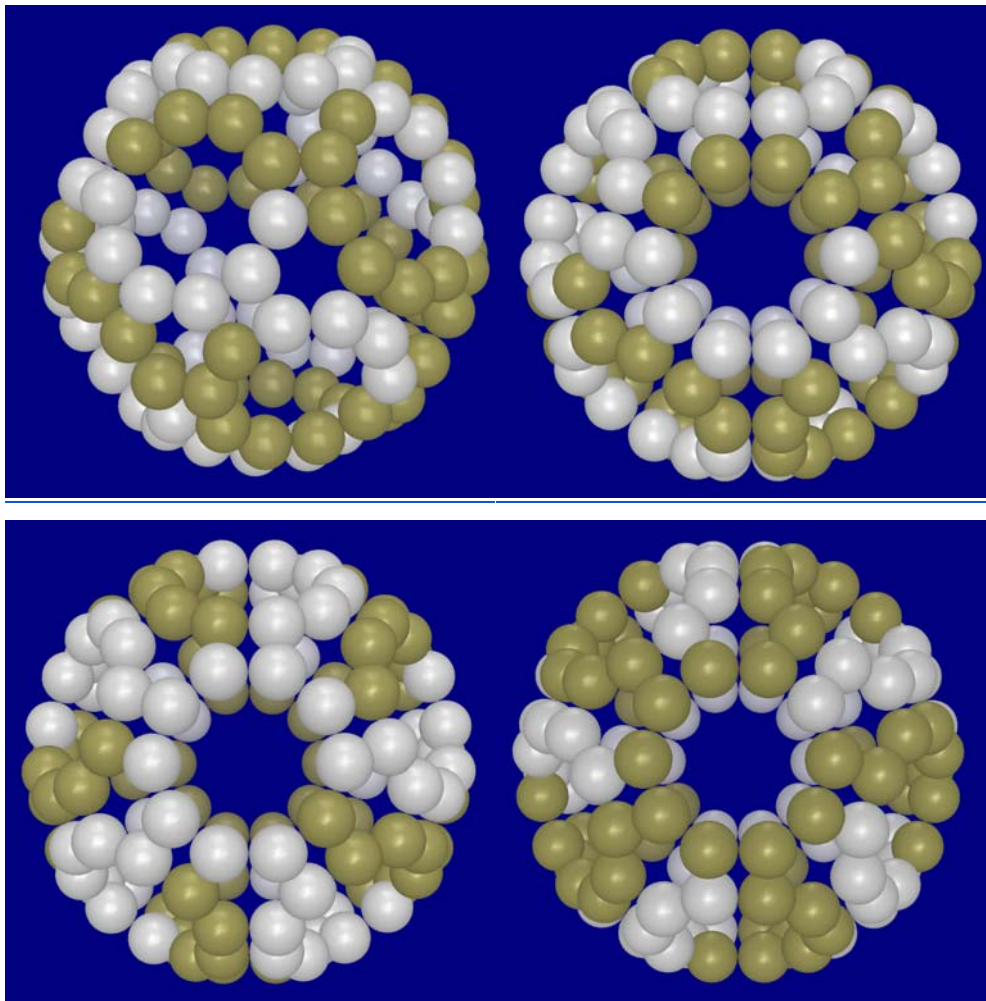


Fig. 5 The Yin-Yang Spider

3-COLOR SYMMETRY

Pattern 1-each color having only a single pair of antipodal 4-cycles

The pattern is suggested by the configuration of the largest cube placed inside a wire-frame dodecahedron.

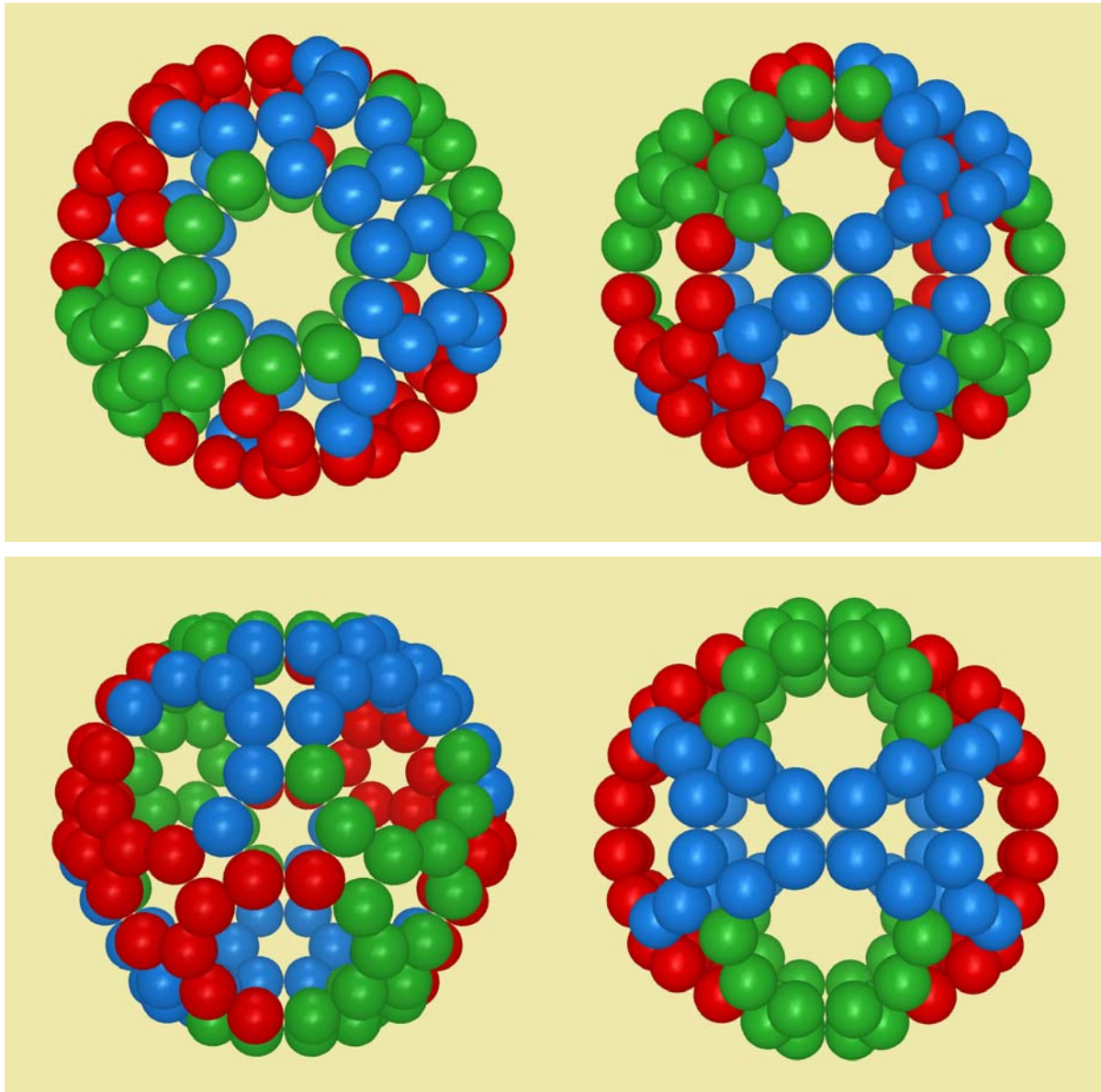


Fig. 6 An octahedral symmetric pattern

Pattern 2- each color consisting of ten 4-cycles

This pattern is a variation of Pattern 1.

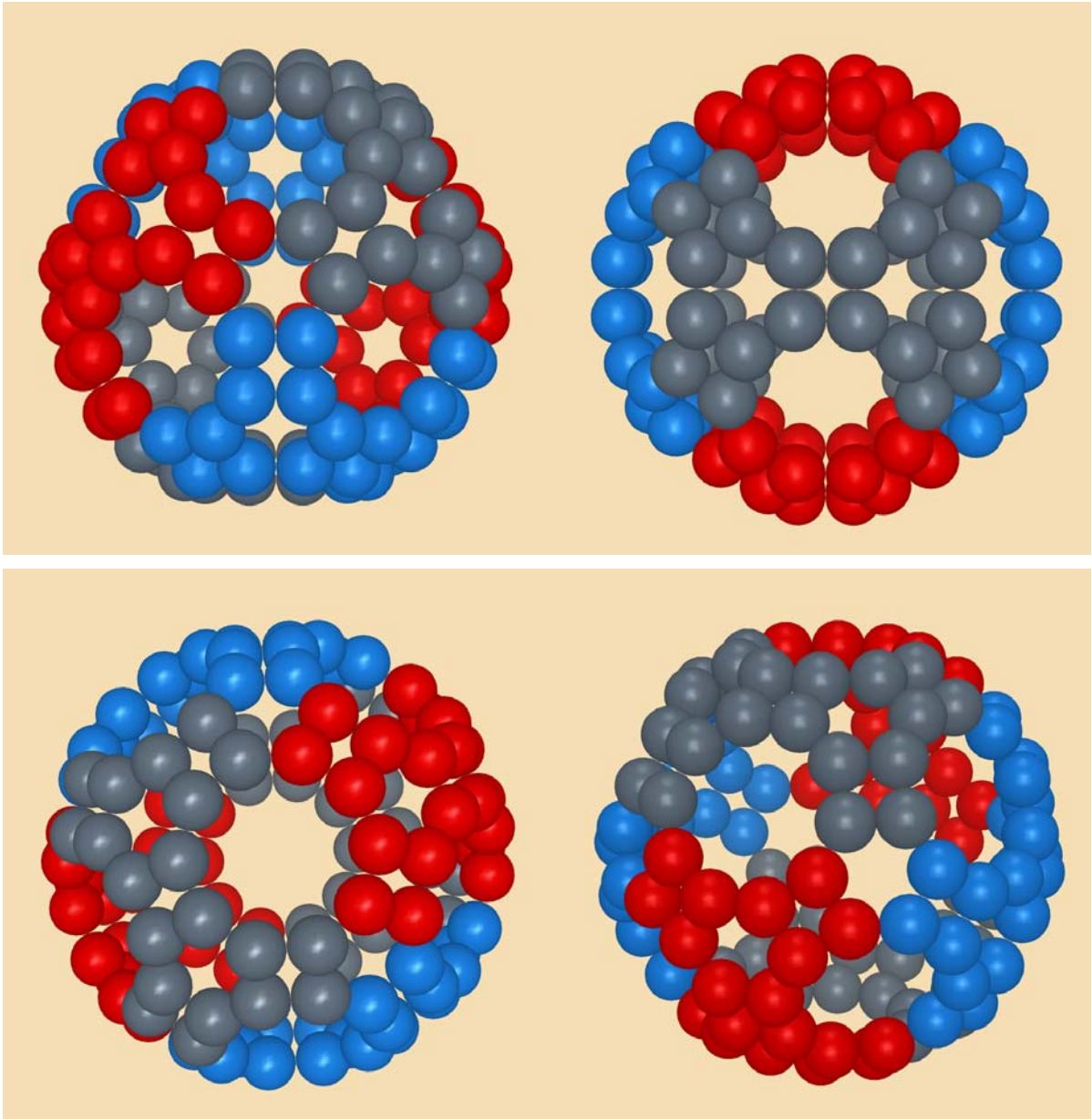


Fig. 7 Another octahedral symmetric pattern

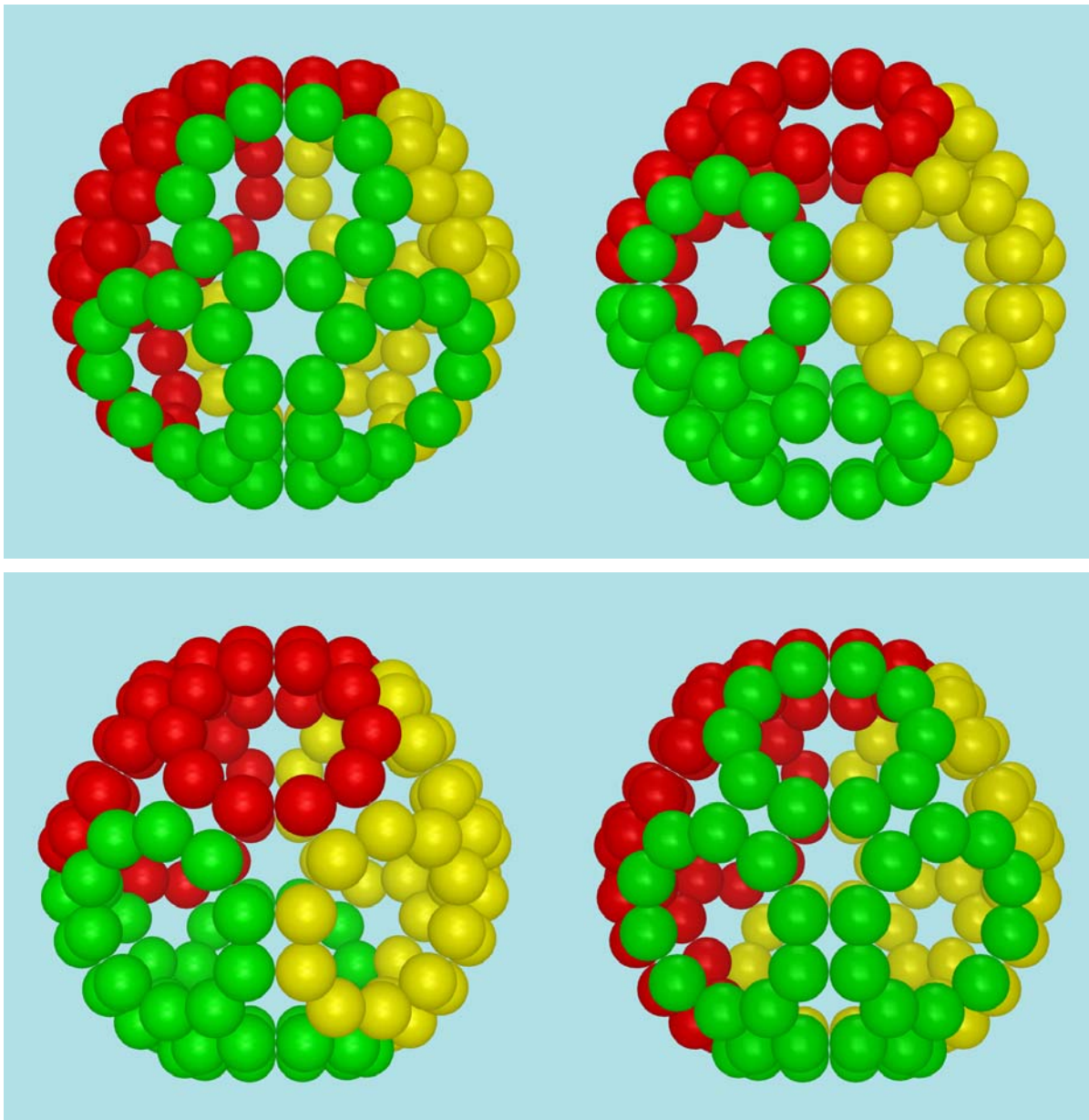


Fig. 8 Each color consisting of four 10-cycles

4-COLOR SYMMETRY

Pattern 1- Spheres of the same color have the graph structure of a tree

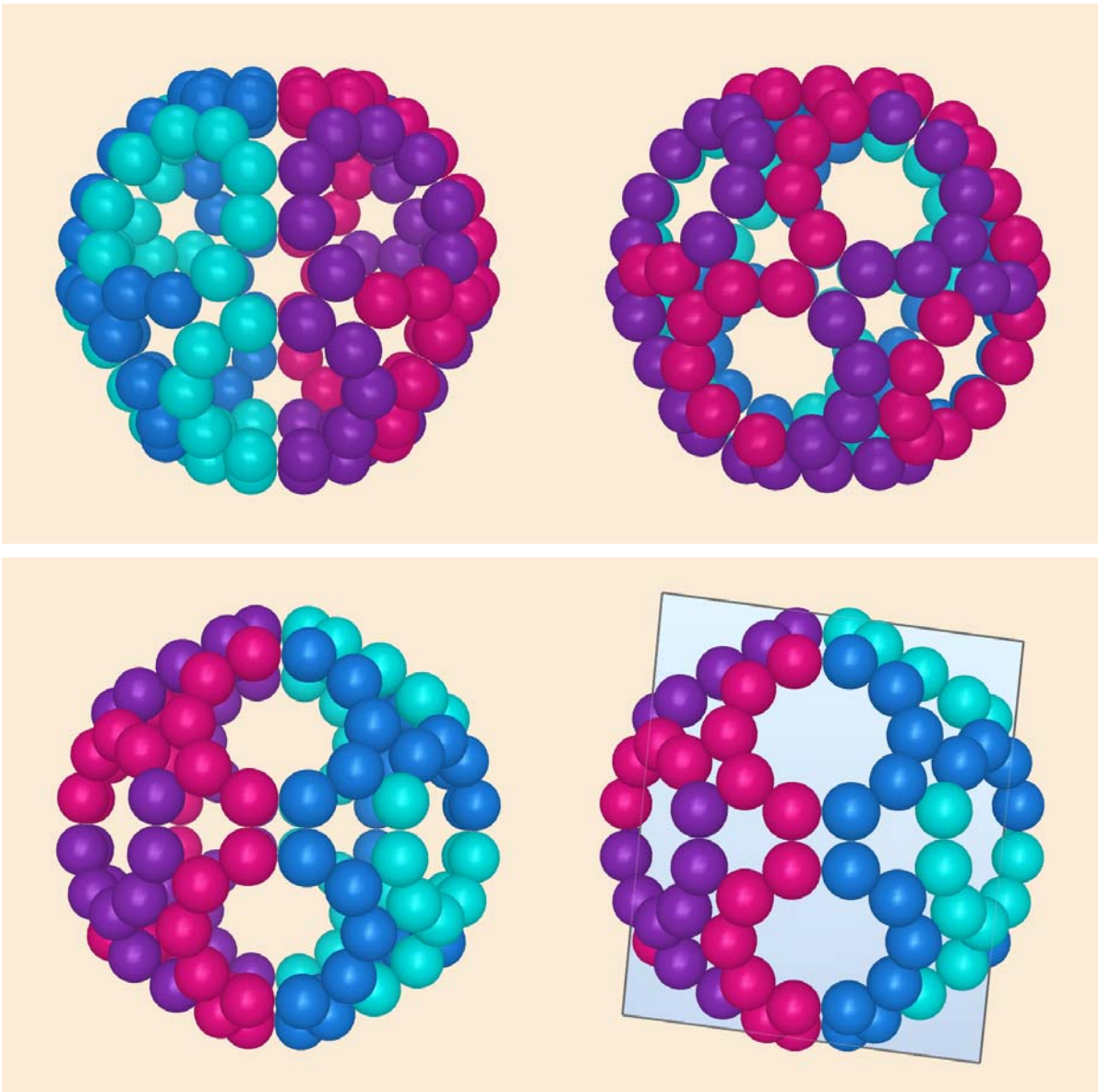


Fig. 9 Each color containing no cycles

Pattern 2- Each set of the same color has 3-fold rotational symmetry

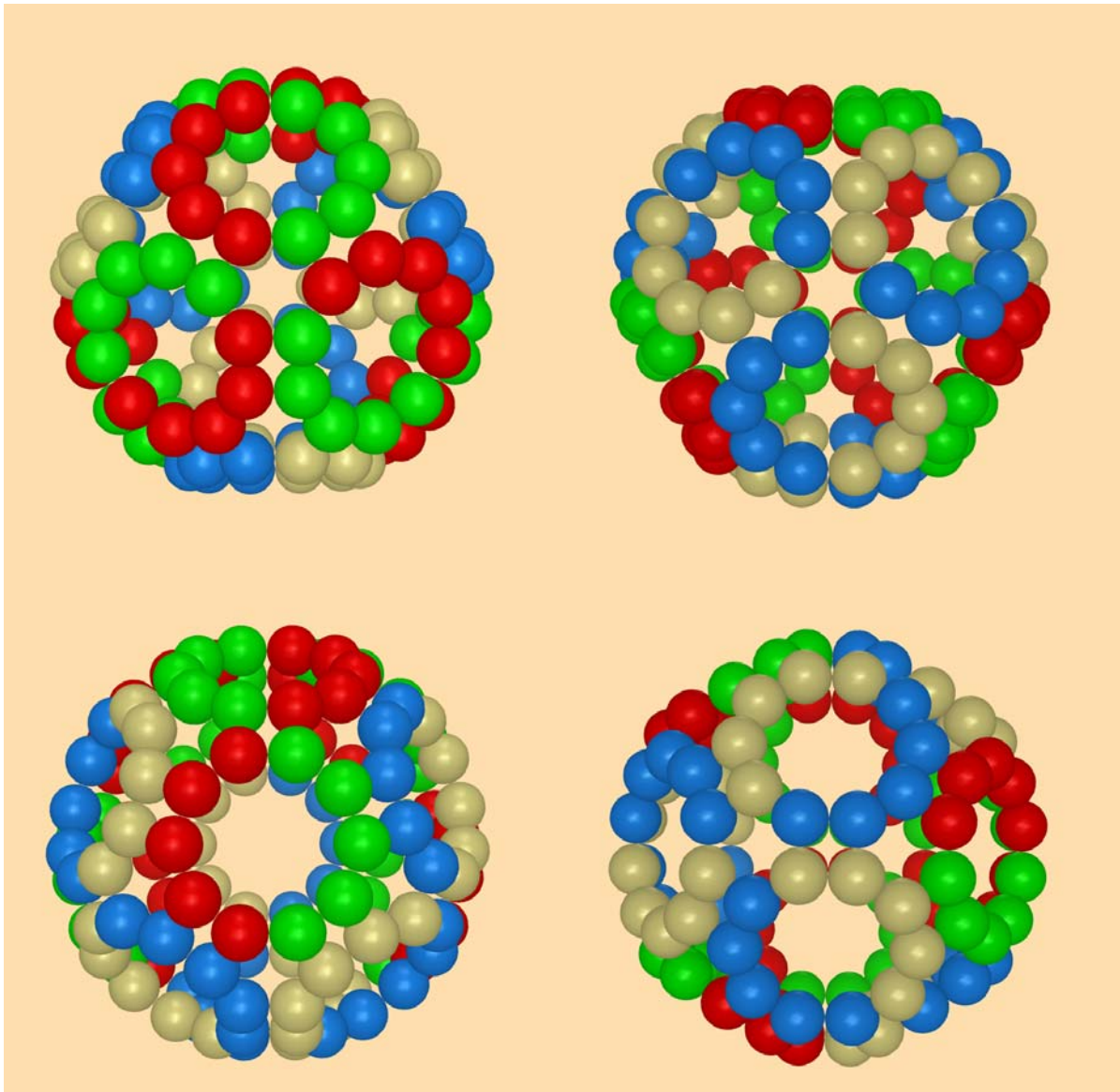


Fig. 10 One-Halves of 10-cycle distributed evenly

Pattern 3- Each set of the same color has no forks

Among 30 spheres of the same color, each touches two others except the two at each end. The author proposes to call it an ATCM Snake.

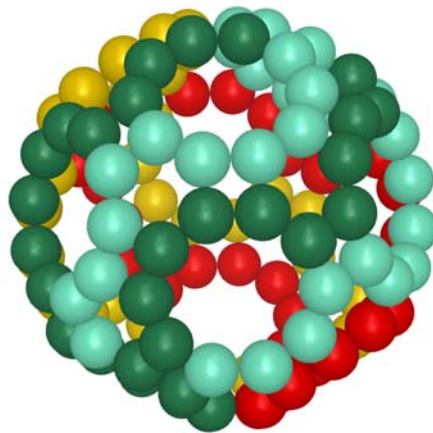
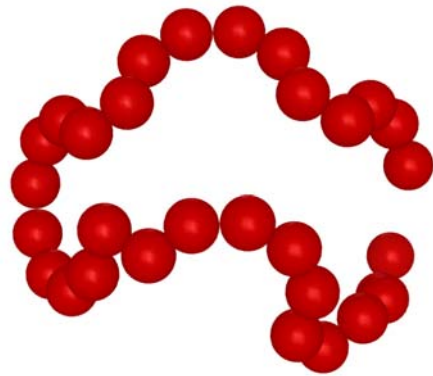
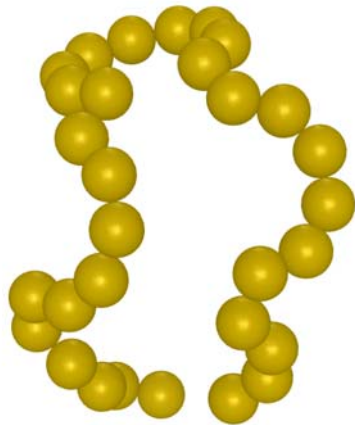


Fig 11 Four snakes covering all

5-COLOR SYMMETRY

Pattern 1-each color formed by 24 connected spheres

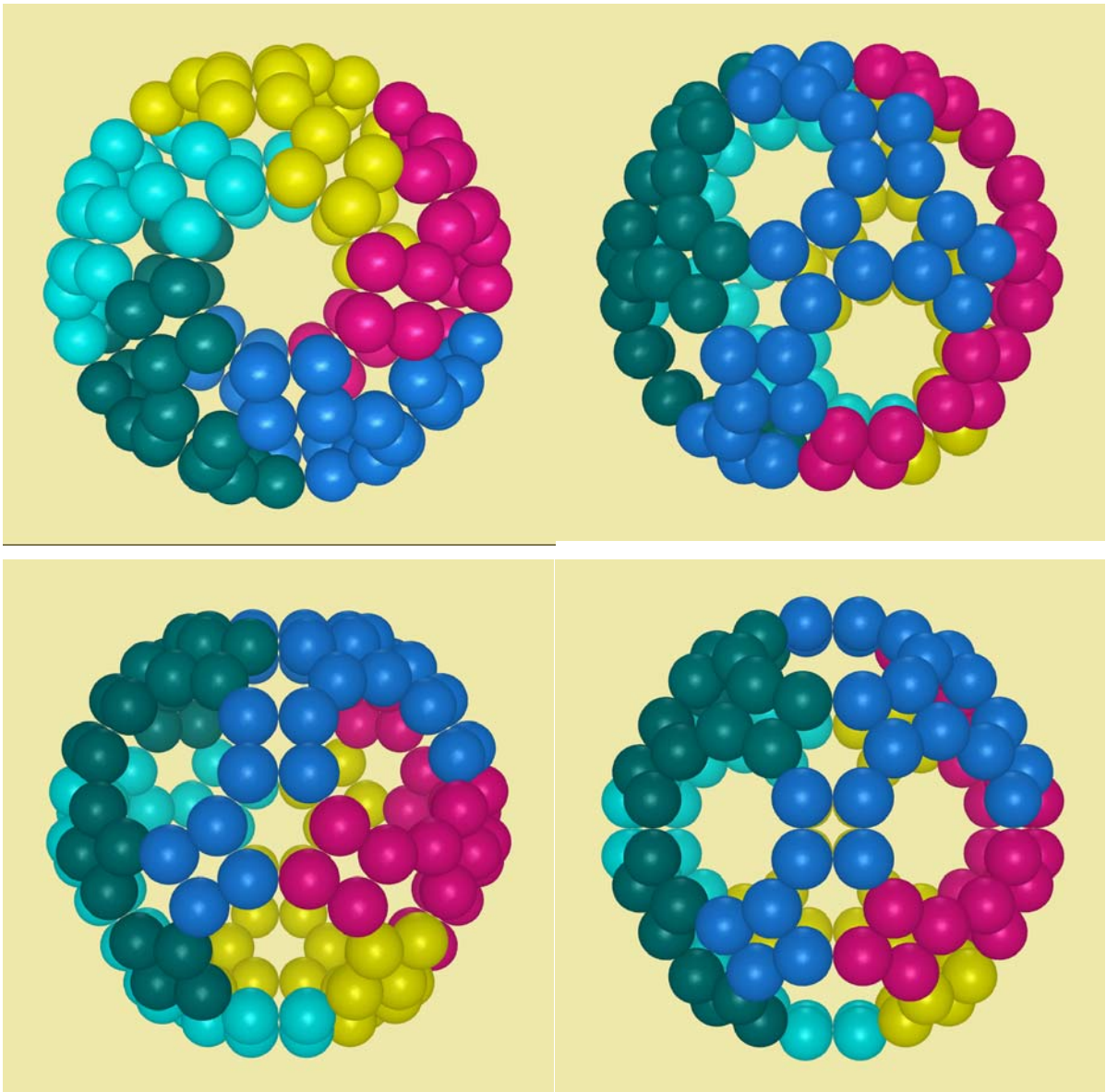


Fig. 12 4-cycles covering all

Pattern 2-each color consisting of an antipodal pair of 3 connected 4-cycles

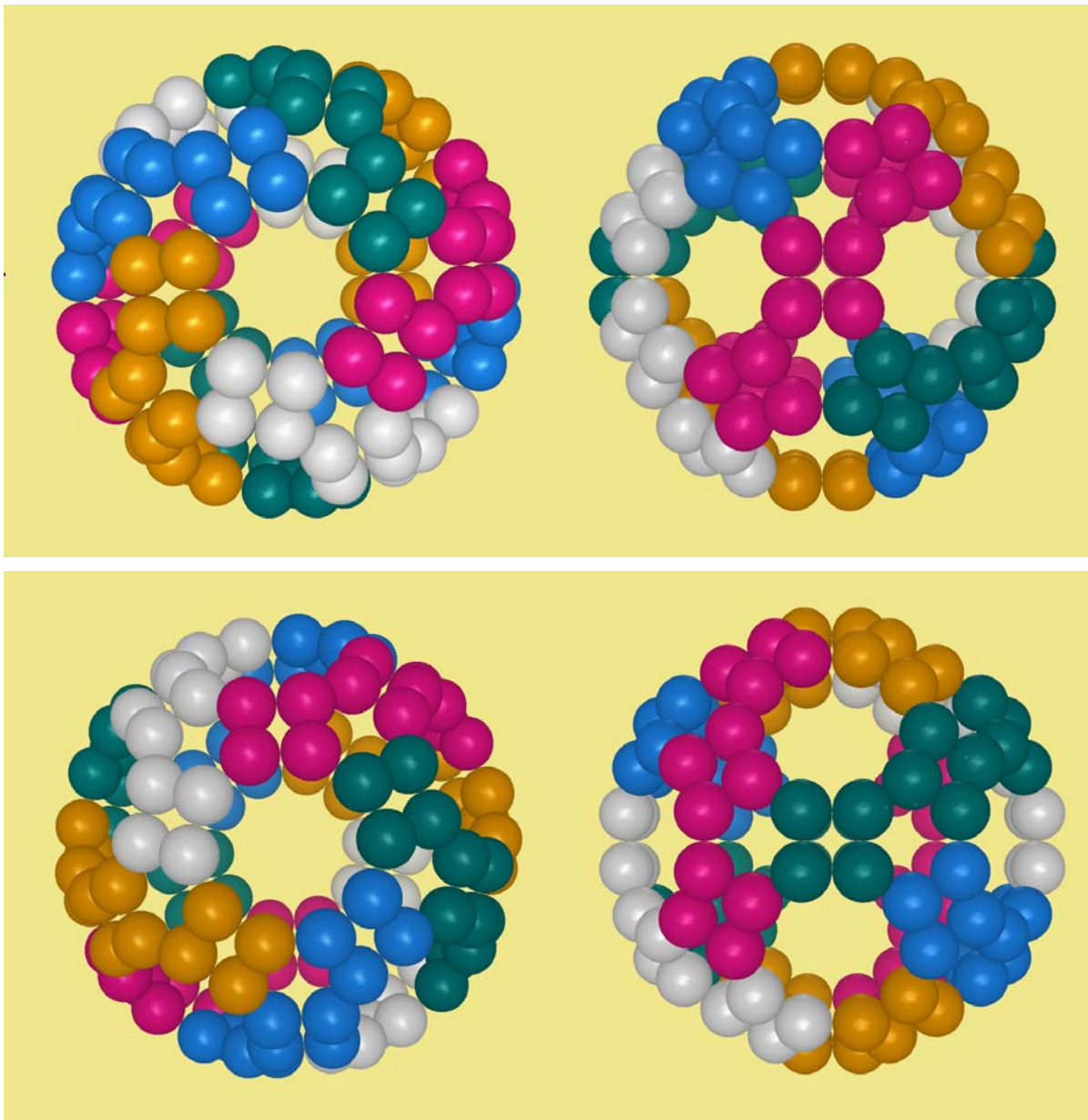


Fig. 13 Art of counting 4-cycles

Pattern 3-each color consisting of four 6-cycles

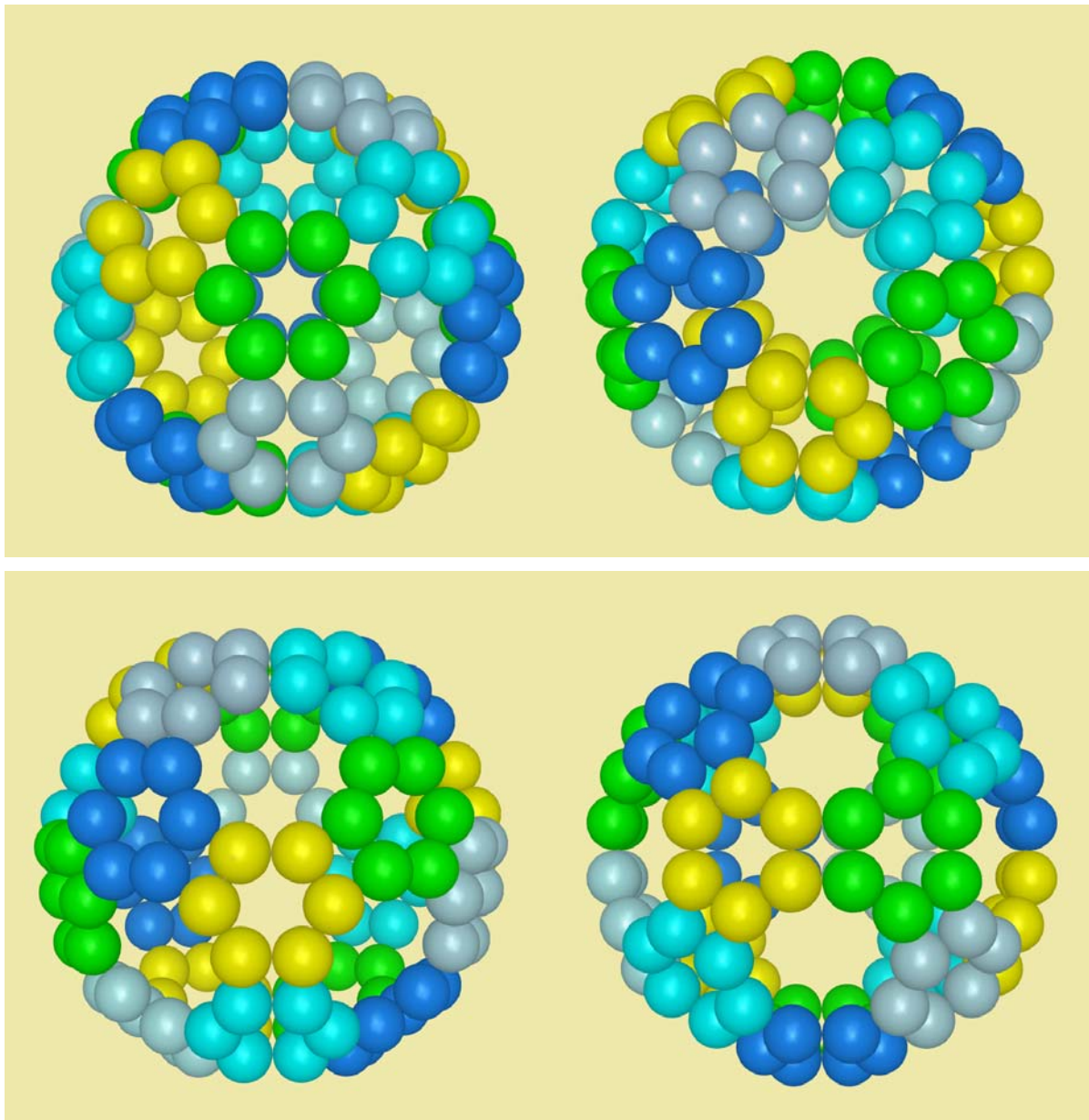


Fig. 14 6-cycle of the same color are centered at vertices of a tetrahedron 5-compound

Pattern 4-each color consisting of six 4-cycles

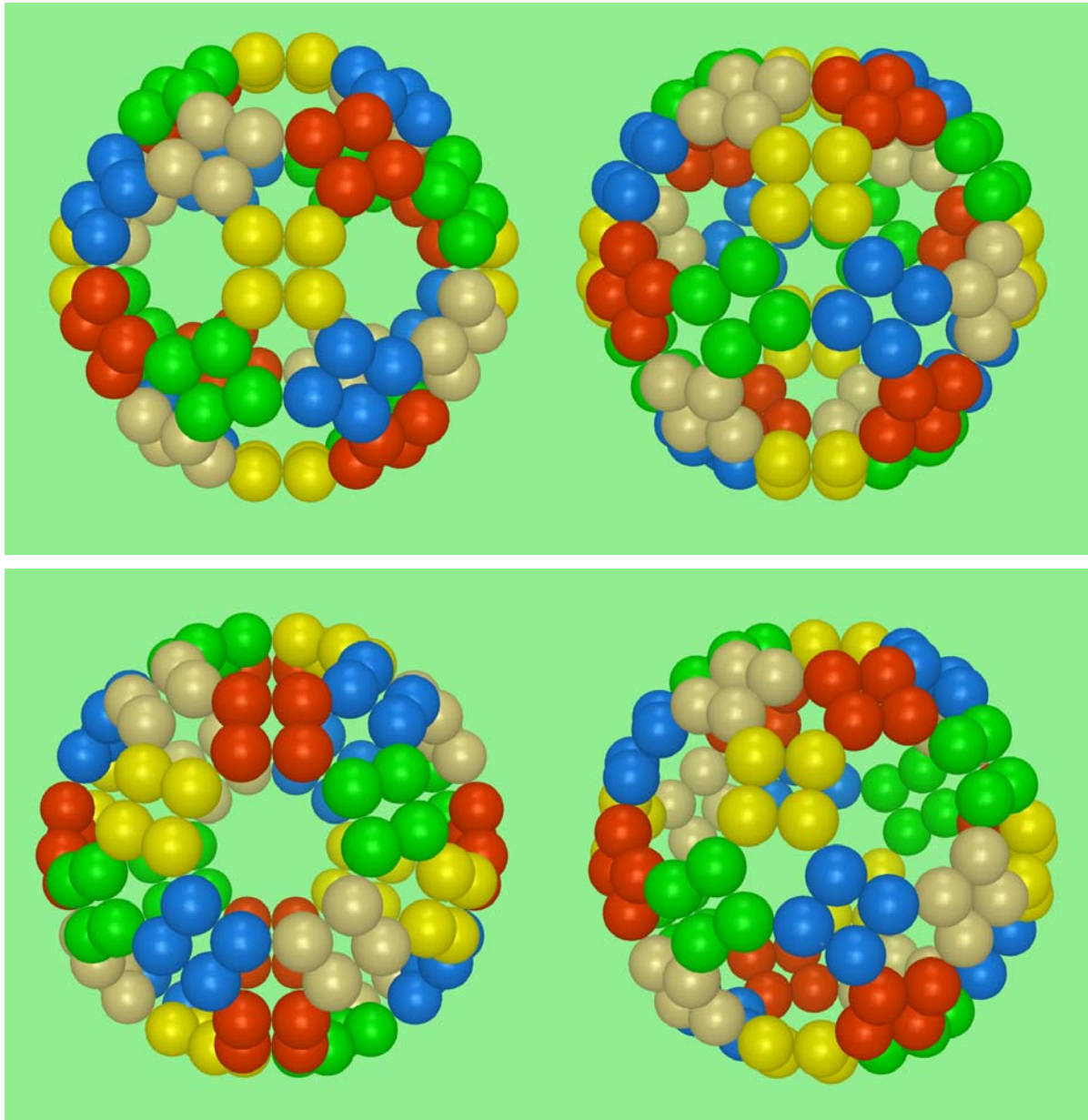


Fig. 15 4-cycles of the same color are centered at a vertex of an octahedron 5-compound

6-COLOR SYMMETRY

Pattern 1- each color consisting of 10 connected couples

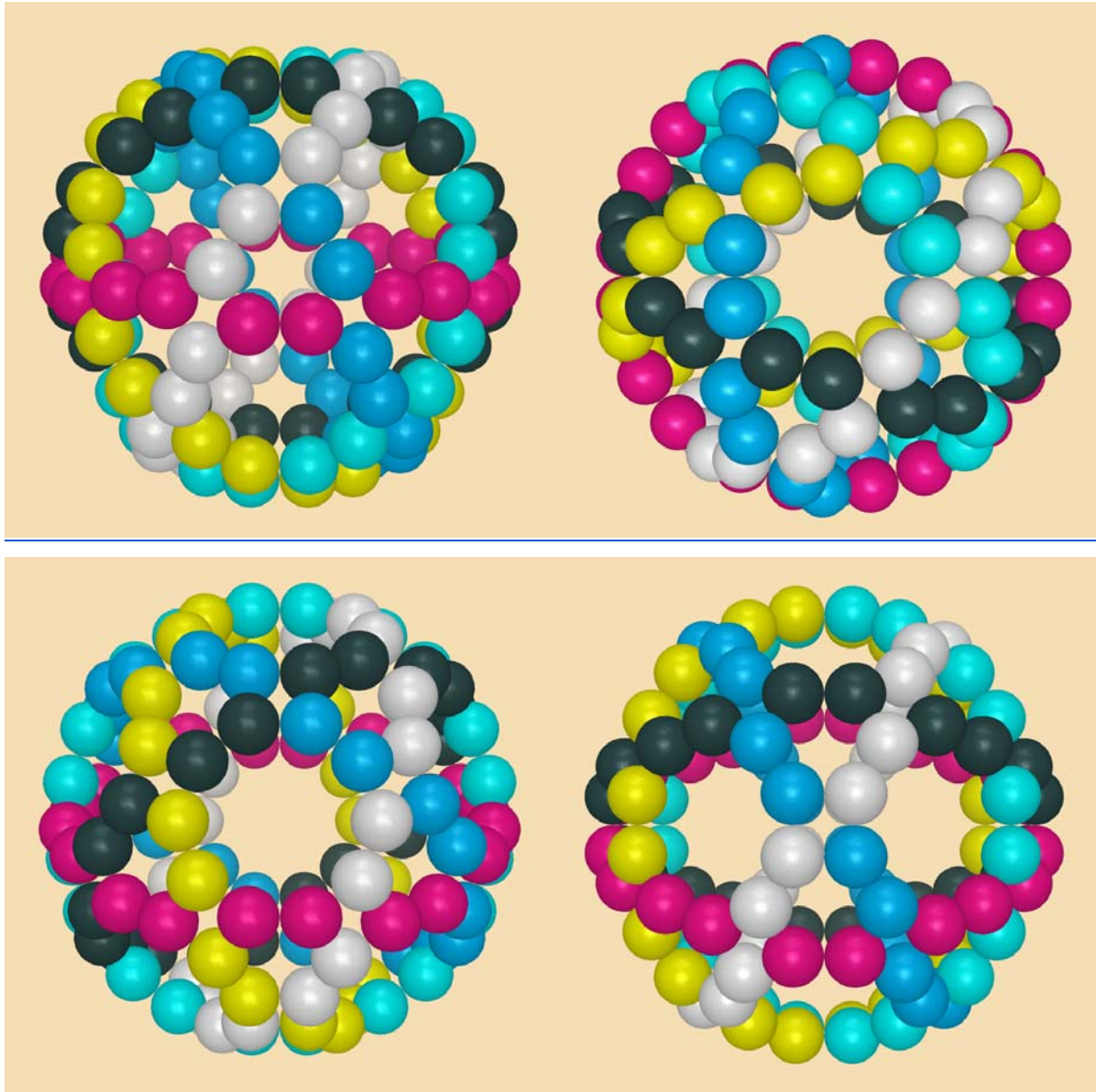


Fig. 16 Ten pairs of kissing spheres near a great circle

Pattern 2 - each color consisting of a pair of connected 10-cycles

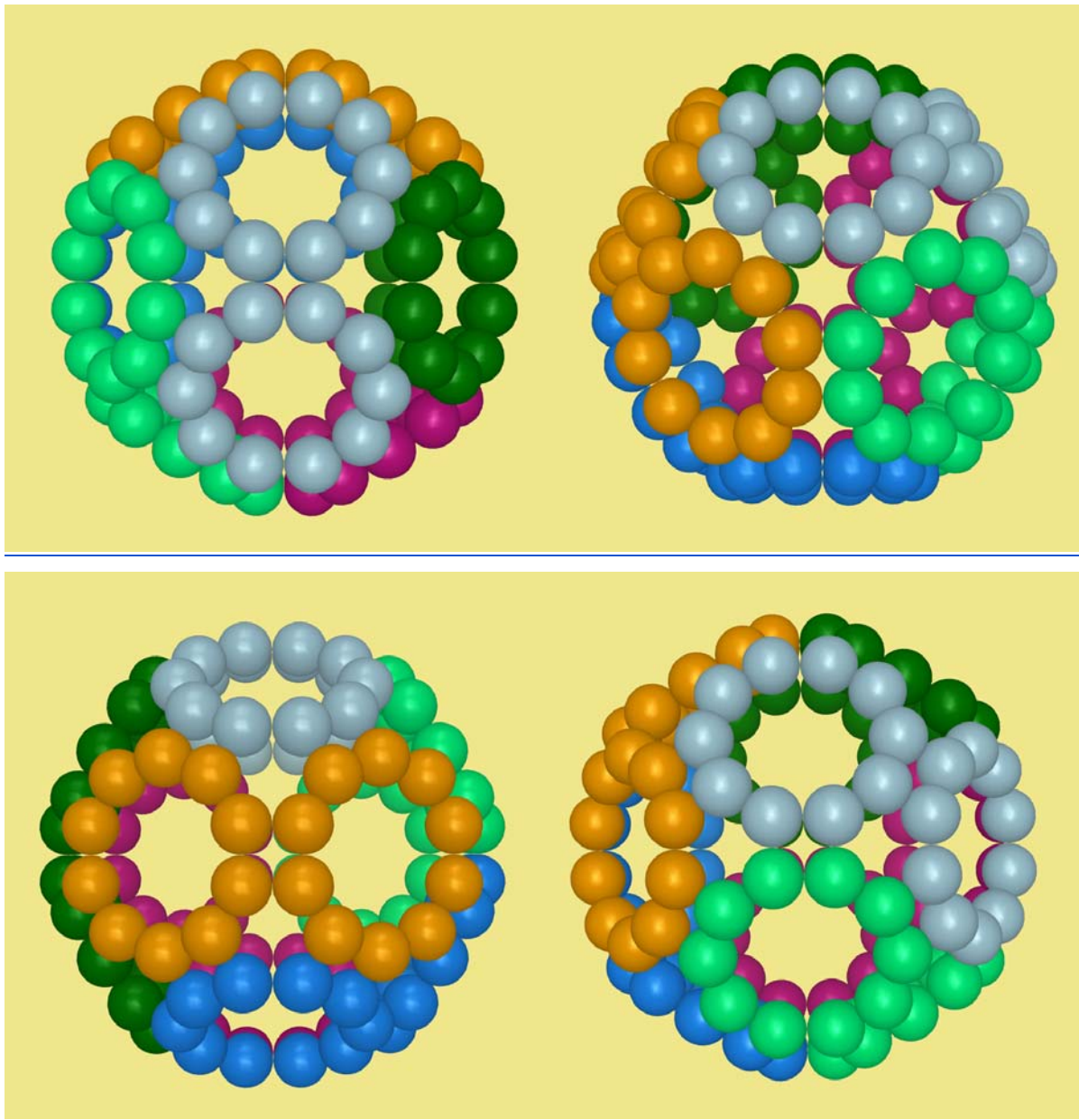


Fig. 17 Asymmetric coloring of 6 pairs

Pattern 3 - each color consisting of two antipodal 10-cycles

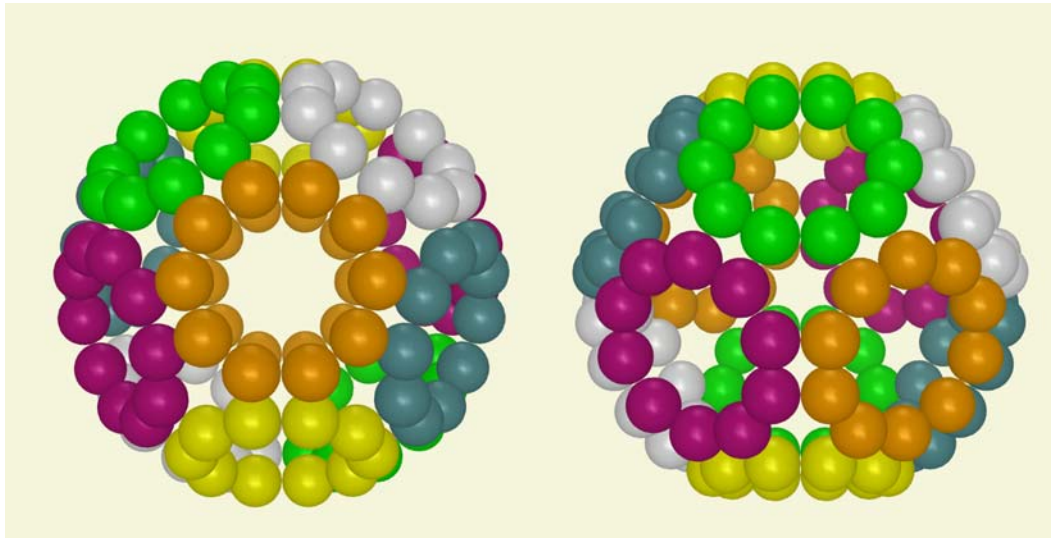


Fig. 18 Six antipodal pairs of 10-cycles

Pattern 4 -120 spheres formed by 6 snakes

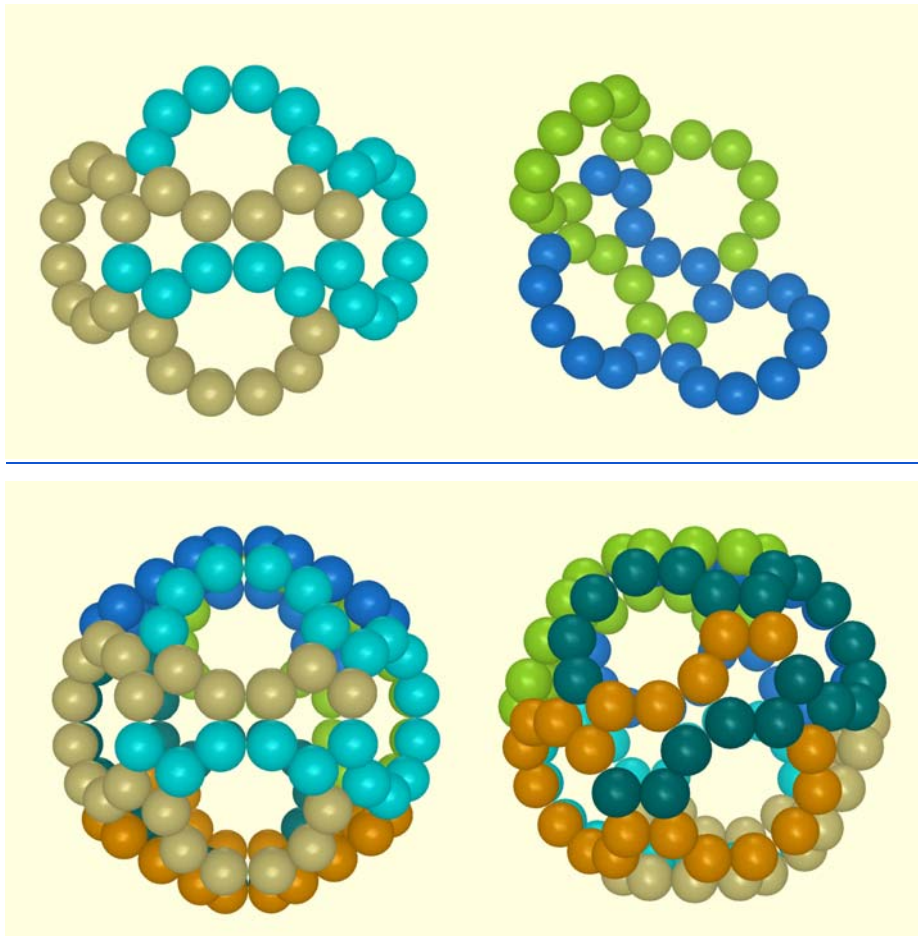


Fig.19 120 spheres formed by 6 snakes

8-COLOR SYMMETRY

Patent 1-coloring by Northants

This appears to be the only interesting pattern, created by dividing 120 spheres into 8 Northants:

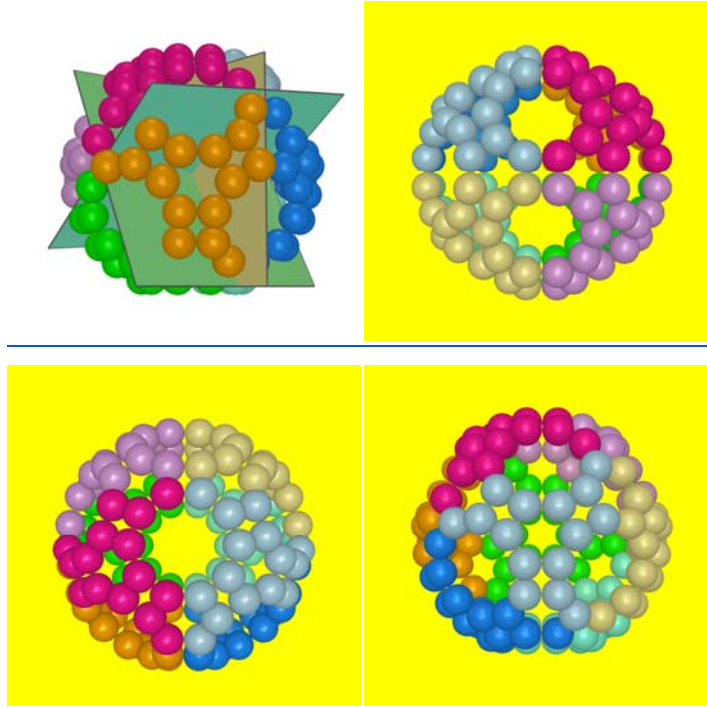


Fig. 20 Same Northants, same color

CONCLUSION

G.H. Hardy once said: “A mathematician, like a painter or a poet, is a maker of patterns.” The reader is invited to download the cg3 file of the monochrome 120-Sphere model as a template to make many more interesting patterns under the Cabri 3D environment.

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https://drive.google.com/drive/folders/1z9C6IibpW8EvSNT024_Eu5WpKdMX76Gs?usp=sharing

Technologies and laboratorial activities for a robust understanding approach: from closed to open laboratories

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Abstract. *The aim of this paper is to show the evolution of a meta-methodology adopted by the authors in some of the activities they proposed over the last 10 years to students and teachers. All the activities, called laboratories, involve the use of technology and are based on laboratorial methodology aiming to a robust understanding of mathematics. We describe, as example, five laboratories and provide a discussion on our meta-methodology as researchers on such laboratories finding a meta-methodological evolution. In particular, by analysing this evolution, we detect an initial attitude towards “closed” laboratories (rigidly structured, letting no freedom of choice for students and teachers) evolving into an attitude to “open” laboratories (letting freedom of choice to students and/or teachers).*

1. Introduction

Mathematics learning in schools is subject to numerous national and international surveys (<https://www.oecd.org/pisa/>; <https://timssandpirls.bc.edu/timss-landing.html>) which often reveal difficulties encountered by students in this discipline. In order to deal with these problems, it is advisable to influence the attitude that students have towards mathematics, which consists of three interacting components: emotional disposition, vision of mathematics and sense of self-efficacy [8; 9]. In particular: the *emotional disposition* is the set of emotions (fear, anxiety, frustration, anger, pride, satisfaction, excitement, joy, to name a few) that are awakened by an activity; the *vision of mathematics* is the set of beliefs the person holds about it; the *sense of self-efficacy* is people’s beliefs in their ability to organise and carry out the actions necessary to deal adequately with the situations one encounters in order to achieve the desired results. Beliefs of efficacy influence the way people think, feel, find personal motivation and act [3]. The stronger the sense of efficacy is, the more vigorous people are in dealing with problematic situations and the more successful they are in changing them.

The improper or negative manifestation of the various components generates in the student a closure towards the discipline which cannot be removed [8; 9]. The teacher can influence this attitude by preparing specific activities aimed at achieving concrete objectives that respect the abilities of individual students and of the class.

A methodology for fostering a positive attitude towards mathematics that has been developed by the mathematics education research community since the beginning of this century is the *mathematics laboratory* [1], which aims to increase students’ sense of self-efficacy by providing a correct view of the discipline and promoting positive emotions.

The mathematics laboratory is a stimulating and engaging environment for the students in which the role of the teacher is fundamental in conveying the activity. We will see in this paper the evolution of the mathematics laboratory in student/teacher training activities, promoted by the Mathematics Education Research Group (MERG) at the University of Catania. These activities are carried out in a STEM (acronym for Science, Technology, Engineering, Mathematics) approach because of a large use of Technology in view of Mathematical learning. In recent years, the activities are more oriented towards a STEAM approach. In the acronym STEAM, evolution of STEM [19; 16], the A stands for “Arts”, meaning not only arts themselves (paintings, literature, ...) but also an artistic attitude: creativity, openness, observation of reality and self-observation, passion, attention to beauty,

In this paper, we ask whether math education researchers can evolve from a STEM to a STEAM approach in designing laboratorial activities in mathematics for teachers and students.

In section 2, we illustrate the theoretical framework, composed of several theoretical lens that frame all the activities: Laboratory of Mathematics and Teaching for Robust Understanding as pedagogical methodology; the Technological, Pedagogical and Content Knowledge as a super-structure which combines pedagogical and content knowledge with technological knowledge; and the Meta-Didactical Transposition, because of the joint and synergistic work of two communities, the researchers and the teachers, in implementing activities for students. In section 3, we describe five of the many activities carried out in the last 10 years, calling them laboratories. Many teaching experiments were done with students and teachers; we chose these laboratories to show the evolution of our methodology as researchers. In section 4 some discussions and conclusions on the laboratories described in section 3 and on their evolution are given. In particular, our considerations involve the methodology adopted by us, as researchers and designers of the laboratories, that we call meta-methodology. It is the result of a reflection that we are able to deduce by considering all the activities that we have been designed over the years. Today, they have undergone a methodological transformation, moving from what we will call *closed laboratories* to what we call *open laboratories*.

2. Theoretical framework

The mathematics laboratory is a “phenomenological space to teach and learn mathematics developed by means of specific technological instruments and structured negotiation processes in which math knowledge is subjected to a new representative, operative and social order to become object of investigation again and be efficaciously taught and learnt” (translated from [6]). At national level, the Italian commission for the teaching of mathematics, back in 2003, suggested: A mathematics laboratory is a methodology, based on various and structured activities, aimed at the construction of meanings of mathematical objects. Such activities involve people (students and teachers), structures (classrooms, instruments, organisation and management), ideas (projects, didactical planning and experiments). We can imagine the laboratory as a learning environment in which students learn by doing, seeing, imitating, communicating with each other, in a word: practising, with the aid of instruments, and interactions between people working together in a collaborative/cooperative modality [1; 23].

In particular, elements (Table 2.1) that characterize an activity in a mathematics laboratory are [22]:

Table 2.1 Mathematics laboratory elements

<i>A Problem to solve (P)</i>	<i>Instruments that can be used/manipulated (I)</i>	<i>Working Method (relationship-interaction) (WM)</i>	<i>The role of the Teacher (T)</i>
<p>The activity in the laboratory starts with a problem. In order to create an atmosphere of research and discovery, the situations proposed must be “new” to the students, they must involve problems that are neither too easy nor too difficult and their resolution must require knowledge they have already acquired.</p>	<p>The activity has to foresee exploration instruments to be manipulated, which, by facilitating the development of thinking, allow to elaborate conjectures, to verify properties, to enhance argumentative and deductive skills, to suggest possible demonstration strategies. These instruments can be either old technology (pen and paper, ruler, compass ...) or new technology (Dynamic Geometry Systems, Computer Algebra Systems, Electronic Sheets, Calculators...).</p>	<p>In the laboratory, students compare ideas, intuitions, arguments, and collaborate/cooperate to achieve results using their critical skills: they explore, formulate conjectures, check their validity and then, eventually, demonstrate them.</p>	<p>It is the teacher's role to guide students to achieve results: he/she validates correct proposals, questions proposals that still need to be improved, encourages students to pursue them, rewards students when they achieve a significant result. He/she also sets the pace, creates a positive setting and is attentive and ready to transform the frustration resulting from failure into a moment of re-evaluation of the objectives set.</p>

In the mathematics laboratory students do not study mathematics, but rather *do* mathematics, i.e. they:

Pose/Deal with a problem, Explore/manipulate, Conjecture, Verify, Prove, Apply.

The mathematics laboratory is therefore suitable for promoting *ambitious* and *robust* teaching, aiming at a deep understanding [25; 26; 27]: in it, students work on a problem (Content), which must be neither too easy because they might get bored, nor too difficult because they might get discouraged (Cognitive Demand), comparing themselves with students from the whole class (Equal Access to Content) and with the teacher (Formative Assessment), affirming and discussing their choices (Agency, Ownership, and Identity). That is, the 5 dimensions of Teaching for Robust Understanding come into play [25; 26; 27], described in Table 2.2.

Table 2.2 The Five dimensions of the TRU Framework (<https://truframework.org/>)

The Five Dimensions of Powerful Classrooms				
The Content (C)	Cognitive Demand (CD)	Equitable Access to Content (EA)	Agency, Ownership, and Identity (AOI)	Formative Assessment (FA)
The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful disciplinary thinkers. Discussions are focused and coherent, providing opportunities to learn disciplinary ideas, techniques, and perspectives, make connections, and develop productive disciplinary habits of mind.	The extent to which students have opportunities to grapple with and make sense of important disciplinary ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding. The level of challenge should be conducive to what has been called “productive struggle”.	The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core disciplinary content being addressed by the class. Classrooms in which a small number of students get most of the “air time” are not equitable, no matter how rich the content: all students need to be involved in meaningful ways.	The extent to which students are provided opportunities to “walk the walk and talk the talk” – to contribute to conversations about disciplinary ideas, to build on others’ ideas and have others build on theirs – in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners.	The extent to which classroom activities elicit student thinking and subsequent interactions respond to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction “meets students where they are” and gives them opportunities to deepen their understandings.

The activities we will present, that foster a Robust Understanding, use technological instruments to manipulate, according to the Laboratory of Mathematics. We strongly believe, in fact, that nowadays the Content and Pedagogical Knowledge have to be joined with a Technological Knowledge, as suggested by Koehler and Mishra [17; 18; 20]. In the Technological, Pedagogical, And Content Knowledge framework (TPACK) they highlight the complex interplay of the three primary forms of knowledge in the learning and teaching process (Figure 2.1).

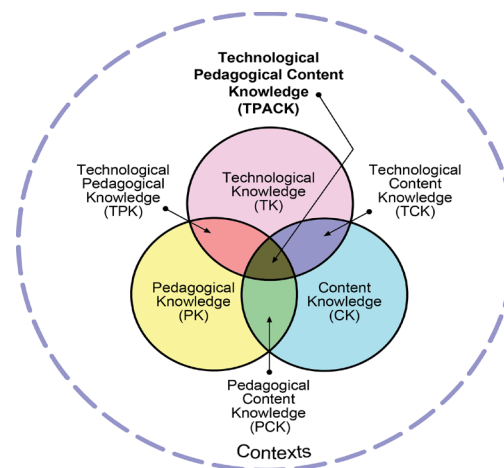


Figure 2.1 TPACK image (from <http://tpack.org/>)

Here we briefly report the description of the three primary components:

- **Content Knowledge:** “Teachers’ knowledge about the subject matter to be learned or taught. The content to be covered in middle school science or history is different from the content to be covered in an undergraduate course on art appreciation or a graduate seminar on astrophysics... As Shulman (1986) noted, this knowledge would include knowledge of concepts, theories, ideas, organizational frameworks, knowledge of evidence and proof, as well as established practices and approaches toward developing such knowledge” [18, p. 63].
- **Pedagogical Knowledge:** “Teachers’ deep knowledge about the processes and practices or methods of teaching and learning. They encompass, among other things, overall educational purposes, values, and aims. This generic form of knowledge applies to understanding how students learn, general classroom management skills, lesson planning, and student assessment.” [18, p. 64].
- **Technology Knowledge:** Knowledge about certain ways of thinking about, and working with technology, tools and resources. This includes good understanding information technology broadly enough to apply it productively at work and in everyday life, being able to recognize when information technology can assist or impede the achievement of a certain goal, and being able continually adapt to changes in information technology [18].

The activities carried out within our research group involve a constant practice: we are used to working in synergy with teachers. Then, two communities emerge, that of researchers and that of teachers. A theoretical framework that describes and analyses the relationship and reciprocal influence between these two communities – involved in a course in mathematics education for professional development, with respect to their professional practices – is the Meta-Didactical Transposition (MDT), a theoretical model elaborated by Arzarello and colleagues [2], expanding on Chevallard’s concept of didactical transposition and praxeology [5].

In the MDT, the researchers have the objective of transposing a certain piece of knowledge, related to the teaching and learning of mathematics, to favour the professional development of the teachers, according to the reference institutions (national curricula, textbooks, ...). In this case, Arzarello and colleagues [2] introduce the notion of meta-didactical praxeologies: they consist exactly of the tasks, techniques, and justifying discourses that develop in teacher education processes. In fact, an educational course generally aims – with the engagement of researchers as trainers – at developing teachers’ existing praxeologies, transforming them into new ones, for example targeted to the introduction of new technologies, or teaching practices, or theoretical frames by research in

mathematics education, or new curricula, and so on, according to the aims of the programme. This evolution in the praxeologies, if happen, is therefore the result of an interaction between the community of researchers and that of teachers. The tangible result of the evolution of teachers' praxeologies is their application in the classroom, with their students, who benefit from the professional development that their teachers have experienced.

3. Mathematics laboratories at MERG

In this section, we describe some of the laboratories that we, as researchers in mathematics education, have proposed over the last 10 years. We will describe 5 activities, one addressed to students (Laboratory 1) and the others also to teachers (Laboratories 2-5).

The activities are framed in the TPACK as described in Table 3.1, differentiating in Content and Technology.

Table 3.1 Laboratories in the TPACK framework

	Technology Knowledge	Pedagogical Knowledge	Content Knowledge
Laboratory 1	yEd, Icosien game	Mathematics laboratory; TRU framework	Elements of Graph Theory
Laboratory 2	GeoGebra		Surprising properties of centroids
Laboratory 3	GeoGebra, Excel		Brahmagupta theorem and consequences
Laboratory 4	Moodle platform; MathCityMap		Outdoor math trails
Laboratory 5	GeoGebra classroom		Irrational numbers and connection with philosophy

3.1 Laboratory 1: Elements of Graph theory

Laboratory 1 was part of the *Percorsi per le Competenze Trasversali e l'Orientamento (PCTO)* project, promoted by the Italian Ministry of Education and compulsory for all high-school students, which aims to bring students closer to the world of work.

Several laboratories dealing with graph theory topics have been experimented, from primary school to secondary school [10]. Here we focus on a proposal carried out at secondary school that involved students of the first two years (4 meetings of 2.5 hours each). The course aimed, on the one hand, to introduce basic concepts of graph theory up to Eulerian graphs and Fleury's algorithm and, on the other hand, to develop skills in modelling problems through a graph; in expressing conjectures, arguing them, comparing one's own hypotheses with one's peers in order to reach shared results; in reflecting on the fact that mathematical objects are “hidden” in various situations and objects of everyday life.

Concepts were always introduced by first dealing with a list of problems. For example, problems that can be schematised by means of a graph were initially proposed, before the students knew the

mathematical topic of graph (Problem 1), or Eulerian graphs were presented by first solving Problems as Problem 2.

Problem 1. The coach in the ball

Few weeks ago a football tournament between the following schools started: Archimedes, Descartes, Euclid, Fermat, Pythagoras, Thales. The rules of the tournament provide for a single round of matches, i.e. each team meets all the others only once.

The following matches have been played so far: Archimedes-Thales, Euclid-Archimedes, Thales-Pythagoras, Thales-Cartesian.

How can you summarise and clearly express the current situation of the tournament?

Please express the situation at the end of the tournament succinctly and clearly.

Problem 2. The Pentagon's routes and...

a) Can you draw a pentagon and all its diagonals without ever lifting the pencil from the paper or going over the same line twice?

b) Can you draw a square and all its diagonals without ever lifting the pencil from the paper or going over the same line twice?

At the end, students are guided to recognise the characteristics of Eulerian/Semieulerian graphs and to use Fleury's algorithm to solve problems. The activity ends with a real problem that deals with routes of airlines companies: Preamble: *Air transport is a complex business. It involves major investments (aircraft and maintenance infrastructure), highly qualified staff (pilots and flight attendants) and precise information in real time (reservation systems, for example). The costs associated with "air traffic" are huge and waste must be avoided. For example, an aircraft on the ground does not provide any revenue, so the amount of time each aircraft is stationary must be reduced. To this end, some airlines identify routes and design circular routes for individual aircraft. A circular route is defined as one that covers all routes, once and only once.* Problem: *Can you help the Eurofly company to organise its routes?* (and a table containing the covered routes was given).

During the activity, students played an online game (Icosien), experiencing the difficulty of solving a problem (in this case finding Eulerian and semi-Eulerian paths) without having a mathematically founded strategy (up to date this game is no longer available). They also used the yEd software (<https://www.yworks.com/products/yed>), through which graphs can be drawn, manipulated and analysed.

3.2 Laboratory 2: Magic of centroids

Laboratory 2 was part of the *Piano Nazionale Lauree Scientifiche (PNLS)* project, promoted by the Italian Ministry of Education, which plays an important role in teachers training and students' orientation and self-assessment in scientific subjects. The activities of the PNLS are laboratory activities [1], designed and carried out jointly by teachers and university researchers. This joint work of teachers and researchers is the element that characterises the PNLS and promotes the development and strengthening of relations between the school and university systems.

During this Laboratory, some properties of centroids of geometric figures, such as triangles, quadrilaterals and tetrahedra are investigated. In particular, the properties are proved by means of geometric transformations and by introducing extensions of triangles and quadrilaterals, i.e. by adding one, two or three new vertices to the figure [13].

Laboratory 2 is structured in two steps: (1) design of laboratories and preparation of materials necessary for the construction of laboratories; (2) implementation of laboratories in class. Step 1 and 2 were carried out both as mathematical laboratory type activities: the first one for teachers and the second one for students. So, a so-called *double-laboratory* was proposed [11]. This terminology comes from us of the MERG.

Both steps were arranged so that teachers first and students then were just guided, in order to become independent in developing the activity.

We underline that, in Step 1, teachers and researchers decided to structure the work on the use of GeoGebra and through worksheets allowing students to work independently. These worksheets were elaborated from a sample worksheet prepared by the researchers. Worksheet had a tabular layout with two columns: the left column indicated the action that was then made explicit in the right column [12]. Some of the actions that have been used in the worksheets are: Construction, Exploration, Definition, Observation, Control... Most of the actions are carried out on GeoGebra. Technology, and GeoGebra in particular, help teachers in implementing the “explore-discover-test-conjecture-proof” model [12] at every level. The use of Dynamic Geometry Systems “has totally changed the way Euclidean geometry can be studied. With a minimal introduction, students may explore and discover dynamically relevant properties rather than being told about them” [21, p.1]. For example, we ask students to explore the following situation. Let ABCP and ABCQ be two pyramids with the same base ABC. Referring to the Figure 3.2.1, let $P_1P_2P_3$ be the triangle with vertices the centroids of the faces of ABCP, and let $Q_1Q_2Q_3$ be the triangle with vertices the centroids of the faces of ABCQ. Students are asked to explore and conjecture on the two triangles also referring to the medial triangle of ABC (Table 3.2.1).

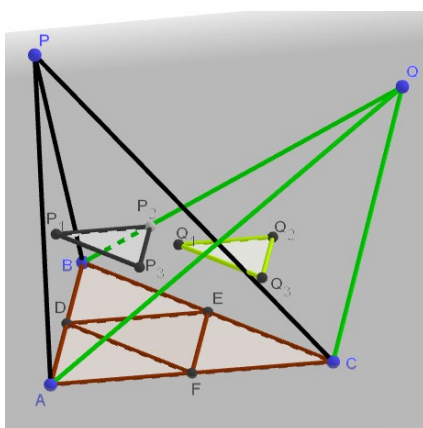


Figure 3.2.1 Centroids' triangles

Table 3.2.1 Exploring centroids' triangles

Exploration	What relationships do you think there are between the sides of $P_1P_2P_3$ and those of $Q_1Q_2Q_3$? To check your assumption, use the appropriate tools in GeoGebra or the Algebra view. Move the points P and Q, the property still holds?
Conjecture	Let us call <i>centroid triangle</i> the triangle with vertices the centroids of the faces of a pyramid with triangular basis. The centroid triangles of two pyramids with the same triangular basis are

Table 3.2.1 is only part of a longer activity where students are guided to construct the objects in Figure 3.2.1 and to explore them, in order to conjecture first and prove afterward that

Given a pyramid with vertex P and base a triangle T , the medial triangle of T is the correspondent of the centroid triangle of the pyramid in the homothetic transformation with centre P and ratio $3/2$.

3.3 Laboratory 3: Brahmagupta theorem and consequences

Also this laboratory, as Laboratory 2, is a double-laboratory carried out within the *PNLS* project: teachers are first called to work together with researchers in writing the worksheets, and then students work in class. The laboratory activity is aimed at students in their second year of high school and is based on a geometric problem already posed in 600 AD: Brahmagupta's theorem [7]. It aimed to cover the topic in modern terms through the use of GeoGebra, in order to renew students' interest in the study of geometry through an innovative experience.

In the first laboratory, teachers and researchers design and create digital worksheets, implemented in Excel, at various levels of difficulty: *Helps* and *Enhancements* are provided on the worksheet and can be viewed by the student if necessary (Figure 3.3.1a). In this way, each student can customise his/her own learning path, proportionate to his/her skills and abilities. These sheets allow a formative evaluation to be carried out, since the teacher can trace the path taken by each student. The student can also carry out a self-assessment of the course he/she has completed (Figure 3.3.1b).¹

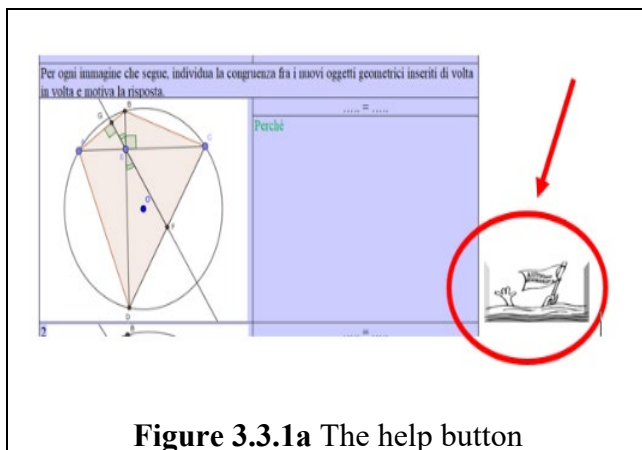


Figure 3.3.1a The help button

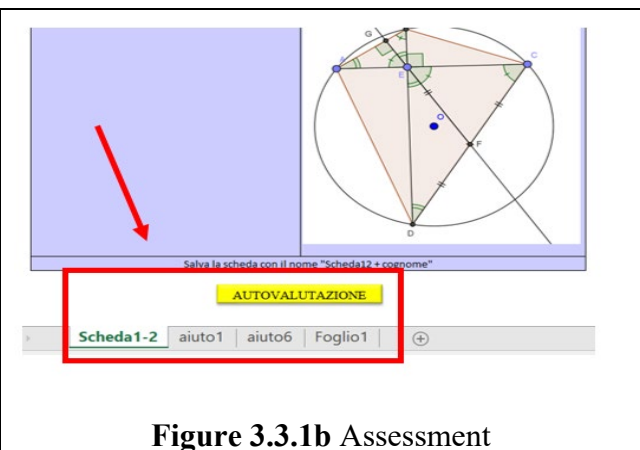


Figure 3.3.1b Assessment

3.4 Laboratory 4: A MOOC for mathematics teacher education

The Laboratory 4 is part of the European project, MaSCE³: “Math Trails in School, Curriculum and Educational Environments of Europe” (<http://masce.eu/>). The project promotes the adoption of math trails, a collection of tasks, located with walking distance, that are useful to discover and solve mathematical problems on real objects [28]. Math trails can be used in the school contexts to offer a real life experience besides textbooks [24]. Nevertheless, they require teachers' preparation and a solid post-processing in the classroom. MathCityMap (MCM, <https://mathcitymap.eu/en/>), a math trail management system, facilitates this process by the benefits of technology: on MCM users can create tasks and trails and share them among themselves or with the public [15]. In fact, it is already

¹ The oval in Figure 3.3.contains the picture of the Help button. The rectangular box in Figure 3.3.1b specifies the number of the worksheet (1-2 in this case) and the Helps that have been used (1 and 6 in this case).

possible for teachers to create trails in a web portal and for students to run them on a smartphone app. Studies show that MCM math trails have a positive effect on student motivation and learning when run regularly [4]. However, math trails are still used sporadically by teachers and not systematically within learning curricula. Within the MaSCE³ project, in order to make teachers autonomous in the use and creation of math trails with their own students, a 12 weeks MOOC was delivered online, in English, via the DI.MA. platform (<http://dimamooc.unict.it/>) managed by the University of Catania.

During the MOOC, the enrolled teachers, by means of specific digital resources (videos, tutorials, ...) received indications on how to design math trail tasks on MCM. In particular, in addition to the types of format of a math task (for more details see: <https://mathcitymap.eu/en/the-mathcitymap-task-formats-2/>), specific design criteria were provided that had to be respected to produce a suitable MCM task. The criteria indicated to the teachers were as follows [14]:

- *Clarity*: For each task, a picture must be created that allows the clear identification of the situation or the object the task is about.
- *Presence*: The task can only be solved on site, i.e. the task data must be collected on site. This also means that the picture or the task description must not be sufficient to successfully solve the task.
- *Activity*: The person who solves the task must be active and do something (e.g. measuring or counting).
- *Multiple solutions*: The task should be solvable in different ways.
- *Reality*: The task should be application-oriented, realistic and not too contrived.
- *Graduated hints*: At least two hints should be added to each task.
- *School mathematics and “tags”*: The task should have a clear relation to school mathematics: Use the prepared tags or add new terms. The task should also be assigned to a class level.
- *Solution formats*: The solution of the task should be presentable as a solution interval (good and medium interval), as an exact number, as multiple choice or as a GPS task.
- *Tools*: No special tools should be required to solve the task.
- *Sample solution*: One should offer a solution and hints (only visible in the portal) for teachers.

The design of the tasks took place on the MCM web portal, filling in a template structured in the light of the design criteria. For the final homework, teachers had to run their own math trail with their students and report back on this experience.

3.5 Laboratory 5: Order and Disorder.

Laboratory 5 is part of the *Liceo Matematico* project. This project aims to enhance mathematical skills in high-school students, through laboratorial activities, pointing mathematics as a glue among different subjects. In particular, here we refer to a module entitled “Order and Disorder”, aiming to underline connections between mathematics and philosophy. The laboratory we present here is again a double-laboratory. The teachers’ laboratory (4 meetings of 2 hours each) was held by the researchers at distance on an online collaboration app, because of the COVID-19 pandemic situation. The mathematical content was irrational numbers, and aimed to make students owners of *reductio ad absurdum* proof. The students’ laboratory started with a philosophical text, orienting students to the questions “Who am I? Am I ordered or disordered?” and “When can we define that a number is disordered?”. The laboratorial methodology in class was used not only for the mathematical part, but also for the philosophical one, because classes were set as research communities on the Socratic model, typical of the ancient mathematical and philosophical Greek schools. There was not a telling of history of philosophy, but rather students philosophized themselves on the topic of Order and Disorder, and thought about irrational numbers, originally seen (by Pythagoras) as disordered

numbers. The initial task of searching a “definition” of Disorder was faced by analysing several situations, from paintings (Velázquez’ and Picasso’s “Las Meninas”) and geometry in Arabian ceramic (a mosaic in Samarcanda, where many irregular polygons are arranged for a final regular structure), to nature (irrational numbers useful in plants to guarantee light to all the leaves). Then, the attention was focused on the geometric construction of the square root of two, by using GeoGebra to manipulate the diagonal of a square, in phase of conjecturing that the number $\sqrt{2}$ has an infinite number of digits without a repeating sequence. Students worked at distance in the *GeoGebra classroom* platform (implemented by the GeoGebra developers during the spring 2020 in view of distance teaching). Thanks to this environment the teacher can see in sync how students are working. While working with GeoGebra, students had ideas exchanges with classmates and the teacher. Then teachers and students read an excerpt of Plato’s “Menone”, in which a maieutic process is used by Socrates to prove that everyone (even the slave of the famous dialogue) can learn something originally unknown (the irrationality of $\sqrt{2}$, this time). Finally, students were guided to the famous proof of irrationality of $\sqrt{2}$, made by *reductio ad absurdum*, again in the *GeoGebra classroom* platform. During the activity, students were invited to express their own thoughts and feelings on the theme of Order and Disorder with poems, drawings or whatever they wanted. We note that teachers gave the possibility to use both technological and classical tools, but students preferred to use paper. In Figure 3.5.1 and Figure 3.5.2 two of their works.



Figure 3.5.1 My Order and Disorder



Figure 3.5.1 My Order and Disorder

The whole materials (considerations about paintings, mosaic, irrationality in nature, reads, *GeoGebra classroom*'s activities) were prepared by the researchers. But teachers (both mathematics' teachers and philosophy's teachers), who attended the first laboratory of the double-laboratory were free to adapt the contents to their classes, given only the indication to use laboratorial methodology. During the first laboratory, teachers were open and interactive, and proposed some changes and improvements of the second laboratory.

4. Discussion and conclusions

The proposed activities, when experimented in the classroom, from a methodological point of view, are based on the *mathematics laboratory*: Students *work together* (WM in Table 2.1) on *problems* (P in Table 2.1), discover concepts by themselves, experiment and manipulate instruments (I in Table 2.1), guided by the *teacher* (T in Table 2.1). All laboratories are oriented to a *robust teaching*: the *Contents* (C in Table 2.2) are rich because either the mathematical topic is not trivial

(as in Laboratories 1, 2 and 3), or it is standard but connected with reality (Laboratory 4) or to other disciplines (Laboratory 5). Moreover, all the contents are chosen to develop productive disciplinary habits of mind, such as modelling (Laboratories 1 and 4), exploring, conjecturing and proving (Laboratories 2, 3 and 5). The activities have task difficulty ranging from moderate to demanding in such a way the *Cognitive Demand* (CD in Table 2.2) is challenging. Anyway, thanks to the support of the teacher and the classmates, given the working together modality, no student is left behind, guaranteeing an *Equitable Access to Content* (EA in table 2.2). With the laboratorial methodology, by seeing mathematics in real-life objects (as in Laboratory 4) and/or thanks to the use of technological tools (yEd, GeoGebra, MathCityMap), able to make students “see” and “manipulate” mathematical objects, students feel themselves within the mathematical environment, perceiving graphs, geometric figures and numbers as “alive” objects next to them. This causes engagement, *Agency, Ownership* over the content and the development of a positive *Identity* (AOI, in table 2.2) as thinkers and learners. Teachers do not evaluate students, but rather help them to adjust or refine their reasoning, when they are wrong. In such a way, a *Formative Assessment* (FA in table 2.2) takes place. Mistakes are not judged, but rather taken as a pretext to discuss and argue, helping the comprehension of the content.

Moreover, we observe how, even in a totally online context (such as Laboratory 4), the two communities are present in most of the proposals illustrated: the community of the researchers and that of the teachers. As we have already pointed out, working in synergy with teachers is a strength for us at MERG. In each of the workshops (except Laboratory 1 that is directly intended for students), the researchers were engaged in transposing meta-didactical praxeologies to teachers. These praxeologies concerned new mathematical contents that were not strictly curricular (properties of centroids, Brahmagupta’s theorem, ...), new teaching methodologies (e.g. conscious use of laboratory methodology, outdoor mathematics, connection with other disciplines ...) and new technological tools that teachers may use in their own teaching (e.g. yEd, GeoGebra, Excel, MathCityMap, ...). The teachers who took part in the laboratories worked in an active way: they not only followed the training sessions, but then implemented these proposals in class with their students. Sometimes they faithfully reproduced the activities (e.g. Laboratory 2, 3), sometimes they also customised the activities in terms of the choice of contents to be proposed (e.g. Laboratory 4) or the way in which they were proposed in class (e.g. Laboratory 5). These are, therefore, testimonies of evolution in the teachers’ didactical praxeologies, as they have benefited from training and modified their usual teaching practices.

As said, all activities share the mathematical laboratory approach aimed at a Robust Understanding of mathematics, but they differ in the way of proposing the laboratory. In Laboratory 1 students work solving a series of problems in order to introduce concepts and apply acquired knowledge. In Laboratory 2 students start using a two columns worksheet, where they are invited to explore, observe, manipulate, conjecture. Worksheets are quite guided and they are the same for all students. In Laboratory 3 the 2 columns worksheets become “flexible”: students can personalise it, freely deciding whether to use helps or not. A new opening appears in Laboratory 4, the MOOC. In fact, on the MCM web portal, on the one hand, the teachers had to follow the “rigid” structure of the template, but on the other hand, they had a lot of freedom in the choice of contents to be inserted. In fact, although a precise typology of the task had to be respected, the choice of the mathematical object to be considered and the related mathematical problem to be associated with it, together with hints and solution strategy, were freely chosen by the teacher. If we wanted to draw a parallel with the worksheets adopted in the laboratories described above, we could say that here the worksheet (i.e. the MCM template) is a container and there is openness towards teachers, while there the openness is towards students. In Laboratory 5, in the end, there is an opening towards both, students and teachers: the laboratory is open for teachers, because in the first laboratory the researchers introduced the

contents and teachers not only proposed changes valid for the whole activity, but also could decide how to guide their own classrooms' activities. The only materials equal for all the involved students were the GeoGebra classroom activities, agreed with all the teachers. The laboratory is open for students because the questions were related to students' lives (Are you ordered?) and their points of view (What is order?), rather on fixed topics, in such a way the classroom discussion was free to flow, and no classroom had the same discussion of another one. Moreover there is openness for students because they were free to produce their thoughts and feelings toward the theme with every way they preferred (discussion, draw, poems, ...).

Then, the flow of the evolution of our approach over the years, as researchers, that we call meta-methodology, is summarized in the following table:

Table 4.1 From closed to open mathematics laboratories

Activity	Meta-Methodology
Laboratory 1: <i>Elements of Graph Theory</i> , addressed to students. Worksheets: list of problems.	Closed
Laboratory 2: <i>Magic of centroids</i> , double laboratory addressed to teachers and students. Worksheets: tabular two columns layout.	Closed for students; Open for teachers (that work on the construction of the worksheets).
Laboratory 3: <i>Brahmagupta's theorem and consequences</i> , double laboratory addressed to teachers and students. Worksheets: tabular two columns layout, with help options.	Flexible for students (because of the help options in the two columns worksheets); Open for teachers (that work on the construction of the worksheets).
Laboratory 4: <i>MOOC on Math Trails</i> , laboratory addressed to teachers. Worksheets: MCM container.	Open for teachers (that can choose contents and problems).
Laboratory 5: Order and disorder, double laboratory addressed to teachers and students Worksheets: GeoGebra Classroom Activities	Open for students (that were free to express themselves); Open for teachers (that decide how to bring the content in class).

We are working in the direction of what we can call *open Labs*: the path taken by the students and by the teachers is not fixed, students can take different paths and teachers are free on how to present the topics. In the introduction of the activities, many hints are shown and the students choose which aspect to take care of. This is the direction of a laboratory proposed last year on *Mathematics and reality* (mathematical content: *The golden ratio*). Awareness of the strong link between mathematics and real life is unfortunately often lacking in students. The activity intends to make the connection between mathematics and reality more evident. In particular, the ideas offered are aimed at promoting a more lively and attractive view of mathematics, directed at meaningful learning of mathematics

which helps students to develop skills adapted to the demands of society. The activity consisted of a double laboratory: teaching materials on “Mathematics and reality” with particular reference to the golden ratio and its countless applications were prepared and presented, together with some worksheets, at the first laboratory with teachers. Teachers and researchers elaborated together the final version of the material to be presented in class for starting the activity. Digital worksheets, implemented in Excel with the possibility of *asking for helps*, were used so that each student could customise their use, thus working at various levels of difficulty. Subsequently, taking their cue from the initial presentation, the students were free to deepen or integrate certain aspects of it, freely choosing their own learning path to follow. Changing the curricular teaching approach to more engaging paths and methodologies linked to the use and integration of digital technologies also means taking into account the important role that emotions and motivation play in learning, to develop a positive attitude towards mathematics.

Then, a STEM approach is slowly evolving mutating to a STEAM approach, where the A, as said in the introduction, is not just arts, but it is meant in a broad sense of *creativity*: on the one hand, teachers are “granted” freedom to customise and re-adapt the content or materials offered by researchers, on the other hand, students are encouraged to be systematic and experimental, as well as to use their imagination and make new connections among ideas. Students can play with concepts of aesthetics and with sensory and emotional engagement, in the context of critical thinking, logical inquiry, or creative production about the world around them.

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FLOWGORITHM: A possible passage from algorithmic reasoning to creatively founded mathematical reasoning

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September 12, 2021

Abstract

This paper argues that algorithm design in the sense of computational thinking (CT) does not involve only routinized procedural applications void of deep conceptual understanding of mathematics. By using the programming language FLOWGORITHM, we demonstrate how classroom tasks centred around algorithm design may be used to activate creatively founded mathematical reasoning (CMR) in mathematics students.

1 Introduction

1.1 Mathematical competencies and abilities

The mathematics teacher's primary role is to help students develop their *mathematics competencies*, i.e., abilities to “understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role” [12, pp. 6-7]. In particular, [10] expanded mathematics competencies into six *abilities*: (1) problem solving ability, (2) reasoning ability, (3) applying procedures ability, (4) representation ability, (5) connection ability and (6) communication ability.

^{*}This work is supported by funding OER 10/18 LCK for the project “How to bring Computational Thinking (CT) into Mathematics classrooms: Designing for disciplinary-specific CT”.

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Worldwide, classroom tasks are thus designed to develop these six abilities. This paper proposes the use of algorithm as nexus connecting the abilities of problem-solving (how to solve tasks without knowing a solution method in advance), reasoning (the ability to justify choices and conclusions) and applying procedures (the ability to execute known procedures that are learnt by heart).

Applying procedures is familiar to mathematics teachers. In mathematics and mathematics education, such procedures are called *algorithms*. An *algorithm* is a set of rules to be followed in calculations or other problem-solving operations, be it by a machine or a human being. In mathematics education, this definition is broadened below:

Definition 1 (Algorithm) *An algorithm is a set of finite sequences of executable instructions enabling one to solve a given set of tasks ([3]).*

Common algorithms encountered by Singapore secondary school students (aged from 13 to 16¹) in the Singapore Mathematics syllabuses ([11]) include, but are not limited to, multiplication and long division (involving integers and/or polynomials), prime factorisation, obtaining highest common factors and least common multiples of natural numbers, quadratic formula, mensuration formulae, Pythagoras Theorem, plotting graphs, using formulae in coordinate geometry (e.g., gradient of line segment, length of a line segment, area of a triangle the vertices of which have given coordinates), partial fractions decomposition, finding derivatives and antiderivatives of special functions, etc. In short, the ability of applying algorithms in mathematics is tied on to routinized procedures. Imagine a continuum where a mathematical task is placed based on the availability of known mathematical procedures to the student engaged in that task. Then a task which merely requires the student to invoke a known algorithm (i.e., to mimic the teacher's demonstration of the same algorithm) will be placed on one end of this spectrum.

Thus, problem-solving tasks are placed on the other end of this spectrum. By a *problem*, we mean a task in which a student who is engaged in solving it does not have a readily available method or procedure to solve it. According to [8, p. 50], successful problem solving involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve some tension or ambiguity (i.e., lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity. A problem solver inevitably engages in the Pólyan cycle of understanding the problem (UP), devising a plan (DP), carrying out the plan (CP) and checking and/or extending the solution (CE) ([13]).

1.2 Concerns about extensive use of algorithm

Several studies ([2, 4]) have revealed that the majority of the class-time had been spent on learning and rehearsing algorithms which, when mastered, are intended to ensure students' efficiency and accuracy in solving mathematics problems and tasks that they tackle at some later stage. Algorithms form an essential body of knowledge for mathematics students because they are meant to yield quick and reliable answers to those tasks that can be solved by following a 'fixed' solution path. Because situations where a certain algorithm can be employed are completely deterministic and predictable, tasks that involve merely repeated usage of the

¹Typically, Secondary One students are aged 13, Two 14, Three 15 and so on.

algorithm then provide students plenty opportunity to practise. In themselves, there is nothing innately wrong about tasks that involve students' practice on applying algorithms but it is the extensive use of such tasks, that some studies, suggest may be counterproductive [4]. The danger here occurs when a classroom situation is typified by (a) a teacher providing students a set of mathematical tasks that have a common solution template, and (b) students repeatedly applying the same algorithm in solving this set of tasks. Then the students would have no opportunity to reflect on their use of the algorithm. It is this unreflective use of the algorithm which detaches the meaning of the algorithm from its application ([3]).

In summary, if there is an unchecked and extensive use of algorithmic learning that does not invoke the learner's reflection about the meaning of the algorithm then there is concern whether or not the learner's mathematical reasoning abilities will be hampered.

1.3 Proposed approach

The problem lies not with algorithmic learning but instead the way classroom tasks are designed (and implemented). In this paper, we suggest various ways whereby algorithms can in fact be exploited to create opportunities for deep conceptual learning of mathematics and active engagement in problem solving. This approach gives a counter-argument against the usual acceptance that algorithmic learning deprives student of creatively founded mathematical reasoning.

2 Algorithm design in Computational Thinking

2.1 Algorithmic and creatively mathematically founded reasonings

Two types of reasoning are specifically mentioned in [10, 9]: *algorithmic reasoning* (AR) and *creative mathematically founded reasoning* (CMR). The notion of AR, first coined by [10], refers to the state in which a problem solver employs repetitive numerical task-solving method – such a method must make use of an algorithm that can be employed to solve the problem (and one which is provided together with the task). As for CMR, [9] defines it to embody all of the following attributes: (a) *creativity* – a reasoning sequence new to the reasoner is created, or a forgotten one is re-created; (b) *plausibility* – arguments supporting both the strategy choice and implementation must be there to provide plausible or correct reasons to reach the conclusion(s); and (c) *anchoring* – placing arguments at the intrinsic mathematical properties of the components that are involved in the reasoning required to solve the problem.

Regarding AR and CMR, two significant findings reported in a recent work ([7]) by B. Jonsson and his team motivate our current approach: (1) AR leads to better students' performance during practice sessions when compared to CMR, as a function of the algorithmic support that was provided in the task (i.e., solution method and/or template is available to the students when given the task). (2) CMR outperforms AR whenever a task involves (i) conceptual understanding, (ii) memory retrieval and/or (iii) (re)construction of solution methods.

From the above finding (1), we gather that AR leads to positive learning outcomes during the initial stage of student's familiarization with key definitions, methods, calculations, algorithms and solution templates through practice. However, finding (2) informs us that the learner must move on from this initial stage of drill-and-practice to engage with (cognitively) more

demanding tasks that require genuine conceptual understanding, retrieval of useful information and/or (re)construction of solution methods.

2.2 Algorithm design

In an earlier work [6] presented at the 24th ACTM, the authors gave domain-specific interpretation for the four components of Computational Thinking (CT), namely, Decomposition, Pattern Recognition, Abstraction and Algorithm Design. Additionally, they argued for the use of task design principles grounded in Computational Thinking (CT) to construct meaningful tasks which exploit CT in mathematics learning.

The specific component of Computational Thinking we are focusing in this paper is *Algorithm Design*, which according to that same paper,

“involves the planning and development of a set of precise and step-by-step instructions for solving the problem.”

Such a finite sequence of instructions is termed as a *program* which can either be carried out by a computer or a human being in an ‘insightless’ manner. Notice here that running the algorithm or program requires no insight and hence unlikely to result in meaningful learning. Indeed, a proficient use of algorithms can at best reduce both the cognitive demands of complicated calculations ([5]) and the cognitive load on the learner’s working memory ([15]). But *planning and development* of such an algorithm requires not only insight but a lot of creativity and deep conceptual understanding of the mathematics that learners are engaging with.

Clearly, it is the way tasks are designed to train learners in planning and developing algorithms with the intention of activating students’ reasoning that really matters. In fact this view is not new as already [9] reported that the reasoning that students activate as a result of active planning and developing (in relation to specific tasks) is one key variable in learning mathematics through task solving.

2.3 The role of algorithm design

This is where the element of Algorithm Design lend itself to transit the learner smoothly from the initial stage of applying the algorithm or solution template to the more cognitively demanding engagement with the higher-order thinking processes. To operationalize this transition, there are two distinctive task labels associated to the task we design around the element of Algorithm Design. (1) Implementing a procedure: Present the key algorithm/solution template in a *standard* format that can be communicated by the teacher, and can be understood by the student. (2) Problem solving: Construct (or co-construct) a novel algorithm which is used to solve a problem embedded in the mathematical task.

In order that a standard format be established to allow communication between the teacher and the student, we have chosen a free programming language called FLOWGORITHM. This software requires little overhead in programming knowledge. It allows its user to construct a program in the form of a flowchart and – the best part – to run the program.

Typically, in an ‘implementing a procedure’ task the teacher will display a flowchart program (that implements some mathematical procedures) for students to read. This is in line with the pedagogical approach called *PRIMM* ([14]), that is, (P) Predict: look at a short program and

try to guess what it will do; (R) Run: run the codes of the program, observe the outcome and check whether the prediction made earlier is correct; (I) Investigate: carry out investigations about the different part of the codes, e.g., tracing the program flow, making annotations, etc.; (M) Modify: changing the code to effect desired changes; (M) Make: make a brand new program by making use bits of the codes from the original program to solve a new problem. We shall explain the details of how the transition takes place from ‘implementing a procedure’ to ‘problem solving’ in the next section.

3 FLOWGORITHM as passage from AR to CMR

3.1 Flowchart and FLOWGORITHM

When a student first learns a text-based programming language, it is not uncommon that he or she be required to enter several lines of syntax (i.e., commands) before something simple can be achieved, e.g., to print “Hello, world!”. One advantage of the using flowchart is to take away the cognitive load of familiarizing with the nuances of the text-based programming language, and let the learner focus on the visual flow of information, and thereby acquire a quick understanding of the underlying programming concepts. To print out “Hello, world!”, the important point here is appreciate that “Hello, world” is a string (i.e., literally a string of characters that can be input through the keyboard) to be *output*. Figure 1 depicts the FLOWGORITHM program (on the left) to print in the chat bubble “Hello, world!” (on the right).

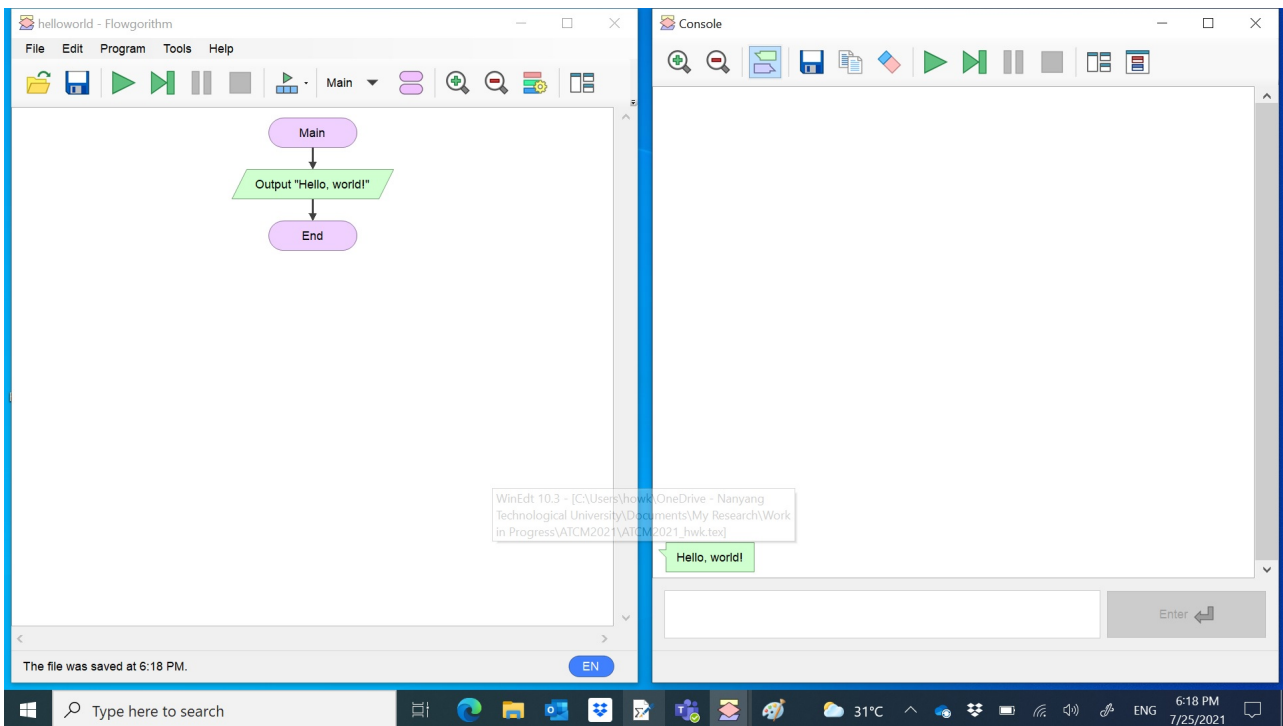


Figure 1: FLOWGORITHM program helloworld.fprg

On the official website <http://www.flowgorithm.org/download/index.htm>, one can find

the installation instructions for FLOWGORITHM, together with an introduction to how FLOWGORITHM programs are written.

3.2 Looking for the right topics

To illustrate how the aforementioned transit can be made from AR to CMR in an authentic classroom situation, we invoke a running example through this section. The first point to be made here is that it is important to identify the topics in which algorithms already reside.

The running example we have chosen sits in the topic of quadratic equations and expressions, which in the mathematics syllabus usually taught to Secondary Three (Express) students. We give some background of this topic below.

When students first encounter quadratic equations in Secondary Two, they will be taught how to recognize one, i.e., the students would check that a quadratic equation has only one variable, say x , and is of the form $ax^2 + bx + c = 0$, where a , b and c are constants and crucially $a \neq 0$. Usually these coefficients are given as integers, though not necessarily so.

Students are expected to be able to apply their prior knowledge in algebraic manipulation to reduce equations to a quadratic one, e.g., to be able to prove that the equation $\frac{10}{x} - \frac{10}{x+2} = \frac{1}{5}$ can be reduced to $x^2 + 2x - 200 = 0$.

At Secondary Two, students would be taught how to factorize a quadratic expression into its linear factors (over the polynomial ring $\mathbb{Z}[x]$) using trial-and-error. The method is known as the ‘cross-method’. For instance, to factorize $2x^2 + x - 3$ into its linear factors, a student tries all possible integer factorization of the constant term “ -3 ” (e.g., 3×-1 and -3×1). Figure 2 shows a typical working carried out by a student when employing the “cross-method”. Notice that cross-multiplying x -terms with the factors yield $+3x$ and $-2x$ in the add-column on the right, where the sum of $+3x$ and $-2x$ is checked to be $+x$, as desired.

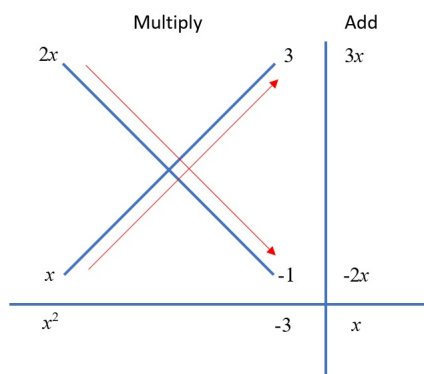


Figure 2: “Cross-method” for factorization

At Secondary Two level, students would realize that not all quadratic expressions can be factorized into linear factors with integer coefficients by solely applying the “cross-method”. They are not expected to know the necessary and sufficient conditions for a quadratic equation with integer coefficients to be factorized into two linear factors with integer coefficients. At that level, the nature of roots, i.e., real or complex roots, is also not in the syllabus. According to the Secondary Mathematics Syllabus ([11]), students at Secondary Three are expected to

apply the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (1)$$

to obtain the roots of the quadratic equation $ax^2 + bx + c = 0$, assuming the real roots exist. Though the formula is available in the formula booklet, it is often advised that students memorized the formula and apply it at will. In a typical lesson based on the AR approach, the teacher will demonstrate the solution of a quadratic equation using a solution template typified in Figure 3.

In the equation $x^2 + 2x - 200 = 0$, the coefficients are $a = 1$, $b = 2$, $c = -200$.
Applying the quadratic formula,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-200)}}{2(1)} \\ &= -11.06 \text{ and } 9.06, \end{aligned}$$

correcting the answers to two decimal places.

Figure 3: Solution template for using the quadratic formula

3.3 Starting with flowchart

The teacher can set a few AR-tasks for the student to drill-and-practice and familiarize with the above solution template. With the goal of transiting to the use of CMR approach in teaching and learning, we do not stop here. We suggest that the teacher introduces what an algorithm is in the sense of Definition 1. This can be done by inviting students to examine the structure of the solution template in Figure 3, using the following prompt:

Teacher: In the solution template that we have been using to solve those quadratic equations,

1. identify those key input data you must use to calculate the roots of the given equation $ax^2 + bx + c = 0$;
2. write down a step-by-step set of instructions using those data to output the required roots.

In the ensuing conversation between the teacher and the students, the discussion will likely culminate with the following points:

- Input(s): The key input data identified are the values of a , b and c – but not x .

- Process(es): Although the mathematical syntax employs a single variable x to denote the two possible roots, the roots are actually calculated using the input data and two separate formulae:

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

- Output(s): The first root can be stored as x_1 , and the second as x_2 .

The students' realization that there are in fact two outputs x_1 and x_2 deepens his or her understanding of the quadratic formula, where, in particular, the sign ' \pm ' really denotes two different operations (plus and minus) performed separately. Apart from this realization, one of the end-goals of the above task is to heighten the student's awareness to the 'Input-Process-Output' sequence in an algorithm, i.e., given some inputs (declared by the programmer, and later supplied by the user), some processes are then invoked to act on the supplied inputs, and these will finally yield some outputs. This observation then lends itself naturally into a flowchart – the paradigm of information-flow from the input source, through the processes (i.e., computations), to the output target.

At this point of the discussion, a flowchart that enacts the algorithm of calculating the two roots of a quadratic equation is displayed (see Figure 4).

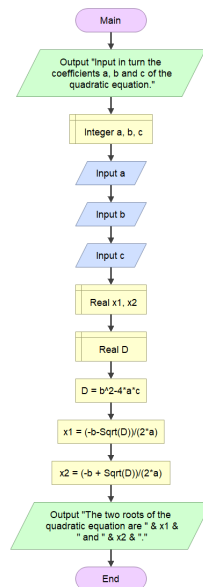


Figure 4: Flowchart for calculating the roots x_1 and x_2

The follow-up activity that makes use of the flowchart in Figure 4 would involve the students to predict what the flowchart does (given that they do not know some of the syntax in the flowchart) and communicate the intention of the key inputs, processes and outputs appearing in the flow of information from Main (input) to End (output). This activity is identified with the 'Predict' phase of the PRIMM framework.

The purpose of this segment is for the teacher to use the diagrammatic nature of the flowchart to explain what the flowchart does, and simultaneously introduce the fundamental computing concepts of variables as placeholders for storing values, e.g., a , b and c have been set as variables taking on integer values, for example; and x_1 and x_2 are variables taking on

real number values. Additionally and concurrently, the programming concept of data and data types can be introduced to the students within the disciplinarity of mathematics.

3.4 Using FLOWGORITHM to build flowcharts

The next teaching move would be to show how FLOWGORITHM can be employed to build the flowchart shown in Figure 4. This demonstration will in addition introduce the following essential concepts: (1) String data type, e.g., the words appearing within the double inverted commas, and the concatenation operator “&”. (2) Assign a value to a variable, e.g., defining D , x_1 and x_2 using formulae. (3) Re-use codes or variables in the program to avoid redundant repeated computations, e.g., calling up the variable D and using it later in x_1 and x_2 .

Each of these teaching points will involve different actions (e.g., Input, Declare, Assign, Output) which are distinguished by the different shapes. The directed arrows reinforce the concept of information-flow.

3.5 Running the flowchart in FLOWGORITHM

The role of the machine enters the scene when the teacher demonstrates that the constructed flowchart in Figure 4 is in fact *executable*, i.e., it can in fact be operated as a computing machine! Pressing the play button sets the program running, i.e., (i) the program interacts with the user by asking for inputs to a , b and c , and (ii) the program returns with the output embedded in the sentence “The roots of the quadratic equations are ... (first root x_1) and ... (second root x_2)”. Figure 5 (the console on the right) shows an instance of running the flowchart in Figure 4 with the inputs $a = 1$, $b = 2$ and $c = -100$ as in the quadratic formula.

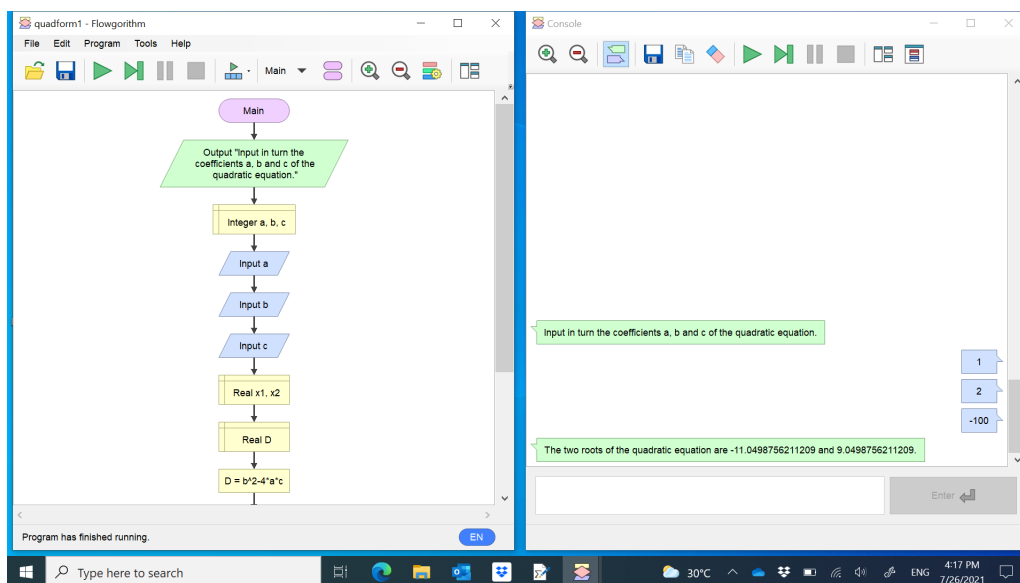


Figure 5: Running the flowgorithm program quadform1.fprg

The teacher may now invite the students to use their flowchart as a quadratic formula calculator to check that the answers obtained in the AR-tasks completed a priori are indeed correct. The end-goal of writing a FLOWGORITHM program in this lesson is to culminate with

a useful product, i.e., a calculator that can calculate the roots of a quadratic equation. Thus, this task is aligned with Papert’s original conception of what Computational Thinking should achieve, i.e., the learner acquires knowledge by constructing a useful product (see [16, 6]).

We classify the above task as an “Implementing a procedure”-task because the main objective of this type of task is to get the students to appreciate that the solution template associated with solving a quadratic equation can in fact be implemented as an algorithm (in this case using FLOWGORITHM as the programming language). Additionally as the teacher walks the students through the flowchart (in Figure 4), the students are given another chance to internalize the quadratic formula on top of the drill-and-practice tasks completed prior to this flowchart activity. Also, a pleasant side-effect of this approach is that students acquire the necessary programming concepts through the use of FLOWGORITHM in the context of solving quadratic equations *without* being burdened by the syntax of a text-based programming language.

3.6 Moving on to problem solving

We recall from our earlier discussion in Section 3.2 that Secondary Three students “are not expected to know the necessary and sufficient conditions for a quadratic equation with integer coefficients to be factorized into two linear factors with integer coefficients.” This now can be turned into an opportunity for the students to engage in authentic problem solving.

Moving forward, the teacher can craft the problem-solving task as follows:

Problem. Given a quadratic equation $ax^2 + bx + c = 0$, with integer coefficients, i.e., a , b and c are all integers, devise a test involving a , b and c that would help us determine whether this equation can be solved by using the ‘cross-method’ (i.e., guess-and-check in the sense illustrated in Figure 2) involving integer coefficients for the linear factors.

Although the problem stated is not unfamiliar to students at this level, they do not have a ready solution to it. The teacher can guide the students through the flowchart and ask them to identify which part of the program is instrumental in calculating the roots of the equation. In other words, we are engaging the students in the ‘Investigate’ phase in the PRIMM framework. Suitable scaffolding can be designed to lead the students to examine the different sub-expressions or terms appearing in the quadratic formula.

The students can employ the FLOWGORITHM program (i.e., `quadform1.fprg`) shown in Figure 5 on several sets of integral values of a , b and c to explore the outcomes. A closer examination then reveals that the discriminant $D := \sqrt{b^2 - 4ac}$ must be a perfect square in order that $\sqrt{D} = \sqrt{b^2 - 4ac}$ is an integer so that the final expression $\frac{-b \pm \sqrt{D}}{2a}$ is rational.

For the ‘Modify’ or ‘Make’ phase, the teacher can continue to challenge the students to devise a FLOWGORITHM program to determine whether a given quadratic equation has rational roots.

4 Task design schema

In the preceding Section (i.e., 3.2–3.6), the sample tasks sequence is built around a selected mathematics topic, and algorithm design is embedded into the different tasks that help students transit from Algorithmic Reasoning (AR) to Creatively Mathematically Founded Reasoning

(CMR). We now abstract the details of this tasks sequence in the form of a task schema. Recall that schemas are categories of information stored in long-term memory ([1]). In other words, a schema contains groups of linked memories, concepts or words. The purpose of having a task design schema, in our case, is to create a cognitive shortcut for the task designer so that the specific task design we recommend here can be stored and retrieved from one's long-term memory much more quickly and efficiently. Because the classroom task is designed for teaching and learning of Mathematics via Computational Thinking, we employ the four task design principles proposed in [6]. For the reader's convenience, we briefly recall what these four design principles are, and then apply them as we describe the task schema proposed herein.

4.1 Four task design principles

The original intention of bringing in this set of task design principle is to address teacher's concern about instructional design centred around crafting Mathematics lesson that activate Computational Thinking. Because Computational Thinking comprises four key components, i.e., *decomposition*, *pattern recognition*, *abstraction* and *algorithm design*, the four design principles have been targeted to address each of these components in the form of questions to which the task designer must answer. Below, we quote these questions verbatim from [6, pp. 5-6].

Complexity Principle. “Does the mathematical concept give rise to sufficiently complex problem?” The problem should involve the use of the identified concept, and be complex enough so that decomposition of this main problem into sub-problems is a needful step. If the problem or task is routine or too simple, e.g., there exists a ready-made solution or method, then decomposition is uncalled for.

Data Principle. “Can the mathematical concept occur in various forms so that it is possible to collect data for its occurrence?” The topic should involve observable and quantifiable data that can be collected, created, analyzed, and shared.

Mathematics Principle. “Can the problem associated to the mathematical concept be mathematized?” Mathematization is the formulation of the problem using mathematical terms. It turns a problem in real world context in an abstract and precise manner to a mathematics problem. We do not restrict mathematics to mean only numbers, algebra, geometry, and so on. Rather, mathematics can have a more inclusive meaning of encompassing abstract concepts and structures which are definable, representable, and can be reasoned about within some logical framework.

Computability Principle. “Does there exist an effectively calculable solution to the mathematized problem?” By ‘effective calculable’ we use it in the sense of Recursion (or Computability) Theory, that is, there exists a computer program that can calculate a solution to the problem through a finite procedure via a physical agent (e.g., machine, human being).

It is however to be noted that there is *no* need for the above principles to be applied in any prescribed order.

4.2 Task schema proposed and validated

From the preceding section, we see that there are four key stages involved, and these are presented in the form of a schema shown in Figure 6.

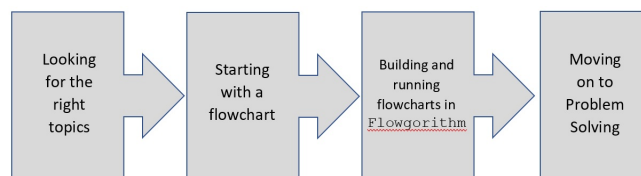


Figure 6: Task design schema for transiting from AR to CMR

Looking for the right topics. Since we are advocating the situating of algorithms as the connecting piece between AR and CMR, the task designer should naturally browse through the potential topic and look out for all the existing procedures, solution templates to standard questions, standard calculations involved, etc.

We now apply the *Computability Principle* at this juncture to check if those identified procedures or solution templates can be implemented as an algorithm. Even when these procedures are programmable, there is the question of whether the programming concepts or techniques are too difficult for the students at that level. In the present proposal, we recommend the use of FLOWGORITHM as a simple programming language with a low overhead of programming syntax. The question here is whether the algorithm when implemented in this language is simple enough for students of that level to understand.

Starting with a flowchart. At this stage, the student would be engaged in understanding what a flowchart involves, e.g., the key variables and their associated data types. In other words, the student becomes cognizant of the different components of the problem or algorithm at hand. This step of starting with a flowchart helps the student stay focused on the different inputs, the various decision-making and the calculations together with all the variables. As the student moves on from AR to CMR, he or she would be expected to be more independent in deciding which are the variables to be defined, and what data types they belong. All these are possible if the task designer factors in the process of Decomposition, i.e., breaking down the complex task into smaller sub-tasks which are more tractable. Thus, the *Complexity Principle* validates this part of the task schema in that this task engages the students in identifying the various parts of the flowchart – the students are moved towards appreciating this part-whole relationship between the individual parts of the algorithm (e.g., variables, sub-processes) and the whole algorithm itself. In addition, *Data Principle* can be applied here to ask for evidence of students' working out which input data are used and how these are transformed by the subsequent information-flow directed by the flowchart.

Building and running flowcharts in FLOWGORITHM. This is closely tied to the component of Algorithm Design in Computational Thinking, and we are thus invoking the *Computability Principle* again. At this stage, we check whether the task requires that the algorithm can be effectively implemented in the chosen language. In this part of the task schema, the students must

construct the given flowchart using FLOWGORITHM. Like any other programming language, students at this stage are bound to make mistakes and hence we expect them to be debugging their programs. While training them to pay attention to each of the constituent parts of the entire algorithm, we are indirectly helping the engaged students to internalize the algorithm. The authors are in the view that algorithmic learning implemented through this approach is far more meaningful than a mere memorization of formulae or routinized procedures.

Moving on to problem solving. Remembering that Computational Thinking is evidence-based because there are clear deliverables and observables tied to it – meaningful learning yields useful products. At the end of the AR-focused segment of the lesson, a useful product is obtained – the resulting FLOWGORITHM. Now the transit from AR to CMR crucially hinges on how the task requires the student to make use of the program that he or she has just constructed. In order that CMR be invoked, the activities at this stage must involve the learner to think in a deeper manner through the mathematical concepts involved. In our running example, the FLOWGORITHM program that implements the quadratic formula serves as an object of study in itself: which part of the formula determines the rationality of the roots of the equation given that all the coefficients are integers? The FLOWGORITHM program has the advantage that it can be run on input data so as to create visible patterns for students to engage in *Pattern Recognition* and hence to formulate relevant conjectures towards solving the problem at hand. Hence we see that the task is centrally rooted in a mathematized problem – the *Mathematical Principle* thus validates this last but critical part of the task schema. Indeed without this part, the transition from AR to CMR can never take place. This is where the students’ mathematical creativity emerges out of the situationally induced need to reasoning out for themselves what was going on in the heart of the algorithm.

5 Discussion

Having presented the task design schema that is targeted to transit the learner from AR to CMR, it is time to reflect on our proposed approach. Let us begin with some limitations.

5.1 Limitations

Flowcharts, even in the present form written in FLOWGORITHM, are hard to modify. Although the flow structure appears easy to understand, a working program is often completed *all at once* – one can hardly modularize by making the small parts work first. Of course, FLOWGORITHM allows one to call up functions (which can be coded separately from the main program) but it becomes intractable once the program involves too many function calls. Additionally, debugging can be challenging because one just cannot take apart the flowchart and reassemble the parts again. Our proposed approach relies heavily on the choice of the right topics. Even when there are fixed procedures, e.g., finding shortest distance of a point from a line requires one to identify the foot of perpendicular from the point to the line, sometimes there is such a great variety of situations (in which the same procedure is to be applied) that it is simply not possible to code the procedure as a computer program. This is especially the difficulty encountered when one is dealing with geometry problems.

5.2 Implications

Although the task design schema looks largely plausible to apply, it is far from being clear how the mathematics teacher can use it to generate specific lesson tasks on specifically selected topics. For example, can one incorporate this proposed approach systematically into a professional development program for mathematics teachers? At the moment of writing this paper, the first author is in the process of introducing the proposed approach of using algorithms to mathematics teachers at an in-service training program that is intended to equip them with basic programming skills. In this course, one special feature of FLOWGORITHM is frequently emphasized and exploited, i.e., all programs written in FLOWGORITHM can be exploited as programs written in a variety of different programming languages (e.g., C++, Fortran, Python, Javascript, etc.). From a practitioner's point of view, what one needs is a rich collection of exemplars, be they flowcharts of all the commonly encountered algorithms that are associated to implementing a procedure or those used to encourage problem solving or making conjectures. The first author, together with some beginning teachers, is building up a library of FLOWGORITHM (as well as others) programs for the purpose of teaching mathematics at the secondary level.

6 Conclusion

In this paper, we have argued that Algorithm Design – one of the key components of Computational Thinking – is *not* the same as *Algorithmic Reasoning*. In particular, we proposed a type of task schema that is crafted around the essential entity of algorithms and demonstrated how such a task schema can be uniformly applied to give rise to a sequence of tasks that brings the learner through the journey of Algorithmic Reasoning to advance to Creatively Founded Mathematical Reasoning. Crucially, this proposed task schema has been rigorously checked against the four CT+Math task design principles put forth in an earlier ATCM conference. Our present work can be seen as a further application of this set of design principles, thus demonstrating its versatility in the domain of task design that is specific to Computational Thinking used in Mathematics teaching and learning. The constraint of space in this paper has not allowed us the luxury of explaining how the activities described above can be carried out in an authentic classroom situation, what methodology we employ and how the desired positive results may be achieved with the students. Thus, one important future work would be to actualize what has been proposed in this paper.

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Some affordances and opportunities of advanced scientific calculators for learning introductory calculus

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Abstract: *Advanced scientific calculators, developed in recent years, include more sophisticated capabilities than original models, including some of direct relevance to elementary calculus. In this paper, some relevant calculator capabilities are described and their potential use for students and teachers of elementary calculus are illustrated and evaluated, against the framework of a model for learning mathematics with calculators. Despite the name of such devices, calculators derive their educational value from processes other than mere calculation. Key ideas of continuity, convergence, limits, differentiation and integration can be both represented and gainfully explored on an advanced scientific calculator, supporting student learning in powerful ways not otherwise available to many students in Asian countries.*

1. Introduction

It is now around forty years since the use of scientific calculators has become routinely accepted in secondary school mathematics education in many countries, including Australia, the USA and many European countries. The use of calculators in those countries was originally focused on computational use – since the devices afforded ready access to otherwise troublesome computations – but recently attention has shifted to educationally more important aspects. Over that time, other forms of technology – most noticeably computers and the Internet – have been developed and achieved some popularity, leading to the suggestion in [1] that the scientific calculator has appeared to have become a ‘forgotten technology’.

In many affluent countries, the scientific calculator seems indeed to have almost become forgotten, as mathematics students routinely use graphics calculators and even graphics calculators with computer algebra system (CAS) capabilities over the last quarter of a century, both in classrooms and in high stakes external examinations. Experience in Asian contexts, however, has been different to date, most likely for economic reasons, and it seems still to be the case that the scientific calculator continues to be the most appropriate tool for many classrooms, as suggested in [2]. Recent developments in scientific calculators have increased their significance for learning, as functionality has increased to match the school curriculum and usability has been improved. Modern advanced scientific calculators like the *CASIO fx-991EX* calculator (*ClassWiz*) have many capabilities shared with graphics calculators – without the graphing capabilities and limited by smaller screens – and thus have significant potential for school and early undergraduate mathematics education.

In this paper, we focus on the productive use of such calculators for the particular context of calculus, drawing in part on earlier work highlighting their educational significance [3]. Although other calculators might be used, for convenience we illustrate the arguments with the *ClassWiz*.

2. Learning calculus with calculators

In part to place calculation in context, [4] proposed a model for the educational use of calculators, identifying four kinds of activity: representation, computation, exploration and affirmation. While many productive uses of calculators in school involve several of these components simultaneously, those that involve *only* calculation are usually of very limited importance. Many activities include computation in some way – in precisely the same way that computer software such as that for

spreadsheets, word processing, or dynamic geometry or access to the Internet via web browsers always involve computation in some way, although the computation is not the key feature for learning, and indeed is often hidden from the direct view of the user.

To illustrate this model, consider the command on *ClassWiz* for determining the numerical value of a derivative of a function at a point. As the command is actually written on the keyboard, and thus always visible to a user, it provides unmistakable evidence that the calculator has links to the calculus. Figure 1 shows an example of this command in use. This screen reflects, potentially, all four of the educational uses of the calculator, as described below.

$$\frac{d}{dx} (x^2+x+1) \Big|_{x=3} = 7$$

Figure 1: Using the numerical derivative command on *ClassWiz*

Representation: Like many modern calculators, the *ClassWiz* represents mathematical ideas using conventional mathematical syntax, so that the ideas are familiar to users. In this case, the representation of d/dx is shown as a fraction; the function (of x) involved is represented conventionally using an expression in which indices are raised, smaller, numbers and variables are shown in italics; the point at which the derivative is evaluated is shown using the conventional notation of a vertical line; the resulting derivative is shown as an exact number, in this case the integer 7. In short, the calculator represents the result in a way that is perfectly acceptable in everyday conventional representation, and thus immediately recognisable.

Computation: The *ClassWiz* has computed the value of the numerical derivative, saving the labour involved for the user of finding a (symbolic) general term for the derivative and then substituting the particular value of $x = 3$. It is not clear at all *how* the calculator has done this, but the result is immediate and the calculation is correct, both of which are helpful for students.

Exploration: As suggested later in this paper, a screen like that shown in Figure 1 might have arisen in several contexts, only one of which is the determination of a derivative of this function at this particular point. Student users of the calculator can use this numerical derivative function to determine related derivatives and use these to discover, or confirm, some properties of derivatives. For example, the above screen might have arisen following earlier numerical derivatives, such as those shown in Figure 2, as part of different sorts of student explorations:

$$\frac{d}{dx} (x^2) \Big|_{x=3} = 6$$

$$\frac{d}{dx} (x^2+x) \Big|_{x=3} = 7$$

$$\frac{d}{dx} (x^2+x+1) \Big|_{x=2} = 5$$

Figure 2: Related derivatives to that shown in Figure 1

Affirmation: By its nature a scientific calculator is a responsive device, driven by the user. So it has the potential to affirm (or to contradict) student preconceptions about an outcome, both of which are very helpful to learners. In this sense, it can be regarded as a form of personal hypothesis-testing device, of considerable value to a student thinking about the mathematics involved. To illustrate this

aspect, consider the situation before a student taps the equals key to evaluate the numerical derivative in Figure 1, as depicted below in Figure 3.

$$\frac{d}{dx}(x^2 + x + 1) \Big|_{x=3}$$

Figure 3: *ClassWiz* screen immediately prior to executing the command

Students’ preconceptions and expectations will depend on the context and on the student concerned. But to illustrate, a student who has just determined that the derivative of $f(x) = x^2 + x$ is 7 when $x = 3$ (as in the middle screen of Figure 2) might naively expect that the result in Figure 3 to be $7 + 1 = 8$, and find their thinking *contradicted* by the calculator; a fellow student might expect that the result will be the same as that in Figure 2, and find their thinking *affirmed* by the calculator. In either case, the calculator can contribute to student’s learning, provided it is used thoughtfully.

In summary, to consider the relevance of the calculator to a student learning calculus, we need to pay attention to the context of the learner, not simply to the technical capabilities of the device.

3. Closer and closer

A major conceptual jump for mathematicians since antiquity and thus for students learning the calculus today is the need to deal with infinite processes, as brilliantly revealed in a remarkable recent book [5]. By definition, we need to think in approximations to infinite processes, and many teachers will be familiar with trying to help students to understand the idea of getting ‘closer and closer’ to something – without ever reaching it. While technologies can appear to overcome this hurdle (for example, as computer algebra systems generate symbolic results of infinite processes, such as those involved with differentiation or integration), there are several ways in which calculators can help students to see for themselves what is happening behind the symbolism. In this section, we explore briefly some of these connections.

3.1 Continuity and discontinuity

Introductory calculus deals mostly with continuous functions; one way of thinking of these is that small changes in the variable lead to small changes in the function itself. Figure 4 shows an example of this for the continuous function $f(x) = x^3 + x^2$ near $x = 1$. Examining functions in this way is made possible by a tabulate facility available on advanced scientific calculators like *ClassWiz*; such facilities were not typically available on earlier scientific calculator models.

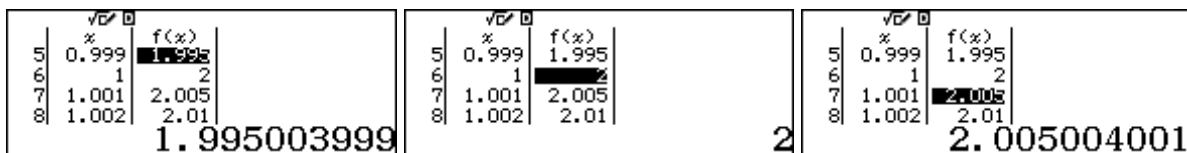


Figure 4: Exploring the continuity of $f(x) = x^3 + x^2$ near $x = 1$

These screens suggest that the function is continuous at $x = 1$. Although the screen is small (and thus the table size small), students can see for themselves that choosing still smaller intervals for x (i.e. those for which the values of x get closer and closer to 1) results in a similar phenomenon for

$f(x)$, which is the essential idea of continuity. The calculator is facilitating an exploration of this kind by permitting fast and efficient calculation – much more so than would be possible without the aid of the technology.

In contrast, when a function is discontinuous, function values can change dramatically with only small changes in the value of the variable. This is illustrated by setting up a table of values as shown in Figure 5.



Figure 5: Setting up *ClassWiz* to explore a discontinuous function

The resulting screens in Figure 6 show a different kind of pattern than that seen in Figure 4. As x gets closer and closer to 3, the function values change dramatically, and the function is not defined when $x = 3$, as indicated by the error message on the screen.

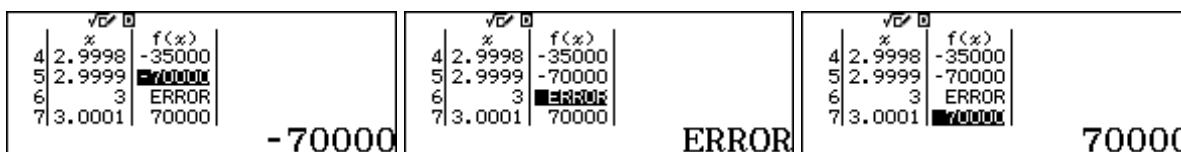


Figure 6: Exploring the continuity of $f(x) = \frac{7}{x-3}$ near $x = 3$

The flexibility of the calculator makes it relatively easy for students to explore jump discontinuities like this, and to compare their behaviour with that of continuous functions. Getting ‘closer and closer’ involves merely changing the interval values for the variable of interest. Elementary functions can be discontinuous in other ways of course, such as the concept of a removable discontinuity, for which a function is continuous everywhere except for one point. A typical example, often studied by beginning calculus students is shown in Figure 7.

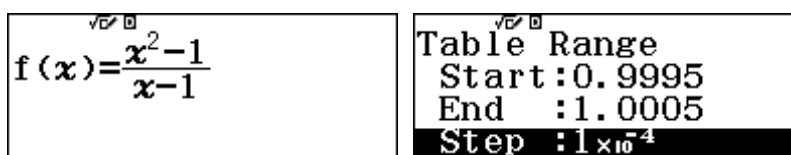


Figure 7: Preparing *ClassWiz* to explore the removable discontinuity of $f(x) = \frac{x^2-1}{x-1}$

Figure 8 shows the function values as x gets closer and closer to 1:

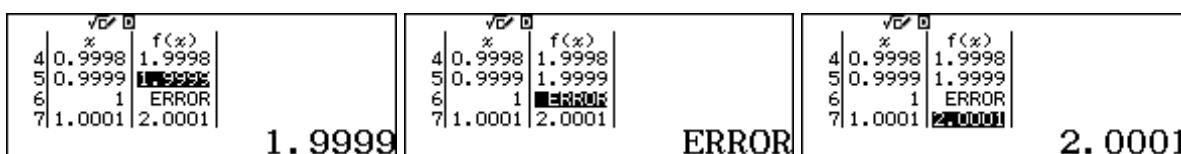


Figure 8: Understanding a removable discontinuity

Introductory calculus students can see that, while the value of the function is undefined at $x = 1$, it seems to behave elsewhere as if it were the continuous function, $f(x) = x + 1$. They can scroll tables like these, and adjust them to get closer and closer to $x = 1$ (by adjusting the interval appropriately) and see for themselves what is happening: that the value of the function is $f(x) = x + 1$ almost everywhere, except at the point $x = 1$; getting closer and closer doesn't affect this result. Students will of course usually be given more formal treatments of continuity and discontinuity, but the calculator might serve a useful role to build invaluable insight into these formal treatments; it is not suggested that they should replace these, but rather that they should supplement them.

3.2 Limits at a point

The idea of a limit at a point is a critical one, commonly emphasised in introductory calculus courses (although not formally treated by Newton or Leibnitz in their early development of the calculus). Although treatments are often very formal (including proofs with $\epsilon - \delta$ definitions), an intuitive understanding is important also and a calculator is a useful tool for developing this. Figure 9 suggests how the calculator might be used for this purpose, by constructing a table of values for a function, with an increasingly small interval for the variable x . As x gets closer and closer to 0 – from either side – the tables suggest that the ratio gets closer and closer to 1, and that it is not defined for $x = 0$.

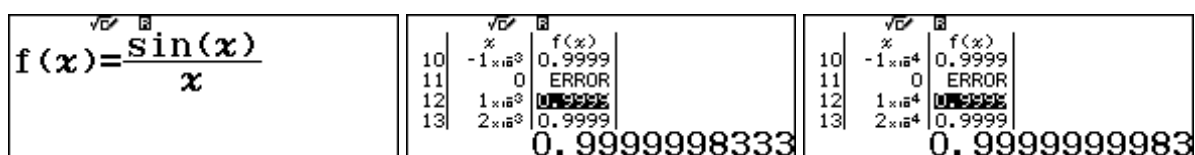


Figure 9: Using *ClassWiz* to explore $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

Calculators like *ClassWiz* are limited in their display capabilities, with narrow columns, but Figure 9 shows that more precision is available by highlighting a given value.

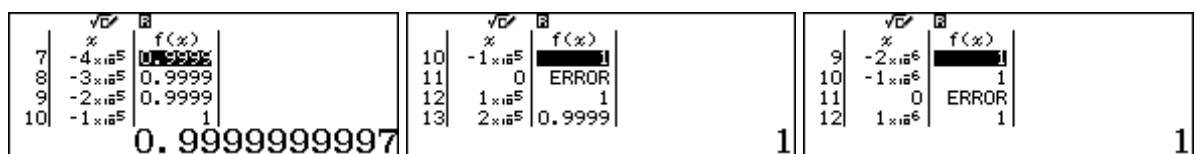


Figure 10: Getting closer and closer with smaller intervals

As students explore limits of this kind, with x getting ‘closer and closer’ to 0, eventually the calculator gives the impression that the result is 1, as shown in Figure 10 – a good opportunity for a discussion regarding the numerical processes involved and the limitations of a finite display of infinite decimals; although the result is technically incorrect, it is shown by the calculator as the result is rounded to the limitations of the calculator. However, such conversations are an important part of dealing with the many ideas of the infinite that are involved with the calculus.

3.3 Gradient at a point

A major motivation for studying limits in introductory calculus is to deal with the idea of a gradient at a point and thus of the derivative function. The derivative is typically defined as a limit and sometimes interpreted as a gradient of a secant or a tangent:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Advanced calculators like *ClassWiz* allow students to explore suitable gradient functions to see informally the limiting processes involved. An example is shown in Figure 11, to see the gradient of a secant to the curve of $f(x) = x^2$, at various values for x and as x gets closer to $x = 2$ (when h gets closer and closer to 0.)

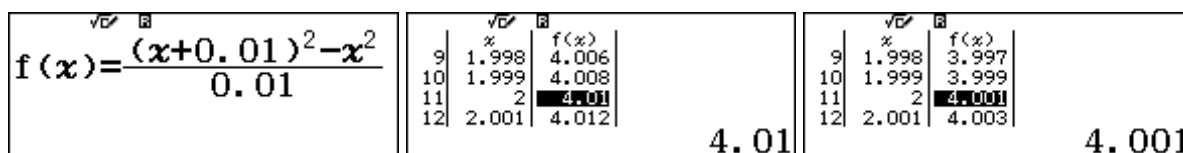


Figure 11: Evaluating gradient functions for $h = 0.01$ and $h = 0.001$

In Figure 11, the first screen shows that the ratio has been evaluated for $h = 0.01$, as shown in the middle screen. A smaller value of $h = 0.001$ is used to generate the third screen. In this way, students can see that the gradient gets closer and closer to 4 as h gets closer and closer to 0. The calculator is used to quickly perform the necessary calculations, allowing students to interpret the results. The first screen in Figure 12 shows the case for a very small positive value of $h = 0.000001$.

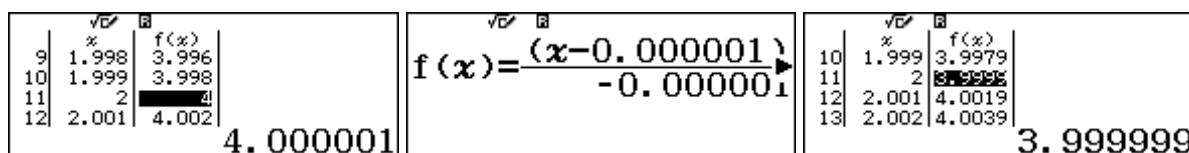


Figure 12: Exploring the gradient function as h approaches zero

Figure 12 also shows the results for a value of h that is close to zero, but is negative rather than positive, to show that the limit approaches the same value (of 4) from both above and below. The middle screen shows the case of $h = -0.000001$ and the third screen shows the resulting table. Using and continuing these kinds of explorations, students can get a strong sense of the idea that the limiting value is 4 in this case, suggesting that the derivative is $f'(2) = 4$.

3.4 Infinite limits

Limits to infinity are involved when a variable increases without bound. In such situations, ‘closer and closer’ usually means in effect ‘larger and larger’, so can be explored by comparing results as the value of a variable increases. An example is shown in Figure 13, to evaluate

$$\lim_{x \rightarrow \infty} \frac{5x - 1}{2x + 7}$$

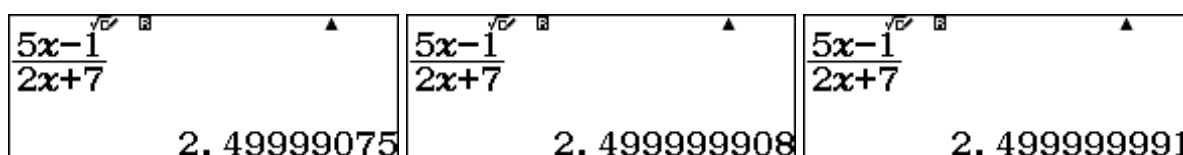


Figure 13: Exploring an infinite limit as x gets larger and larger

Although this sort of process can be explored on any scientific calculator, efficient explorations are facilitated by modern calculators like *ClassWiz*; the screens in Figure 13 evaluate the expression for successively larger values of x : respectively 1 million, 100 million and 1 billion, using a calculator command for easy substitution of values. Again, the calculator rounds results to its level of precision, so that eventually what might seem to be an incorrect result is provided, as shown in Figure 14. While this result makes clear that the limiting value seems to be 2.5, as suggested by the succession of screens in Figure 13, it is of course only a (very good) numerical approximation to the value of the expression for $x = 100$ billion.

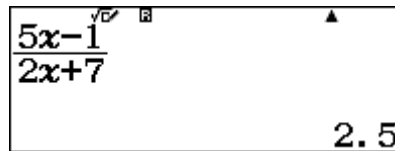


Figure 14: The limiting value of an expression suggested for $x = 100$ billion

3.4 Convergence of a series

Some infinite series can also be explored fruitfully on advanced scientific calculators, using special commands. A good example involves the infinite series for the exponential function, e :

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

As more terms of the series are included, the series can be seen to converge (in this case, very quickly) to a good numerical approximation to e , rather than the exact value.

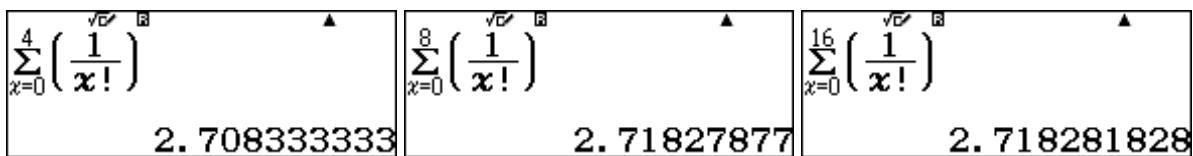


Figure 15: Exploring the limit of an infinite converging series for e

The *ClassWiz* environment makes it easy for students to edit the parameters of the (finite) series to see the effects of taking more and more terms, so that attention is readily focused on the output. Figure 15 shows a typical progression of series values as the number of terms increases, showing a rapid convergence of the result to the well-known value of the transcendental number e .

4. Using numerical calculus commands

One of the distinguishing features of advanced scientific calculators like *ClassWiz* is the presence of numerical calculus commands for differentiation and integration. While these are numerical in nature – and thus do not permit the user to determine either a derivative function or an indefinite integral, unlike computer algebra system (CAS) calculators – they can be used to support student learning, in addition to the computational task of determining numerical answers. There are a number of ways in which this can be done, some of which are illustrated below.

4.1 Patterns for differentiation

When a calculator is used to determine a derivative at many points, not just a single point, the underlying patterns can be seen by students, and help them to understand the idea of a derivative function. An example is shown in Figure 16, where the derivative of $f(x) = x^2 + 3x$ is tabulated and shown at a succession of points. (To do this, note that the tabulated function determines the derivative at each of the chosen table points, using the symbol $x = x$, rather than just a single point).

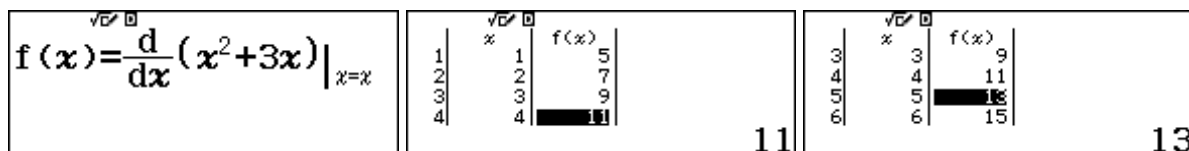


Figure 16: Exploring numerical derivatives at a point to generate a derivative function

For an exploration of this kind, students generating the table of derivatives shown in the middle screen of Figure 16 can be encouraged to look for a pattern in the results. Most students studying introductory calculus will readily see the linear function $f(x) = 2x + 3$, and can be further encouraged to predict the next terms (as the table shows only the first four values). So, an activity of this kind offers an opportunity for affirmation of the result, readily checked by scrolling down the table, as the third screen in Figure 16 shows. The transition from the idea of the derivative at a particular point to the derivative at all points (i.e., a derivative function) is a critical one, and the calculator offers students a way to understand the concepts involved. Of course, it is an easy matter for students to explore other functions in a similar way.

4.2 Patterns for integration

In a similar way, the numerical integral commands can be used to generate a pattern of results to understand integration better. While a singular numerical integral allows a student to determine the area under a curve between two points, a succession of integrals allows for a powerful pattern to be seen. An elementary example is shown in Figure 17, using the numerical integral command to generate a table of values.

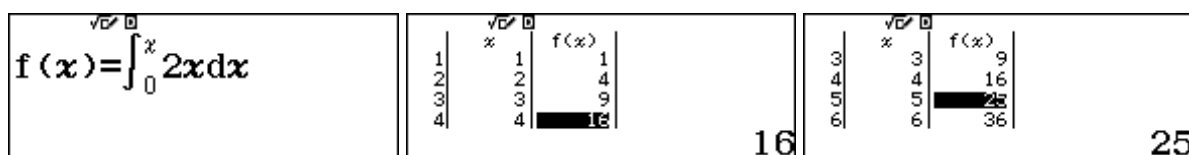


Figure 17: Exploring numerical integrals

Again, the pattern of results from this exploration will be familiar to most students studying elementary calculus: it seems (from the middle screen in Figure 17) as if the integral is well represented by the function $F(x) = x^2$. Again, the calculator environment offers an easy way to confirm this hypothesis by scrolling down a little further, as the third screen in Figure 17 shows. While this is not of course sufficient, and it is assumed that students will encounter the Fundamental Theorem of the Calculus in other ways as well, the calculator provides an opportunity to see important connections in a fresh way. Students can, of course, explore further by considering other functions to integrate, and to strengthen their sense of the important relationships involved.

4.3 Properties of derivatives

When ready access to numerical derivatives is available, an advanced calculator allows for derivatives to be compared and for some properties of derivatives to be highlighted. This is relatively easy for learners to do when a calculator like *ClassWiz* allows a pair of tables to be produced, rather than just a single table. To illustrate this, Figure 18 shows how the tables can be used to compare derivatives of one function, $f(x)$ with those of another function $g(x) = 5f(x)$.

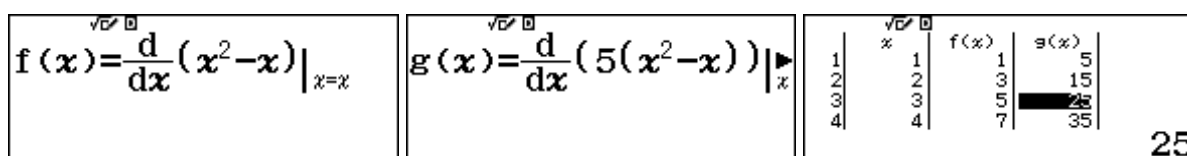


Figure 18: Comparing derivatives to illustrate a multiplicative property

In this case, users will be able to see that multiplying a function by a constant will result in a derivative that has been similarly multiplied. The calculator environment makes it easy for students to check this relationship with other constants or even other functions. Of course, such work does not mathematically establish a result, but helps students to appreciate it by considering some examples of their own choosing.

4.4 Special derivatives

While access to numerical derivatives can provide both practical results as well as opportunities for insight for any functions, students can use calculators to explore two special functions with surprising derivatives – at least surprising to novice students. One of these is the exponential function $f(x) = e^x$, for which the derivative at a point is the same as the value of the function at the same point. Figure 19 shows that this unexpected result is easily verified on the calculator.

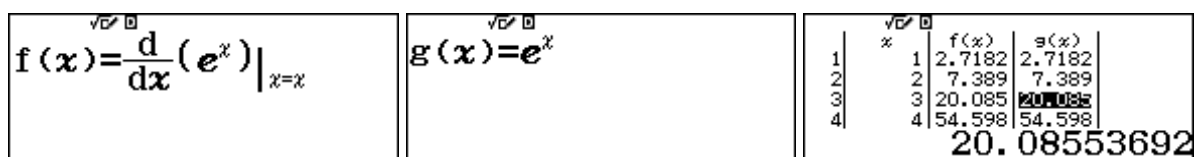


Figure 19: Affirming a special property of the exponential function

Again, a capacity to tabulate a pair of functions is key to this kind of exploration. Students can readily see for themselves that similar results do not hold for powers of numbers other than e , by tabulating them for themselves. As an illustration, Figure 20 shows that the derivatives of $f(x) = 2.7^x$ are generally lower than the corresponding values of the function, helping to make clear how special is the transcendental number e .

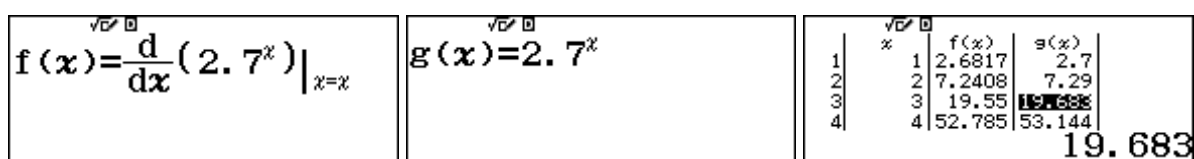


Figure 20: Exploring another exponential function to reveal a special property of $f(x) = e^x$

Another surprising result, readily affirmed with numerical derivatives, concerns the derivative of the natural logarithmic function. Again, students can use a suitable table of values to see for themselves that the derivative of the function is merely the inverse of the associated number.

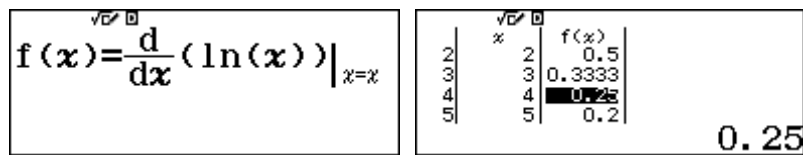


Figure 21: Demonstrating a special property of the natural logarithmic function

At this level of study, students would hopefully recognise the reciprocals in their decimal forms, but they could of course include them as a second column to the table if necessary.

5. Conclusions

The paper demonstrates that it is too limiting to regard calculators as merely devices for computation. The proposed model [4] for an educational role of calculators suggests that representation, exploration and affirmation are also important processes for student learning. Some examples of these processes are provided for the case of calculus, indicating two important consequences of advanced scientific calculator capabilities, using the CASIO *ClassWiz* as an example.

Firstly, they permit the exploration of the idea of ‘closer and closer’, regularly used by mathematics teachers as a helpful idea for both teaching and learning calculus. Consequently, they offer opportunities to attain further insight into the infinite processes involved than would otherwise be possible, as generations of teachers can attest.

Secondly, the availability of commands for numerical differentiation and integration offer some opportunities for the exploration of important calculus ideas, unburdened by the requirement for by-hand computation. The calculator provides a way for students to explore for themselves some properties and relationships associated with differentiation and integration.

In most introductory calculus courses, students will be expected to learn more formal and more symbolic aspects of the subject. The paper does not suggest that these are to be replaced by numerical opportunities provided by advanced scientific calculators; rather it is suggested that the calculator offers fresh opportunities to develop insight into the ideas involved with an affordable and available technology, well-suited to many classrooms across the region served by ATCM.

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Computational Discovery with Newton Fractals, Bohemian Matrices, & Mandelbrot Polynomials

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Abstract

The authors have been using a largely algebraic form of “computational discovery” in various undergraduate classes at their respective institutions for some decades now to teach pure mathematics, applied mathematics, and computational mathematics. This paper describes what we mean by “computational discovery,” what good it does for the students, and some specific techniques that we used.

1 Setting the stage

“The imparting of factual knowledge is for us a secondary consideration. Above all we aim to promote in the reader a correct attitude, a certain discipline of thought, which would appear to be of even more essential importance in mathematics than in other scientific disciplines.” Pólya & Szegő vol I. [28, p. VII]

The preface quoted above from the classic book cited, which is nearly a hundred years old now, opens with an epigraph which we further paraphrase, as follows: “What is good education? Giving students systematic opportunities to discover things for themselves.” Indeed, *Computational Discovery*, also called “Experimental Mathematics,” is also very familiar to the research mathematician, not just mathematics educators: nearly everyone uses it (even if they say that they don’t, or don’t say that they do). There can be no shame in it, if the likes of Gauss and Euler used the technique [6, 7]. See also the excellent book [14]. The most basic idea is, after all, very simple: one computes a few cases, tries to guess a pattern, and if successful,

tries to prove it¹. However, there is the all-too-common attitude that says “Once one has a proof, who really cares how the theorem was discovered?”

Quite a few people care, actually. First, it’s much easier to get students to succeed in learning mathematics if we teach them this powerful method, which many mathematicians actually *use*. Second, the world of mathematics is larger than the human mind can normally conceive of: computational experiments can genuinely surprise even experienced mathematicians. Third, mathematics is changing and has changed with the introduction of new powerful computational tools (for instance, the ordinary differential equation solvers in Julia are extremely impressive: see [29]). We have a duty to train our students in the tools used to explore that larger world. We also have a duty to try to keep up with the changes in these tools: see for instance that aforementioned paper on the new solvers in Julia, but this has been known for a long while [1].

Speaking of Julia, the phrase “computational thinking” has been given new life by an MIT course of that name, using Julia, offered by David Sanders, James Schloss, and Alan Edelman, and given away for free on YouTube. Julia is sometimes referred to as a “Python killer,” because it is very nearly as easy to use as Python but also offers very nearly the same performance as C or Fortran. Fortran, by the way, is very much still alive in the HPC world, for example in weather prediction. Python is by no means dead yet either, and has several advantages for use in mathematics instruction. Chief among these advantages is its very large user base amongst computational scientists: students are motivated to learn Python because it is directly used by computational neuroscientists, financial mathematicians (who may also use C), for machine learning, and many other buzzword-compliant research areas. The Open Educational Resource (OER) called “Computational Discovery on Jupyter” is intended to take advantage of these technologies [10].

One user interface for Python, namely the Jupyter Notebook, seems to be dominating at present. This allows access to Python, to Julia, to Matlab, and to SAGE Math, which offers an interface to many computer algebra systems. A Jupyter Notebook allows combined use of text, formulas (through a flavour of “Markdown” which is kind of a blend of HTML and L^AT_EX that is lighter weight than either), figures, data, and programs. Those people who are used to Maple Documents (or Worksheets or Workbooks) will be able to see the similarity, and indeed Python is rather structurally similar to Maple as well.

An experienced experimental mathematician or instructor will know that the platform for teaching has some impact, both good and bad, on the quality of teaching. One does not want the platform to *interfere* with the teaching. One does not want to spend too much time in class explaining the platform, instead of using that valuable time explaining the concepts one is trying to get across. At the same time, one wants to use the “shiny” technology to help motivate the students (and, to be honest, the instructor). By using current software, and using tools that the student can be confident will be used elsewhere, the expense of time needed in explaining the platform can be amortized. However, some classic tools, such as Maple, can still be very much worth the effort, as we will demonstrate by examples.

This represents both an opportunity and a challenge for the instructor. There are enough new features in Maple that are useful in instruction that for many of us, some upgrading of

¹In the hands of Pólya, of course, this most basic idea was refined and extended almost beyond recognition. Similarly, Bill Bauldry advocates in [3] a variation of this idea, which he terms the Action–Consequence–Reflection Principle; this is very similar to the advanced ideas that Pólya advocates in [26]. Still, it is astonishing how powerful this idea is, even at its simplest, given modern computational tools.

our training (or even retraining!) is necessary. One might question the effort needed, especially given the rise of Python, and the rise of Julia; but we think it is very worthwhile.

What about Mathematica? We have heard that it, too, has added significant and interesting new features; but we work in an institution with a Maple site license, and in an institution that uses SAGE Math; we will let our comments on Maple stand for (what we believe will be) similar comments for Mathematica.

A bigger question is what about GeoGebra (or Desmos, or Maple Learn). Our approach has largely been algebraic, or programmatic, and not explicitly visual, although we do use visualization of algebraic and functional concepts. We do not weigh here the comparative merits of this approach versus the more visual approach of the geometrically-flavoured teaching tools just mentioned; for the sake of argument, we ask the reader to accept (if only temporarily) that an algebraic and programmatic approach can be useful. We acknowledge that the geometric approach can also be useful, of course: see for example [8] or [32].

So: the basic point of view of this paper—which is shared by a great many other papers, including many at this ATCM and at previous ones—is that instructors can, and should, use current technology to help teach mathematics. In particular, we advocate using *computational discovery* with the help of the technology to strengthen students' knowledge of *both* mathematics and of computational technology. We'll now give specific (algebraic or programmatic) examples of how to do this, and discuss our experiences in the classroom and out.

The literature on using technology for teaching is very, very large. We will try to put our efforts in perspective in that literature in the final section of this paper, “Concluding remarks and Further Reading.” We will include technical references in each specific section.

2 Specific goals

2.1 Course objectives for the students

1. To acquire facility in using the computer as a tool for solving and exploring mathematical problems.
2. To learn the fundamental concepts and techniques of procedural and object-oriented programming. These include: flow control, modular construction, elementary data structures, recursion, and graphics
3. To develop problem-solving and communication skills by solving programming projects.
4. To learn to ask new questions, and to be a bit more comfortable with not being able to find out a definitive answer.
5. To become familiar with other computational resources for mathematics, in particular, but not limited to: \LaTeX , the Online Encyclopedia of Integer Sequences (OEIS), and the Inverse Symbolic Calculator (“identify” in Maple).

2.2 Learning Outcomes

On completing this course, the student will be expected to be able to:

1. Take a mathematical question and write it as a computational question.
2. Give examples of student-generated questions, including “what if?” questions
3. Give a pseudo-code version of an algorithm to solve the computational question

4. Convert pseudo-code into Sage or Maple or Python code
5. Comment code cleanly and effectively
6. Obtain and interpret output from their code
7. Give visualization and other methods of viewing output effectively.
8. Write up lab reports in L^AT_EX or Markdown on their investigations

2.3 Assessment

The course that depends on the resources we describe here is largely project based: there typically are small projects (especially at the beginning of the course) to ensure that everyone is coming up to speed with coding. Then there can be several more substantial projects, taking a mathematical question, and learning how to explore it in an experimental fashion. We make no assumptions about level of computer experience: if the student has written code before that will help, but if they haven't we will get them up to speed quickly.

Their projects are assessed using the following criteria:

- Does the project contain new² questions, new thoughts, or new answers?
- Do they have code that *runs*?
- Does the code produce *output*?
- Is the output *correct*?
- Is the output *complete*?
- Is the code *clean*? Are the variables suitably named? Are functions, control structures, data types, etc used appropriately?
- Is the code *documented*?
- Is the project written up as a lab report?

If the project passes all these criteria, it will get a good grade!

In some situations (e.g. for oral presentations) *peer-assessment* may be used: that is, part of your grade will be determined by fellow students. Each student will be expected to supply assessments of their fellow students in turn³.

3 The first example

Consider the sequence

$$1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \dots \quad (1)$$

Plunked down in front of the students without explanation, it is pretty mysterious. Explicitly telling them to square each entry generates a more intelligible sequence, namely

$$1, \frac{9}{4} = 2\frac{1}{4} = 2\frac{1}{2^2}, \frac{289}{144} = 2\frac{1}{12^2}, \frac{332929}{166464} = 2\frac{1}{408^2}, \frac{443365544449}{221682772224} = 2\frac{1}{470832^2}, \dots \quad (2)$$

²By this we mean *new to the student*, ideally questions that they came up with themselves. If they are genuinely open questions, or genuinely novel questions not in the literature, so much the better; but this is not required.

³Telling the students that the peer-assessed marks will be blended with the instructors' assessment, because students are sometimes harsh on each other, tends to improve things. We also ask students to justify their assessments.

Although more intelligible, this sequence is *still* mysterious. At this early point in the course, students are reluctant to ask questions (not just for fear of looking foolish in the eyes of their peers, but also for fear of being wrong [19]), but the urge to ask is, for many, irresistible here. We get questions like, “where did this sequence come from?” and “is the sequence of squares really going to 2?” and, if we are lucky, “does this mean that the original sequence tends to $\sqrt{2}$?” When we ask the Online Encyclopedia of Integer Sequences (OEIS) about the sequence of (say) denominators of the original sequence, we get [25, A051009], the “Reduced denominators of Newton’s method for $\text{sqrt}(2)$,” and at that point we can explain that the rule we were using to generate the sequence was

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{2}{x_n} \right) ; \quad (3)$$

that is, we start with an initial estimate ($x_0 = 1$), and then we average it with what we get when we divide it into 2. If our initial estimate of the square root had been correct, then we would stay the same; if our estimate was too small, then dividing it into 2 would give a larger number and averaging is plausibly a way to generate a better answer. It’s still mysterious to the students, but they are beginning to get interested.

It gets better: we then expand into continued fractions. For space reasons we use a compact notation here. The original sequence is 1 , $1 + [2]$, $1 + [2, 2]$, $1 + [2, 2, 2]$, $1 + [2, 2, 2, 2, 2, 2]$, and so on; the students are astonished, perhaps a little appalled. We do the first few by hand, and then the rest by using Maple’s “convert” facility (to “confrac”).

We can then digress to the other continued fractions $1 + [2, 2, \dots, 2]$ where the number of 2’s is not one of 1, 3, 7, 15, and so on. This is already a very fruitful line of investigation.

Indeed most students are stunned to discover that the square root of 2 has a perfectly predictable infinite continued fraction. Getting them to make the claim that

$$\sqrt{2} = 1 + [2, 2, 2, \dots] , \quad (4)$$

even given the numerical evidence above, takes encouragement from the instructor. The students are afraid, at this point, of a trap, of guessing wrong, especially when they “know” (i.e., have been taught) that the decimal expansion of the square root of two has no pattern. This naturally leads to a discussion what it means to perform an infinite number of operations. The OEIS proves useful here, as well; perhaps simply by letting the students in on the infinitely wonderful world of sequences.

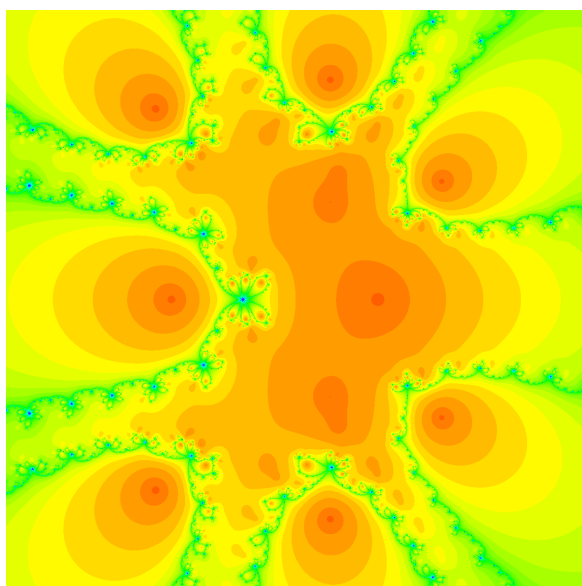
From here, we can go to true Newton’s method, in general, and not just for square roots.

4 Visualizing Newton’s Method

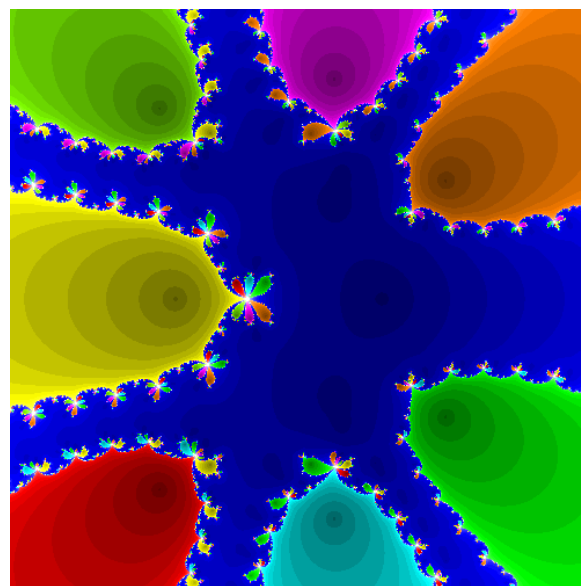
The chaotic dynamics of Newton’s method are very well-studied: see for instance [33], [31], [12], [21], [11], or indeed [23]. They prove perennially popular with students, in part because they produce such pretty pictures. See Figure 1, which shows the basins of attraction for Newton’s method on the polynomial

$$z^8 + 3z^7 + 5z^6 + 5z^5 + 4z^4 + 2z^3 + z^2 + z . \quad (5)$$

We want the students to learn to produce their own such pictures, and to begin to understand what they mean. To this end, we use both built-in procedures and hand-crafted procedures.



(a) Maple version



(b) Python version

Figure 1: Basins of attraction for Newton's method on a certain degree 8 polynomial, displayed in equation (5), as produced by the `Fractals:-Newton` command in Maple (left) and in by a custom Python program (right). The view in the complex plane is from $-1.8 \leq x \leq 1.0$ and $-1.4 \leq y \leq 1.4$.

The best ones of all are the ones the students learn to write for themselves, but we have to work up to this.

4.1 Summary of what we teach in this section

We teach the students

1. The special Newton iteration $z_{n+1} = (z_n + a/z_n)/2$ for finding square roots.
2. The general Newton iteration formula $z_{n+1} = z_n - f(z_n)/f'(z_n)$; we usually give a graphical demonstration similar to the usual one sketched in a first calculus class.
3. What happens when you apply Newton iteration to $f(z) = z^2 - 2$, using exact rational arithmetic and $z_0 = 1$, and its connection to continued fractions (as previously discussed).
4. A game, which we call "pass the parcel," which cements the idea of iteration: the students must take turns evaluating the Newton iteration and then pass their answer on to the next group to compute the next iterate.
5. The importance of the initial estimate z_0 (more on this, below, but we try to get the students to realize this without us telling them; this actually happens, especially after playing "pass the parcel," and the students do ask how to choose the initial estimate).
6. That z_n exactly solves the *nearby* equation $f(z) - f(z_n) = 0$ and show that $f(z_n)$ gets very small very rapidly if the initial estimate is good. We interpret this as a change in the question: e.g. $17/12$ is not just an approximation to the square root of two, it is the exact square root of $2^{1/144}$; that is, it is the exact square root of an approximation of two, and we can compute this *residual error* even if we don't know the true value of the square root of two.

7. The very beginnings of floating-point arithmetic and its peculiarities. This is actually very hard to teach at this level, but we believe it is necessary. We also teach the beginnings of complexity theory, by showing that exact arithmetic is costly; floating-point is vastly faster, *because its memory usage is predictable*.
8. How to use the Online Encyclopedia of Integer Sequences (OEIS) at <http://oeis.org/>.
9. What happens when you apply Newton iteration with a *real* initial guess to $f(z) = z^2 + 1$; see [31] and [21].
10. How to use the Maple `Fractals` package to make fractals of their own.
11. Variations on Newton iteration including the secant method, Halley’s method, Householder iteration, and Schröder iteration; and that all of these (apart from the secant method) can be considered as Newton iterations of *some* function (although the proof of that requires calculus).

4.2 Behind the curtain: why we teach what we do, and what the students get out of it

The main pedagogical purpose of this section is to reinforce the notion of a *function*. Students usually have been taught functions and even differentiation rules in high school, but frequently they differentiate only by rote and their notion of a function is weak: they typically only think of functions as expressions. By getting them to play “pass the parcel” (with the support of a Maple operator or procedure for the function) we emphasize the *active* nature of a function. This also is an easy way to break the “inactivity barrier” because they have to interact with their fellow students.

We use *scaffolding*: we supply the functions $f(z) = 0$ to solve and the initial estimates z_0 , at first; but then we give some functions *without* initial estimates. The students quite rightly find it difficult to construct their own (although some ingenuity is often displayed) and are naturally led to ask the crucial question of what is the influence of the initial estimate. This leads directly to the Newton fractal pictures.

The secondary pedagogical purpose is a gentle introduction to the software tools. We also use Python for this section, but in that case there is more programming involved⁴, up to and including *automatic differentiation* (as opposed to *symbolic* differentiation, or as opposed to differentiation by hand, or as opposed to numerical differentiation by finite differences or other numerical techniques).

A third pedagogical point is to introduce the beginnings of complexity theory—computation with exact rationals is expensive and the growth of the length of the exact rational answers is remarkable; students are quickly converted to the worth and utility of floating-point arithmetic. Newton’s method is relatively benign for use in floating-point, usually; this is a help. We only introduce as much numerical analysis of floating-point numbers as we are forced to do.

There are many directions to take this unit: we make connections to number theory via the theory of continued fractions, for instance. We choose to do this in part because continued fractions are not part of (most) students’ curricular choices, and therefore we are not “stealing

⁴What the students gain from this increase in programming effort includes greater control: for instance, the picture in Figure 1b includes more intelligible detail, in that different colours correspond to different limits in the iteration, whereas the colours in the Maple fractal were chosen generically by the developer (David W. Linder at Maplesoft) to indicate iteration count only.

the thunder” from subsequent courses⁵. The main reason, though, is that the results are surprising and beautiful.

Using the OEIS, for instance, the students can be guided to discover that the n th iterate of Newton’s method for $f(z) = z^2 - m$, starting at the rational initial estimate $z_0 = a/b$, is

$$z_n = \sqrt{m} \left(\frac{1 + r^{2^n}}{1 - r^{2^n}} \right) \quad (6)$$

where $r = (a - b\sqrt{m})/(a + b\sqrt{m})$. Since $0 < r < 1$ if a , b and m are positive, this shows the quadratic convergence of Newton’s method very well⁶. There are other number-theoretic directions to go, as well: one can use this as a springboard to study Pell’s equation. There are higher-dimensional versions, and indeed the *matrix* square root is a subject of much research—see for example [17] and [18].

We try to avoid using the word “convergence,” however, and satisfy ourselves by computing the exact square roots of numbers near to 2 (for instance). This is a surprisingly useful intermediate step.

4.3 Suggestions for Assessment

It is straightforward to ask the students to compute Newton iterations for other quadratic irrationals, and for other algebraic numbers. It is also straightforward to get them to use Maple’s built-in Newton fractal package to find basins of attraction for functions of their own choice. Our most successful exercise here so far, however, has been to hold a contest to see who could find a function that produced the “best” picture (by choosing their own functions and regions). Unsurprisingly, the students overwhelmingly produce excellent pictures, and we have sometimes settled on making a collage of them all. The students are then asked to make a similar collage of the basins of attraction of Halley’s method⁷ on the *same* functions; by asking Maple to do Newton iteration on $f(x)/\sqrt{f'(x)}$ one is actually asking for Halley iteration⁸. The resulting pair of collages is quite instructive in that one can understand some differences between Newton iteration and Halley iteration intuitively thereby.

5 Bohemian Matrices

The section of the course on **Bohemian Matrices** is popular, in part because there are so many interesting pictures already, but mostly because there is so much unknown. Many of the pictures in the gallery at www.bohemianmatrices.com were actually produced by students in the course, and never seen by anyone before those students made them. In Figure 2 you can

⁵This curious English expression means, in this context, to use the highlights; “stealing thunder” would make this particular course more exciting, but would make subsequent courses more boring, which is unfair to the downstream teacher.

⁶This purely algebraic formula is equivalent to the cotangent formula of [31] and [21].

⁷In some classes we covered quite a few iterative methods: see [21] for a case where a student solved a problem (in class!) that was believed to be open.

⁸This fact is in the literature, as is the more general fact previously mentioned that every iteration can be cast as a Newton iteration for *some* function.

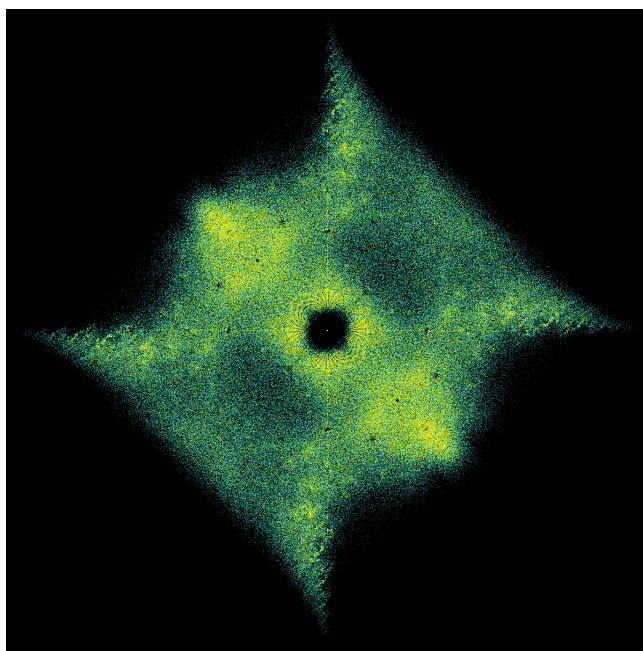


Figure 2: Eigenvalue density plot of all $2^{17} = 131072$ ten by ten Bohemian skew-pentadiagonal matrices with population $\{1, i\}$, plotted on $-3.25 \leq \Re(\lambda) \leq 3.25$ and $-3.25 \leq \Im(\lambda) \leq 3.25$ in the complex λ -plane. Hotter colours have higher density. At this time of writing we have no explanation whatever for the distribution pattern visible here.

see a new picture made specifically for this paper⁹, and likewise never seen before by anyone (there are infinitely many such; welcome to the party).

5.1 Summary of what we teach

We don't wish to steal the thunder of linear algebra classes, so we teach eigenvalues from a determinantal point of view. We begin with the single linear equation $t_1x_1 = 1$ in the single unknown x_1 where t_1 is known to be drawn from our finite population (say, t_1 must be a member of $P = \{-1, 0, 1\}$). Then, unless $t_1 = 0$, we can solve this system of equations; that is, whether $t_1 = 0$ or not *determines* whether or not this equation can be solved for x_1 . We then move on to a related two-by-two system of equations in the two unknowns x_1 and x_2 :

$$\begin{aligned} t_1x_1 + t_2x_2 &= 1 \\ -x_1 + t_1x_2 &= 0 \end{aligned} \tag{7}$$

⁹This uses *skew-symmetric pentadiagonal* matrices; the five by five case with generic entries is

$$\begin{bmatrix} 0 & t_1 & t_5 & 0 & 0 \\ -t_1 & 0 & t_2 & t_6 & 0 \\ -t_5 & -t_2 & 0 & t_3 & t_7 \\ 0 & -t_6 & -t_3 & 0 & t_4 \\ 0 & 0 & -t_7 & -t_4 & 0 \end{bmatrix}.$$

Each t_i must be chosen from the finite population; all possible choices make up the Bohemian family. Here if $P = \{1, i\}$ there are $2^7 = 128$ possible matrices in this family with this dimension.

Only one new member of the population, t_2 , is involved. By multiplying the second equation by t_1 and adding the first equation, we get $(t_1^2 + t_2)x_2 = 1$, and now $t_1^2 + t_2$ determines whether or not we can solve the equations for x_1 and x_2 .

The three-by-three case is analogous:

$$\begin{aligned} t_1x_1 + t_2x_2 + t_3x_3 &= 1 \\ -x_1 + t_1x_2 + t_2x_3 &= 0 \\ -x_2 + t_1x_3 &= 0 \end{aligned} \tag{8}$$

and the students are easily guided to discovering that the determinant of this (Toeplitz) system is $t_1^3 + 2t_1t_2 + t_3$. The determinant of the 4 by 4 version is $t_1^4 + 3t_1^2t_2 + 2t_1t_3 + t_2^2 + t_4$, and the 5 by 5 version has determinant $t_1^5 + 4t_1^3t_2 + 3t_1^2t_3 + 3t_1t_2^2 + 2t_1t_4 + 2t_2t_3 + t_5$. The students have an interesting time trying to guess the pattern here, and even experts can be stumped: but we will leave it as an exercise.

Replacing t_1 with $t_1 - \lambda$ is straightforward; one then has a collection of polynomials to solve, which the students are happy to leave to the machine. One can then show them an “eigenvalue” solver without shocking them too much, or doing too much violence to the later curriculum.

5.2 Why we teach it

Eigenvalues are often given short shrift in a first linear algebra course, and the students need more practice with them. We can also use the occasion to show the students different matrix structures including symmetric matrices, skew-symmetric matrices, and others. The structure of the introductory example is a special case of a Toeplitz matrix structure. We even get the students to invent their own: the “checkerboard” matrix picture at www.bohemianmatrices.com is an example of a student-generated matrix structure, which we’ve never seen elsewhere and seems unlikely to have any application, but we like it anyway.

But the biggest reason to have this as part of the course is that the research area is so new that even many easy questions are as yet unanswered, and the students have quite a good chance to contribute something new. Frankly, since there are many open problems here, and we are just scratching the surface of this new field, we can use the students’ creativity, too.

5.3 Suggestions for Assessment

Asking students to count quantities of interest and to try to find patterns can be quite accessible, for some matrix structures. For instance, one can ask how many m by m Toeplitz matrices of the form used above there are, with population $P = \{-1, 0, 1\}$ (this is easy); one can ask how many distinct characteristic polynomials there are at dimension m ; one can ask how many different eigenvalues there are at dimension m (this is quite a bit harder, because some eigenvalues may be shared between more than one matrix in the Bohemian family); one can ask how many multiple eigenvalues there are, or how many matrices are singular, and so on. One can ask *programming* questions: how quickly can you generate all 3^m matrices, for instance.

Our favourite assessment here, however, came in the form of a contest: who could produce the most interesting pictures? As a “filtering” mechanism, this failed utterly: the students all loved each others’ pictures, and refused, quite rightly, to rank them. We were quite happy to give everyone full marks for this exercise.

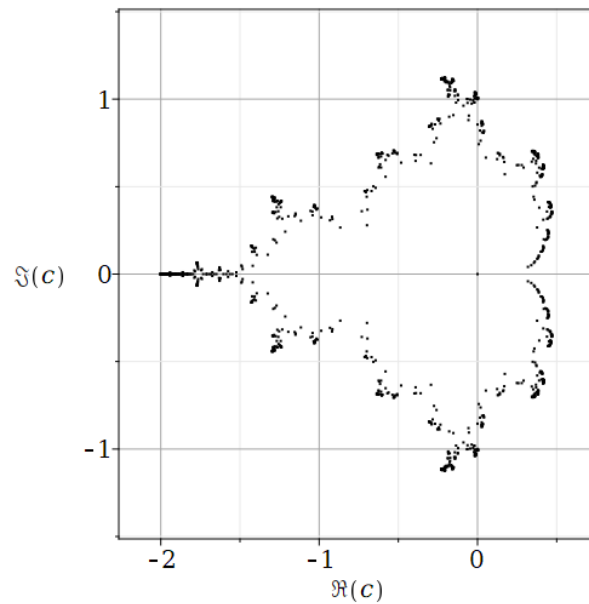


Figure 3: Zeros of $z_{11}(c)$, a polynomial of degree 1024. Each of these zeros is a value of c which leads to a periodic orbit of period 11, namely $0, z_1(c) = c, z_2(c), \dots, z_{10}(c), 0$. Such points are also called *hyperbolic centers* in the Mandelbrot set and are actually quite sparse in the Mandelbrot set.

6 The Mandelbrot Polynomials

For general material on the *Mandelbrot set*, the Wikipedia page on the subject provides a good starting point to learn more. Indeed many students are familiar with the definition of the Mandelbrot set, but few of them have seen the *Mandelbrot polynomials*, defined by $z_{n+1} = z_n^2 + c$ where $z_0 = 0$ and c is a symbol or variable representing an as yet unknown complex number. The first few of these polynomials are listed in [9]: $0, c, c^2 + c, c^4 + 2c^3 + c^2 + c$, and so on.

By studying the *zeros* of these polynomials we are studying the *periodic points* of the Mandelbrot iteration. This turns out to be surprisingly fruitful, in that there is a connection to eigenvalue problems, as detailed in the aforementioned paper and its references, as well as a connection to the modern theory of dynamical systems.

6.1 Summary of what we teach

We use this example to teach that computing roots of polynomials can be numerically difficult: the condition number $B_n(c) = \sum_{k=0}^{d_n} |a_k| |c|^k$ of the Mandelbrot polynomials $z_n(c) = \sum_{k=0}^{d_n} a_k c^k$ grows *doubly exponentially* with n (singly exponentially with the degree $d_n = 2^{n-1}$) which means that the number of decimal digits needed to compute the zeros grows linearly with n . In contrast, the matrix eigenvalue problem is well-conditioned, and can be used in ordinary double precision floating-point arithmetic for remarkably large dimension of the matrix and degree of the polynomial.

6.2 Why we teach it

The Mandelbrot polynomials give a good excuse for studying *nonlinear* iterations in general; it is a very great surprise that, for c not in the Mandelbrot set, this recurrence relation can be “solved” exactly (well, by a convergent series); see [9] and [20]. We did not know this ourselves when we were teaching from this material, and we now wonder just how much of this we can include in a first course. But Peter Taylor of Queen’s University¹⁰ has been able to successfully use a similar iteration and its asymptotics to study a bacterial growth model in a first-year context, and so we hope to be able to do so as well.

6.3 Suggestions for Assessment

Students enjoy writing fast programs to draw the Mandelbrot set; getting them to carefully draw pictures of the periodic points in the Mandelbrot set is not difficult.

There are 21 “facts” listed in [9, Sec. 2], some of which make good questions for students to attempt; in the following section there are seven *conjectures*, open as we write this. The last two, asking if Mandelbrot polynomials are *unimodal*¹¹, seem as if they ought to be accessible. However, we don’t know how to prove either, and one of us tried quite hard. Perhaps a student could do it. If so, they would get a paper out of it.

Getting them to *come up with their own questions*, however, is one of the goals of the course. By this time in the course, we hope that the students are doing so: perhaps wondering what happens if the constant c is replaced by a pre-specified sequence, or a random sequence; or what happens if c is not a scalar but a matrix; or if the iteration is $z_{n+1} = z_n^\alpha + c$ for some noninteger α , possibly even complex α . Some of these questions are sterile, as will be some of the questions the students come up with; but some might not be, and the adventure begins.

7 Concluding Remarks and Further Reading

Our aim is to try to teach an experimental approach to mathematics, *without* either stealing the thunder of courses already in the standard curriculum, or requiring significant prerequisites. Mathematics is vast and the standard undergraduate curriculum only scratches its surface. Nonetheless, those scratches are somehow *fractal* and it is difficult to avoid them while remaining relevant and interesting. As presented here, this material comes right up to and touches on several fundamental notions of the standard mathematics curriculum: for example, we want the students to *experience* convergence, which motivates the theory of limits. We also touch on existence and uniqueness of solutions of equations, and the meaning of proof. We believe that this extra motivation for standard material is to the good.

The use of advanced mathematics to teach early undergraduate students was advocated by Mandelbrot himself. See for instance [22], [15], and [23].

We are not trying to get the students to a destination: we want them to experience the journey. It is less like using a GPS, and more like an extended Sunday exploration, perhaps with a paper map.

¹⁰This is from a talk he gave at the Summer Canadian Math Society Meeting 2021.

¹¹A polynomial is unimodal if its nonzero monomial basis coefficients are all positive integers, and those coefficients increase towards a unique maximum and then decrease [34].

When introducing students to the ideas of experimental mathematics, learning how to ask questions is not just a part of the process, it *is* the process.

Therefore, we concentrate much more heavily on things *not* in the standard curriculum: such as actual mathematically-oriented computer programming, which strengthens the notion of a function. This kind of programming also both motivates, and partially replaces, the notion of proof, for this course. A working program helps you check that you have an effective definition.

Other authors have tried active non-standard approaches before, of course. One that comes to mind is Z. A. Melzak's *Companion to Concrete Mathematics* [24], which is a rich source of nonstandard tricks and the problems they apply to. The most venerable of such works, however, is the magnificent two-volume set by Pólya and Szegő [27, 28].

Why use active learning at all? The evidence for its effectiveness now is so strong—see, for example, [16]—that it is unethical *not* to use it, if you can. It is as simple (and as difficult) as that. The mathematics education literature is very nearly as vast as the mathematics literature¹², but there are few ideas so well-supported in that literature as the now-established fact that it is better for students to *do mathematics* than it is for them to listen or watch it and then regurgitate it on an exam. Indeed, quoting from [35], “Creativity does not come from drills.”

There are principles that help the teacher design good exercises to increase the activity of the student: a good exercise should be connected to great currents of mathematical thought (so as not to waste the student's time); it should be engaging, perhaps by having an element of surprise about it; it should be accessible at the student's level of education; it should be rich and allow open-ended exploration. Those two last criteria are sometimes termed “low floor, high ceiling,” meaning the exercise should be easy to start, but have many further levels. We hope that you agree that the examples presented here fit these criteria.

Where the literature diverges, however, is on how best to actually *get* the students to be active. For example, the cumulative online proceedings of this conference since at least 1997 show a very wide variety of approaches, and it is quite possible that they *all* work, perhaps to varying degrees. What we are recommending here, like many others before us, such as [2], is to *enrich the content* [13]; as in [5] and [4] we are describing courses specifically designed to encourage active learning of mathematics. Given that many curricula are already overfull, the question of where this will fit in the student's timetable is a difficult one, but it seems clear that to make room for this course, some other less-useful course will need to be pushed aside. We, innocently, will look the other way, whistling, as you decide just which.

Acknowledgements. We thank Professor Wei-Chi Yang for the invitation to present this work, and the anonymous referees for very helpful remarks. This work was partially supported by an award from the Centre for Teaching and Learning at Western University.

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¹²In addition to the many volumes of ATCM proceedings, see, for instance, the chapters in [30] and the references therein, as a slice of the intersection of the math ed literature with AI literature.

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On the locus related to chords of conic sections

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Abstract

The paper deals with the locus of points related to chords of conic sections. Firstly the locus is explored using dynamic geometry software, particularly for displaying it, secondly, using elimination in computer algebra software, the locus equation is derived. However this elimination leads to the zero elimination ideal. It is shown how to compute the searched equation in such a case. Further the locus is applied in the proof of the theorem which is related to the Frégier point. Finally, connection between the original formulation of the locus and the formulation by an envelope is demonstrated.

By solving the problem we mainly use dynamic geometry software GeoGebra and computer algebra program CoCoA and Singular.

1 Introduction

In the paper we investigate the locus of points related to chords of conic sections. Firstly, the locus is explored using dynamic geometry program GeoGebra [3], particularly for displaying it, secondly, using elimination in computer algebra program CoCoA [1] and Singular [6], the locus equation is derived, see Theorem 1. However this elimination leads to the zero elimination ideal. It is shown how to compute the searched locus equation in such a case.

Next, we will show a connection of the Theorem 1 to the theorem on Frégier point [7]. Finally a correspondence between the original formulation of the locus and a formulation by an envelope is discussed, see Theorem 2.

Let us start with a few examples of the locus in various types of conics.

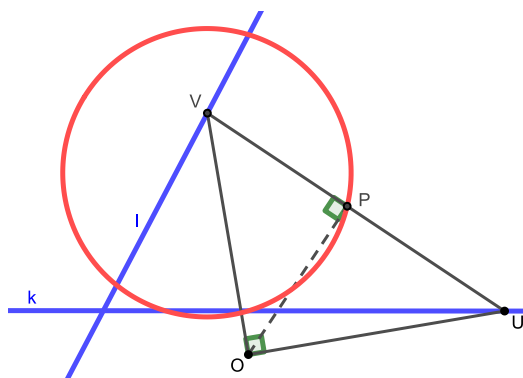


Figure 1: Determine the locus of P when U moves along the line k

Consider two lines k, l , a point U in k and an arbitrary point O . Construct the point V in l such that $OV \perp OU$. Determine the locus of the foot P of the perpendicular from O to the chord UV when U moves along the line k .

To display the locus we use GeoGebra command `Locus`, which works on numerical basis. It seems that the point P lies in a circle, Fig. 1.

Similarly, let us show another example:

Given a hyperbola centred at its centre O and a point U in it. Construct the point V in the hyperbola such that $OV \perp OU$. Determine the locus of the foot P of the perpendicular from O to the chord UV when U moves along the hyperbola.

Using the command `Locus` it seems that the point P lies in a circle as well, see Fig. 2.

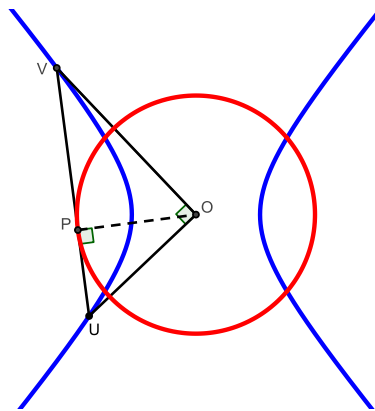


Figure 2: Determine the locus of P when U moves along the hyperbola

The construction is also valid when the point O is not at the centre of a conic. It may happen

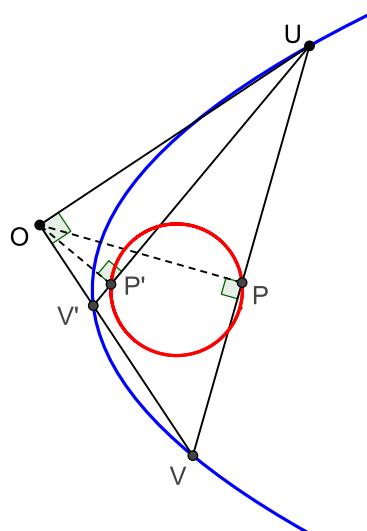


Figure 3: The locus of P is a circle

that sometimes we get only a part of the circle. To get the whole circle, realize that there are two points V, V' in the conic such that $OU \perp OV$ and $OU \perp OV'$. This leads to two points P, P' that together trace the circle, see Fig. 3 in the case of a parabola. This enables us to use this construction for all conics.

In all examples above we get the same locus. In the next section we will formulate the related theorem and prove it.

2 Chord of conics

In accordance with the previous constructions we formulate the theorem:

Theorem 1: *Given a conic κ , a point U in κ and an arbitrary point O . Let V be a point in κ such that $OV \perp OU$. Then the foot P of the perpendicular from O to the line UV when U moves along the conic lies on:*

- a circle if κ is not an equilateral hyperbola or a pair of mutually orthogonal lines,
- a line if κ is an equilateral hyperbola or a pair of mutually orthogonal lines.

Proof: We will find the locus equation. Consider a conic

$$\kappa : ax^2 + cy^2 + dx + ey + f = 0. \quad (1)$$

Adopt a rectangular system such that $O = [r, s]$, $U = [u_1, u_2]$, $V = [v_1, v_2]$ and $P = [p, q]$,

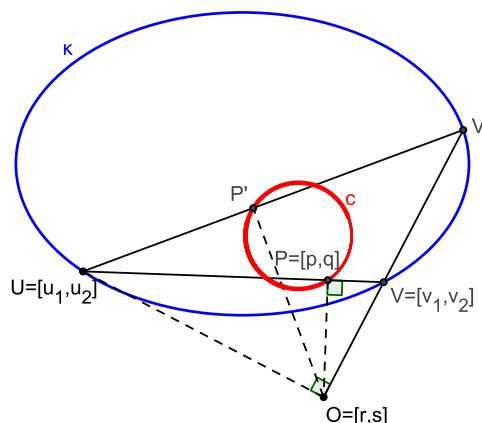


Figure 4: Determine the locus of P when U moves along the conic κ

Fig. 4. Then:

$$U \in \kappa \Leftrightarrow h_1 := au_1^2 + cu_2^2 + du_1 + eu_2 + f = 0,$$

$$V \in \kappa \Leftrightarrow h_2 := av_1^2 + cv_2^2 + dv_1 + ev_2 + f = 0,$$

$$OV \perp OU \Leftrightarrow h_3 := (u_1 - r)(v_1 - r) + (u_2 - s)(v_2 - s) = 0,$$

$$OP \perp UV \Leftrightarrow h_4 := (p - r)(u_1 - v_1) + (q - s)(u_2 - v_2) = 0,$$

$$P \in UV \Leftrightarrow h_5 := pu_2 + u_1v_2 + qv_1 - u_2v_1 - pv_2 - qu_1 = 0.$$

Elimination of u_1, u_2, v_1, v_2 in the system $h_1 = 0, h_2 = 0, \dots, h_5 = 0$ yields

```
Use R:=Q[a,c,d,e,f,p,q,r,s,u[1..2],v[1..2]];
I:=Ideal (au[1]^2+cu[2]^2+du[1]+eu[2]+f,av[1]^2+cv[2]^2+dv[1]+ev[2]+f,
(u[1]-r)(v[1]-r)+(u[2]-s)(v[2]-s),(p-r)(u[1]-v[1])+(q-s)(u[2]-v[2]),
pu[2]+u[1]v[2]+qv[1]-u[2]v[1]-pv[2]-qu[1]);
Elim(u[1]..v[2],I);
Ideal(0);
```

the zero elimination ideal, see [2]. This could be a problem. I am not sure whether a general solution of this problem (in the case of zero elimination ideal) is known. We could tackle the problem in the following way.

First compute the Hilbert dimension of I (cardinality of the maximal independent set of variables for I) in CoCoA. We get

```
Dim(R/I)=9;
```

But in standard cases we would expect that $\text{Dim}(R/I)=8$ since we have 13 variables and 5 equations. Then there must be a component of dimension 9 that is degenerate.

Further we will proceed by a heuristic approach. Let us suppose, that $U \neq O$, i.e. $((u_1 - r)^2 + (u_2 - s)^2)t - 1 = 0$, where t is a slack variable. Realize that if $U = O$ then the line OU is not defined. We add this condition to the ideal I and eliminate variables u_1, u_2, v_1, v_2, t . One obtains

$$(a + c)(p^2 + q^2) + (d - 2cr)p + (e - 2as)q + cr^2 + as^2 + f = 0 \tag{2}$$

which is a desired locus equation. In (2) we distinguish two cases:

- a) If $a + c \neq 0$ then the locus is a circle.
- b) If $c = -a$, i.e. if κ is an equilateral hyperbola or two orthogonal lines, the locus equation

$$(d - 2cr)p + (e - 2as)q + cr^2 + as^2 + f = 0,$$

represents a line, Fig. 5. □

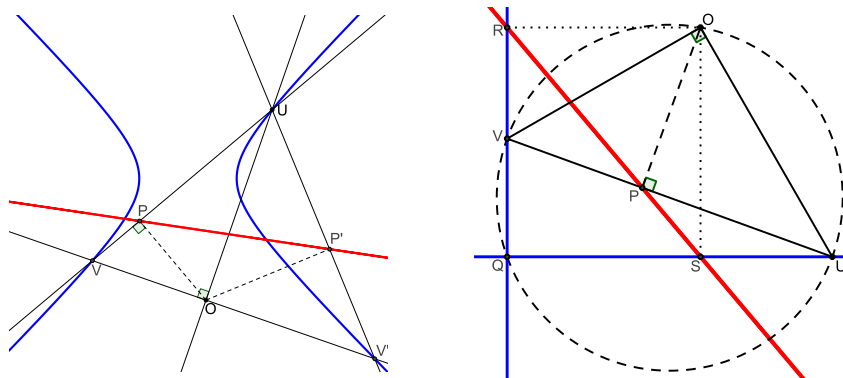


Figure 5: For equilateral hyperbola or two orthogonal lines the locus of the point P is a line

Remark: 1) We could also compute the characteristic series of the system of equations I [5]

using the command `char_series` in Singular [6].

2) One can easily check using the command `NF(I)` (normal form of I) in CoCoA that the product

$$((a + c)(p^2 + q^2) + (d - 2cr)p + (e - 2as)q + cr^2 + as^2 + f)((u_1 - r)^2 + (u_2 - s)^2)$$

really belongs to the ideal I , whereas the polynomial in (2) not.

3) The case of two orthogonal lines can also be proved classically. Applying the Simson–Wallace theorem on the triangle QUV and O in its circumcircle, the points P, S, R are collinear (the Simson line), Fig. 5 right.

2.1 Connection to the Frégier’s theorem

About in 1815 M. Frégier published the following theorem [7], [9]:

Given a conic κ and a point O on κ , then the hypotenuses of right-angled triangles inscribed to κ and having common right-angle vertex O intersect at one point F , the Frégier point to O with respect to κ .

We will prove the theorem using the Theorem 1 when the point O lies in the conic, Fig. 6. The hypotenuse UV of the right triangle UOV intersects the locus circle c at the points P and F .

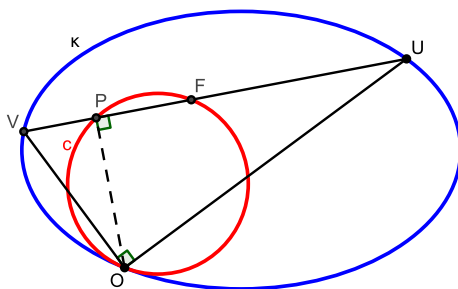


Figure 6: The point F is fixed for all positions of U

To show that the point F is fixed for all positions of U , realize that P lies in the circle c and hence the segment OP must be its diameter. Since O and c are fixed the Frégier theorem follows.

3 Formulation of the locus by an envelope

In this section we arrive at the locus above using envelopes [8]. Let us briefly describe what is the envelope and how to obtain it [4].

The envelope of a one parameter family of curves $F(x, y, t) = 0$, is a curve which is tangent to every curve of the family.

The equation of the family may be given in an implicit form as $F(x, y, t) = 0$, where t is a parameter. To find the equation of the envelope, it is necessary to eliminate the parameter t both from the equation of the family and its partial derivative with respect to the

parameter $\partial F(x, y, t)/\partial t = 0$. This is guaranteed for those points for which $(\partial F(x, y, t)/\partial x)^2 + (\partial F(x, y, t)/\partial y)^2 \neq 0$. If both $\partial F(x, y, t)/\partial x$ and $\partial F(x, y, t)/\partial y$ are zero, then the envelope can have a singular point here.

We show a connection between the above formulation of the locus and its formulation by envelopes in the following Theorem 2, where a conic is for simplicity presented by two lines.

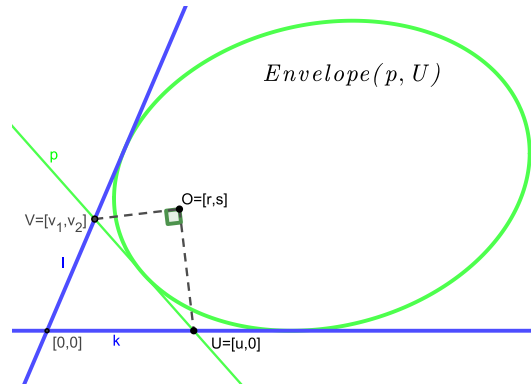


Figure 7: The envelope of lines p when U moves along the line k is a conic

Theorem 2: *Given two lines k, l containing a point U in k and an arbitrary point O . Let V be such a point in l that the lines OV and OU are orthogonal. For U moving along the line k the envelope of the family of lines $p = UV$ is a conic.*

Proof: Choose a rectangular coordinate system such that $k := y = 0$, $l := ax + by = 0$, $O = [r, s]$, $U = [u, 0]$, $V = [v_1, v_2]$, Fig. 7. Then:

$$V \in l \Leftrightarrow h_1 := av_1 + bv_2 = 0,$$

$$OV \perp OU \Leftrightarrow h_2 := (r - v_1, s - v_2) \cdot (r - u, s) = 0,$$

$$X \in UV \Leftrightarrow h_3 := xv_2 + uy - uv_2 - yv_1 = 0.$$

Elimination of variables v_1, v_2 in the system $h_1 = 0, h_2 = 0$ and $h_3 = 0$ gives

Use $R := \mathbb{Q}[a, b, r, s, x, y, u, v[1..2]]$;

$J := \text{Ideal}(av[1]+bv[2], (r-v[1])(r-u)+s(s-v[2]), xv[2]+uy-uv[2]-yv[1])$;

$\text{Elim}(v[1..2], J)$;

a one parameter family of lines $p(x, y, u)$ with the parameter u

$$p(u) := arxu - ar^2x - as^2x + 2bryu - asyu - br^2y - bs^2y + ar^2u + as^2u - aru^2 - byu^2 = 0.$$

Partial derivative of $p(x, y, u)$ with respect to the parameter u yields

$$\frac{\partial p}{\partial u} := ar^2 + as^2 + arx + 2bry - asy - 2aru - 2byu = 0.$$

Finally, eliminating u in the system $p = 0, \partial p/\partial u = 0$ we get the equation of the envelope of the family of lines $p(x, y, u)$

$$a^2r^2x^2 - 2as(ar + 2bs)xy + s(a^2s - 4abr - 4b^2s)y^2 - 2a^2r(r^2 + s^2)x - 2a^2s(r^2 + s^2)y + a^2(r^2 + s^2)^2 = 0$$

which is a conic, see Fig. 7. Note that the point O is the focus of the conic, as we can compute

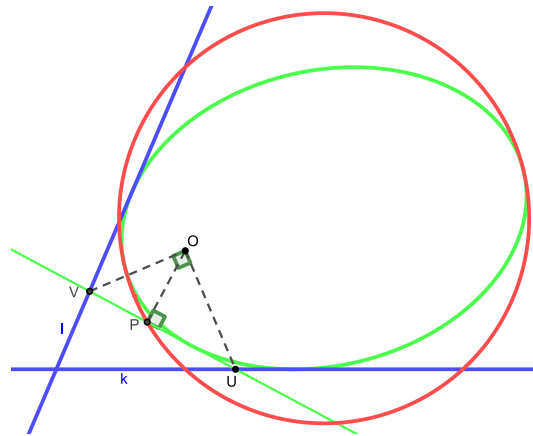


Figure 8: The locus of points P is the director circle of the conic

from the equation above. □

To display the envelope of the family of lines p we can use GeoGebra command `Envelope`, Fig. 7.

Now it is easy to arrive at the circle. It is well-known that feet of perpendiculars from the focus of a conic to all its tangents form the director circle of the conic, see Fig. 8.

In the case when the lines k and l are mutually orthogonal, the envelope is a parabola, with the focus at O . Then the feet of perpendiculars from the focal point O to tangents of the parabola

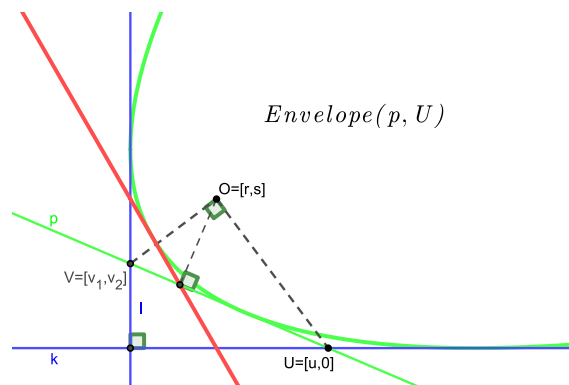


Figure 9: If $k \perp l$ then the envelope of the family of lines p is a parabola

form the directrix line, see Fig. 9.

Similarly, the director circle appears in the case of hyperbola.

4 Conclusions

In the paper locus of points related to chords of conic sections and its properties are described. The locus is explored using both dynamic geometry and computer algebra software. It is

shown how to compute the locus equation in the case when we obtain the zero elimination ideal. Finally, connection of the locus to the envelope of a parametric family of lines and its relation to the Frégier point is given.

There are some questions for a future work. The first one is relating to generalization of the locus construction for an arbitrary angle. The second one relates to the 3D version of the construction.

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The Mathematics of QR Codes

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Abstract: *Quick Response (QR) codes are two-dimensional bar codes that have become a common medium for easily accessing information such as URLs, phone numbers, and small amounts of text. The creation of QR codes requires placing data in a prescribed format so that it can be displayed on a surface and then detected by common QR code scanner software. Like other physical means of storing data, QR codes are prone to errors when their data is interpreted in a digital format. Reed-Solomon codes provide a mechanism for ensuring that QR code scanners can reliably process information when errors occur. This involves sending information in the form of polynomial coefficients using finite field arithmetic. Utilizing Reed-Solomon codes allows logos and emblems to be embedded within QR codes to advertise their purpose. In this paper we will present how QR codes are constructed, and how Reed-Solomon codes are incorporated into them to provide error correction. To assist in demonstrating this, technology involving Maplets will be used.*

1 Introduction

Since their 1994 invention by the DENSO Corporation in Japan, QR codes have provided a widely used method for quickly accessing information. Originally, QR codes were used for parts manufacturing inventory tracking, but they can now be found on advertisements, web pages, business cards, any many other mediums. QR codes are notable for their long-term stability and higher capacity for storage than other bar codes. However, like other physical means of storing data, QR codes are prone to errors. Reed-Solomon codes provide a mechanism for ensuring that QR code scanners can reliably process information when errors occur.

In this paper we will give an overview of how QR codes are created, and how data is integrated into their construction. As part of this, we will describe the basics of how Reed-Solomon codes work, and how Reed-Solomon codes are incorporated into QR codes to increase the likelihood that they are able to transmit data reliably. This information provides a means for demonstrating a hands-on method for the use of mathematics in a practical real-life application.

2 Data Representation and Finite Fields

QR codes encode text represented as binary numbers. The QR code standard allows for text strings to be created using four encoding modes: numeric, alphanumeric, byte, and kanji. In this paper we will describe the byte mode. Descriptions of the other three modes can be found in [5].

The byte mode of encoding text for QR codes utilizes the ASCII character set. A list of the ASCII character set and the correspondences between characters and decimal numbers in ASCII for the printable characters on a modern keyboard is shown in Table 1.

Char	Num	Char	Num	Char	Num	Char	Num	Char	Num
(space)	32	3	51	F	70	Y	89	l	108
!	33	4	52	G	71	Z	90	m	109
"	34	5	53	H	72	[91	n	110
#	35	6	54	I	73	\	92	o	111
\$	36	7	55	J	74]	93	p	112
%	37	8	56	K	75	^	94	q	113
&	38	9	57	L	76	_	95	r	114
'	39	:	58	M	77	`	96	s	115
(40	;	59	N	78	a	97	t	116
)	41	<	60	O	79	b	98	u	117
*	42	=	61	P	80	c	99	v	118
+	43	>	62	Q	81	d	100	w	119
,	44	?	63	R	82	e	101	x	120
-	45	@	64	S	83	f	102	y	121
.	46	A	65	T	84	g	103	z	122
/	47	B	66	U	85	h	104	{	123
0	48	C	67	V	86	i	105		124
1	49	D	68	W	87	j	106	}	125
2	50	E	69	X	88	k	107	~	126

Table 1: Correspondences between characters and decimal numbers in ASCII.

Each character in ASCII corresponds to a decimal number, which can then be expressed as a string of eight binary digits, or a *byte*. For example, the character M corresponds to the decimal number 77, which can be expressed in binary as 1001101, or the byte 01001101. This byte can then be represented as a polynomial, by using these digits in reverse order as coefficients on powers of the variable a , with terms of increasing degree. For example, the byte 01001101 can be represented as the polynomial $1 + 0a + 1a^2 + 1a^3 + 0a^4 + 0a^5 + 1a^6 + 0a^7 = 1 + a^2 + a^3 + a^6$. Similarly, every character in ASCII can be expressed as a byte, and as a polynomial in a of maximum degree 7. Since all characters can be expressed as binary numbers, all computations that follow will be done using modulo 2 arithmetic. This means that all numerical computations will result in a 0 or 1. Specifically, if a result gives a number that is divisible by 2, it can be reduced to 0, and if it gives a number that is not divisible by 2, it can be reduced to 1. For example, the number 6 with modulo 2 arithmetic reduces to 0; that is, $6 \bmod 2 = 0$. The number 7 with modulo 2 arithmetic though reduces to 1; that is, $7 \bmod 2 = 1$. This can also work with coefficients of polynomials. For example, with modulo 2 arithmetic, the polynomial $7a^3 + 4a^2 - a + 2$ reduces to $(7a^3 + 4a^2 - a + 2) \bmod 2 = 1a^3 + 0a^2 + 1a + 0 = a^3 + a$.

A particular finite field provides the method with QR codes for performing operations on polynomials that represent information. The finite field used with QR codes contains $2^8 = 256$ elements, and uses the polynomial $p(x) = x^8 + x^4 + x^3 + x^2 + 1$ to generate the 255 of these field elements which are nonzero. These 255 nonzero field elements can be expressed as all nonzero polynomials in the variable a of maximum degree seven and with coefficients that are all either 0 or 1, which can be generated as follows. Let $x = a$ be a root of $p(x)$, called a *primitive* element in the field. To generate all of the nonzero elements in the field, we can raise a to consecutive integer powers, from the first through the 255th. That is, the 255 nonzero field elements are $\{a, a^2, \dots, a^{254}, a^{255}\}$, the last of which will equal 1 (an explanation for which can be found in [3]). The finite field is then completed by including 0 with this set. To represent the 255 nonzero field elements as polynomials in a of maximum degree seven and with coefficients that are all either 0 or 1, we can use the fact that a is a root of $p(x)$. As such, after the first seven nonzero field elements have been formed (which are simply a, a^2, \dots, a^7) and a^8 is reached, since a is a root of $p(x)$, then $p(a) = a^8 + a^4 + a^3 + a^2 + 1 = 0$. Solving for a^8 and reducing the coefficients with modulo 2 arithmetic gives $a^8 = -a^4 - a^3 - a^2 - 1 = a^4 + a^3 + a^2 + 1$. That is, $a^4 + a^3 + a^2 + 1$ is the polynomial of maximum degree seven that can be used to represent a^8 in the field. Subsequent nonzero field elements can be generated similarly.

$$\begin{aligned}
 a^9 &= aa^8 = a^5 + a^4 + a^3 + a \\
 a^{10} &= aa^9 = a^6 + a^5 + a^4 + a^2 \\
 a^{11} &= aa^{10} = a^7 + a^6 + a^5 + a^3 \\
 a^{12} &= aa^{11} = a^8 + a^7 + a^6 + a^4 = a^4 + a^3 + a^2 + 1 + a^7 + a^6 + a^4 = a^7 + a^6 + a^3 + a^2 + 1 \\
 &\vdots
 \end{aligned}$$

This process could be continued until the last nonzero element, $a^{255} = 1$, were reached. Since doing this would generate all 255 nonzero polynomials in the variable a of maximum degree seven and with coefficients that are all either 0 or 1, then $p(x)$ is also characterized as a *primitive* polynomial.

We will now demonstrate a Maplet¹ written by the authors that can be used to generate the elements in this finite field, with the nonzero elements expressed as polynomials in a of maximum degree seven and with coefficients that are all either 0 or 1. Specifically, Figure 1 shows how the Maplet **FiniteFieldGenerator** can be used for this purpose.

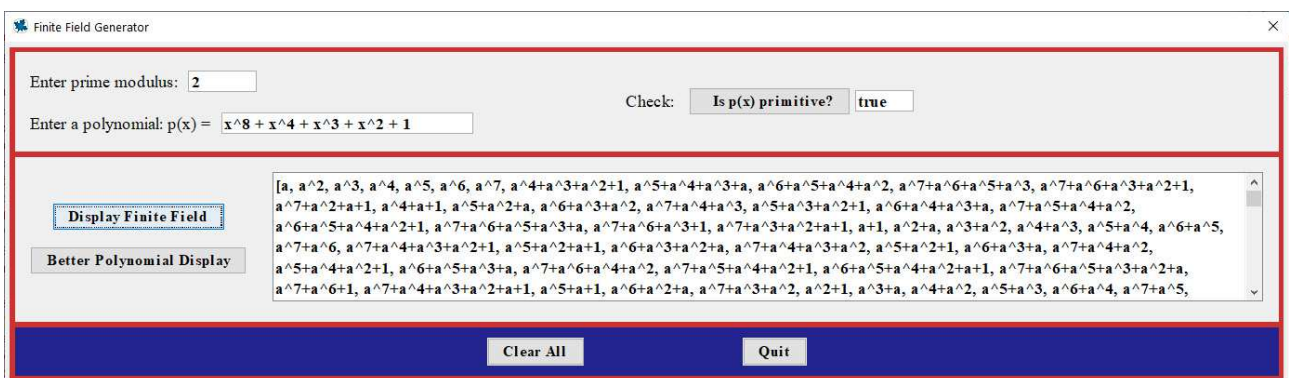


Figure 1: Finite field elements for primitive polynomial $p(x) = x^8 + x^4 + x^3 + x^2 + 1$.

¹A Maplet is like an applet, but uses (and requires) the engine of the computer algebra system Maple, and is written using Maple functions and syntax.

A QR code formats a message with specifications, and places the result on a two-dimensional grid made up of pixels. QR codes can have grid sizes ranging from the smallest for version 1 at size 21×21 , to the largest (as of this writing) for version 40 at size 177×177 . Each new QR code version after the first uses a grid size with 4 more rows and 4 more columns than the grid size used in the previous version. The size of the grid for a particular message depends on the number of characters in the message and the error correction level specified for the code. Error correction is included to help ensure that messages can still be read correctly even if part of them are unreadable. Reed-Solomon codes, as described in section 3, are used for this purpose.

To demonstrate how the characters in a message can be formatted for a QR code, consider a QR code that when scanned, portrays the message <https://atcm.mathandtech.org/>. To begin, an error correction level is selected. There are four possible error correction levels with QR codes: L, which allows the information to be accessible if up to 7% is unreadable; M, which provides up to 15% error correction; Q, which provides up to 25% error correction; and H, which provides up to 30% error correction. For our example, we will use error correction level H.

The next step is to determine the smallest possible grid size that can be used to encode the message with the specified error correction level. Table 2 shows the character capacities for each QR code version in byte mode and with error correction level H. A table showing the character capacities for each QR code version in all modes and with all error correction levels can be found in [5].

Version	Char Cap	Version	Char Cap	Version	Char Cap	Version	Char Cap
1	7	11	137	21	403	31	790
2	14	12	155	22	439	32	842
3	24	13	177	23	461	33	898
4	34	14	194	24	511	34	958
5	44	15	220	25	535	35	983
6	58	16	250	26	593	36	1051
7	64	17	280	27	625	37	1093
8	84	18	310	28	658	38	1139
9	98	19	338	29	698	39	1219
10	118	20	382	30	742	40	1273

Table 2: QR code character capacities in byte mode and with error correction level H.

Since <https://atcm.mathandtech.org/> contains 29 characters, the lowest QR code version that could use to encode it in byte mode and with error correction level H is version 4, with a 33×33 grid.

Next, information that will be encoded into the grid is represented by a string of binary digits. The first four digits are the *mode indicator*, which for byte mode are 0100. The four mode indicator digits for all mode types can be found in [5]. A *character count indicator* is then included, which gives the number of characters in the message. Since <https://atcm.mathandtech.org/> contains 29 characters, and the decimal number 29 is 11101 in binary, these digits are included next, although padded with three 0s at the start to have full length 8, as is required for the character count indicator for all QR code versions 1–9. That is, the character count indicator for our message would actually be included as 00011101. QR code versions 10–40 also require character count indicators of a specified length, although this required length is 16 bits rather than 8. Next, the characters in the actual message are converted into their ASCII code decimal representations, which are then converted into binary and themselves each padded with 0s at the start to have full length 8. These bytes are

then included in order after the character count indicator. Using Table 1, we see the characters in <https://atcm.mathandtech.org/> correspond to the following ASCII decimal representations.

104 116 116 112 115 58 47 47 97 116 99 109 46 109 97
 116 104 97 110 100 116 101 99 104 46 111 114 103 47

These decimal representations can then be converted to binary and padded to give the following bytes.

01101000 01110100 01110100 01110000 01110011 00111010 00101111
 00101111 01100001 01110100 01100011 01101101 00101110 01101101
 01100001 01110100 01101000 01100001 01101110 01100100 01110100
 01100101 01100011 01101000 00101110 01101111 01110010 01100111
 00101111

For each version and error correction level, QR codes require that binary strings completely fill the *total capacity* of the code. For codes of version 4 and with error correction level H, the total capacity is 288 bits. As such, after obtaining the bits for a mode indicator, character count indicator, and message, it is almost always necessary to add bits, in particular, some 0s and some padding bytes. First, a *terminator* would be included at the end of the binary string. If the string were four or more bits shorter than the total capacity, then the terminator would consist of exactly four 0s. If the bit string were fewer than four bits shorter than the total capacity, then the terminator would consist of only the number of 0s needed to reach the total capacity. Since in our example, the mode indicator, character count indicator, and message give a total of 244 bits, the terminator 0000 would be included, raising the total length of the binary string to 248 bits.

If the length of a string with the terminator is not a multiple of 8, then additional 0s would be included to bring the length up to a multiple of 8. In our example, since 248 is a multiple of 8, additional 0s are not required. Since 248 is short of the 288-bit total capacity of a code of version 4 and with error correction level H though, the string must still be padded to reach 288 bits. For this, bytes alternating between 11101100 and 00010001, representing 236 and 17 in decimal, respectively, are included until the total capacity is reached. For our example, since we need 40 additional bits to extend our 248-bit string to the required 288 bits, we include 11101100 00010001 11101100 00010001 11101100 to complete the data. A final list of the 288 bits for our example is shown in Table 3.

Mode	Char Count	Message	Terminator	Pad Bytes
0100	00011101	01101000 01110100 01110100 01110000 01110011 00111010 00101111 00101111 01100001 01110100 01100011 01101101 00101110 01101101 01100001 01110100 01101000 01100001 01101110 01100100 01110100 01100101 01100011 01101000 00101110 01101111 01110010 01100111 00101111	0000	11101100 00010001 11101100 00010001 11101100

Table 3: QR code example for the message <https://atcm.mathandtech.org/>.

Regrouping these 288 bits into blocks of 8 bits each gives the following full data binary string for our example.

```

01000001 11010110 10000111 01000111 01000111 00000111 00110011
10100010 11110010 11110110 00010111 01000110 00110110 11010010
11100110 11010110 00010111 01000110 10000110 00010110 11100110
01000111 01000110 01010110 00110110 10000010 11100110 11110111
00100110 01110010 11110000 11101100 00010001 11101100 00010001
11101100
    
```

To work with this sequence of bytes mathematically, we convert each byte into its representation as a polynomial and then as a power of a in the finite field used with QR codes. For example, the byte 11010110 has polynomial representation $a^7 + a^6 + a^4 + a^2 + a$, which turns out to be a^{85} .

We will now demonstrate a Maplet written by the authors that can be used to convert bytes into their corresponding powers of a in the finite field used with QR codes. Specifically, Figure 2 shows how the Maplet **FFConversion** can be used for this purpose.

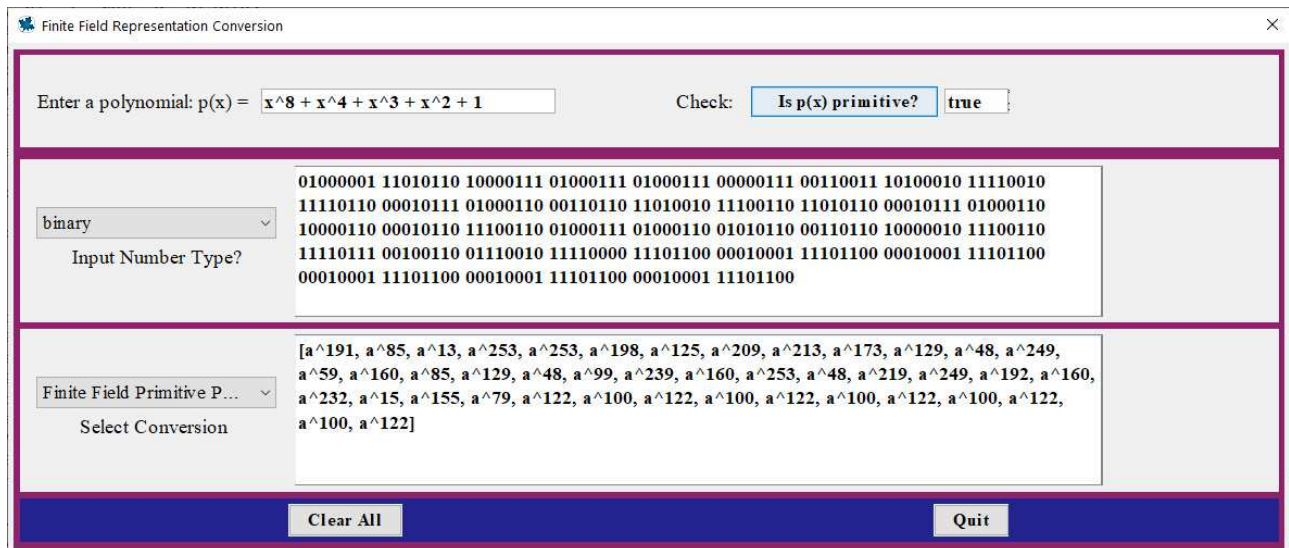


Figure 2: Primitive element power representation of data.

Summarizing this result, the full data binary string for our example can be represented as the following string of powers of a .

$$\begin{matrix}
 a^{191} & a^{85} & a^{13} & a^{253} & a^{253} & a^{198} & a^{125} & a^{209} & a^{213} & a^{173} & a^{129} & a^{48} & a^{249} \\
 a^{59} & a^{160} & a^{85} & a^{129} & a^{48} & a^{99} & a^{239} & a^{160} & a^{253} & a^{48} & a^{219} & a^{249} & a^{192} \\
 a^{160} & a^{232} & a^{15} & a^{155} & a^{79} & a^{122} & a^{100} & a^{122} & a^{100} & a^{122} & & &
 \end{matrix} \tag{1}$$

Next we will consider error correction in QR codes.

3 Reed-Solomon Codes

Since their description in a 1960 paper by Irving Reed and Gustave Solomon, Reed-Solomon codes have been widely used to ensure reliable transmission of data. In addition to their utility in QR codes,

Reed-Solomon codes have been used extensively to correct data transmission errors in mobile phones, compact discs, satellites, space probes, and numerous other examples.

The objects in which Reed-Solomon codes can actually correct errors are finite field elements, in a field containing 2^m elements for some positive integer m , generated by a primitive polynomial $p(x)$ of degree m , with primitive element we will again denote by a . To create a Reed-Solomon code, the desired maximum number of position (i.e., polynomial coefficient) errors guaranteed to be correctable upon the arrival of information at its destination must first be chosen, and specified. Let t be this desired maximum number of position errors guaranteed to be correctable, which can be any positive integer t with $2t < n$, where $n = 2^m - 1$ is the number of nonzero elements in the field.

With any error correcting code, transmitted messages are called *codewords*. Reed-Solomon codewords are polynomials consisting of a *prefix* and a *suffix*.² The prefix of a codeword is constructed starting with a polynomial of the form

$$m(x) = b_{k-1}x^{k-1} + \dots + b_1x + b_0,$$

whose coefficients are understood to contain the information actually needing to be sent. As such, we will call $m(x)$ the *message polynomial*. To form the prefix, we use the polynomial

$$g(x) = (x - 1)(x - a) \dots (x - a^{u-1}),$$

where $u = 2t$ or $u = 2t + 1$. The polynomial $g(x)$ is called the *generating polynomial*. We then form the prefix by computing $x^u m(x)$.

The suffix polynomial $s(x)$ is the remainder when the prefix polynomial is divided by $g(x)$. That is, if

$$x^u m(x) = q(x)g(x) + s(x) \tag{2}$$

with $\deg s(x) < \deg g(x)$ or $s(x) = 0$, then the suffix is $s(x)$.

For the various QR code versions and error correction levels, different numbers of prefix polynomials of given degrees along with generating polynomials of specified degrees are used to create codewords. Table 4 summarizes these numbers for versions 1–10 with error correction level H.

Version	Num Codewords	Num $m(x)$	Deg $m(x)$	Deg $g(x)$
1	1	1	8	16
2	1	1	15	27
3	2	2	11	21
4	4	4	8	16
5	4	2; 2	10; 11	22
6	4	4	14	28
7	5	4; 1	12; 13	26
8	6	4; 2	13; 14	26
9	8	4; 4	11; 12	24
10	8	6; 2	14; 15	28

Table 4: Codeword formulations for QR code versions 1–10 with error correction level H.

²Other descriptions of Reed-Solomon codes specify codewords as single polynomials. Separating the prefix and suffix is useful though, because it allows for the coefficients of the data to be visible in the prefix.

Codeword formulations for all of the various QR code versions and error correction levels can be found in [5].

From Table 4, we can see that a QR code of version 3 with error correction level H requires that information be formulated into two codewords, both with prefixes of degree 11, resulting in 12 polynomial coefficients containing data. The generating polynomial is then of degree 21, and has the form $g(x) = (x - 1)(x - a) \cdots (x - a^{20})$. Since this code is guaranteed to correct $t = 10$ position errors, each suffix corresponding to each prefix will have maximum degree 20, resulting in 21 polynomial coefficients.

Also from Table 4, we can see that a QR code of version 8 with error correction level H requires that information be formulated into six codewords, four having prefixes of degree 13, resulting in 14 polynomial coefficients containing data, and two having prefixes of degree 14, resulting in 15 polynomial coefficients containing data. The generating polynomial is then of degree 26, and has the form $g(x) = (x - 1)(x - a) \cdots (x - a^{25})$. Since this code is guaranteed to correct $t = 13$ position errors, each suffix corresponding to each prefix will have maximum degree 25, resulting in 26 polynomial coefficients.

In Section 2, the data given by (1) for the message <https://atcm.mathandtech.org/> was represented for a QR code of version 4 with error correction level H. Table 4 indicates that this data should be formulated into four codewords, all with prefixes of degree 8, resulting in 9 polynomial coefficients containing data. The generating polynomial for the code is then of degree 16, and has the form $g(x) = (x - 1)(x - a) \cdots (x - a^{15})$. Since this code is guaranteed to correct $t = 8$ position errors, each suffix corresponding to each prefix will have maximum degree 15, resulting in 16 polynomial coefficients. The data given by (1) shows the 288-bit full data string expressed as 36 primitive power finite field elements. Since there are four prefixes, we must divide these 36 powers of a into four equal parts, with each part representing the 9 coefficients of each of the prefix polynomials of degree 8. Once these prefixes are formed, we divide each by the generating polynomial $g(x)$, and using (2), find the four corresponding suffixes.

To demonstrate this process, for the 36 primitive power finite field elements given by (1), we take the first 9 of these elements, which are

$$a^{191}, a^{85}, a^{13}, a^{253}, a^{253}, a^{198}, a^{125}, a^{209}, a^{213},$$

and form the following message polynomial in descending degree order.

$$m(x) = a^{191}x^8 + a^{85}x^7 + a^{13}x^6 + a^{253}x^5 + a^{253}x^4 + a^{198}x^3 + a^{125}x^2 + a^{209}x + a^{213} \quad (3)$$

Since the degree of the generating polynomial is $u = 16$, we multiply $m(x)$ by $x^u = x^{16}$ to form the following prefix.

$$x^{16}m(x) = a^{191}x^{24} + a^{85}x^{23} + a^{13}x^{22} + a^{253}x^{21} + a^{253}x^{20} + a^{198}x^{19} + a^{125}x^{18} + a^{209}x^{17} + a^{213}x^{16}$$

To form the suffix, we must divide this prefix by the generating polynomial $g(x)$. To do this division, we will use another Maplet written by the authors, **ReedSolomonCodewordGeneratorATCM**, which has been designed to do this division and form the prefix/suffix codeword pair. Figure 3 shows how this Maplet can be used for this purpose. The Maplet specifically allows a user to enter the primitive polynomial, the degree of the generating polynomial, and the message polynomial, and then by clicking the appropriate buttons, output the generating polynomial in expanded form and the prefix/suffix codeword pair.



Figure 3: Prefix/suffix Reed-Solomon codeword generation.

From Figure 3, we can see that the prefix/suffix pair corresponding to the message polynomial (3) is given by the following.

$$\begin{aligned}
 x^{16}m(x) &= a^{191}x^{24} + a^{85}x^{23} + a^{13}x^{22} + a^{253}x^{21} + a^{253}x^{20} + a^{198}x^{19} + a^{125}x^{18} + a^{209}x^{17} + a^{213}x^{16} \\
 s(x) &= a^{252}x^{15} + a^{145}x^{14} + a^{49}x^{13} + a^{146}x^{12} + a^{165}x^{11} + a^{39}x^{10} + a^{136}x^9 + a^{97}x^8 + a^{55}x^7 \\
 &\quad + a^{181}x^6 + a^{21}x^5 + a^{21}x^4 + a^{45}x^3 + a^{100}x^2 + a^{77}x + a^{83}
 \end{aligned}$$

The following summarizes all four prefix/suffix codeword pairs, which can be found using the Maplet.

$$\begin{aligned}
 x^{16}m_1(x) &= a^{191}x^{24} + a^{85}x^{23} + a^{13}x^{22} + a^{253}x^{21} + a^{253}x^{20} + a^{198}x^{19} + a^{125}x^{18} + a^{209}x^{17} + a^{213}x^{16} \\
 s_1(x) &= a^{252}x^{15} + a^{145}x^{14} + a^{49}x^{13} + a^{146}x^{12} + a^{165}x^{11} + a^{39}x^{10} + a^{136}x^9 + a^{97}x^8 + a^{55}x^7 \\
 &\quad + a^{181}x^6 + a^{21}x^5 + a^{21}x^4 + a^{45}x^3 + a^{100}x^2 + a^{77}x + a^{83}
 \end{aligned}$$

$$\begin{aligned}
 x^{16}m_2(x) &= a^{173}x^{24} + a^{129}x^{23} + a^{48}x^{22} + a^{249}x^{21} + a^{59}x^{20} + a^{160}x^{19} + a^{85}x^{18} + a^{129}x^{17} + a^{48}x^{16} \\
 s_2(x) &= a^9x^{15} + a^{83}x^{14} + a^{148}x^{13} + a^{241}x^{12} + a^{94}x^{11} + a^{42}x^{10} + a^{74}x^9 + a^{161}x^8 + a^{52}x^7 + a^{82}x^6 \\
 &\quad + a^{188}x^5 + a^4x^4 + a^{167}x^3 + a^{241}x^2 + a^{111}x + a^{97}
 \end{aligned}$$

$$\begin{aligned}
 x^{16}m_3(x) &= a^{99}x^{24} + a^{239}x^{23} + a^{160}x^{22} + a^{253}x^{21} + a^{48}x^{20} + a^{219}x^{19} + a^{249}x^{18} + a^{192}x^{17} + a^{160}x^{16} \\
 s_3(x) &= a^{146}x^{14} + a^{84}x^{13} + a^{32}x^{12} + a^{206}x^{11} + a^{47}x^{10} + a^{109}x^9 + a^{36}x^8 + a^{151}x^7 + a^{208}x^6 \\
 &\quad + a^{105}x^5 + a^{65}x^4 + a^{134}x^3 + a^{74}x^2 + a^{61}x + a^{38}
 \end{aligned}$$

$$\begin{aligned}
 x^{16}m_4(x) &= a^{232}x^{24} + a^{15}x^{23} + a^{155}x^{22} + a^{79}x^{21} + a^{122}x^{20} + a^{100}x^{19} + a^{122}x^{18} + a^{100}x^{17} + a^{122}x^{16} \\
 s_4(x) &= a^{151}x^{15} + a^{48}x^{14} + a^{123}x^{13} + a^{235}x^{12} + a^{234}x^{11} + a^{25}x^{10} + a^{68}x^9 + a^{45}x^8 + a^{169}x^7 \\
 &\quad + a^{245}x^6 + a^{219}x^5 + a^{138}x^4 + a^{43}x^3 + a^{51}x^2 + a^{229}x + a^{63}
 \end{aligned}$$

To check whether the prefix and/or suffix contain any errors after their transmission, (2) is solved as $q(x)g(x) = x^u m(x) - s(x)$. Since $x^u m(x) - s(x)$ is a multiple of $g(x)$, the roots of $g(x)$ must also

be roots of $x^u m(x) - s(x)$. As such, when $x^u m(x) - s(x)$ is evaluated at the roots of $g(x)$, which are $1, a, \dots, a^{u-1}$, each result must be 0. These roots, $1, a, \dots, a^{u-1}$, are called the *syndromes* of $x^u m(x) - s(x)$.

As shown in the Maplet window in Figure 3, the syndromes for $x^{16} m_1(x) - s_1(x)$, as evaluated for the roots $1, a, \dots, a^{15}$ of the generating polynomial $g(x)$ of degree 16, are all 0, indicating that the prefix and suffix in that example are correct. When errors occur in either the prefix or suffix coefficients, Reed-Solomon codes provide an error-correction process through which the location of the errors can be found, and the errors corrected. We provide some details of this process in [2].

After the prefix/suffix codeword pairs are created, the data is then interleaved together. The purpose of this interleaving is so that when the data is placed into the QR code, localized damage to the QR code and an overwhelming of its capacity to correct errors can be prevented. This interleaving of the data occurs as follows.

- The coefficients of each prefix are written consecutively in order across the rows of a matrix, and then taken in order down the columns of this matrix, skipping over any blank entries that may have resulted from prefixes of different lengths.
- The coefficients of each suffix are written consecutively in order across the rows of a matrix, and then taken in order down the columns of this matrix.
- The interleaved prefixes and suffixes are then joined together to represent the data.

As an example of this, with the four prefix/suffix pairs we gave previously, we first write the coefficients of the prefixes in order across the rows in a 4×9 array.

$$\begin{array}{ccccccccc} a^{191} & a^{85} & a^{13} & a^{253} & a^{253} & a^{198} & a^{125} & a^{209} & a^{213} \\ a^{173} & a^{129} & a^{48} & a^{249} & a^{59} & a^{160} & a^{85} & a^{129} & a^{48} \\ a^{99} & a^{239} & a^{160} & a^{253} & a^{48} & a^{219} & a^{249} & a^{192} & a^{160} \\ a^{232} & a^{15} & a^{155} & a^{79} & a^{122} & a^{100} & a^{122} & a^{100} & a^{122} \end{array}$$

We then interleave these prefix coefficients by taking them in order down the columns of this array. This results in the following.

$$\begin{array}{cccccccccc} a^{191} & a^{173} & a^{99} & a^{232} & a^{85} & a^{129} & a^{239} & a^{15} & a^{13} & a^{48} & a^{160} & a^{155} \\ a^{253} & a^{249} & a^{253} & a^{79} & a^{253} & a^{59} & a^{48} & a^{122} & a^{198} & a^{160} & a^{219} & a^{100} \\ a^{125} & a^{85} & a^{249} & a^{122} & a^{209} & a^{129} & a^{192} & a^{100} & a^{213} & a^{48} & a^{160} & a^{122} \end{array}$$

We next write the coefficients of the suffixes in order across the rows in a 4×16 array.

$$\begin{array}{cccccccccccccccc} a^{252} & a^{145} & a^{49} & a^{146} & a^{165} & a^{39} & a^{136} & a^{97} & a^{55} & a^{181} & a^{21} & a^{21} & a^{45} & a^{100} & a^{77} & a^{83} \\ a^9 & a^{83} & a^{148} & a^{241} & a^{94} & a^{42} & a^{74} & a^{161} & a^{52} & a^{82} & a^{188} & a^4 & a^{167} & a^{241} & a^{111} & a^{97} \\ 0 & a^{146} & a^{84} & a^{32} & a^{206} & a^{47} & a^{109} & a^{36} & a^{151} & a^{208} & a^{105} & a^{65} & a^{134} & a^{74} & a^{61} & a^{38} \\ a^{151} & a^{48} & a^{123} & a^{235} & a^{234} & a^{25} & a^{68} & a^{45} & a^{169} & a^{245} & a^{219} & a^{138} & a^{43} & a^{51} & a^{229} & a^{63} \end{array}$$

We then interleave these suffix coefficients by taking them in order down the columns of this array. This results in the following.

$$\begin{array}{cccccccccccccccc} a^{252} & a^9 & 0 & a^{151} & a^{145} & a^{83} & a^{146} & a^{48} & a^{49} & a^{148} & a^{84} & a^{123} & a^{146} & a^{241} & a^{32} & a^{235} \\ a^{165} & a^{94} & a^{206} & a^{234} & a^{39} & a^{42} & a^{47} & a^{25} & a^{136} & a^{74} & a^{109} & a^{68} & a^{97} & a^{161} & a^{36} & a^{45} \\ a^{55} & a^{52} & a^{151} & a^{169} & a^{181} & a^{82} & a^{208} & a^{245} & a^{21} & a^{188} & a^{105} & a^{219} & a^{21} & a^4 & a^{65} & a^{138} \\ a^{45} & a^{167} & a^{134} & a^{43} & a^{100} & a^{241} & a^{74} & a^{51} & a^{77} & a^{111} & a^{61} & a^{229} & a^{83} & a^{97} & a^{38} & a^{63} \end{array}$$

Combining the interleaved prefix and suffix coefficients gives the following full data for the QR code.

$$\begin{matrix}
 a^{191} & a^{173} & a^{99} & a^{232} & a^{85} & a^{129} & a^{239} & a^{15} & a^{13} & a^{48} & a^{160} & a^{155} & a^{253} & a^{249} & a^{253} & a^{79} & a^{253} \\
 a^{59} & a^{48} & a^{122} & a^{198} & a^{160} & a^{219} & a^{100} & a^{125} & a^{85} & a^{249} & a^{122} & a^{209} & a^{129} & a^{192} & a^{100} & a^{213} & a^{48} \\
 a^{160} & a^{122} & a^{252} & a^9 & 0 & a^{151} & a^{145} & a^{83} & a^{146} & a^{48} & a^{49} & a^{148} & a^{84} & a^{123} & a^{146} & a^{241} & a^{32} \\
 a^{235} & a^{165} & a^{94} & a^{206} & a^{234} & a^{39} & a^{42} & a^{47} & a^{25} & a^{136} & a^{74} & a^{109} & a^{68} & a^{97} & a^{161} & a^{36} & a^{45} \\
 a^{55} & a^{52} & a^{151} & a^{169} & a^{181} & a^{82} & a^{208} & a^{245} & a^{21} & a^{188} & a^{105} & a^{219} & a^{21} & a^4 & a^{65} & a^{138} & a^{45} \\
 a^{167} & a^{134} & a^{43} & a^{100} & a^{241} & a^{74} & a^{51} & a^{77} & a^{111} & a^{61} & a^{229} & a^{83} & a^{97} & a^{38} & a^{63} & &
 \end{matrix} \tag{4}$$

This interleaved prefix and suffix data gives a total of 100 primitive element powers. Each power can be converted into a byte, giving a total of $8 \cdot 100 = 800$ bits to represent the data in full for the code. Some QR code versions also require a string of 0s to be included to completely fill out the data. This number of additional 0s, which depends on the version, is given in [5]. QR codes of version 4 require seven additional 0s. Thus, for our example, the string 0000000 would be appended to the 800 bits representing the data, giving a total of 807 bits for the data section of the QR code. Next we will describe how this data is placed into the QR code, and the code itself created.

4 Data Placement and QR Code Creation

QR codes include several components placed in precise locations in a two-dimensional array. Figure 4 gives a summary of the placement of components in a QR code of version 4.

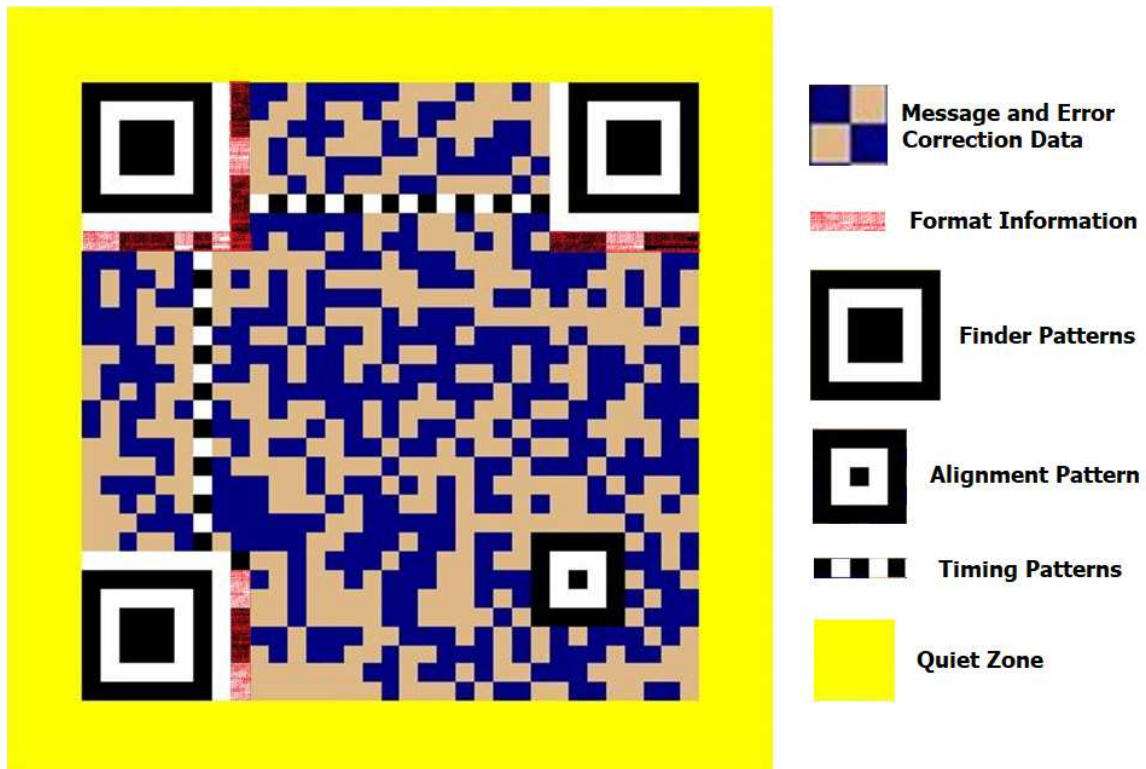


Figure 4: Component summary for a QR code of version 4.

A QR code of version 4 is essentially a 33×33 matrix. In Figure 4, the message and error correction data is located in the small dark blue and light brown squares. Each square in this data

placement represents a binary digit, for which normally a 1 is represented by a darker color (dark blue, in this case), and a 0 by a lighter color (light brown, in this case). The data is placed in the image starting in the lower right corner. The regions outlined in red in Figure 4 are reserved for the *format information*, which is a string of bits giving, among other things, the error correction level (L, M, Q, or H) for the code. The format information also gives the *mask pattern*, which we will describe in more detail later. Error correction bits are also included as part of the format information in case any of the information is lost. The *function patterns* consist of the *finder patterns*, *alignment patterns*, and *timing patterns*, as well as a *dark module*, which is a single black square located directly above the format information block near the lower left corner of the matrix. *Separators* shown as solid white rectangular strips in Figure 4 detach each finder pattern from the rest of the QR code. The function patterns are designed to be placed in specific areas of the QR code, and their purpose is to ensure that QR code scanners can correctly identify and orient an image for decoding. More information about how the function patterns are aligned in a QR code can be found in [5]. Finally, a *quiet zone* is included, as indicated in yellow in Figure 4, as an area of lighter color designed to enclose the main QR code matrix.

QR codes of versions 7–40 require more alignment patterns and an additional 18-bit block placed in two parts of the matrix containing information about the QR code version. More information can be found about their generation and placement in [5].

For the message <https://atcm.mathandtech.org/>, Figure 5 shows how the data given by (4) is placed in a QR code of version 4.

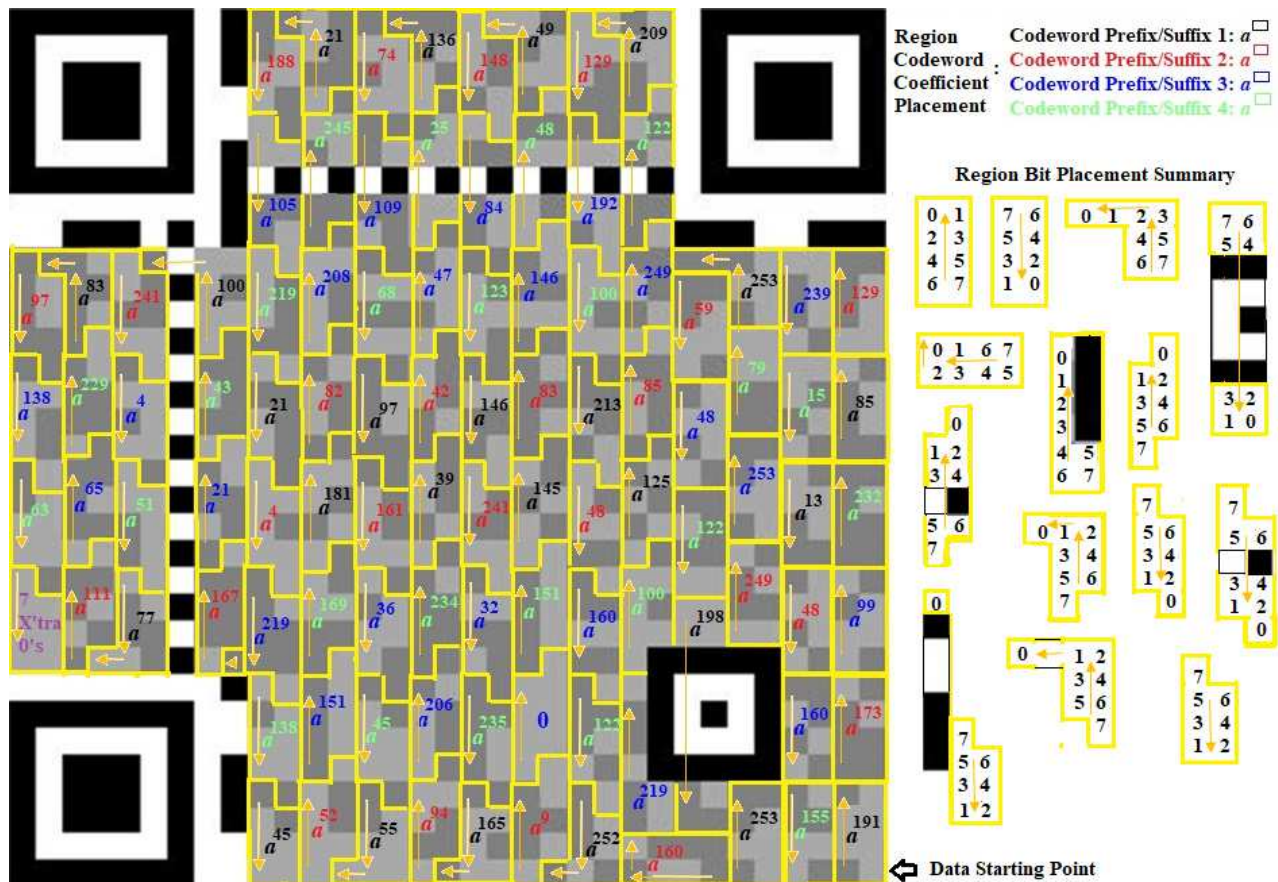


Figure 5: QR code data placement for the message <https://atcm.mathandtech.org/>.

Starting in the lower right corner of a QR code matrix, the data for each codeword prefix/suffix is placed in blocks in an up-and-down pattern. Each primitive power element is color-coded to illustrate the codeword prefix/suffix with which it is associated, as well as how the interleaving spreads each codeword prefix/suffix throughout the matrix. For each individual block, the eight binary digits representing the primitive power element are placed in a zig-zag pattern. The region bit placement summary section in Figure 4 indicates the order in which the binary digits are placed within similarly shaped blocks, with the coefficients of each polynomial of degree 7 representing each primitive element power placed in descending degree order. For example, the first primitive power element placed in the lower right corner, a^{191} , is represented by the polynomial $a^6 + 1$. The coefficients of this polynomial are read in descending order, 01000001, and placed in the first rectangular block. The order of the placement is indicated by the first rectangular block under the region bit placement summary section in Figure 4.

After the data is placed into the QR code matrix, to make the data as readable as possible by a QR code scanner, the process of *masking* is applied. When data in a QR code is masked, particular data squares, depending on their location in the matrix, are flipped, with a dark square becoming light, and vice versa. Only data squares are flipped though. Squares involving function patterns, version, and format information are left intact. QR code specifications define eight mask patterns that can be applied to a QR code. The mask pattern chosen is dependent on a calculated penalty score from certain observed square occurrences in the QR code matrix. The different type of mask patterns and how the penalty score is calculated for each are summarized in [5].

For the QR code of version 4 representing <https://atcm.mathandtech.org/>, mask pattern 2 is chosen. Numbering the 33 columns of this matrix from 0, 1, ..., 32, mask pattern 2 flips all of the data squares in all columns with (column number) mod 3 = 0. Figure 6 illustrates this QR code before and after masking, with the column numbers labeled where the masking occurs.

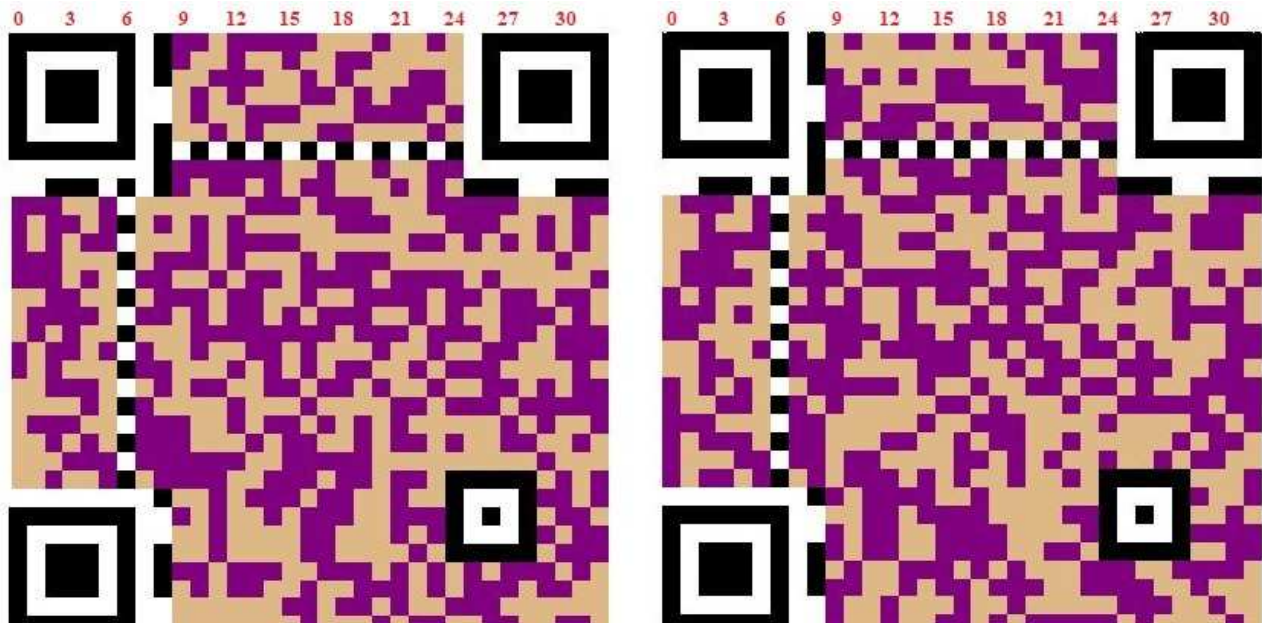


Figure 6: QR code example before and after masking.

After masking, a QR code is ready to be output. For generating QR codes, we will use another Maplet written by the authors, **QRcode**. This Maplet allows users to input a message to be included,

select an error correction level, and then generate the resulting QR code. The Maplet automatically generates the QR code with a specific version based on how much and what type of data is entered and the error correction level selected. Figure 7 shows the output for a QR code of version 4 and with error correction level H that will scan to the message <https://atcm.mathandtech.org/>.

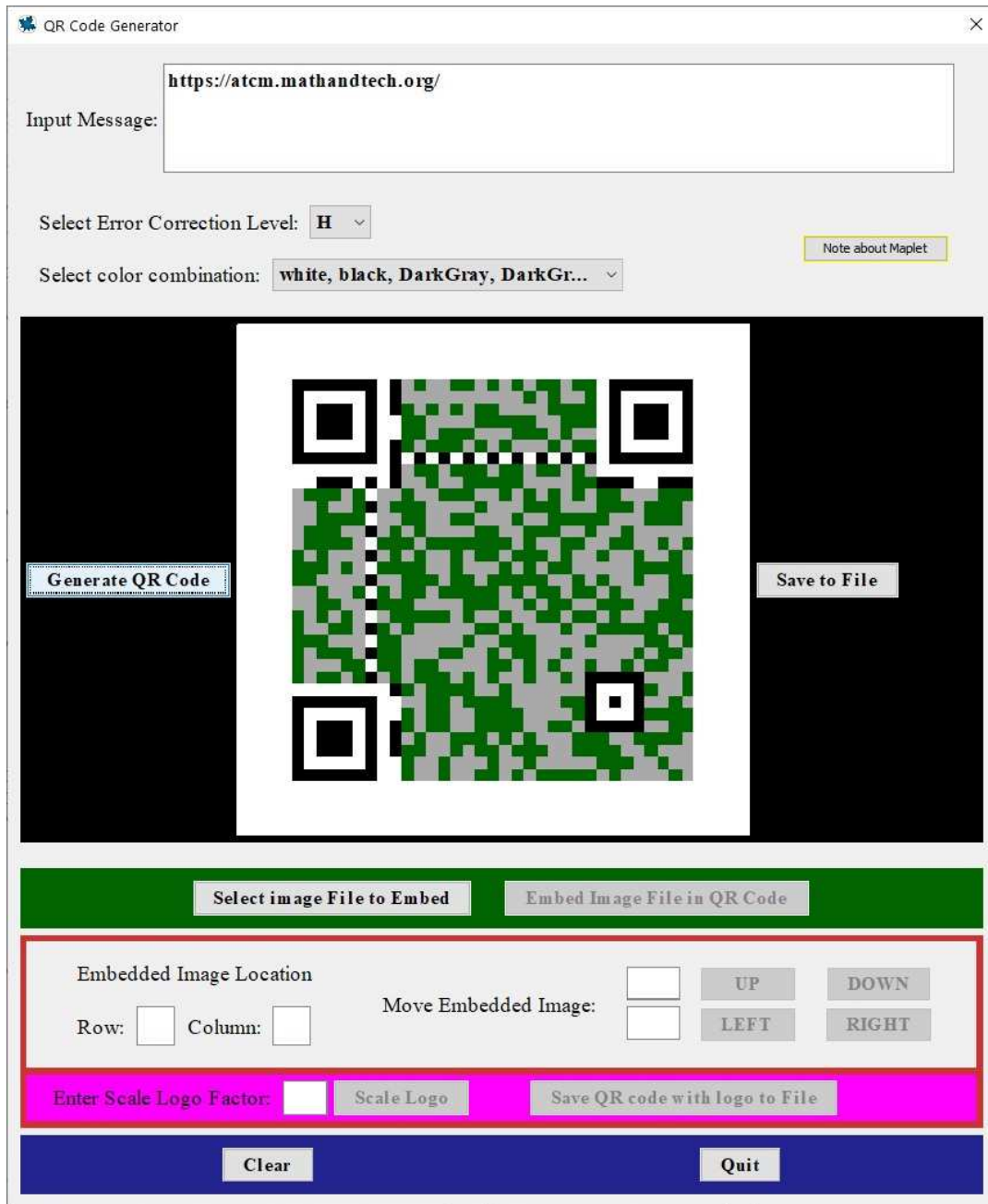


Figure 7: QR code Maplet generation example for the message <https://atcm.mathandtech.org/>.

With error correction included, images up to a certain size can be overlaid onto a QR code to help identify its message contents. With error correction level H, even with 30% of the message data squares covered by the image, the QR code would still scan correctly. The **QRcode** Maplet can select a jpg file of a user's choice and overlay it onto a generated QR code. Figure 8 shows the result with the ATCM logo overlaid onto a QR code that would still scan to <https://atcm.mathandtech.org/>.

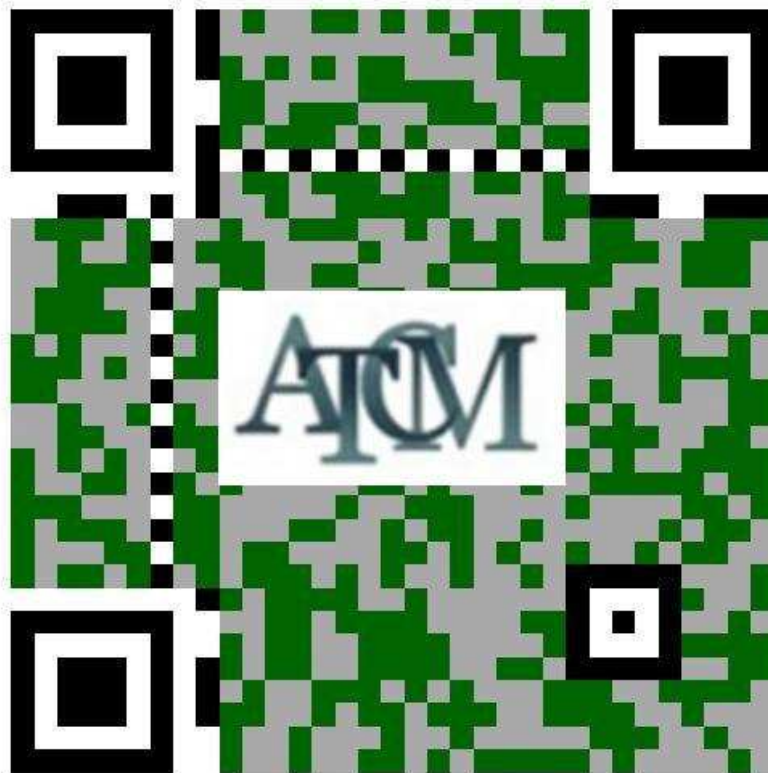


Figure 8: QR code with an overlaid image.

5 Conclusion

In this paper, we showed the basics of how data is formatted and placed into QR codes, with error correction included via Reed-Solomon codes. Maplets written by the authors were used to demonstrate this. These Maplets are available for download at [4].

The QR code generated in this paper focused on specific data that was generated for a particular version. Additional explanations for how other data can be formatted and placed into QR codes of other versions can be found in [5].

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Approaching Cesàro's inequality through GeoGebra Discovery

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Abstract

We illustrate the possibility of developing, using the automated reasoning tools implemented in the dynamic geometry program GeoGebra Discovery, a certain parallelism with Michael de Villiers reflection on the 'discovery function' of proof, as described in his 2012 paper concerning the formulation by one student of a certain geometric conjecture (Clough's conjecture).

1 Motivation and goals

Michael de Villiers is a well know emeritus professor from the University of KwaZulu-Natal and an honorary professor at the University of Stellenbosch, both in South Africa, with worldwide leading research contributions along the past 30 years on Dynamic Geometry and on the role of proof and reasoning in mathematics education. Visit, for example, his web page <http://dynamicmathematicslearning.com/homepage4.html> for a relation of his publications with links to some of them.

In 2004 he conducted workshops at two different conferences (AMESA¹, South Africa, and NTCM², USA), dealing, in particular, with what he labeled as “Clough's Conjecture” [6]. Later on, based on a presentation by himself at the 12th International Congress on Mathematics Education (ICME, July 2012, Seoul, Korea), de Villiers published an article [7], mentioning his contributions to the above Conferences and describing its content as follows ([7], p.3; Figure 1 below is reproduced from there):

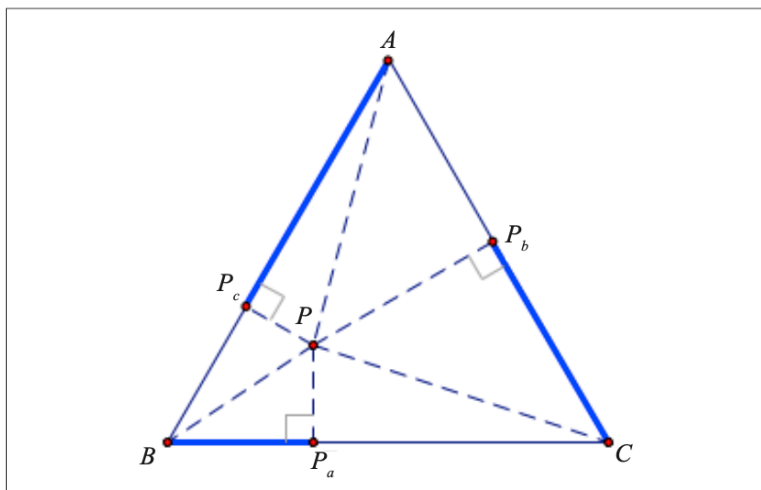
The main purpose of this article is to contribute further to the theoretical aspects of the role of proof by providing a heuristic description of some of my personal experiences of the explanatory and discovery functions of proof with a geometric conjecture made by a Grade 11 student.

[...]

¹10th Association for Mathematics Education of South Africa (AMESA) Conference, 30 June-4 July 2004, Potchefstroom, South Africa, <http://www.amesa.org.za/AMESA2004/>

²National Council of Teachers of Mathematics (NCTM) Annual Meeting, April 2004, Philadelphia, USA

During 2003, a Grade 11 student from a high school in Cape Town was exploring Vivianis theorem using dynamic geometry. The theorem says that the sum of distances of a point to the sides of an equilateral triangle is constant (i.e. in Figure [below] $PP_a + PP_b + PP_c$ is constant, irrespective of the position of point P inside triangle ABC). The students further exploration led him to measure the distances AP_c, BP_a and CP_b , and then add them. To his surprise, he noticed that $AP_c + BP_a + CP_b$ also remained constant no matter how much he dragged P inside the triangle. However, he could not prove it.



His teacher eventually wrote to me to ask whether I could perhaps produce a simple geometric proof, as he himself could only prove it algebraically by means of coordinate geometry. Below is the geometric proof I first produced, followed by further proofs, explorations and different generalisations of what has become known as Clough’s conjecture ([6]).

[...]

The underlying heuristic reasoning is carefully described in order to provide an exemplar for designing learning trajectories to engage students with these functions of proof.

In summary: in [7] de Villiers describes a Grade 11 student (Clough) experiment with Dynamic Geometry, initially aiming to prove Viviani’s theorem, but at some point deriving towards the formulation of a different statement, conjectured by the student. And de Villiers develops the different steps (proving, discovering, conjecturing, generalizing, proving again ...) involved in this experience, as a useful example for the analysis and design of learning paths concerning the role of proof in mathematics.

It is our goal here to follow, in some sense, the same storyline, but replacing Clough’s protagonist role by the performance of our dearest “personal geometry assistant”, the program *GeoGebra Discovery*³, whose main characteristics and features have been already described in

³See <https://github.com/kovzol/geogebra-discovery>, and <http://autgeo.online/geogebra-discovery/>. *GeoGebra Discovery* is available in two options: GeoGebra Classic 5, for Win-

[2], [16] or in the Asian Technology Conference in Mathematics (ATCM) 2020 invited lecture, entitled “GeoGebra Reasoning Tools for Humans and for Automatons” [11]. See the next section for a short collection of basic examples illustrating the different tools and possibilities of *GeoGebra Discovery*.

On the other hand, as a more challenging instance of the performance of this program, Figure 1 shows the answer of *GeoGebra Discovery* to Clough’s conjecture, namely, when asked for the relation between

- the sum of segments $l = EC$, $m = FB$, and $n = GA$, where E, F, G are the feet of the perpendiculars to the sides of the equilateral triangle from an arbitrary point D ,
- and $3p/2$, where p is the length of side AB , i.e. $3p/2$ is half the perimeter $3p$ of the equilateral triangle ABC .

Here *GeoGebra Discovery* declares that such relation is “true on parts, false on parts”, an answer that we will explain and comment in the last part of this paper and that illustrates some of the issues we would like to address here.

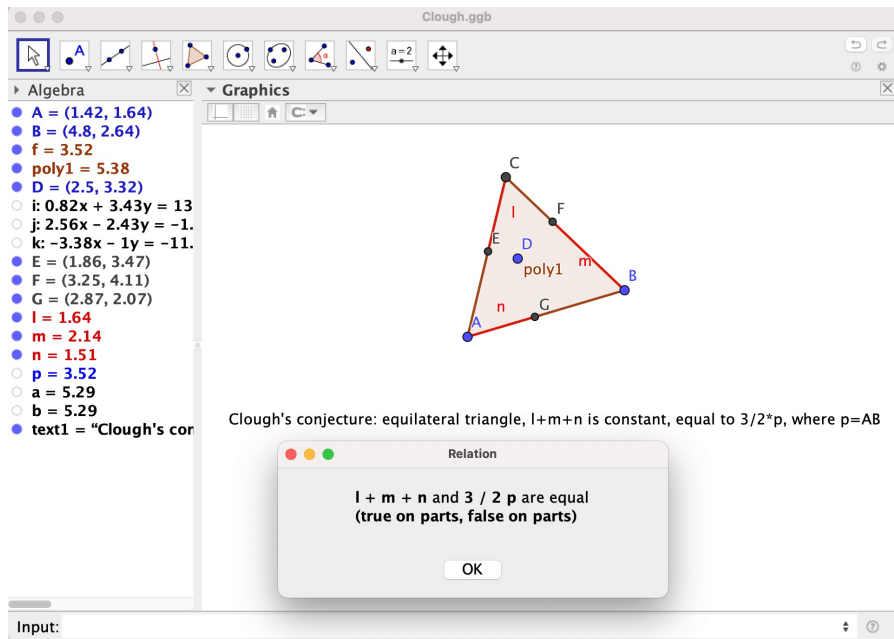


Figure 1: Checking Clough’s conjecture yields ”’true on parts”

Of course, we are aware that the intended parallelism between de Villiers’ mentioned paper [7] and our work here, has several limitations. Leaving aside the obvious differences of expertise from the authors of both articles, it is clear that de Villiers’ aims to exemplify the educational relevance of the heuristics involved in the production of traditional proofs. This interaction: proof/heuristics, i.e. what he calls “...the ’looking back’ discovery function of proof...” [7], p. 7, could

dows, Mac and Linux systems; and GeoGebra Classic 6, made for starting it in a browser, mainly ready for use on tablets and smartphones

... at least acquaint students with the idea that a deductive argument can provide additional insight and some form of novel discovery ... Problem posing and generalisation through the utilisation of the 'discovery' function of proof is as important and creative as problem-solving itself, and ways of encouraging this kind of thinking in students need to be further explored.

And then he emphasizes that this 'discovery' function of proof is, in his own words, something to remark over the more traditional functions, such as 'verification':

... Instead of defining proof in terms of its verification function (or any other function for that matter), it is suggested that proof should rather be defined simply as a deductive or logical argument that shows how a particular result can be derived from other proven or assumed results; nothing more, nothing less. It is not here suggested that fidelity to the verification function of proof is sacrificed at all, but that it should not be elevated to a defining characteristic of proof. Moreover, the verification function ought to be supplemented with other important functions of proof using genuine mathematical activities as described above.

On the contrary, as described in the ATCM 2020 proceedings [11], *GeoGebra Discovery* facilitates

... the exploration, by humans, of geometric tasks by using GeoGebra as a kind of "symbolic geometry calculator": the user poses a concrete geometric task and GeoGebra provides a mathematically sound answer.

That is, as a "symbolic geometry calculator", *GeoGebra Discovery* main feature is, precisely, the verification of geometric statements, without bringing any human readable argument for their truth or falsity. Roughly speaking, as we will show in the next section, it provides just a "yes/no" answer to a certain query posed by the user. Therefore, in some sense, the use of this technological tool can not enhance the 'discovery function' of proof, since there is no proof at all!

Thus, bearing in mind this drastic affirmation, why do we regard in this paper the possibility to follow with *GeoGebra Discovery* a parallel path to the one established in de Villiers' cited work?

Answer: in three different ways. First of all, trying to imitate de Villiers' discourse on the opportunities brought by the 'discovery function' of proof, but now concerning the 'extended discovery opportunity' of geometric properties that comes when having an 'oracle' at hand, just as having a numerical calculator at our disposal can help us finding out numerical properties. This has already been sufficiently argued in some recent works of ours, such as [13], [10], [18], [12], [16], thus we will not address in more detail this issue here.

Second, replicating de Villiers' route towards proving and generalizing Clough's conjecture, now through the analysis of the problems and difficulties shown by the algorithms involved in the *GeoGebra Discovery* commands when dealing with different statements. We will exhibit how these elements conform a sort of 'discovery function' of automated proof, describing a helpful learning path to discover algebro-geometric properties of the involved figures for more advanced mathematics students, teachers, researchers. ...

Third, as a way to describe, again, a learning path, this time not for the human user, but for the "personal geometry assistant", i.e. for the researchers and programmers involved in its debugging and improvement.

2 GeoGebra Discovery: a short digest

We have described *GeoGebra Discovery* as a kind of 'oracle'. Indeed, the most obvious command in this context (see [13] for a sort of tutorial) is the one that answers to the quest to *Prove*... a certain relation between two geometric objects in a figure. For example⁴ consider a parallelogram $ABCD$ and let E, F, G, I be the midpoints of the sides. Then we would like to verify if the line $j = EF$ and the line $k = IG$ are parallel. The answer is displayed in Figure 2.

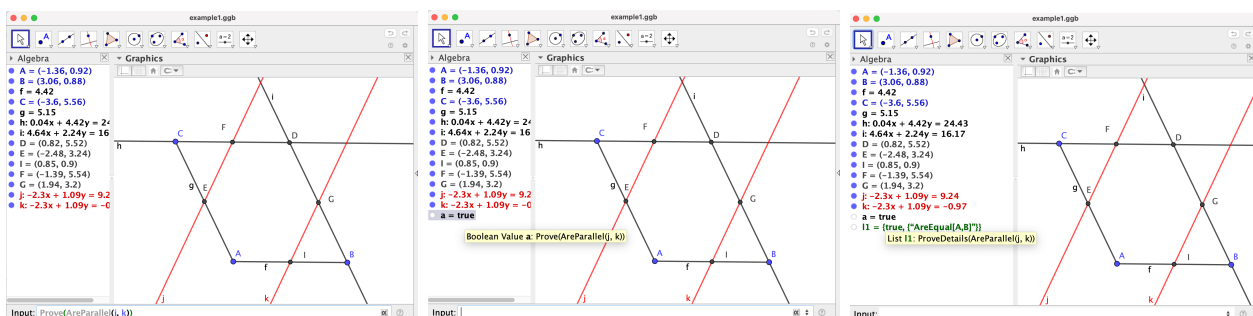


Figure 2: Left, input: $Prove(AreParallel(j,k))$. Center, output: $True$. Right, details: $true$ in the case of non-degeneracy (i.e. with non-coincident initial points, $A \neq B$).

To the left of the Figure, in the input line, the command $Prove(AreParallel(j,k))$. The output is presented in the central image, with the concise $true$ reply. To the right, after introducing the user the demand for details through the $ProveDetails(AreParallel(j,k))$ order, *GeoGebra Discovery* points out that it is required –for the parallelism of j and k – that the construction does not collapse, i.e. that points A and B , that are the starting points for constructing the parallelogram (together with point C , displaying some segments $f = AB, g = AC$ and then some lines h, i parallel to f, g (respectively) passing through C and B , etc.), are different.

Now, a key feature of *GeoGebra Discovery* is the *Relation* command, that allows the user to ask the program to formulate possible relations holding between two elements of a figure. That is, while the *Prove* command requires the user to “guess” a statement that will be confirmed or denied by the program, the *Relation* automatically tests, numerically, a collection of potential statements involving two selected elements and outputs, in a first step, some relations that seem (apparently) to hold true. Then, after clicking on the *More...* button, *GeoGebra Discovery* confirms or denies the rigorous truth of the automatically suggested statement. See Figure 3.

Next, an ample generalization of the *Relation* command is the *Discover* command, that does not even require the user to point out two possible elements, but just one, for example, point E in Figure 4. Then, $Discover(E)$ launches a collection of *Relation* tests between E and other elements in the geometric construction, yielding, and visually highlighting, a list of statements that are true involving point E . See Figure 4.

Finally, let us mention (and exemplify with an elementary, yet surprising, result) the recent extension of these *GeoGebra Discovery* tools to deal with statements involving inequalities [17].

⁴Following de Villiers’ inspiring paper, we will on purpose restrict in general to simple examples, of school level, although *GeoGebra Discovery* is able to deal with quite complicated ones, arising for instance, in mathematical contests, university entrance or professional selection exams, see [14], [15], [16].

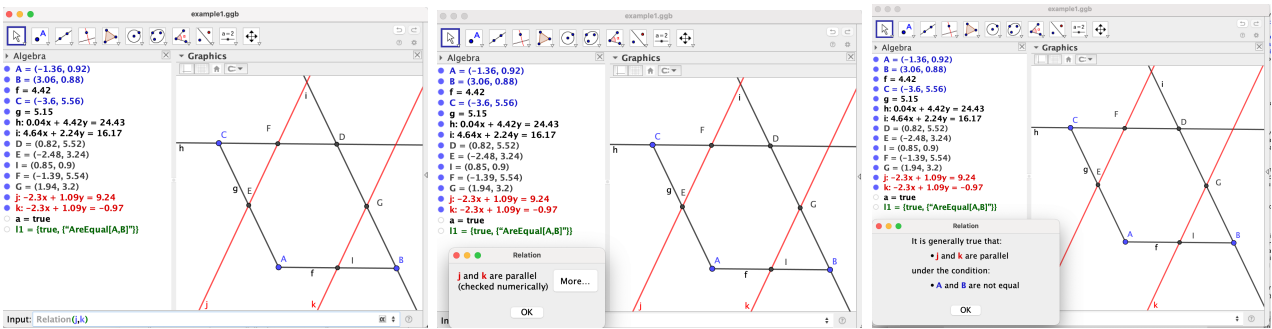


Figure 3: Left, input question: is there any Relation(j,k)? Center, initial output: numerical parallelism of the two lines. Right, symbolic verification: parallelism of j, k is a mathematically sound statement.

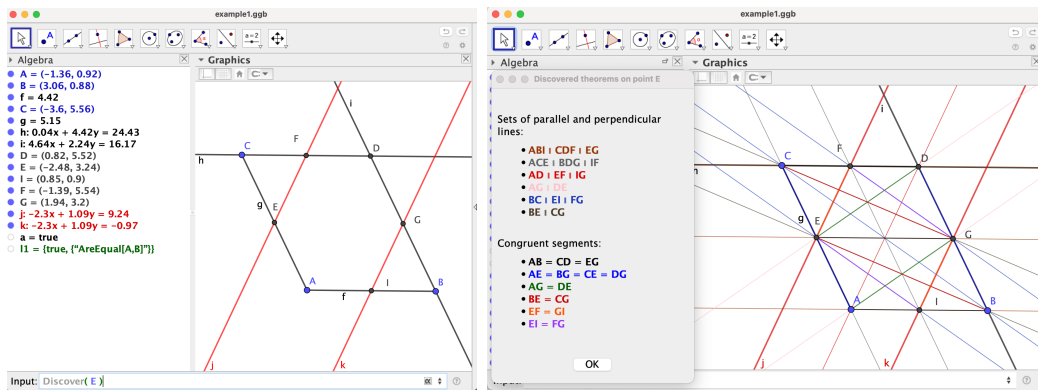


Figure 4: Left: input, Discover(E). Right: output, a collection of true statements on parallel or perpendicular lines or congruent segments, involving E.

Thus, in the next section we will describe an attempt to investigate the basic inequality number 1.4 (from E. Cesàro’s [4], p. 140) included in the classic book by Bottema et al. [3], p. 12. Let us remark that the original proposal by Cesàro (see Figure 5) includes, as a footnote, the fact that for equilateral triangles the inequality is an equality, while Bottema’s formulation of the same inequality adds that this only holds in this equilateral case.

How can *GeoGebra Discovery*’s very concise answers help developing ’learning paths’ associated to some kind of ’discovery’ function of proof in this context? We will attempt to exemplify our point of view on this issue in the next section.

3 Cesàro’s inequality

Let us start by asking *GeoGebra Discovery* for the relation between the products $(a + b) \cdot (a + c) \cdot (b + c)$ and abc , where a, b, c are the lengths of the sides of a triangle. *GeoGebra* replies almost immediately (over an old MacBookPro 2.5 GHz) presenting the –perhaps– unexpected inequality $(a + b) \cdot (a + c) \cdot (b + c) \geq 8abc$. See Figure 6.

Next, we try to provide some reasons that justify *GeoGebra*’s answer. Our first idea is to

Question 529.

THÉORÈME. — Dans tout triangle, le produit des rapports de chaque côté à la somme des deux autres, ne surpasse pas $\frac{1}{8}$.

(E. CESARO.)

Le théorème revient à prouver l'inégalité

$$\frac{abc}{(a+b)(b+c)(c+a)} < \frac{1}{8}. \quad (1)$$

Or, la moyenne géométrique de deux nombres est plus petite que leur moyenne arithmétique; ainsi :

$$\sqrt{ab} < \frac{a+b}{2}, \quad \sqrt{bc} < \frac{b+c}{2}, \quad \sqrt{ac} < \frac{a+c}{2}.$$

On conclut, de ces inégalités,

$$abc < \frac{(a+b)(b+c)(c+a)}{8} \quad (*).$$

(E. FAUQUEMBERGUE.)

Autres solutions par MM. Torrès, élève du Lycée de Bordeaux, Leinekugel, étudiant (Paris) et Cochetoux, élève de l'École des Mines (Liège).

(* Il est bon d'observer que, si le triangle est équilatéral,

$$\frac{abc}{(a+b)(b+c)(c+a)} = \frac{1}{8}.$$

(E. C.)

Figure 5: Original formulation of E. Cesàro's inequality.

consider (and verify with GeoGebra) a much simpler case, the well known triangle's inequality: every side is smaller than the sum of the other two. Thus, $(a+b) \geq c$, and, likewise, $(b+c) \geq a$ and $(a+c) \geq b$. See Figure 7. But, using these inequalities, we would just obtain abc as a lower bound for $(a+b) \cdot (a+c) \cdot (b+c)$, not $8abc$. We wonder, then, if it could be true in general over any triangle that $(a+b) \geq 2c$, $(b+c) \geq 2a$ and $(a+c) \geq 2b$. After a moment's thought we discard this hypothesis, thinking, for instance, of the right triangle with sides $a = 3, b = 4, c = 5$, where $a+b \not\geq 2c$. We do not even need to recall such Pythagorean triple; we could just ask *GeoGebra Discovery* for the locus of, say, vertex C such that $a+b = 2c$. The answer 'seems' to be an ellipse, see Figure 7. Placing C inside the ellipse would yield $a+b \leq 2c$; and placing it outside would imply $a+b \geq 2c$.

But we would like to confirm this visual impression. Thus, we select some simple coordinates for $A = (0,0), B = (1,0)$, as the analysis of the investigated equality can be reduced, without loss of generality, to this particular case, by homothecy. Now we can handle easily the displayed output equation, which is, indeed, an ellipse with foci in A, B , center in the midpoint of AB , axis of size 2 and $\sqrt{3}$, respectively. And the ellipse includes the points $C = (1/2, \sqrt{3}/2)$ and $C = (1/2, -\sqrt{3}/2)$, corresponding to an equilateral triangle.

Now, in the next Figure 9 we show how we have extended this computation and displayed the locus of C for $2c = a+b$, the red ellipse; the locus of C for $2a-b-c$, the blue quartic;

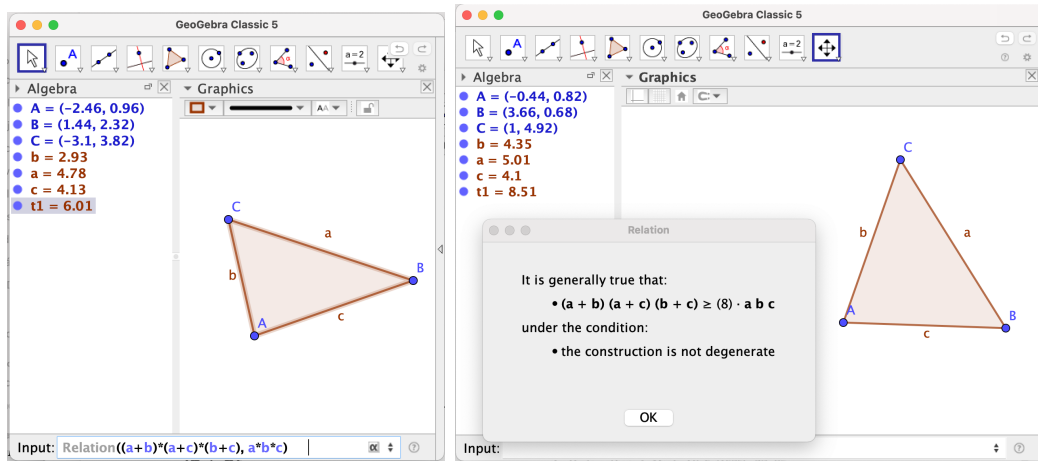


Figure 6: Left: input, asking for the relation between $(a + b) \cdot (a + c) \cdot (b + c)$ and $a \cdot b \cdot c$. Right: output, the inequality $(a + b) \cdot (a + c) \cdot (b + c) \geq 8abc$.

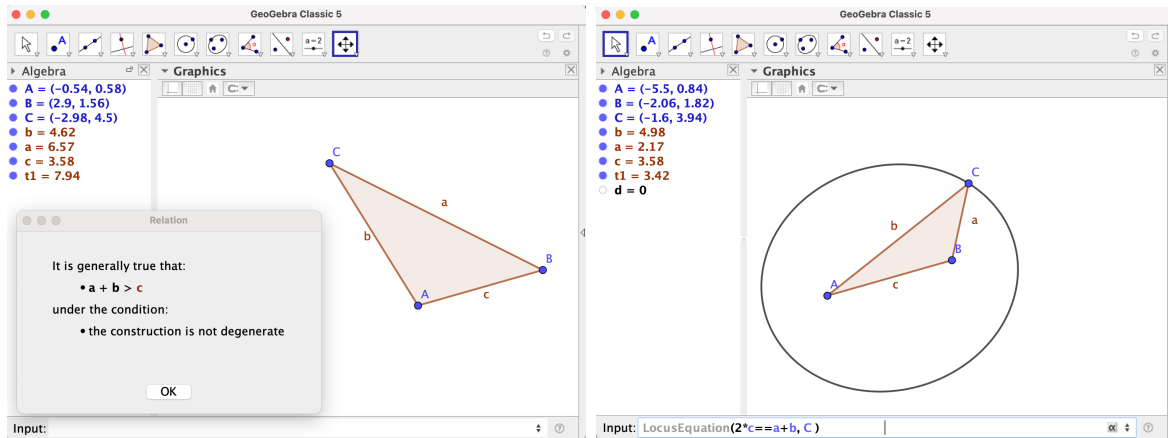


Figure 7: Left: A side of a triangle is smaller than the sum of the other two. Right: Locus of C for $a + b = 2c$.

and the locus of C for $2b - a - c$, the black quartic. We have labeled as d, e, f the values of $2c - a - b$, $2b - a - c$, $2a - b - c$, respectively. Notice that placing C inside the red ellipse makes d positive (and negative outside of the ellipse), while placing this point inside the blue or black quartics makes both e, f negative (positive, otherwise).

But finding in a geometrically precise way the intersection of the three curves seems challenging for GeoGebra (there is no command for finding the intersection of three curves). Yet, in this case the intersection is easy to describe: indeed, the conjunction of $\{a + b - 2 * c = 0, a + c - 2 * b = 0, b + c - 2 * a = 0\}$ yields as solution $a = b = c$, which means that C should be the third vertex of one of the two equilateral triangles with vertices at $A = (0,0), B = (1,0)$. Notice that, conversely, for any equilateral triangle it is true that $\{a + b - 2 * c = 0, a + c - 2 * b = 0, b + c - 2 * a = 0\}$, since $a = b = c$. In particular this means that over such triangles $(a + b) \cdot (a + c) \cdot (b + c) = 8abc$, as remarked by Cesàro.

Obviously, this success also means that our initial conjecture about "... if it could be true in

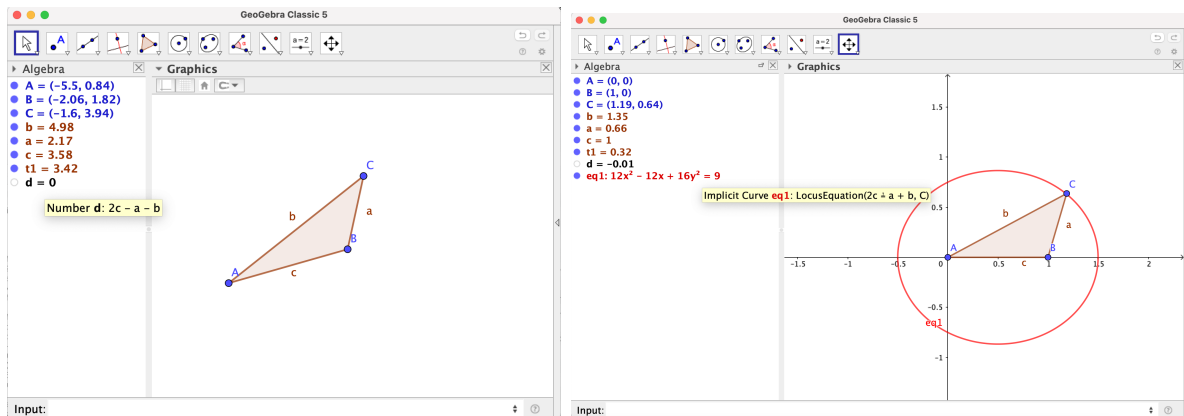


Figure 8: Left: $d = 2c - a - b$. Right: Equation of the locus of C for $a + b = 2c$, showing it is an ellipse.

general over any triangle that $(a+b) \geq 2c$, $(b+c) \geq 2a$ and $(a+c) \geq 2b$." is false. Thus, we have to look for a different approach towards proving Cesàro's inequality $(a+b) \cdot (a+c) \cdot (b+c) \geq 8abc$. Since it involves the sum and the product of every two sides of the triangle, we might try to find out if there is some simpler inequality holding between, say, $a + b$ and ab . The initial problem is that both expressions are not of the same degree and, thus, they can not be reduced to the case $A = (0, 0)$, $B = (1, 0)$. Yet, we ask GeoGebra, through the *Relation* tool, for the relation between $a + b$ and ab , yielding that $(a + b)^2 \geq 4ab$, see Figure 10.

In order to have some explanation for this inequality (let us remark that GeoGebra's output is the result of some symbolic computation with real algebraic geometry tools, such as quantifier elimination, so it is already mathematically sound) we observe that the inequality is equivalent to $((a + b)/2)^2 \geq ab$. Thus, in Figure 10, right, following https://en.wikipedia.org/wiki/Inequality_of_arithmetic_and_geometric_means, we have constructed a circle with diameter AB and radius $i = (a + b)/2$. We place now a point C on this circle in such a way that the foot of its height is the point where the segments a, b meet, yielding the triangle ABC and apply the altitude (or geometric mean) theorem $h^2 = ab$ (see https://en.wikipedia.org/wiki/Geometric_mean_theorem, and [8] for a proof with GeoGebra and a generalization), yielding that $i \geq h$, and thus $i^2 \geq h^2$, i.e. that $((a + b)/2)^2 \geq ab$.

Now, this inequality implies that $(a + b) \geq 2\sqrt{ab}$ and, likewise, $(a + c) \geq 2\sqrt{ac}$ and $(b + c) \geq 2\sqrt{bc}$. It is now clear that multiplying all these expressions we will arrive to Cesàro's inequality, Q.E.D.

4 Cesàro's Equality locus

Finally, we wonder about when the equality $(a + b) \cdot (a + c) \cdot (b + c) = 8abc$ holds, beyond the already analyzed equilateral case. In Figure 11 we show how computing the locus of C verifying this equality yields an empty graph, associated to a complicated equation: $eq4 := \{252 * x^{10} + 1264 * x^8 * y^2 + 2536 * x^6 * y^4 + 2544 * x^4 * y^6 + 1276 * x^2 * y^8 + 256 * y^{10} - 1260 * x^9 - 5056 * x^7 * y^2 - 7608 * x^5 * y^4 - 5088 * x^3 * y^6 - 1276 * x * y^8 + 2151 * x^8 + 6596 * x^6 * y^2 + 6866 * x^4 * y^4 + 2548 * x^2 * y^6 + 127 * y^8 - 1044 * x^7 - 2092 * x^5 * y^2 - 1052 * x^3 * y^4 - 4 * x * x$

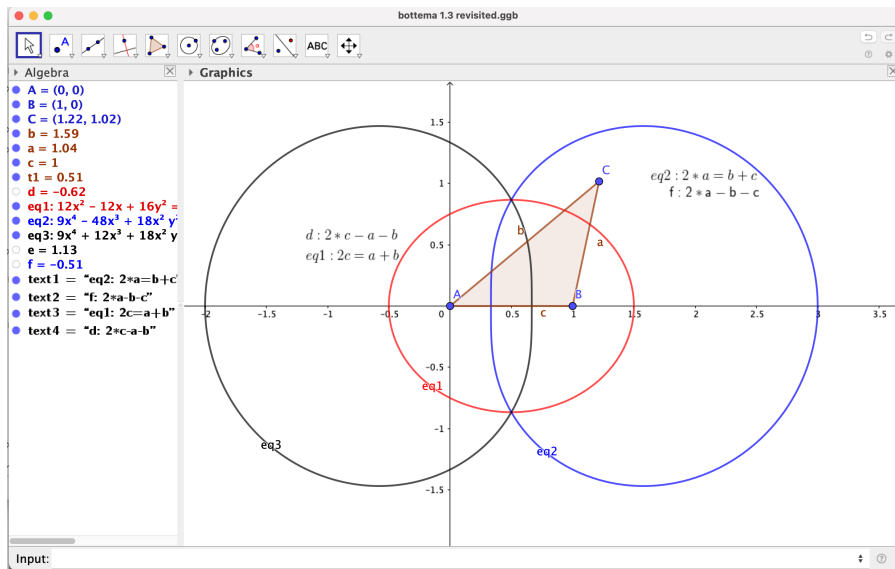


Figure 9: Intersection of the three curves corresponding to the loci of C for $2c - a - b = 0$, $2a - b - c = 0$, and $2b - a - c = 0$, respectively.

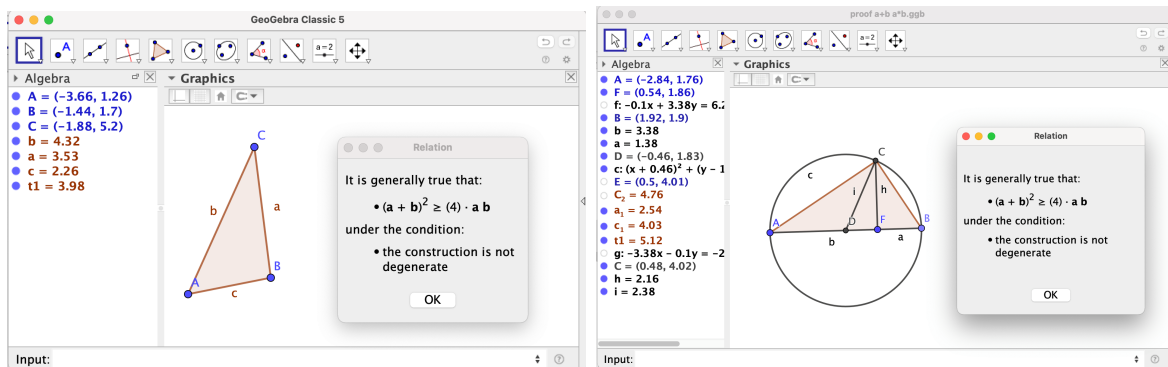


Figure 10: Left: $(a+b)^2 \geq 4ab$. Right: Explaining this inequality through the altitude theorem.

$$y^6 - 198 * x^6 - 682 * x^4 * y^2 - 738 * x^2 * y^4 - 254 * y^6 - 1044 * x^5 - 1048 * x^3 * y^2 - 4 * x * y^4 + 2151 * x^4 + 2294 * x^2 * y^2 + 127 * y^4 - 1260 * x^3 - 1276 * x * y^2 + 252 * x^2 + 256 * y^2 = 0\}$$

This “missing real points” problem (empty or not complete graphic output in GeoGebra, of the equation of a curve) happens in many other cases. For example, if we input *ImplicitCurve*($x^2 + y^2$), we get just the empty set. Or, if we input *ImplicitCurve*($x^2 + y^2 - x^3$), we get a branch of the curve, but not the origin – a point that obviously belongs to the curve. Thus we need to find specific ways to deal with this issue in this case.

To begin with, we know that this equality locus *eq4* should contain the points $C = (1/2, \sqrt{3}/2)$ and $C = (1/2, -\sqrt{3}/2)$, and this can be easily verified using the *Substitute* command in GeoGebra CAS view, over the above equation *eq4*. It is also easy to imagine (and to check) that the curve contains the degenerate instances $C = A, C = B$. Any other

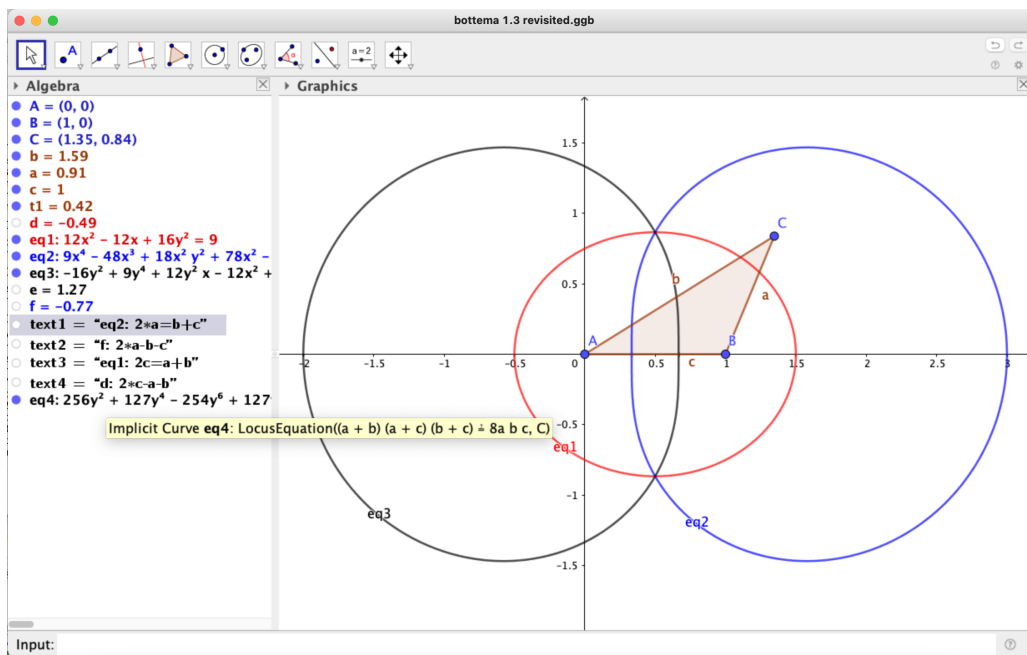


Figure 11: Computing the locus of C for $(a + b) \cdot (a + c) \cdot (b + c) = 8abc$

points? To answer this question we proceed as follows, using some tools from the CAS Maple⁵, such as *sturm*, that computes the number of real roots of univariate polynomial on an interval (eventually, the interval $(-\infty, +\infty)$), or *realroot*, that computes isolating intervals for each real root:

- We collect *eq4* as a polynomial in the variable y with coefficients in x
- We compute the discriminant of this polynomial in y , getting a univariate polynomial in x .
- We compute the different real roots of this discriminant (11 in total), including 0, $1/2$ and 1.
- We compute the real roots of the curve over each of these roots of the discriminant, finding only the already known ones: $y = 0$ over $x = 0$, $y = \sqrt{3}/2$ and $y = -\sqrt{3}/2$ over $x = 1/2$, $y = 0$ over $x = 1$.
- For each interval conformed by a pair of consecutive roots of the discriminant (or between minus infinity and the smallest root, and between the largest root and plus infinity) assign a test number in the interval. For example, we know that the smallest root is $1/2 - \sqrt{(63618 + 7446 \cdot \sqrt{73})}/4$, thus we can choose -90 as a smaller, test number. As a more automatic alternative, use *realroot* for handling intervals enclosing the roots of the discriminant.

⁵<https://www.maplesoft.com>

- Over each assigned real number between the real roots of the discriminant, compute the number of real roots in the curve (using, again, the *sturm* command). For example, substitute $x = -90$ in $eq4$, yielding a polynomial in y , that has no real roots.
- As there are no real roots over each one of these test points, we conclude that the only points in the curve are $\{(0, 0), (1/2, \sqrt{(3)}/2), (1/2, -\sqrt{(3)}/2), (1, 0)\}$. In fact let us recall that the number of real roots of $eq4(x, y)$ does not vary as x moves on an interval between consecutive roots of the discriminant. So, since in the test points of the intervals we have selected there are no roots, it follows the same happens all over the interval. So the only roots are the ones we have already previously found.

Of course, the obtained computation confirms Bottema’s assertion that Cesàro’s inequality holds as an equality only in the equilateral case, leaving aside the degenerate instances when C coincides with A or B .

5 Conclusions

The last two sections show, in our opinion, a way to approach geometry learning that combines empiric experimentation and formal reasoning, in which man-machine interaction is fundamental, not just auxiliary, following the declaration of Corless [5]: “Any tool should always be used to expand the users capabilities, and not as a crutch to prop up weak skills”. See [8] for another recent example in the same direction.

We could say that “computer mediated thinking” (again, a formulation from [5]) should have a parallel status to the traditional “writing mediated thinking”: how could we think of developing a sound geometry reasoning without using symbols and writing skills? How can we nowadays think of developing geometry reasoning without using “personal geometry assistants” such as *GeoGebra Discovery*? The evolution of our approach to the proof of Cesàro inequality (and equality) shows well –at least in this particular example– the great advantages (and specific difficulties) of having this tool at our disposal. It would have been very difficult to replicate our way of proving these statements without the concourse of GeoGebra. Of course, it is possible that there is an alternative way, but ... will it be relevant in mathematics education in the digital era?

We are aware that this is just one isolated example, and it is one addressed to persons with some skills on higher mathematics. But it also intends to support the urgent need for “Opening a discussion on teaching proof with automated theorem provers” [9], the title of a very recent paper by Hanna –one of the world most reputed experts in the topic– and Yan. In that paper there is a section specifically dealing with GeoGebra’s automated reasoning tools, and the authors conclude that

It is perhaps too early for empirical studies of classroom experience using the enhancements to GeoGebra. In this respect the situation of GeoGebra is similar, but not identical, to the proof technology in general. While it is reasonable to expect proof technology to foster students proving abilities, and there is certainly supporting anecdotal evidence, its potential advantages have not yet been systematically assessed.

[...]

We know that automated proof assistants are designed to provide a guarantee of correctness, and indeed they are very good at establishing the validity of a proof. The question, then, is to what degree these tools can also be helpful in explaining why it is that a theorem is true.

We agree about the need to seriously start investigating these issues about how the use of *GeoGebra Discovery* can improve mathematics learning, not of the traditional curriculum, but of one that already takes into account the existence of new possibilities associated to the 'discovery function' of computer-enhanced proof processes.

It is also true that this research must go in parallel, involving, on the one side, to curriculum decision makers, teachers, students [1] and, on the other, proof assistant developers. As stated in [9]

Proof assistants . . . will never be developed in the absence of initiative on the part of mathematics educators and a demonstrated demand fueled by increased use. Secondly, success also requires new and effective teaching strategies. These two efforts stand in a reciprocal relationship, so that the full benefit of proof assistants will be seen only over time as new teaching strategies effect the demand for new tool features and vice versa. The responsibility for both efforts rests squarely on the shoulders of educators. The key is to make a start, beginning with exploratory studies of the potential of these new tools at both the secondary and post-secondary levels.

Indeed, our verification of Clough's conjecture, see Figure 1, shows the need (and the involved mathematical, algorithmic and user-interface difficulties) to improve proof assistants to output some answer that could be clearly understood by most users. Here let us just succinctly state that 'true on parts, false on parts' refers to the fact that the algebraic translation of the construction involves different components (but, for a standard user, there is only one, the one that is graphically and intuitively perceived), because the idea of 'length' of a side is, in the complex geometry algorithmic background for *GeoGebra Discovery*, some square root that can take positive or negative values. And, of course, the involved conjecture is true for the component where these roots are positive, and fails for the others.

The option to associate signs to the lengths of segments involves real algebraic geometry and it is on-going work [17]; but it is much less efficient at this moment, so it would be more useful to develop –with the cooperation of teachers, experimenting with students, etc.– some user-interface modifications to avoid such confusing answers for the expected users of the program in the educational world, perhaps implementing two kind of versions: for 'experts' and for 'students'.

6 Acknowledgements

The authors were partially supported by the grant PID2020-113192GB-I00 (Mathematical Visualization: Foundations, Algorithms and Applications) from the Spanish MICINN.

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Blending Knowledge and Technology to Construct Steiner Chains

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Abstract: *The aim of this paper is to show the crucial role of knowledge to tackle some geometric problems. We have chosen to show in detail how to tackle the problem of construction of Steiner chains, that is to say, closed chains of tangent circles also tangent to two given nested circles (to simplify the presentation and get understandable figures). Accordingly, a great part of the paper is a reminder of the necessary knowledge used for the techniques of construction of such chains: notion of harmonic division, notion of bundle of circles, notion of polar of a point with respect to a circle and above all the notion of inversion and the link between this transformation and the previous notions. The paper shows how these techniques can be used with some dynamic geometry software such as the New Cabri (containing the tool “macro” but not the tool “inversion”), Cabri 3D (containing the tool “inversion” with respect to spheres) or TI-NSpire (less efficient for complex figures because the tools “macro” and “inversion” are not available). Finally, we will perform very precise figures mostly with the New Cabri where Steiner chains associated to two given nested circles are generated and can be animated when their conditions of existence are satisfied.*

1. Link between bundle of circles at base points and bundle of circles at Poncelet points ([4])

1.1. Definitions (figure 1 left):

1.1.1. The bundle of circles at base points A and A' with $A \neq A'$, is the set of circles centered on the perpendicular bisector of $[AA']$ and passing through A and A' .

1.1.2. The bundle of circles at Poncelet points A and A' with $A \neq A'$ is the set of circles centered on (AA') cutting this line in two points C and D , these points cutting the segment $[AA']$ under the same ratio or such that (A, A', C, D) is an harmonic division (see §2) or such that D is the image of C with the inversion (see §4) of circle the circle with diameter $[AA']$

1.2. Orthogonality of such bundles (Figure 1 left)

Let us prove the following theorem

Theorem: Each circle of the bundle of circles at base points A and A' is orthogonal to every circle of the bundle at Poncelet points A and A' .

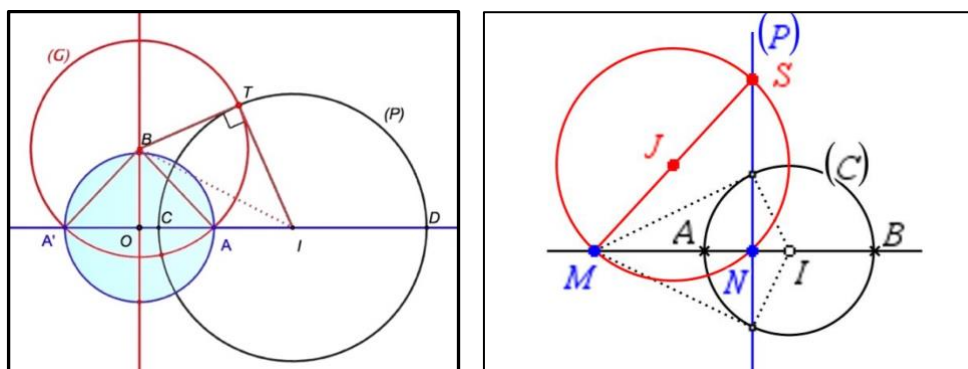


Figure 1: Orthogonality of bundles and construction of a polar

Let us give A and A' two points symmetric with respect to point O on the horizontal blue line. (G) is one of the circles of the bundle at base points A and A' , centered at B on the red line (perpendicular bisector of $[AA']$). In the system of axes supported by the blue and the red lines, $A(a,0)$, $A'(-a,0)$ and $B(0,\beta)$. Let us find all the circles centered on (AA') (center called I), orthogonal to (G) and passing through a point T of (G) . Let us consider one of these circles (P) . The radius of (G) is equal to BA' or BT or BA : we know that $BA^2 = a^2 + \beta^2$. Let us evaluate BI^2 in two different ways:

$$BI^2 = BT^2 + TI^2 = BT^2 + CI^2 = BA^2 + CI^2 = a^2 + \beta^2 + CI^2 \text{ and also } BI^2 = BO^2 + OI^2 = \beta^2 + OI^2.$$

From these we get $a^2 = OI^2 - CI^2$ which is the power of O for (P) ([9]) that can also be written $OC \cdot OD$. So, D is the image of C by the inversion centered at O which circle is the blue circle (center O and radius $OA = a$). Another consequence is that (A', A, C, D) is an harmonic division and circle (P) is one of the circles of the bundle at Poncelet points A and A' .

When T is moving on (G) , we get all the centers I on the blue line and therefore all the circles of the Poncelet bundle. That completes the proof.

2. Harmonic division

Definition: if A, B, C and D are four points belonging to the same line, (A, B, C, D) is an harmonic division if C and D cut segment $[AB]$ under the same ratio. In particular: $\frac{CA}{CB} = -\frac{DA}{DB}$

Property 1: If a, b, c and d are the abscissa of A, B, C and D with respect to an axis supported by (AB) and where the origin is located at the midpoint of $[AB]$, the previous condition can be written: $c \cdot d = a^2$ or $\overline{OC} \cdot \overline{OD} = \overline{OA}^2 = a^2$ where D (respectively C) is the image of C (respectively D) by the **inversion centered at O which ratio is a^2** .

Property 2: If the origin of the axis is located at A , this relation becomes:

$$\frac{2}{b} = \frac{1}{c} + \frac{1}{d} \text{ or } \frac{2}{OB} = \frac{1}{OC} + \frac{1}{OD}$$

3. Polar of a point for a circle (Figure 1 right) ([2])

3.1. Définition: The polar of point M for a circle (C) (center I and radius r), is the set of points S such that the circle of diameter $[MS]$ is orthogonal to (C) .

3.2. How to characterize the polar of a point for a circle

3.2.1 This polar is not an empty set because point N inverse of M by the inversion centered at I and which circle is (C) belongs to this polar. In fact, $IM \cdot IN = r^2$ means that the power ([9]) of I for this circle is equal the square of the radius of (C) , which means also that the circle with diameter $[MN]$ is orthogonal to circle (C) . With respect to a system of axes with origin I , supported by (IA) and its perpendicular at I , if $M(x_M, 0)$ then, $N(\frac{r^2}{x_M}, 0)$.

3.2.2. Let us find now the set of points $S(x_S, y_S)$ such that the circle of diameter $[MS]$ is orthogonal to (C) . If J is the midpoint of $[SM]$, which means the center of such a circle, the necessary and sufficient condition for this property is $IJ^2 = JM^2 + r^2$.

With $J(\frac{x_M+x_S}{2}, \frac{y_S}{2})$, $IJ^2 = (\frac{x_M+x_S}{2})^2 + (\frac{y_S}{2})^2$ and as $JM^2 = (\frac{x_M-x_S}{2})^2 + (\frac{y_S}{2})^2$, the necessary and sufficient condition can be written: $x_M \cdot x_S = r^2$ or $x_S = \frac{r^2}{x_M}$ which is the equation of the line passing through N (inverse of M with respect to (C)) and perpendicular to (IM) . So:

Theorem: The polar of a point M of a circle (C) centered at I is the perpendicular at N to (IN) where N is the image of M by the inversion of circle (C) .

Remark: if M' is a point of the polar of M for (C) and if (MM') cuts circle (C) at points U and V , it is equivalent to say that (M, M', U, V) is an harmonic division (Figure 2 left).

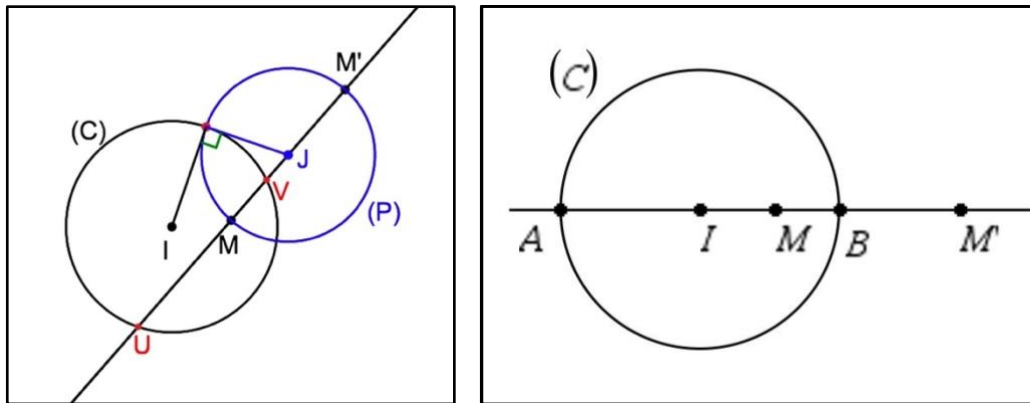


Figure 2: Property of polars, definition of inversion and construction of polar

4. Inversion ([7], [8]) (Figure 2 right)

4.1. Definition: If (C) is a circle centered at I with radius r , we call inversion with respect to this circle, the plane transformation mapping each point M different from I onto a point M' belonging to the ray $[IM)$ such as $IM \cdot IM' = r^2$ and letting I be invariant.

As I is the midpoint of $[AB]$, the equality of definition can be interpreted differently: (A, B, M, M') is an harmonic division or M' is the orthogonal projection of M on its polar for (C) .

4.2. Images of lines (Figure 3)

Let us consider the inversion of circle (C) (center I , radius r) and a line (D) .

The image of a line passing (D) through I is globally invariant by such an inversion.

The image by such an inversion of a line (D) which does not contain I is a circle containing I :

We give a proof for the two possible cases. The first case is when the line cuts circle (C) and the second one is when it does not. In these two cases, we call m the orthogonal projection of I on (D) and m' its image by the given inversion.

First case “ (D) cuts circle (C) ” (Figure 3 left): point m' belongs to the image of (D) . For each other point n of (D) (different from m) with image n' , we have: $Im \cdot Im' = In \cdot In'$ or $\frac{Im}{In} = \frac{Im'}{In'}$. Triangles Imn and $In'm'$ have a common angle; the previous equality establishes that these triangles are similar. Or Imn is a right-angle triangle at m , so $In'm'$ is a right-angle triangle at n' . therefore, n' belongs to the circle of diameter $[Im']$. Conducting the same reasoning would prove that each point of the circle is the image of a point of (D) by this inversion.

Second case “ (D) does not cut circle (C) ” (Figure 3 right): same reasoning.

Theorem: The image of a line (D) by an inversion (when (D) does not contain the center I of the inversion) is a circle of diameter $[Im']$ where m is the orthogonal projection of I on (D) and m' is the image of m by this inversion.

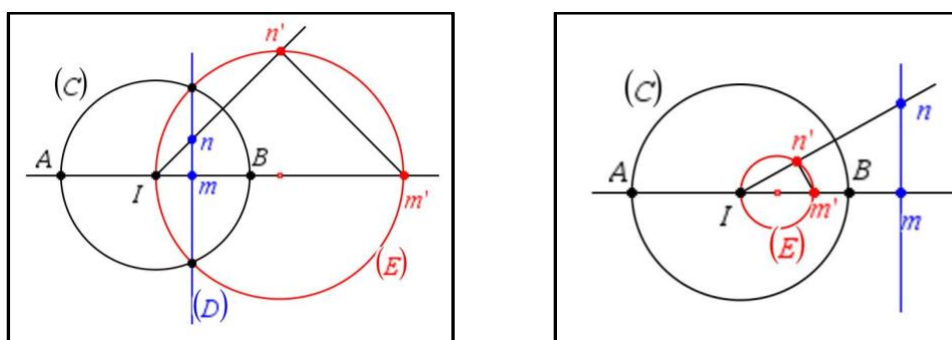


Figure 3: Images of lines by an inversion

4.3. Images of circles

First case “Image of a circle passing through the center of the inversion”: as an inversion is an involution, the image of a circle is a line (use the previous result).

Second case “Image of a circle which does not contain the center of the inversion” (Figure 4):

Let us consider here the inversion I centered at O and whose circle is (C) . We try to find the image of circle (G) whose center is I and diameter $[MN]$ and for which O, M and N are collinear. M' and N' are the images of M and N by the inversion I . Point P is a point generating circle (G) ; its image P' generates the image (G') of (G) by I . Q is the second intersection point between (OP) and circle (G) : this point generates (G) exactly when P generates (G) and therefore Q' its image by I generates (G') exactly when P generates (G) .

If r is the radius of circle (C) , thanks to the definition of the inversion, we obtain:

$$OQ' = \frac{r^2}{OQ} \text{ and } OM' = \frac{r^2}{OM}. \text{ Let us evaluate now the ratio } \frac{OQ'}{OP} :$$

$$\frac{OQ'}{OP} = \frac{\frac{r^2}{OQ}}{OP} = \frac{r^2}{OP \cdot OQ} \text{ or } OP \cdot OQ = OM \cdot ON \text{ (power of point } O \text{ for circle } (G)).$$

$$\text{Therefore, } \frac{OQ'}{OP} = \frac{r^2}{OM \cdot ON} = \frac{\frac{r^2}{OM}}{ON} = \frac{OM'}{ON}.$$

The equality $\frac{OQ'}{OP} = \frac{OM'}{ON}$ reflects the fact that Q' is the image of P by the dilation centered at O and whose scale is $\frac{OM'}{ON}$. So, (G') is a circle of diameter $[M'N']$.

Remark: be careful! The center of (G') is not the image of I by the inversion I . It is the image J of I by the previous dilation. We will show a simple way to construct this center in using the composition of two inversions (it will allow constructions of centers of a Steiner chain easily: see 5.5.).

$$\text{Position of } J: J \text{ is the midpoint of } [M'N'], \text{ so, } OJ = \frac{OM' + ON'}{2} = \frac{\frac{r^2}{OM} + \frac{r^2}{ON}}{2} = \frac{r^2}{2} \cdot \left(\frac{1}{OM} + \frac{1}{ON} \right).$$

If H is the harmonic conjugate of O with respect to M and N , we can write the known equality $\frac{2}{OH} = \frac{1}{OM} + \frac{1}{ON}$. Let us note that H can be considered as the image of O by the inversion of circle (G) or the orthogonal projection of I on the polar of O for this circle.

Eventually: $OJ = \frac{r^2}{OH}$ which means that J is the image of H by the inversion I .

Remark: this proof can be generalized to all circles of the plane which do not contain O in replacing distances by algebraic distances (used in the definition of harmonic division in §2).

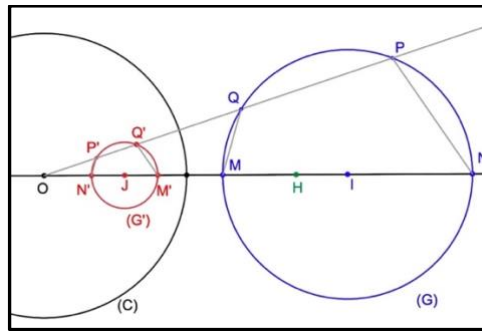


Figure 4: Image of a circle by an inversion

Theorem: The image of a circle (G) centered at I by an inversion I (when this circle does not contain the center O of the inversion) is another circle (G') centered at J where J is the image by I of H , H being the image of O by the inversion of circle (G) .

4.4. Angular property of the tangent lines at a point of two intersecting curves

4.4.1. An angular property of an inscribed quadrilateral (Figure 5 left)

$AA'B'B$ is a quadrilateral inscribed in a circle centered at O . Suppose that (AA') and (BB') intersect at M and (AB) and $(A'B')$ intersect at N . We know that $\angle MB'N$ is equal to $\angle MAB$. If (MI) is the angle bisector of $\angle AMB$ cutting $(A'B')$ at J , then $\angle BMI = \angle IMA$.

Let us consider the two similar triangles MIA and MJB' .

In triangle MJB' , $\angle M + \angle B' = \angle MJN$ (green); in triangle IMA , $\angle M + \angle A = \angle NIJ$ (magenta).

As triangles MIA and MJB' are similar, these sums are equal and so:

$\angle IJN$ (green) = $\angle NIJ$ (magenta), which means that triangle NIJ is isosceles (base $[IJ]$), and also that lines (AB) and $(A'B')$ are symmetric with respect to the perpendicular bisector of $[IJ]$. Eventually:

Theorem: If $AA'B'B$ is a quadrilateral inscribed in a circle, if the angle bisector of $\angle BMA$ cuts (AB) at I and $(A'B')$ at J , therefore (AB) and $(A'B')$ are symmetric with respect to the perpendicular bisector of $[IJ]$.

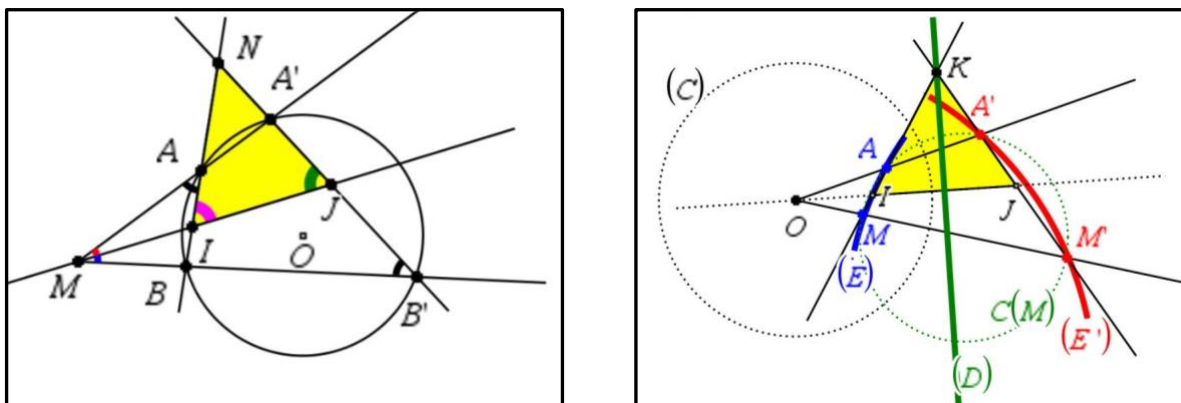


Figure 5: Angular property of inversions

4.4.2. A consequence about the angles of tangent lines to two secant curves and to their images (Figure 5 right)

Let us give the inversion of circle (C) centered at O ; (E') is the image of the blue curve (E) by this inversion; A' is the image on (E') of point A belonging to (E) and M' is the image on (E') of another point M belonging to (E) .

By definition of the inversion, $OA.OA' = OM.OM'$ and therefore points A, A', M' and M belong to the same circle $C(M)$. Thanks to the previous result, triangle KIJ is isosceles (base $[IJ]$) and then lines (AM) and $(A'M')$ are symmetric with respect to the perpendicular bisector (D) of $[IJ]$.

If M approaches A along (E) , the limit of (AM) is the tangent (T) to (C) at A and then, as M' approaches A' along (E') , the limit of $(A'M')$ is the tangent (T') to (C') at A' . In those conditions as I approaches A and J approaches A' , the tangent lines (T) and (T') are symmetric with respect to the perpendicular bisector of $[AA']$ (limit of (D) when M approaches A).

Theorem: If the curve (E) has (E') for image by an inversion, if A is a point of (E) and A' its image by this inversion, therefore, the tangent lines at A and A' respectively to (E) and (E') are symmetric with respect to the perpendicular bisector of $[AA']$.

So, we can deduce from this theorem

Corollary: If two curves intersect at A and if α is the angle of the tangent lines to these curves at this point, the angle between the tangent lines to the curves images of the previous ones by an inversion, at A' image of A by this inversion, is equal to $-\alpha$.

Remark: if the tangent lines to the given intersecting curves are perpendicular, the tangent lines to the curves images of these curves are also perpendicular

4.5. Metric properties

Before the second part of this paper, let us recall how distances and radii of circles are modified by an inversion.

Let us give I the inversion of circle (C) (center I , radius r):

If $[A'B']$ is the image of $[AB]$ by I : $A'B' = \frac{r^2 \cdot AB}{IA \cdot IB}$ (a proof uses the fact that triangle IAB is similar to triangle $IB'A'$)

If a circle C_1 of radius r_1 has for image a circle C'_1 of radius r'_1 , $r'_1 = \frac{r^2 \cdot r_1}{Im \cdot Im'}$ where m and m' are the intersection points between the circle C_1 and the line joining I to the center of C_1 (with the notations of Figure 3 right).

Eventually, in a “certain way”, if two circles are transformed onto two other circles by an inversion, the ratio between the radii of the image circles is “independent” of the scale of the inversion.

5. Images of bundles of circles by an inversion

5.1. Image of a bundle at base points

Figure 6 left: let us consider a bundle of circles at base points A' and A . D is the perpendicular bisector of $[AA']$ passing through the midpoint o_1 of this segment. Every circle C_n of this bundle is centered on D at o_n (in Figure 6 left, we have displayed C_1 centered at o_1 , C_2 centered at o_2 and C_3 centered at o_3). Let us consider now an inversion I centered at A and which circle C passes through a point v of (AA') .

Remark: all the circles of the bundle pass through the center of the inversion.

As a consequence, the images of the circles of this bundle are lines. And, as every circle of the bundle passes through a common point A' , all these lines pass through the image of A' by I . In Figure 6 (left and right), we can see that the images D_1, D_2 and D_3 of C_1, C_2 and C_3 pass really through a common point. This point belongs to the polar of A' with respect to C (circle of the inversion). The direction of each of these lines is respectively the direction of the perpendicular to the line joining A to each center of the given circles of the bundle.

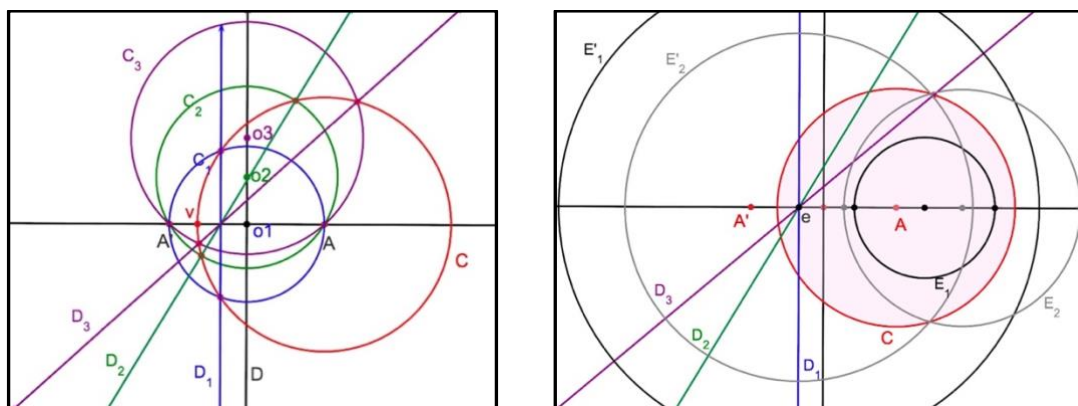


Figure 6: Image of a bundle of circles at base points

Let us consider now the bundle of Poncelet points A' and A whose circles are E_n (in Figure 6 right, only two circles are displayed: E_1 and E_2). We know that these circles are orthogonal to all circles of the previous bundle based at A' and A . As the inversion keeps the orthogonality of tangent lines, the images E'_n by I of the circles E_n are orthogonal to all the images of the circles C_n which means, to all lines D_n . Eventually, as all lines D_n pass through a common point e , E'_n is a circle centered at e .

Theorem: If F is a bundle at Poncelet points A' and A , the image of this bundle by any inversion centered at A is a bundle of concentric circles centered at e image of A' by this inversion.

5.2. Steiner chains of two concentric circles (existence conditions)

From now, our aim is to construct chains of tangent circles but also tangent to two given nested circles. We start by the case when the two given circles are concentric and necessarily the circles of the chain have the same radius.

In Figure 7, (I) and (E) are two concentric circles centered at O with radius respectively r and $k.r$ where $k > 1$. T_1 is a point on (E) and (C_1) is the circle tangent to (I) and (E) respectively at t_1 and T_1 , centered at O_1 .

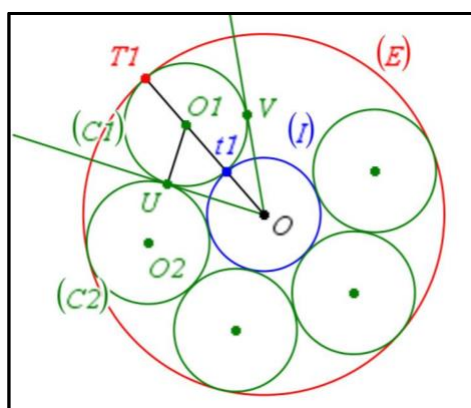


Figure 7: Chain included between two concentric circles

The radius of (C_1) is equal to $(k - 1) \cdot \frac{r}{2}$. U and V are the two contact points of the two tangent lines to (C_1) constructed from O .

The angle $\angle VOU$ is the angle of the rotation centered at O which allows by iteration of the images of (C_1) , a chain of circles (C_n) isometric to (C_1) , respecting our initial constraints.

Let's see now under what conditions for k and the number n of circles of the chain, this chain is closed, which means that the circle (C_n) touches for the first time (C_1) and is tangent to this circle at V .

If $\alpha = \angle UOV$ which is the double of $\angle UOO_1$, we have:

$$\sin\left(\frac{\alpha}{2}\right) = \frac{UO_1}{OO_1} = \frac{(k-1)\frac{r}{2}}{r+(k-1)\frac{r}{2}} = \frac{k-1}{k+1} \text{ and then: } \alpha = 2 \cdot \sin^{-1}\left(\frac{k-1}{k+1}\right).$$

So, the condition expressing that n circles exactly can be inscribed in one turn is:

$$n \cdot \alpha = 2 \cdot \pi \text{ or: } n \cdot \sin^{-1}\left(\frac{k-1}{k+1}\right) = \pi \text{ (Eq1)}$$

Similarly, the condition expressing that n circles exactly can be inscribed in m turns is:

$$n \cdot \alpha = 2 \cdot m \cdot \pi \text{ or: } n \cdot \sin^{-1}\left(\frac{k-1}{k+1}\right) = m \cdot \pi. \text{ (Eq2)}$$

Particular cases:

Chain of 2 circles in 1 turn: $n = 2$ in **Eq1** returns $k = 1$, which means that the two given circles are the same circle. This case is excluded.

Chain of 3 circles in 1 turn: $n = 3$ in **Eq1** returns $k = 7 + 4 \cdot \sqrt{3}$ (Checked in Figure 8 left)

Chain of 4 circles in 1 turn: $n = 4$ in **Eq1** returns $k = 3 + 2 \cdot \sqrt{2}$ (Checked in Figure 8 center)

Chain of 5 circles in 1 turn: $n = 5$ in **Eq1** returns $k = \frac{4+2 \cdot \sqrt{2 \cdot (5-\sqrt{5})}}{4-2 \cdot \sqrt{2 \cdot (5-\sqrt{5})}}$ (Checked in Figure 8 right).

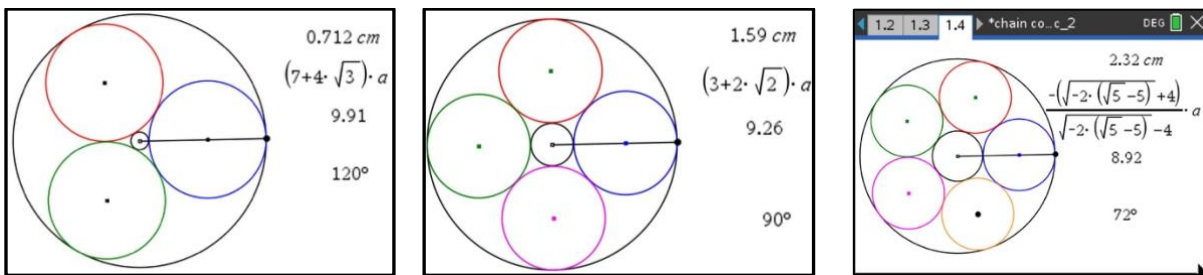


Figure 8: Closed chains in concentric circles

Chain of 6 circles in 1 turn: $n = 6$ in **Eq1** returns $k = 3$ (Figure 9 left)

General case for n circles in m turns: $k = \frac{1+\sin\left(\frac{m\pi}{n}\right)}{1-\sin\left(\frac{m\pi}{n}\right)}$ (particular case $n = 7$ and $m = 2$: Figure 9 right)

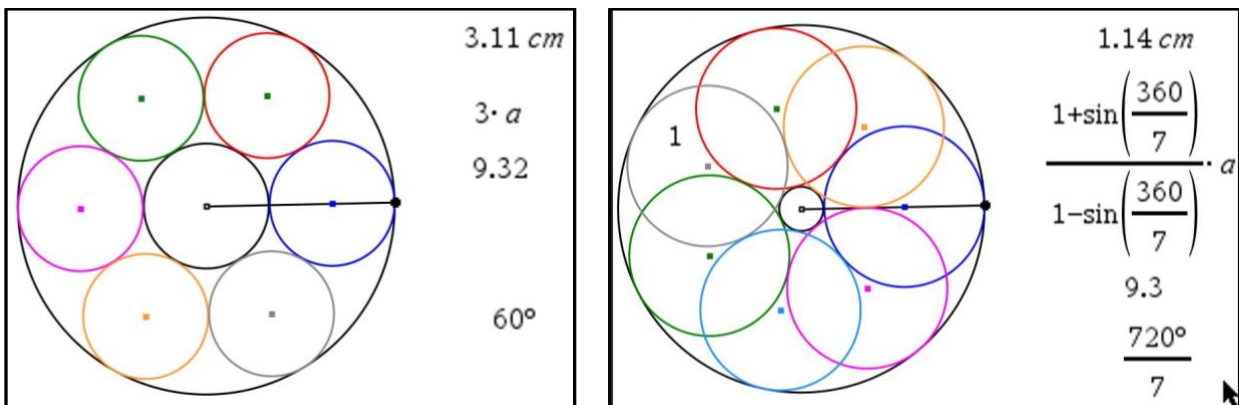


Figure 9: Other closed chains in concentric circles

5.3. Steiner chains: an approach of the general case

We start now from two concentric circles INT and EXT (same center e) which radii are respectively r and $k.r$ where k can be chosen as we want (Figure 10 left). I is an inversion centered at A and of circle C . We know (see 4.2.) that the images of these two circles by this inversion are two circles int and ext belonging to the bundle at Poncelet points A and A' where A' is the image of e by the inversion I (Figure 10 right).

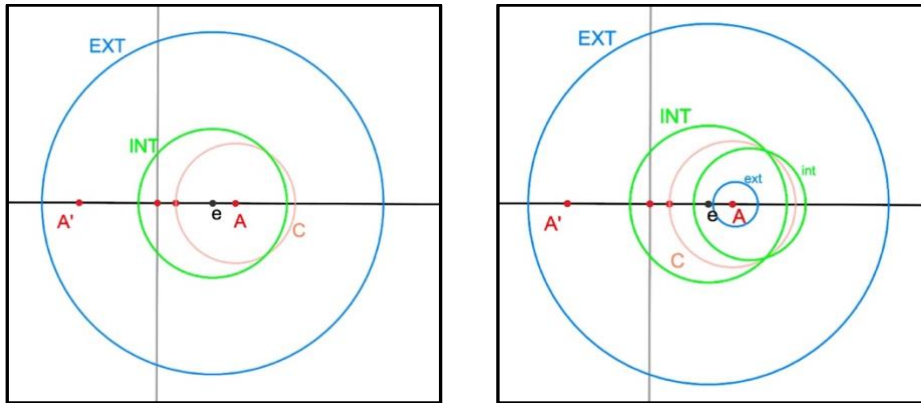


Figure 10: For our first chains in nested circles

From a point T_1 of EXT , let us construct (Figure 11 left), a chain of black circles (a sequence of tangent circles) tangent to EXT and INT (we can see three of these circles in Figure 11 left). The images of these circles by inversion I centered at A , is a chain of circles tangent to the two circles ext and int (here ext is inside of int : due to properties of inversions in this case of figure). If point T_1 is dragged along EXT turning clockwise or anticlockwise inside the concentric circles, we obtain other chains between EXT and INT and by the way chains between int and ext .

Remark: if the value of k is correctly chosen, we can obtain closed chains of circles between the concentric circles and by the way also between their images ext and int . That is displayed un Figure 11 right where we have constructed such a chain of six circles called Steiner chain of circles between ext and int .

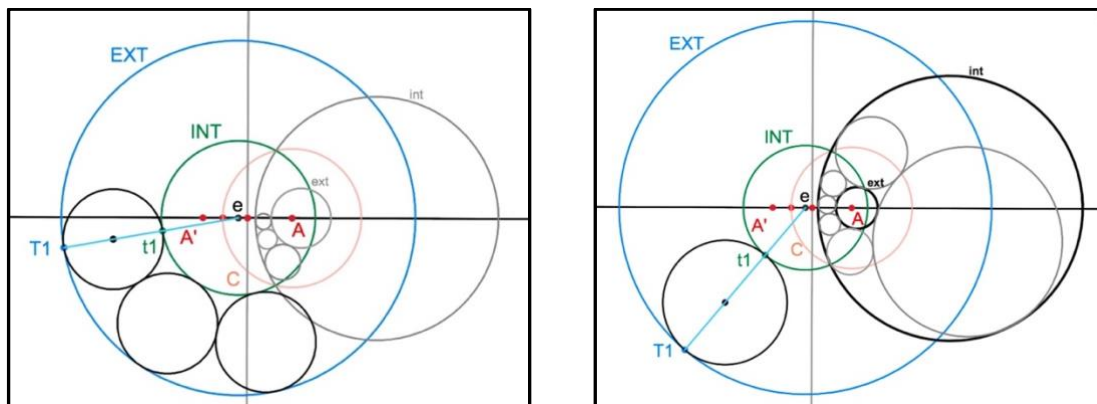


Figure 11: Construction of a Steiner chain

5.4. Steiner chains: general case

5.4.1. Two nested circles belong to the same Poncelet bundle

As shown in Figure 12 left, we consider two nested circles C_1 and C_2 centered respectively at i_1 and i_2 . We show now how to construct the radical axis of these two circles (line containing all the points

having the same power for C_1 and C_2). As this radical axis is perpendicular to (i_1i_2) , we need only to find and construct the point I on (i_1i_2) having the same power for the two circles, which means the point I verifying $Im.Im' = In.In'$ or which is equivalent:

$\frac{In}{Im} = \frac{Im'}{In'}$. This equality means that I is the center of the dilation transforming m onto n and n' onto m' and finally the segment $[mn']$ onto the segment $[nm']$. That justifies the elementary construction of I proposed in Figure 12 left.

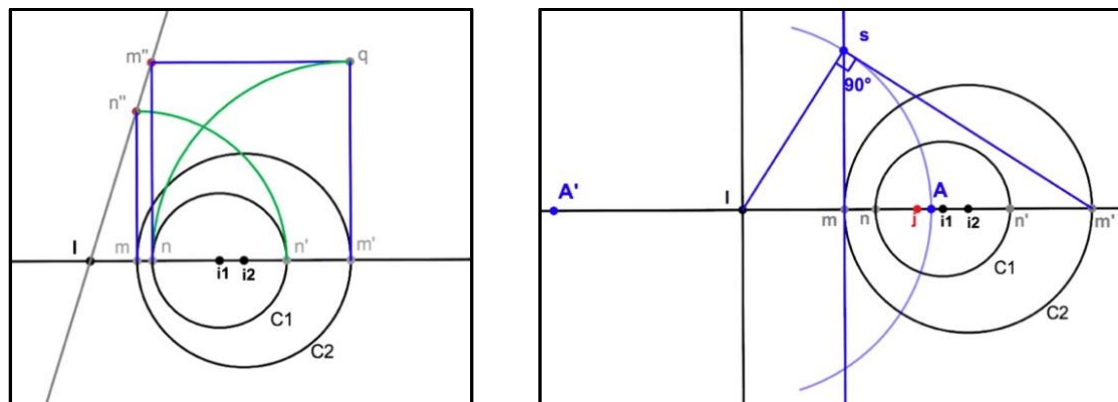


Figure 12: Poncelet bundle including two given nested circles

Figure 12 right shows precisely the construction of point A of the ray $[Im]$ verifying $IA^2 = Im.Im'$ (this construction is based on a known property of right-angle triangles). As I is a point of the radical axis of C_1 and C_2 , $IA^2 = In.In'$ where n and n' are the intersection points between C_1 and line (i_1i_2) . If A' is the symmetric of A with respect to I , the previous relations means that (A', A, m, m') and (A', A, n, n') are harmonic divisions or better that circles C_1 et C_2 belong to the bundle at Poncelet points A' and A . We already know that the images of these two circles by any inversion centered at A or A' are two concentric circles (j is the midpoint of $[Im']$ in the construction of Figure 12 right).

5.4.2. How to construct a chain of circles tangent to two nested circles

Starting from C_1 and C_2 , we construct as we did before point A' and A such that C_1 and C_2 belong to the bundle of points of Poncelet A' and A . We chose an inversion centered at A , for example. We transform these two circles by this inversion to get two concentric circles and their center. Then we construct a chain of tangent circles between the two concentric circles. The image of this chain by the inversion provides a chain of tangent circles between the circles we started from.

Now, the question is: what is the inversion that will generate a value of k compatible with the construction of a closed chain?

5.4.2.1. Complexity of the choice of the parameters of our problem

For such circles C_1 and C_2 whose radii are respectively r_1 and r_2 and centers i_1 and i_2 (where $r_1 < r_2$), let us evaluate the ratio of the radii of the circles images by an inversion I centered at A where A is one of the two points of the Poncelet bundle containing C_1 and C_2 . Recall that this ratio is “independent” of the circle defining the inversion. All the following calculations are conducted in the axis system supported by the line (i_1i_2) oriented from i_1 to i_2 and which origin is i_2 . In this system, point i_1 is given by its abscissa α_1 , with $-r_2 < \alpha_1 < r_2$.

■ We evaluate first the abscissa x_I of point I having the same power with respect to C_1 and C_2 , which means, verifying $\overrightarrow{Im}. \overrightarrow{Im'} = \overrightarrow{In}. \overrightarrow{In'}$ or $(-r_2 - x_I)(r_2 - x_I) = (\alpha_1 - r_1 - x_I) \cdot (\alpha_1 + r_1 - x_I)$ that can be written as a quadratic relation: $2x_I^2 + 2\alpha_1 \cdot x_I + r_1^2 - r_2^2 - \alpha_1^2$ (where $r_1^2 - r_2^2 - \alpha_1^2 < 0$)

If $\Delta = \alpha_1^2 - 2 \cdot (r_1^2 - r_2^2 - \alpha_1^2) = 3\alpha_1^2 - 2r_1^2 + 2r_2^2$, we know that Δ is positive, so our quadratic equation has two solutions with opposite signs. If we add the constraint $\alpha_1 < 0$, then:

$$x_I = \frac{-\alpha_1 - \sqrt{\Delta}}{2}$$

■ The abscissa of A , x_A is evaluated thanks to the condition $IA^2 = \overrightarrow{Im} \cdot \overrightarrow{Im'}$ which is the power \wp of I with respect to circle C_2 . We know that this power is also equal to $Ii_2^2 - r_2^2$ or $x_I^2 - r_2^2$. After computation, we get:

$$IA^2 = \frac{1}{4} (4\alpha_1^2 - 2r_1^2 - 2r_2^2 + 2\alpha_1\sqrt{\Delta}) \text{ and as } IA^2 = (x_A - x_I)^2, \text{ we obtain the abscissa of } A \text{ and } A':$$

$$x_A = x_I + \sqrt{\wp} \text{ and } x_{A'} = x_I - \sqrt{\wp}.$$

■ Finally, if r'_1 and r'_2 are the respective radii of circles C'_1 and C'_2 images of C_1 and C_2 by I , we have (r is the radius of the circle of the inversion):

$$\frac{r'_1}{r'_2} = \frac{r^2 \cdot r_1}{Am \cdot Am'} : \frac{r^2 \cdot r_2}{An \cdot An'} = \frac{r_1}{r_2} \cdot \frac{\overrightarrow{An} \cdot \overrightarrow{An'}}{\overrightarrow{Am} \cdot \overrightarrow{Am'}} \text{ or } \frac{r_1}{r_2} \frac{Ai_1^2 - r_1^2}{Ai_2^2 - r_2^2}.$$

We already know that to construct a Steiner chain with a given number of circles between two nested circles, the previous ratio must be equal to a specific number k (shown in 5.2.).

■ It is easy to imagine the complexity of the evaluation of one parameter with respect of the two other two parameters to obtain a particular value of k . If C_2 , is given, the parameters we can modify are α_1 and r_1 . For example, from the previous equation $\frac{r_1}{r_2} \frac{Ai_1^2 - r_1^2}{Ai_2^2 - r_2^2} = k$, we would have to find a formula giving α_1 with respect to r_1 , r_2 and k , which was unsurmountable at an elementary level. If it had been possible, we would have been able to locate the center of C_1 with respect to the center of C_2 (if the radii of the two circles are fixed to obtain one circle inside the other compatible with the construction of a Steiner chain associated with k). For this reason, we will conclude with a technique of dichotomic approximation of the value of r_1 allowing the best approximation of the given k compatible with the construction of the associated Steiner chain.

5.4.2.2. Solution by successive approximations (figure 13)

On a given line, locate i_2 center of circle C_2 , then i_1 center of circle C_1 with **radius r_1** which is a **displayed number** that can be **modified**. The number k is also a displayed number, evaluated by the software: it is the number which is the ratio of the radii of the concentric circles allowing the construction of a chain of n circles tangent to these concentric circles and by the way the construction of a Steiner chain of n tangent circles to the given circles C_1 and C_2 .

We construct, as done before, point A (and point A') which is the center of the inversion transforming C_1 and C_2 onto two concentric circles (the circle of this inversion is commanded with a point of the initial line).

We construct the images C'_1 and C'_2 of the circles C_1 and C_2 by the inversion and their common center e (which is the image of A' by the inversion of circle C), which allows us to evaluate (with the software we use: the New Cabri) their respective radii as well as the ratio of these radii. Considering the case of our figure, we evaluate r'_1/r'_2 which is displayed.

We construct a point t_2 of C_2 and its image t'_2 on C'_2 by the inversion. Thanks to the ray $[et'_2)$, we construct the first circle tangent to C'_1 et C'_2 (tangent to C'_2 at t'_2) visible in grey in Figure 13 left. Then we construct the image of this circle by the inversion to obtain the first circle tangent to C_1 and C_2 (tangent to C_2 at t_2) visible in the same figure in black.

We construct now a chain of n grey circles tangent to C'_1 and C'_2 starting from the previous grey circle which images by the inversion provide a chain of black circles tangent to C_1 and C_2 starting from the previous black circle (Figure 13 right).

In Figure 13, we have chosen $n = 6$. But we can state that since r'_1/r'_2 is not equal to 3 which is the value of k associated with a closed chain of 6 circles, our chain of 6 grey circles is not closed between C'_1 and C'_2 and so, the black chain we have constructed by inversion is not a Steiner chain of 6 black circles.

The final technique consists in modifying the value of r_1 in a dichotomic way until the displayed value of r'_1/r'_2 is as close as possible to number 3 when using the maximum of digits allowed by the software. If necessary, we can also change the position of i_1 .

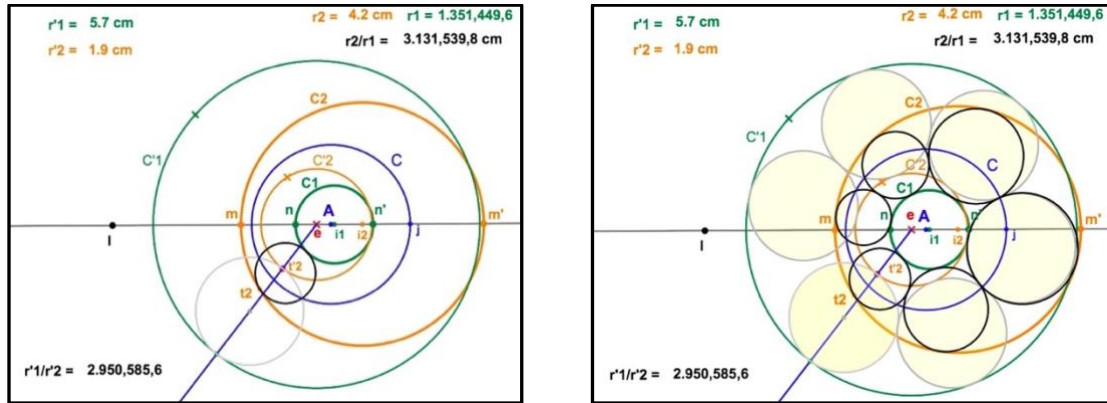


Figure 13: Starting constructions of a Steiner chain in two nested circles

After such an operation we reach a situation visible in Figure 14 left where the Steiner chain of 6 circles is constructed between C_1 et C_2 . The animation of point t_2 along C_2 , allows us to visualize all the Steiner chains of 6 circles between C_1 et C_2 . On the same figure, we can state that, if we modify r_1 until the value underlined in green, we have succeeded to reach for the displayed value of r'_1/r'_2 , 3.000,000,0. In Figure 14 right, we can appreciate the final result after hiding the constructions.

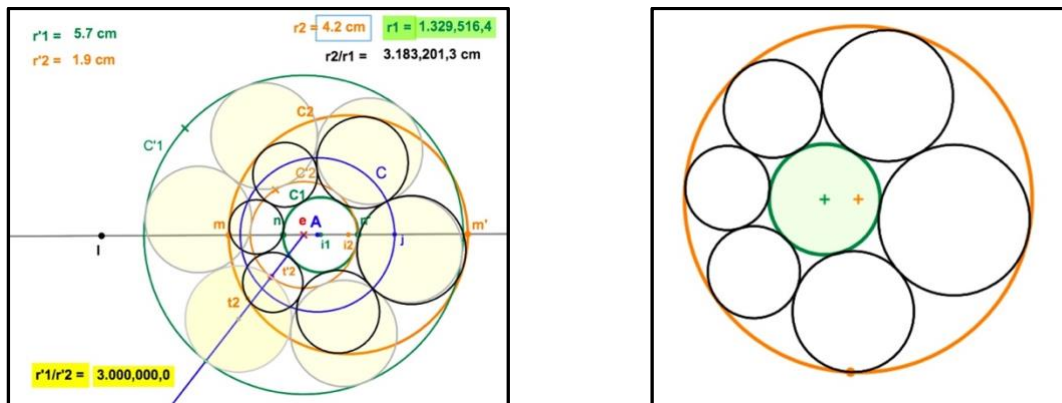


Figure 14: Final construction of a Steiner chain between two nested circles

In Figure 15 left, we show the construction of a Steiner chain of 11 circles closed after one turn (now in red). The value of k that r'_1/r'_2 must reach after an adequate modification of r_1 is equal to $\frac{1+\sin(\frac{\pi}{11})}{1-\sin(\frac{\pi}{11})}$ which value given by the software with its best approximation is 1.7844781. We have reached 1.7844785 that justifies the accuracy of our construction. We can appreciate the result when the constructions are hidden in Figure 15 right.

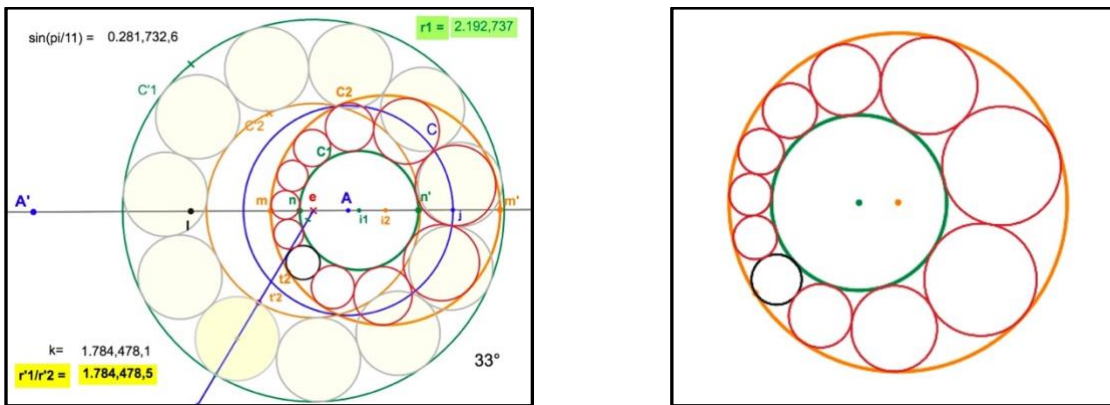


Figure 15: A Steiner chain of 11 circles

In order to construct all possible Steiner chains, we need such a figure in which we have constructed the number of circles corresponding to the expected chain, change the value of k corresponding to this expected chain by using the formula

$$\frac{1 + \sin\left(\frac{m\pi}{n}\right)}{1 - \sin\left(\frac{m\pi}{n}\right)}$$

where, n is the number of the circles of the chain and m is the number of turns necessary for the first closure of the chain

5.5. Steiner chains: some other properties ([6])

Only the use of Dynamic Geometry Software can help us to investigate the following properties.

■ The first one is about the centers of the circles of a Steiner chain and the contact points between them (Figure 16). In Figure 16 left, we have constructed the conic passing through five of the eleven centers of the corresponding Steiner chain: we can easily conjecture that all the centers belong to this conic which is qualified as an ellipse (in red) by our software. In Figure 16 right, we have constructed the conic passing through five of the eleven contact points of the corresponding Steiner chain: we also can easily conjecture that all these points belong to this conic which is qualified as a circle (in blue) by our software. These properties are known properties that can be investigated easily in such figures with the appropriate software.

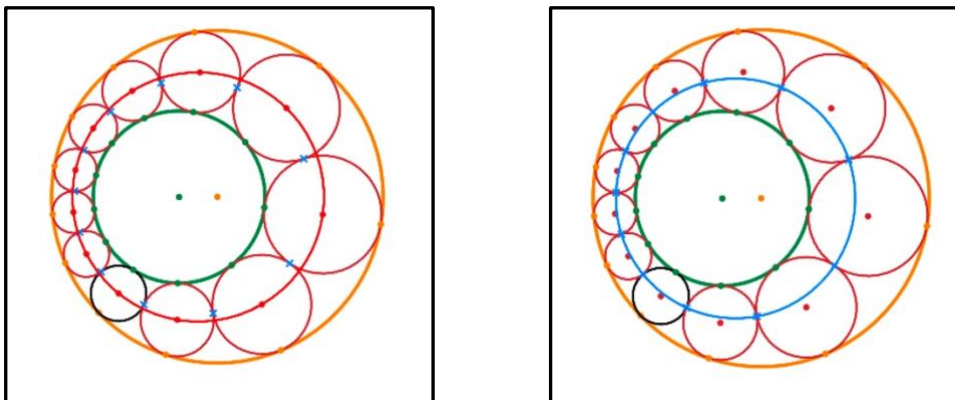


Figure 16: Some points of a chain on the same ellipse or the same circle

■ The second one, in the case of $n = 6$, is illustrated in Figure 17: here we can conjecture another known property stating that the three segments connecting the contact points of the Steiner chain

with the exterior circle (Figure 17 left) pass through a common point belonging to the line of the centers of the two given circles. Same statement for the contact points of the interior circles (Figure 17 right).

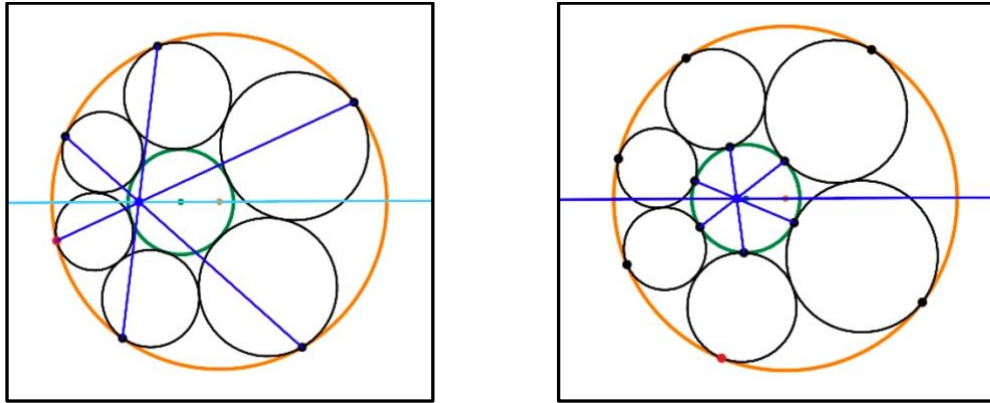


Figure 17: Segments passing through a common point on the line of the centers of the nested circles

6. Addendum

Here are some technological indications about the way we used the inversion of a point or the inversion of a circle under the **New Cabri environment**. This software is the one we used for all our figures even if most of them could have been realized with Cabri 3D (we did it but finally we have chosen the New Cabri for practical reasons). As we had to construct a great quantity of images of points or circles by an inversion, we have defined two macros for such a work:

Macro 1 returning the image of a given point by a given inversion.

Macro 2 returning the image of a given circle by a given inversion.

Creation of Macro 1 (Figure 18 left): let us construct the circle (C) (centered at O) of an inversion, let us create a point M . Using the measurement tool, we display the radius r of (C) and the distance OM . We evaluate with the Calculator in Algebraic mode $\frac{r^2}{OM}$ which is OM' . We construct ray $[OM)$:

point M' is obtained with the tool Measurement Transfer in transferring $\frac{r^2}{OM}$ on ray $[OM)$ (click on the number, click on the ray and click on O). Then we create Macro 1: chose **Initial Object(s)** and click on circle (C) and on point M , chose **Final Object(s)** and click on point M' and finally chose **Define Macro**. A square containing a wheel gear appears. It is possible to change this wheel gear onto another image: it is what I did in including number 1. This square represents **Macro 1**. It is possible now to use it as a tool (inversion of a point): to check it we create a point N , then we select **Macro 1** in clicking on the square and the image of N is created after clicking on (C) first and on N .

Creation of Macro 2 (Figure 18 right): we open another page of the same document, we copy **Macro 1** from the first page and paste it on page 2. We create circle (C) (circle of our inversion), a blue circle (CI) and a point M on (CI). Thanks to **Macro 1** we create the image M' of M (click on **Macro 1**, on circle (C) and on point M). Then ask the software to return the locus of M' when M describes (CI): it is circle ($C'I$) image of (CI) by the inversion of circle (C). We have now to define **Macro 2**: same algorithm than for **Macro 1**: Circle (C) and circle (CI) are the initial objects, ($C'I$) is the final object and clicking on **Define Macro** generates the button (square with a wheel gear) which is **Macro 2**. We have inserted number 2 to replace the wheel gear. At last we can test **Macro 2** in clicking on (C) and on a new blue circle ($C2$) to obtain its image by the inversion which is the orange circle ($C'2$).

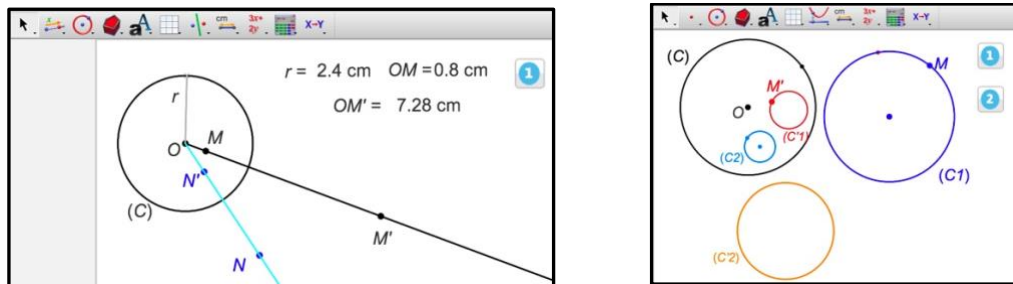


Figure 18: Creation of the two macros related to the tool “Inversion”

7. Conclusion

Thanks to dynamic geometry software allowing easy use of a tool for inversion, the paper has demonstrated how to solve the problem of the construction of Steiner chains as simply as possible. The paper provides to the reader with a wide range of the necessary knowledge to understand the techniques of construction and to show the power of these techniques in the figures created with the particular software used (here the New Cabri). As always in research of this kind, the work involved some questions which could not be solved with elementary tools (such as computations by hand or by CAS), and these are explored in the paper. It is hoped that this paper provides a clear view of the problems of construction of Steiner chains and will inspire teachers with some ideas for experimental investigations for their students even at a highschool level.

Acknowledgment

I am grateful to Barry Kissane for his advice on English expression.

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WIKIPEDIA, *Power of a point*, article at https://en.wikipedia.org/wiki/Power_of_a_point

Software

Cabri “Standard”, version 3.14 and *Cabri 3D* by Cabrilog S.A.S at <http://www.cabri.com>
TI-Nspire™ CX CAS Premium Teacher Software, version 5.2.0.771 by Texas Instruments at <https://education.ti.com/>

Mathematics Learning Strategy Scales for Junior High School Students: Scale Development, Validation and Intelligent Application

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Abstract: *Mathematics learning strategies are an important part of the psychological structure of high-quality mathematics students. How to measure and evaluate students' mathematics learning strategies Level is of great significance in related quantitative research. Based on the existing research on mathematics learning strategies, a set of mathematics learning strategy scales suitable for Chinese junior high school students and with mathematics learning characteristics were compiled. This research has done 4 times of data collection (a total of 959 valid questionnaires), item analysis, confirmatory factor analysis and exploratory factor analysis, which proved the validity of the scale in the measurement of mathematics learning strategies. In this study, the mathematics learning strategies of junior high school students were divided into 3 main aspects: Mathematics Cognitive Strategies, Mathematics Meta-cognitive Strategies, Mathematics Resource Management Strategies; 10 kinds of sub-dimensions: Retelling, Elaboration, Organize, Planning, Monitoring, Feedback-Adjustment, Time-management, Environment-management, Mood-management, Help-seeking. According to the operational definition, test results show that the questionnaire has good reliability and validity, which can be used as an effective measurement tool for mathematics learning strategies of junior high school students. The study has written the scale into an intelligent batch assessment system of mathematics learning quality for primary and secondary school students, providing an accurate and convenient test of mathematics learning strategies for a wide range of junior secondary school students in an automated and intelligent manner, and providing accurate measurement reports and personalized improvement strategies for each student tested.*

1. Introduction

Forty years ago, teachers and scholars have realized that traditional classroom teaching methods are often not the best way for students to acquire knowledge and apply it in practice(see[1]), instead, enabling students to have the ability to learn and master learning strategies has become an important factor affecting their learning achievements(see[2]).Under the circumstance of the new era, UNESCO regards "learn to learn" as the core of education in the 21st century, and "learn to learn" is more important than "learn knowledge". Once learners have the ability to learn, they will be able to learn new knowledge, new technology, accept new challenges and undertake new tasks. Mastering learning strategies is the basis for measuring how to learn, and a good learning strategy is a powerful tool to help students make mathematics achievements (see [3], [4], and [5]). Mathematics learning strategies have always been playing an important role in mathematics learning. Measuring and evaluating the level of students' mathematics learning strategies have also become an important

link in related quantitative research.

Judging from existing researches, many measurement tools such as *Motivated Strategies for Learning Questionnaire* (MSLQ) (see[6]), *Learning and Study Strategies Inventory* (LASSI-HS) (see[7]) and *High School Students Mathematical Learning Strategy Questionnaire* (see[8]) have been compiled by domestic and foreign scholars. However, these researches are not enough to reflect the mathematics learning characteristics of junior high school students in the compulsory education of our country, and because some scales have been formulated for a long time, and many studies have not been revised but used directly, making it difficult to ensure its effectiveness.

In March 2021, China's Ministry of Education and other six departments jointly issued the *Guidelines for Compulsory Education Quality Evaluation*, which pointed out clear instructions and requirements in terms of student development, academic development, and physical and mental development. As a non-intellectual factor, learning strategy has a great influence on the learning process and academic performance of junior high school students. Therefore, based on the purpose of better understanding and measuring the learning strategies of junior middle school students in compulsory education at this stage, this study compiled a set of reliable and valid mathematics learning strategy scale considering the mathematics learning process of junior secondary school students in China. According to the learning characteristics of junior high school students and the actual use of learning strategies, this scale provided targeted theoretical references and evaluation tools for the teaching of junior high school mathematics strategies.

2. Multifaceted definition of learning strategies

2.1. Definition and classification of learning strategies

As an important research direction in psychology and pedagogy, a lot of definitions towards learning strategies have been carried out by scholars. Rigney (see[9]) proposed that learning strategies are various operations and procedures that students use to acquire, maintain and extract knowledge and homework. Bråten and Strømsø (see [10]) pointed out that students who were good at using organizational strategies were better at obtaining good performance. According to Winstein (see [7]), in a broad sense, learning strategies refer to various abilities that are helpful and necessary for effective learning and maintaining information, as assumed by researchers and practitioners. Haga (see [11]) pointed out that learning strategies are strategies affecting learners' self-information processing activities. All activities that can promote learning are learning strategies, such as memory methods, constructing connections between knowledge, taking notes, making comments, drawing marks, etc. According to the definition by Chamot (see [12]), learning strategies are skills, methods, or actions that can be directly carried out, which helps improve the learning effect, optimize the learning process, and strengthen the memory of language knowledge. Generally speaking, learning strategies refer to procedures, rules, methods, techniques, and control methods that learners effectively learn in learning activities, it can be either an implicit rule system or explicit operating procedures and steps.

There are different opinions on the definition of the concept of learning strategy in the academic circle, and they also have different views on the structure of learning strategies. Representative classifications are as follows. Rigney (see [9]) believed that learning strategies consisted of independent and inclusive strategies. Weinstein and Goetz (see [13]) believed that learning

strategies included cognitive information processing strategies, such as finishing strategies, and active learning strategies, such as test-taking strategies. Supplementary strategies, such as strategies for dealing with anxiety; meta-cognitive strategies, like those for monitoring the acquisition of new information. Pintrich (see [14] and [15]) believed that the most basic cognitive strategy was rehearsal, more complex strategies were elaboration and organization; based on learning strategies, Dansereau (see [16]) divided learning strategies into two types: basic strategies and supporting strategies. Basic strategies refer to various learning strategies in which materials can be directly manipulated, which mainly includes information acquisition and storage, information retrieval and application, such as memorization, organization and recalling strategies. Supporting strategies mainly refer to strategies that help learners maintain an appropriate learning mentality, so as to ensure the effective operation of basic strategies, such as focused attention strategies, self-monitoring and judgement strategies. McKeachie etc. (see [6]) sorted out them as cognition based on the components covered by the learning strategy, meta-cognitive strategy, resource management strategy. Cognitive strategies include retelling strategies, finishing strategies, and organizational strategies; meta-cognitive strategies include planning strategies, monitoring strategies, and adjustment strategies; resource management strategies include time management strategies, learning environment management strategies, effort management, and support from others and so on.

2.2. Tools for measuring learning strategies

In the establishing process of many learning strategy measurement tools, there is a certain similarity in the structure of learning strategies by foreign researchers. Among them, the most representative and most widely used is the MSLQ scale established by Pintrich, McKeachie and so on (e.g. [6] and [17]). In terms of the classification of learning strategies, it consists of three categories: mathematical cognitive strategies, mathematical meta-cognitive strategies, and mathematical resource management strategies, in these strategies, 81 items and 15 sub-dimensions were included to evaluate students' learning motivation and learning strategies. Berger (see [18]) established a targeted scale for middle school students' cognitive strategies according to McKeachie's research. Liu (see [19]) established a learning strategy scale for middle school students based on the research of McKeachie et al. Secondly, The Learning and Study Strategies Inventory established by Weinstein (see [20]) divides learning strategies into 10 sub-dimensions, namely: attitude, motivation, time organization, anxiety, concentration, information processing, selection of main ideas, use of techniques and support materials, self-assessment, testing strategies.

Chinese researchers have compiled a series of questionnaires on mathematics learning strategies. For example, Liu's (see [21]) *Mathematics Learning Strategy Scale for Primary School Students* contains two main dimensions: mathematics meta-cognitive strategies (including planning strategies, monitoring and adjustment strategies, evaluation of reflection strategies, strategic awareness) and mathematical cognitive strategies (including mathematics concept strategies, computational learning strategies, application problem solving strategies, geometric knowledge learning strategies). *The Middle School Students' Mathematics Learning Strategy Scale* compiled by Yao (see [22]) contains four dimensions: mathematics meta-cognitive strategy, cognitive strategy, resource management strategy and emotional strategy. The Questionnaire on Mathematics Learning Strategies for High School Students compiled by Wang Guangming divides mathematics learning

strategies into mathematics cognitive strategies, mathematics meta-cognitive strategies and mathematics resource management strategies.

3. Methods

3.1. Model construction

Integrating literature data and existing mature scales, combining expert opinions, dimensions and model of the questionnaire are finally determined (Fig. 3.1.1), which contains 3 primary dimensions and 10 secondary dimensions. The operational definitions of each dimension are shown in Table 3.1.1.

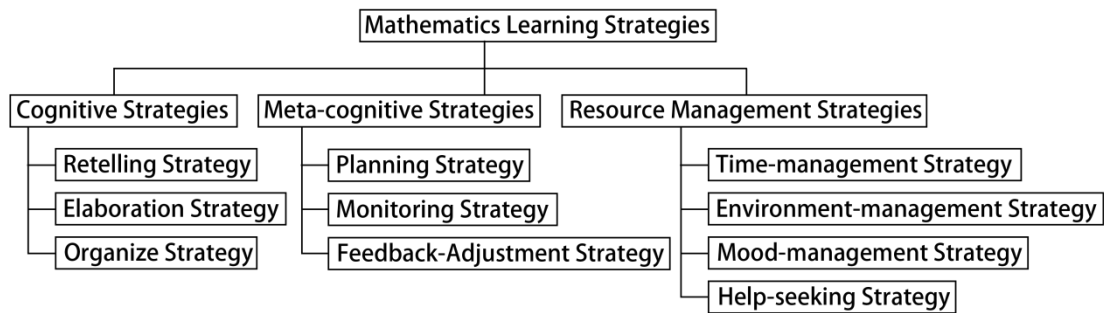


Fig. 3.1.1 Dimension division of mathematics learning strategies

First-level dimension	Second-level dimension	Definition of operation concept
Mathematics Cognitive Strategies	Retelling Strategy	It refers to the strategy of reproducing learning materials or stimuli in order to remember and retain relevant concepts, propositions, properties, inferences, solutions, etc. in mathematics, by means of timely review, repeated practice, multiple representation and elimination of interference in order to form long-term memory.
	Elaboration Strategy	It refers to the strategy of self-coding, translating, explaining and distinguishing different mathematical objects and situations, choosing appropriate ways, such as adding details, giving examples and forming associations, to express mathematical content and construct meaningful learning.
	Organize Strategy	It refers to the integration of mathematical knowledge with a holistic view, sorting and summarizing according to the characteristics or categories of mathematical knowledge, forming a clear knowledge network structure, from which a mathematical method or mathematical model can be selected or designed to

		solve problems through thinking and judgment.
Mathematics Meta-cognitive Strategies	Planning Strategy	It refers to the overall planning of mathematics learning, as well as the strategy of reasonable planning and arrangement of specific learning content, time allocation, difficulty progress, methods and so on.
	Monitoring Strategy	It means that learners are alert to their own mathematical knowledge level, learning state, learning ability, learning effect, etc., and can find out abnormal situations in time, such as realizing that their poor ability of examining problems affects the correct rate of solving problems, realizing that they do not understand the concept of equation clearly, etc.
	Feedback -Adjustment Strategy	It refers to the summary and generalization of learners' success and failure experience in the process of mathematics learning; And when the mathematics learning state and the mathematics learning environment change, in order to maintain the good mathematics learning effect, the learners make changes in behavior, psychology and learning methods.
Mathematics Resource Management Strategies	Time-management Strategy	It refers to the strategy of improving the efficiency of mathematics learning, arranging learning time as a whole, making efficient use of the best time and making flexible use of fragmented time.
	Environment -management Strategy	It refers to the management methods and measures for the preservation of mathematical materials, the placement of mathematical tools and articles, and the creation of mathematical learning atmosphere for the convenience of mathematics learning, such as ensuring the quiet and undisturbed learning environment.
	Mood-management Strategy	It refers to self-motivation, motivation and confidence.
	Help-seeking Strategy	It refers to the strategy of seeking external tools such as concept map software, mind map software, graphic calculator and online resources to assist mathematics learning, or seeking human resources such as teachers and students to assist learning.

Scale pre-establishment

Questions of the initial questionnaire are mainly from:

Learning and Study Strategies Inventory (LASSI-HS) (see[7])

Questionnaire on Mathematics Learning Strategies for High School Students (see[8])

Mathematics Learning Strategies Scale (see[23])

Considering characteristics of Chinese junior high school students' mathematics learning, quoting and adapting the content of the above questionnaires, a 87-question questionnaire was established, including 26 cognitive strategies, 28 meta-cognitive strategies, 27 resource management strategies, and 6 polygraph questions. In the questionnaire, the LIKERT five-point method for scoring is adopted, and the question options "very consistent", "Consistent", "uncertain", "non-consistent", and "very non-consistent" are respectively scored as "5", "4", "3", "2", "1". Random method is adopted to arrange the order of questions.

3.2. Sample and procedure

In the preparation of the questionnaire, a total of 4 tests were carried out using the cluster sampling method. For questionnaires collected in each test, invalid questionnaires are eliminated through the following two steps. Step 1: Visual inspection, eliminate invalid questionnaires with regular, periodic, and uniform answers; Step 2: Continue to screen the rest questionnaires based on the polygraph questions. Remove questionnaire in which there are great differences in answers to polygraph questions.

Sample 1: Pre-test sample. Including students in 8 classes in 4 typical schools in Tianjin, namely, there are students of the seventh and eighth grade of Experimental Middle School, Fangzhou Experimental Middle School, Haihe Middle School, and Tianjin No. 5 Middle School, respectively. A total of 330 questionnaires were issued and 286 were collected. After a two-step questionnaire screening, 209 valid questionnaires were finally obtained.

Sample 2: Used for exploratory factor analysis in the process of scale formulation. Subjects of the survey came from 5 middle schools in Tianjin, Hubei, Liaoning, Gansu, and Jiangsu. A total of 610 questionnaires were issued and 552 questionnaires were collected. After a two-step questionnaire screening, 428 valid questionnaires were finally obtained.

Sample 3: Used for verification factor analysis in the scale formulation process. Participants came from three schools in Tianjin, Shandong and Guangzhou. A total of 300 questionnaires were issued and 264 questionnaires were collected. After a two-step questionnaire screening, 209 valid questionnaires were finally obtained.

Sample 4: Used to calculate the retest reliability of the scale. Students are selected from those who have participated in the second test in two middle schools. A total of 160 questionnaires were issued and 142 questionnaires were collected. After a two-step questionnaire screening, 113 valid questionnaires were finally obtained.

4. Results

4.1. Data analytic of scale

The 81 questions of the questionnaire (not including polygraph questions) were analyzed through the SPSS software: First, analyze the correlation between the scores of each question in sample 1 and the total scores of students' mathematics learning strategies using the total question correlation method, then delete the questions with Pearson difference correlation coefficient of less than 0.4... which are the question 2,7,21,34,40,47,58,61,73,74,77,78 and 80, and a total of 13 questions have been deleted. Secondly, after deleting the above 13 questions, the critical ratio method is adopted to test the significant difference in the mathematics learning strategy scores of the high score group (the last 27%) and the low score group (the first 27%) of sample 1, items with insignificant

differences between high and low groups are excluded. It was found that after these items were deleted using the total correlation of items, there were significant differences between the high score and low score groups in the remaining items. After project analysis, 68 questions were left (not including polygraph questions).

After going through the item analysis to screen the questions, the exploratory factor analysis method was adopted in SPSS to analyze the data, modify and improve questions in the questionnaire. First, perform the Bartlett sphere test on the three main dimensions of the questionnaire, and the results are shown in Table 4.1.1. From the data in the table, we can see that the KMO values of the overall mathematics learning strategy and the three main dimensions are more than 0.89, and the Bartlett sphere test value is significant ($p < 0.01$), indicating that it is suitable for factor analysis of sample data.

Table 4.1.1 Factor analysis test value of the initial data of the scale

	Mathematics Learning Strategies (Total)	Mathematics Cognitive Strategies	Mathematics Meta-cognitive Strategies	Mathematics Resource Management Strategies
KMO	0.891	0.921	0.910	0.918
χ^2	11023.317	2084.042	2180.029	1493.373
Sig.	0.000	0.000	0.000	0.000

Principal component analysis and maximum variance rotation method are adopted in SPSS software to determine the number of questionnaire factors and questions. The following principles are adhered to in terms of keeping questions in the questionnaire.

- (1) The characteristic value of the factor is greater than 1;
- (2) The factor loading value is at least higher than 0.4;
- (3) The load on different factors is no more than 0.4;
- (4) The extracted principal components are consistent with the steep-order test;
- (5) Each factor contains at least 3 questions;
- (6) Only delete one question at a time, and re-examine and analyze the new data after each question is deleted.

Principles of naming after factors are as follows:

- (1) If the topic of a certain factor mainly comes from a certain sub-dimension of the mathematical learning strategy model, it is named after this sub-dimension;
- (2) If the topics that contribute more than half of the variance of a certain factor are scattered from different sub-dimensions of the mathematical learning strategy model, then they should be named after by referring to the common mathematical learning strategies of these topics.

To identify the number of factors in the questionnaire and the number of questions for each factor, the principal component analysis method and the maximum variance rotation method were selected in SPSS. After exploratory factor analysis (refer to Table 4.1.2), the factor structure of the three main dimensions of the mathematical learning strategy is established, and its load value, common factor variance and factor contribution rate are obtained, and the four principal components of the mathematical cognitive strategy are extracted, the three principal components of

cognitive strategy and the three principal components of mathematical resource management strategy.

Table 4.1.2 Rotated factor matrix for Mathematics Cognitive Strategies based on exploratory factor analyses constrained to three factors

Item	Factor loadings			Common factor variance
	1	2	3	
Q75	0.780			0.666
Q30	0.740			0.591
Q67	0.705			0.625
Q51	0.680			0.650
Q83	0.664			0.568
Q53	0.594			0.561
Q8		0.827		0.718
Q28		0.689		0.565
Q50		0.663		0.540
Q81		0.595		0.529
Q3			0.756	0.601
Q43			0.559	0.483
Q24			0.565	0.595
Q27			0.502	0.463
Cumulative percentage of variance (%)	24.182	44.269	58.252	

As is vividly shown in Table 4.1.2, in the factor structure matrix of the mathematical cognition strategy, the load value of each item after rotation is greater than 0.5, indicating that there is a close relationship between the item and the factor to which it belongs; In the common factor variance, the explanation of each question to the questionnaire is 0.4-0.8, indicating that the extracted factors can reflect the information of the original variables. The cumulative contribution rate of the three factors is 58.252%, indicating that this is a first-order three-factor structure, where factor 1 is an organization strategy, with a total of 6 questions; factor 2 is a finishing strategy, with a total of 4 questions; factor 3 is a retelling strategy, a total of 4 questions.

Table 4.1.3 Rotated factor matrix for Mathematics Meta-cognitive Strategies based on exploratory factor analyses constrained to three factors

Item	Factor loadings			Common factor variance
	1	2	3	
Q78	0.747			0.649
Q52	0.714			0.549
Q62	0.706			0.599
Q39	0.698			0.606
Q64	0.696			0.601

Q13	0.628		0.559
Q9	0.614		0.491
Q32	0.505		0.541
Q59		0.834	0.742
Q4		0.773	0.625
Q31		0.447	0.412
Q38			0.829
Q57			0.812
Q29			0.706
Cumulative percentage of variance (%)	28.990	45.846	60.689

As is vividly shown in Table 4.1.3, in the factor structure matrix of the mathematical meta-cognition strategy, the load value of each item is above 0.5, which indicates there is a close relationship between the item and the factor to which it belongs. In the common factor variance, the explanation of each question to the questionnaire is 0.5-0.8, indicating that the extracted factors can reflect the information of the original variables. The cumulative contribution rate of the three factors is 60.689%, indicating that this is a first-order three-factor structure, where factor 1 is a reflective adjustment strategy with a total of 8 questions; factor 2 is a monitoring strategy with a total of 3 questions; factor 3 is a planning strategy, a total of 3 questions.

Table 4.1.4 Rotated factor matrix for Mathematics Resource Management Strategies based on exploratory factor analyses constrained to three factors

Item	Factor loadings			Common factor variance
	1	2	3	
Q44	0.852			0.748
Q42	0.677			0.525
Q33	0.595			0.457
Q22		0.836		0.719
Q72		0.800		0.715
Q84		0.510		0.454
Q56			0.721	0.579
Q5			0.692	0.506
Q37			0.636	0.518
Cumulative percentage of variance (%)	20.533	40.647	58.017	

As is vividly shown in table 4.1.4, in the factor structure matrix of the mathematical resource management strategy, the load value of each item is above 0.5, which indicates that these items are closely related to factors to which they belong; in terms of common factor variance, each item's explanation of the questionnaire are 0.4-0.8, which shows that the extracted factors can reflect the information of the original variables. The cumulative contribution rate of the three factors is 58.017%, indicating that this is a first-order three-factor structure, where factor 1 is a mood management strategy with a total of 3 questions; factor 2 is an environmental management strategy

with a total of 3 questions; factor 3 is the outside world, a total of 3 questions for the help-seeking strategy.

Through item analysis and exploratory factor analysis, a total of 31 questions from the questionnaire were deleted, and 37 questions from the questionnaire on mathematics learning strategies (first edition) were retained. After deleting questions using the software's arithmetic values, it turned out that the division of dimensions and some questions were not ideal. The numerical value calculated by the software could be used as the basis for deleting the question, but the question couldn't be deleted blindly based on the numerical value. It is also necessary to consider whether the semantics of the reserved question are clear and concise, and whether it covers all dimensions. Considering the above reasons, the original dimensions need to be merged or split and renamed. To make questions in the questionnaire more consistent with characteristics of junior high school students' mathematics learning, and compile a questionnaire with good reliability and validity, students' responses to some questions in the mathematics learning strategy questionnaire (first edition) in the process of answering questions, the expression of some questions in the questionnaire is improved and modified. Subsequently, topics are mixed and sorted out, and the junior high school students' mathematics learning strategy level survey questionnaire (second edition) is finally determined, in which there is a total of 46 questions, including 42 formal questions and 4 polygraph questions (reliability questionnaire), the specific distribution of the questions is shown in Table 4.1.5.

Table 4.1.5 Item Distribution of Mathematics Learning Strategy Scale for Junior High School Students (2nd Edition)

Second-level dimension	NO.	Number of item
Retelling Strategy	3,17,24,27,41	5
Elaboration Strategy	2,8,28,46	4
Organize Strategy	1,6,7,15,30	5
Planning Strategy	12,26*,29,36,38,39	6
Monitoring Strategy	4,13,18,31,34,45	6
Feedback-Adjustment Strategy	9,10*,23,32	4
Time-management Strategy	19,25,44	3
Environment-management Strategy	11,16,21	3
Mood-management Strategy	33,40,42	3
Help-seeking Strategy	5*,20,37	3
Polygraph Test	11,14,35,43	4
Total		46

*Remarks: * are items with reversed scoring method*

Investigate sample two using using Mathematics Learning Strategy Scale for Junior High School Students (2nd Edition). First, process the collected data in advance, and then analyze the questionnaire data using use SPSS software based on the data processing method and analysis process in the initial research. The 15th and 17th questions are deleted, and 44 questions were left. Last but not least, the data is analyzed by exploratory factors, and the Mathematics Learning

Strategy Scale for Junior High School Students (3rd Edition) is determined. The specific distribution of questions is shown in Table 4.1.6.

Table 4.1.6 Item Distribution of Mathematics Learning Strategy Scale for Junior High School Students (3rd Edition)

First-level dimension	Second-level dimension	NO.	Number of item
Mathematics Cognitive Strategies	Retelling Strategy	3,24,27,41	4
	Elaboration Strategy	2,8,17,28	4
	Organize Strategy	1,6,7,30	4
Mathematics Meta-cognitive Strategies	Planning Strategy	12,26,29,36,38,39	6
	Monitoring Strategy	4,13,15,18,31,34	6
	Feedback-Adjustment Strategy	9,10,23,32	4
Mathematics Resource Management Strategies	Time-management Strategy	19,25,44	3
	Environment-management Strategy	11,16,21	3
	Mood-management Strategy	33,40,42	3
	Help-seeking Strategy	5,20,37	3
	Polygraph Test	11,14,35,43	4
	Total		44

Investigate the selected sample three through Mathematics Learning Strategy Scale for Junior High School Students (3rd Edition), then perform confirmatory factor analysis on the data collected from the questionnaire using AMOS software. The purpose of the verification factor analysis is to test whether there is a good fitting effect in the measurement results of the questionnaire and the conceptual model. The measurement model in the structural equation model is mainly used to test whether the various questions of the questionnaire can well constitute the 3 main dimensions (second-order factors) and 10 sub-dimensions (first-order factors) in the questionnaire. Before constructing the model, the first-order factors are tested. Then analyze the operation results of the AMOS software, first consider the load value of each topic in the dimension to which it belongs. According to the results, there are 25 questions with a load value of 0.6-0.8, 9 questions with a load value of 0.5-0.6, and 6 questions with a load value below 0.5. Questions with factor loading values less than 0.5 (questions 4, 5, 10, 12, 31, 34) are deleted, and 34 questions were left. Subsequently, observe the MI value of Covariances in Modification Induces. If the MI value is greater than 3.84, it is considered to be large enough. Under the premise that the causality of its parameters is supported by theory, it can be released, namely, deleted, so as to be estimated again. Observe the MI value of Covariances in the Modification Induces report of the result. If it is found that the correction index between 2 pairs of questions in the MI value is above 20, it indicates that there is a certain causal relationship between these questions.

Through the observation of the content of these questions, it is believed that there are indeed repetitive expressions in questions 6 and 7, questions 32 and 33 respectively. For example, question 6 "To consolidate the knowledge of mathematics I have learned, I will make a knowledge network by sorting out relevant knowledge" and question 7 "I will sort out and summarize the common test questions of mathematics in junior high school", there is a certain degree of homogeneity of

expression, and one question can be deleted among them. The factor load of question 6 is less than that of question 7, therefore, delete question 6; the factor load value of question 32 is 0.71, while that of question 33 is 0.53. Therefore, delete question 33 because of its small factor load. Referring to the MI value in Regression Weights and combining the meaning of the sentence, question 29 was classified as a mood management strategy. According to the revised MI value, 2 questions were deleted, leaving a total of 32 questions.

Combining the theoretical framework model, a second-order verification factor analysis was performed on the model. The results of various indicators are shown in Table 4.1.7

Model	χ^2/df	RMSEA	NFI	TLI	CFI	GFI
Second order factor	1.452	0.048	0.866	0.901	0.905	0.802

As is vividly shown in the data in Table 8, the second-order 3-factor model has a better fitting effect. The standardized load value of the first-order factor of 32 questions is between 0.50-0.78, and that on the second-order factor is between 0.50 and 0.78. The standardized load value of first order on the second order is between 0.45-0.68, therefore, we believe that the fitting between the first-order 10-factor and second-order 3-factor model has a good fitting effect and is acceptable. After conducting verification factor analysis, it turns out that the structural validity is good, and the path analysis is carried out to construct the structural model. The structure model of the questionnaire and the correlation coefficient between the standardized load value and the second-order factor are shown in Figure 4.1.1.

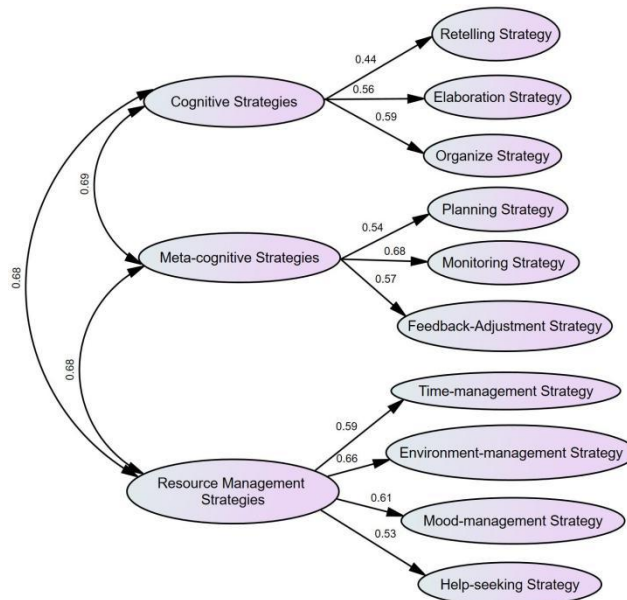


Fig. 4.1.1 Dimension model of mathematics learning strategies scale for junior high school

4.2. Reliability & validity of the scale

Reliability refers to the consistency, stability and reliability of the test results. The higher the

reliability coefficient, the more consistent, stable and reliable the test results. This paper analyzes the reliability of the questionnaire by calculating the internal consistency and test-retest reliability. According to the results (Table 4.2.1), the Cronbach's α coefficients of main dimensions of the questionnaire on the level of mathematics learning strategies for junior high school students are between 0.83 and 0.89, and the Spearman-Brown split-half reliability is between 0.76 and 0.86, indicating that there is a high degree of consistency in the internal questions of main dimensions of the compiled strategy questionnaire. The Cronbach α coefficient of all questions in the questionnaire is 0.970, and the Spearman-Brown split-half reliability is 0.937, indicating that there is a good internal consistency in the compiled mathematics learning strategy questionnaire. In the retest, sample four is selected, and the Pearson product difference correlation method was adopted to calculate the retest reliability of the questionnaire, subsequently, the correlation between the two test scores is analyzed, according to the results, the retest reliability of each main dimension of the questionnaire is between 0.86-0.90, and the retest reliability of the total questionnaire is 0.906, from which we can see that there is a good internal consistency in the questionnaire. Combined with various reliability indicators, it is believed that the *Mathematics Learning Strategies Scale for Junior High School Students* is highly reliable.

Table 4.2.1 Reliability of Mathematics Learning strategies scale

Scale index	Cronbach's α coefficient	Spearman -Brown Split-half reliability	Retest reliability
Mathematics Cognitive Strategies	0.888	0.818	0.891
Mathematics Meta-cognitive Strategies	0.878	0.856	0.858
Mathematics Resource Management Strategies	0.839	0.768	0.792
Mathematics Learning Strategies	0.970	0.937	0.906

Content validity refers to the extent to which questions of the questionnaire can reflect the content being measured, and whether the purpose of the measurement can be achieved. Before the initial test questionnaire was formed, the content of the questions was borrowed from domestic and foreign scales or questionnaire, and the actual situation of mathematics learning of junior high school students in our country was taken into consideration; after that, I discussed with many experts, including the preparation of the questionnaire's guideline and questionnaire. The structure of the questionnaire and precautions for the preparation of the question, etc., are revised or deleted according to opinions of experts, and the survey model is constructed to determine questions of the *Questionnaire on The Level of Mathematics Learning Strategies for Junior High School Students*. In terms of the evaluation of the content validity of the final version of the questionnaire, four experts, Cao Yiming, Yu Ping, Li Hongyu and Wang Xiaozhuang were invited to evaluate the correlation between each topic in the *Questionnaire on the Level of Mathematics Learning Strategies for Junior High School Students* and its respective dimensions (Expert Consultation Questionnaire) See Appendix 5), and calculate the content validity of the questionnaire based on the expert evaluation results, according to the results, the consistency level among the four evaluators is

0.84, indicating that there is good consistency among these evaluating factors. The content validity index of the questionnaire item level is shown in Table 4.2.2:

Table 4.2.2 Content Validity Index of Formal Questionnaire

Number of experts	Number of experts with a score of 3 or 4	Item number	I-CVI	Pc	K*	Evaluation
4	3	7	0.750	0.250	0.670	Good
	4	29	1.000	0.063	1.000	Excellent

There are 28 questions in the questionnaire being unanimously approved by experts, which accounts for 81% of the total number of questions, indicating that each question has good content validity; calculating the average I-CVI value of each question, it can be obtained that the content validity of the questionnaire, the S-CVI/Ave is 0.95, reaching the standard of 0.90, indicating that the *Mathematics Learning Strategies Scale for Junior High School Students* has good content validity.

According to Psychometric related theories, there should be a medium to high correlation between various dimensions of the questionnaire, and each dimension should have a high correlation with the total dimension. If the correlation between the dimensions and the total dimensions is higher than the correlation between the dimensions, it indicates that the dimensions are relatively independent, the structure validity is better; otherwise, the structure validity is poor.

Table 4.2.3 Correlation coefficients between each dimension of the questionnaire and that between each dimension and the learning strategy

	Mathematics Cognitive Strategies	Mathematics Meta-cognitive Strategies	Mathematics Resource Management Strategies	Mathematics Learning Strategies
Mathematics Cognitive Strategies	1			0.942**
Mathematics Meta-cognitive Strategies	0.804**	1		0.915**
Mathematics Resource Management Strategies	0.821**	0.758**	1	0.929**

*Remarks: ** means that there is a significant correlation at a significance level of 0.01 (two-sided).*

Through the observation of the data in Table 4.2.3, it is found that there is a significant correlation among the three dimensions of the questionnaire. The Pearson correlation coefficient between the three dimensions is between 0.75 and 0.83, and the total mathematics learning meta-cognition questionnaire is related to the Pearson correlation coefficient of each dimension. Between 0.90-0.95. The Pearson correlation coefficient between each dimension and the total questionnaire is higher than the correlation coefficient between each dimension, which means that the compiled questionnaire has good structural validity.

4.3. Establishment of the scale

In the *Mathematics Learning Strategy Scale for Junior High School Students*, mathematics learning strategies were divided into three dimensions: mathematics cognitive strategies, mathematics meta-cognitive strategies and mathematics resource management strategies. The details of the scale development are shown in the following three aspects:

1. The number and content of questions. A total of 36 questions were set in the *Mathematics Learning Strategy Scale for Junior High School Students*. The content of the questionnaire is consistent with the characteristics of junior high school students' mathematics learning, which helps prevent students from being distracted due to answering questions for a very long time, leading to distortion in data collection, and simultaneously ensuring the authenticity of the collected data.

2. Expression of questionnaire and instruction. In the compilation of the "Math Learning Strategies Scale for Junior High School Students", in order to clarify the question, words indicating frequency (such as "rarely", "frequently", etc.) were avoided, and definition of options in each level of the item is added to the instruction.

3. Verification factor analysis of the scale. In the compilation of the *Mathematics Learning Strategy Scale for Junior High School Students*, in order to better test whether the various topics of the scale and the theoretical model of the concept have a good fitting effect, sample data different from those in the verification factor analysis were adopted for verification factor analysis.

The questionnaire was developed by combing through the literature and combining expert opinions to construct a model for the initial version of the Mathematics Learning Strategies Scale, which was tested and revised several times to obtain the second, third and fourth versions of the scale respectively. Based on the test data of the fourth version, the survey model was revised to establish the structural model of the scale. A total of 36 questions were contained in *Mathematics Learning Strategies Scale for Junior High School Students*, which were divided into 3 dimensions and polygraph questions. The specific distribution of these questions is shown in Table 4.3.1.

Table 4.3.1 Item Distribution of Mathematics Learning Strategy Scale for Junior High School Students (4th Edition)

First-level dimension	Second-level dimension	NO.	Number of questions
Mathematics Cognitive Strategies	Retelling Strategy	3,10,24,27	4
	Elaboration Strategy	2,8,17,28	4
	Organize Strategy	1,7,30	3
Mathematics Meta-cognitive Strategies	Planning Strategy	26,31,33,36	4
	Monitoring Strategy	13,15,18	3
	Feedback-Adjustment Strategy	9,23,32	3
Mathematics Resource Management Strategies	Time-management Strategy	4,19,25	3
	Environment-management Strategy	16,21,22	3
	Mood-management Strategy	6,12,29	3
	Help-seeking Strategy	5,20	2
Polygraph Test		11--34,14--35	4

 Total

36

Remarks: No.11 and NO.34; NO.14 and NO.35 are two pairwise lie-detect items

5. Development of an intelligent assessment system for mathematical learning strategies

Based on the above learning strategy scales for junior secondary school students, two intelligent assessment tools have been designed and developed based on different scenarios, namely the Individual Student Edition S1.0 and the Integrated School Edition X1.0, which provides reports and recommendations for individual students and schools and districts as a whole.

The software is divided into three modules: the first module is the basic information collection module, the second module is the scale data collection module and the third module is the results and recommendations output module. The research team used technology to improve the compatibility of the scale by upgrading the previous paper-and-pencil test to a combination of online and offline tests; using technology to improve the practicality of the scale by providing timely feedback to students on the test results and giving them strategies for improvement; and using technology to improve the convenience of the scale by collecting the results of each student's test in a coordinated manner to provide a basis for overall proficiency testing in schools and districts.

The software integrates the various dimensions of the mathematics learning strategy scale for junior high school students and the level of learning strategy levels in each region. After uploading the overall data collected by the individual student version S1.0, it can output a customized document for each student with the overall score, First-level dimension score rate, First-level dimension level radar graph, and detailed response recommendations in an intelligent and batch-oriented way. The software uses technology to accurately measure students' level of mathematical learning strategies, enhance students' knowledge of their own level of mathematical learning strategies and greatly improve the quality of their mathematical learning.

6. Conclusion and Discussion

Since Pintrich, Mckeachie et al. (see [6]) developed the MSLQ, scales for measuring students' learning strategies have evolved, but there is a wide variety of dimensional divisions, and this complexity has posed some problems for researchers in selecting and using the scales. According to a literature review of the direction of learning strategy measurement, the learning strategy-related scales developed in recent years are mostly universal and general scales, and there is a lack of special scales for mathematics subjects and junior high school students.

Therefore, this research aims to achieve two goals. One is to establish a dimensional model and operational definition of mathematics learning strategies; the other is to make an authoritative mathematics learning strategy scale suitable for junior high school students based on the established model.

Therefore, this study aims to achieve three objectives: firstly, to establish a dimensional model

and operational definition of mathematics learning strategies; secondly, to produce an authoritative mathematics learning strategies scale for junior secondary school students based on the model developed; thirdly, to develop an intelligent assessment system to provide convenient, fast and accurate conditions for the practical application of the Mathematics Learning Strategies Scale for junior secondary students, and provide students with targeted countermeasure suggestions based on accurate measurements..

In this research, 81 items (belonging to three main dimensions) were first established. Based on the significant difference test, exploratory factor analysis, and expert guidance and suggestions, the items were deleted and adapted to make the items, reduced 81 items to 36 items (belonging to 10 types of sub-dimensions, 3 types of main dimensions), and then the *Mathematics Learning Strategy Scales for Junior High School Students* were established. The reliability of this scale is tested by Cronbach's α coefficient, showing that the scale is very reliable.

Furthermore, verification factor analysis was adopted in this study to evaluate and test the established model. After the model was revised, it was found that all fitness indexes were within the acceptable range, indicating that the structural model was verified. The development process of the "Mathematics Learning Strategy Scale for Junior High School Students" is scientific and objective, with good reliability and validity indicators, and can be used as an effective tool for investigating and evaluating the level of junior high school students' mathematics learning strategy.

Finally, the "Intelligent Batch Assessment of Mathematics Learning Quality for Primary and Secondary Students - Learning Strategies" software, S1.0 for individual students and X1.0 for schools, was developed using the well-developed Mathematics Learning Strategies Scale for junior secondary students. After simulation tests and field applications in Tianjin, Chongqing, Qinghai Province and Henan Province, the software has shown very good results in helping individual students to improve their mathematics learning strategies and learning quality, and in helping schools and districts to accurately control the weaknesses in mathematics education development.

In the follow-up of this study, we will continue to optimize the software, provide students with the measurement of mathematics learning level, provide suggestions for students to learn mathematics according to the measurement results, combine it with curriculum compilation, and make contributions to curricular choices and assessment practices based on experimental data.

Appendix 1

The Mathematics Learning Strategy Scales for Junior High School Students

Dear students:

To explore some of the thoughts and feelings of junior high school students in the process of mathematics learning, we invite you to participate in this survey. Thank you for your cooperation with this survey.

Specific requirements are as follows:

1. Please fill in or select the appropriate answer according to your actual situation. Note that each question needs to be answered, and only one answer can be selected
2. There are five options in each question: A, B, C, D, and E, and the meaning of each alternative answer is as follows

A: Very consistent: it does not mean that the situation described by the fan is always happening to you, but it means that the fan is consistent with you in almost all situations

B: Consistent: it means that this statement is consistent with you under normal circumstances

C: Uncertain: half of the cases where this statement is consistent with you

D: Inconsistent: this statement is inconsistent for you under normal circumstances.

E: Very inconsistent: the statement is inconsistent to you in almost all cases

3. There is no right or wrong answer to the following questions, and the answer results are only for scientific research and not as other basis;

4. The results of this survey will be answered anonymously. We expect to keep the results of the answers absolutely confidential. Please be sure to answer every question carefully and truthfully. Your response is very important to our research.

Basic Information

School: Class: Gender: Age:

Survey item

NO.	Items	Options
1	I will sort out the difficulties in each section of junior high school mathematics.	A B C D E
2	When learning a new mathematical concept, I would contemplate the difference and connection between it and the concept I previously learned.	A B C D E
3	When encountering complex geometric problems, I will draw to assist in the solution.	A B C D E
4	In the process of junior high school mathematics, I can arrange the study time and rest time reasonably.	A B C D E
5	After entering junior high school, I will ask the teacher some math problems.	A B C D E
6	When encountering math problems, I encourage myself to persist in thinking more.	A B C D E
7	I will summarize the question types of the math exams in junior high school.	A B C D E
8	When encountering an unfamiliar mathematical problem, I try to expect it to be transformed into a familiar problem. For example, when solving a system of ternary linear equations, it will be transformed into a familiar system of linear equations in two unknowns, and then solved.	A B C D E
9	I ponder how I can improve my math scores.	A B C D E
10	When studying mathematics, I will mark important content.	A B C D E
11	I hate taking math class.	A B C D E
12	I was able to adjust my mentality during the junior high school math learning process, so that I was neither too slack nor too nervous.	A B C D E
13	I compare the differences in learning behavior between myself and mathematics top students to find ways to improve mathematics	A B C D E

- performance.
- 14 Learning mathematics is helpful to solve problems in real life. A B C D E
- 15 In math class, I will avoid being distracted. A B C D E
- 16 I arranged the mathematics study plans in junior high school in an orderly manner. A B C D E
- 17 Magnet to complex mathematical problems. I will divide it into a small problem. A B C D E
- 18 In order to check whether I have mastered the mathematical knowledge I have learned, I will set up questions. A B C D E
- 19 After entering junior high school, I can arrange time reasonably and complete math homework on time. A B C D E
- 20 I will discuss mathematics issues with my classmates. A B C D E
- 21 I will create a good environment for students to learn mathematics to improve the efficiency of mathematics learning. A B C D E
- 22 I organized the math test papers in junior high school in an orderly manner for easy review and reference. A B C D E
- 23 I reflect on whether my mathematics learning method is effective. A B C D E
- 24 I will describe mathematical concepts in a variety of ways such as text language, graphic language and symbolic language. A B C D E
- 25 In junior high school mathematics study, I can effectively use the break time between classes. A B C D E
- 26 Since entering junior high school, I have perfected my mathematics half-study plan based on actual needs. A B C D E
- 27 I will memorize important mathematical content repeatedly. A B C D E
- 28 When I find similar mathematical knowledge, I will compare their similarities and differences, like concepts of monomials and polynomials. A B C D E
- 29 While solving mathematics problems, I will select the appropriate method of solving the problem according to the question type. A B C D E
- 30 I will sort out the key points of each mathematics knowledge in junior high school. A B C D E
- 31 I will preview and figure out the key content so that I can focus on listening to the teacher's instructions in class. A B C D E
- 32 When I know the math test scores, I will think about why there are fluctuations in my performance. A B C D E
- 33 As the math test papers being issued. I will first have a rough look at what types of questions are included in the test paper. A B C D E
- 34 I like taking math class. A B C D E
- 35 Studying mathematics does not help solve real-life problems. A B C D E
- 36 Before taking math class, I will figure out the difficulties that I will learn so that I can pay attention to teachers' instructions in class. A B C D E

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Designing Performance Tasks in Mathematics Using Technological Tools

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Abstract: *In the Philippines, the performance task is one of the major summative assessments in the K to 12 curriculum. This paper discusses how performance tasks may utilize mathematical apps within the context of blended learning. Guidelines on designing performance tasks, as well as the GRASPS framework, are discussed. Performance tasks that cut across various grade levels and strands of mathematics are presented. These involve divisibility (Numbers), integer and polynomial operations (Algebra), triangle centers (Geometry), and statistics (Statistics). The performance tasks described in this paper can provide an initial idea for the design of other summative assessments and contribute to the literature on the use of technological tools in assessment and evaluation.*

1. Introduction

The Covid19-pandemic has driven a shift of modalities in teaching and learning in schools worldwide. In the Philippines, the Department of Education has called for a blended learning modality for schools in lieu of face-to-face instruction in 2020. These modalities include printed modules,

radio, television, and the Internet. New challenges emerged from the sudden shift in educational practice, and some of these challenges relate to assessment. Traditional paper-and pencil assessment methods are no longer completely applicable. Schools are expected to adopt “assessment practices that can meaningfully support student development and respond to different contexts at this time” [1].

There is already a wealth of strategies for using technology in formative assessments even prior to the pandemic [2-7]. However, new schemes for summative assessments remain to be a key challenge for teachers, especially if these are to be done online [8]. This paper reports on the creation and design of performance tasks utilizing innovative technological tools such as mathematical applications as a potential strategy for summative assessment. Technology can provide new and more dynamic ways to face some mathematical tasks, with respect to paper and pencil methods, such as modeling real-life phenomena, making conjectures and proving properties based on experimentation, predicting, pattern-seeking [7, 9]. As Spector et al. [6, p. 60] said, “In addition to providing the means to support personalized learning and a smart education system responsive to the needs of learners and teachers and their learning environments, new technologies can support key 21st century skills ... – notably critical thinking and problem solving.”

In the Philippines, one mandated summative assessment is in the form of a performance task, which comprises 40% of students’ final grade [10]. This paper reports on the design and construction of sample performance tasks on numbers, algebra, geometry, and statistics using mathematical apps that a mathematics teacher can utilize at appropriate junctures in the curriculum. The use of mathematical apps in the design of each task is deliberate and aims to aid the students to carry out calculations that will be used in the analysis and solutions of the problems indicated in the tasks.

2. Performance Tasks

Philippine policy guidelines [10] define performance tasks as tasks that allow students to demonstrate their knowledge in various ways. These are typically done over a period of time and the purpose is to provide students the opportunity to integrate their understanding of the main topics covered during the academic quarter (about ten weeks). There is a wide range of possible formats, such as skills demonstration, group work, multimedia presentations, research or investigative projects, or written output.

The assessment guidelines were refined in response to the pandemic [1]. Assessment exemplars and rubrics were provided, together with suggestions for students with low, medium, or high access to technology. For mathematics, suggested performance tasks include the following: constructing graphs from conducted surveys, multimedia presentations, outdoor math, probability experiments, problem posing, reasoning and proof, using manipulatives to show math concepts and solve problems using measurement tools and devices. Performance tasks may also integrate multiple competencies not only in mathematics but also in the other subject areas.

Wiggins and McTighe’s GRASPS model [11] is a recommended guide in carrying out performance tasks. This model is applicable when the performance task involves a real-life situation. GRASPS stands for: Goal (a task must state the problem or challenge to be resolved), Role (a task must explain who students are in the scenario and what they are being asked to do), Audience (a task must specify who the students are solving the problem for, who they need to convince of the validity and success of their solution for the problem), Situation (a task must provide the context of the situation and any additional factors that could impede the resolution of the problem), Product, Performance, and Purpose (a task must explain the product or performance that needs to be created and its larger purpose, and Standards and Criteria for Success (a task must dictate the standards that must be met and how the work will be judged by the assumed audience).

3. App-based Performance Tasks

The performance tasks in this section utilize apps that are part of Mathplus resources (mathplusresources.wordpress.com), which are products of an ongoing government supported project “Technology Innovations for Mathematical Reasoning, Statistical Thinking and Assessment” by the Department of Science and Technology- Philippine Council for Industry, Energy and Emerging Technology Research and Development (DOST-PCIEERD). All apps can be accessed through the website <https://mathplusresources.wordpress.com/>.

3.1 Grades 1 to 6

A divisibility rule is a shorthand way of discovering whether a given number is divisible by a fixed divisor without performing the division, usually, just by examining its digits. Knowledge of divisibility and factoring rules can facilitate computations needed for more advanced topics such as fraction operations. This topic is listed in the most essential learning competencies [12] for students in Grades 4 and 5. Students are expected to solve both routine and non-routine problems using rules on factors, multiples, and divisibility of numbers.

Integrating technologies into learners’ activities has the potential to keep them more engaged in studying divisibility rules and factoring. Hence, the assessment activities and performance tasks on divisibility and factoring enumerated in this section begin with the use of technology.

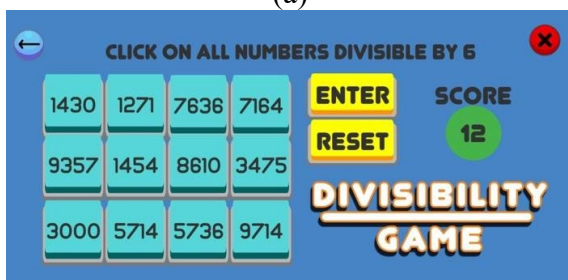
One app designed to facilitate learning number sense is the *Divisibility Game*. Students may choose which divisibility rules they want to practice on and then either play to gain mastery, increase speed in answering questions, or simply engage in drill work. In the Divisibility Game, students can choose the divisor, for example 3, 6, 9 and the number of items (Figure 3.1 (a)). Then students select numbers that are divisible by the divisor.



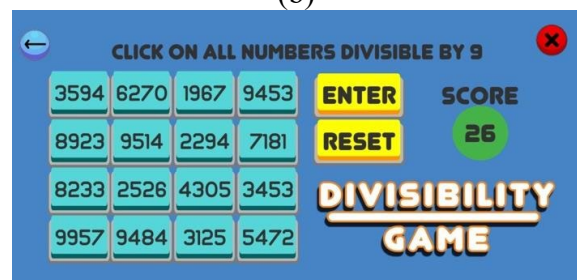
(a)



(b)



(c)



(d)

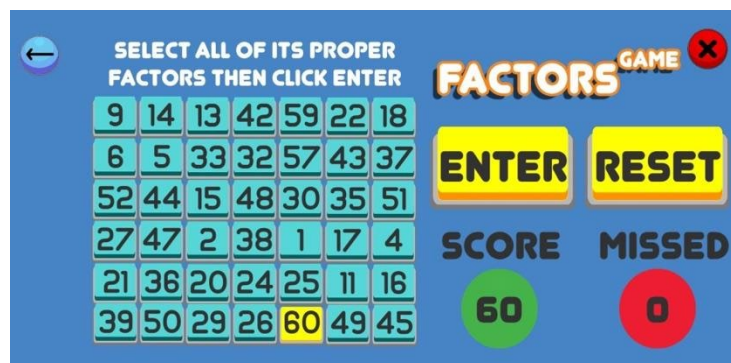
Figure 3.1 Screenshots of the *Divisibility Game*

For example, in Figure 3.1 (b), among the numbers, which are divisible by 3? Students may also choose to play with 12 four-digit items such as in Figure 3.1 (c) or with 16 four-digit items such as in Figure 3.1 (d).

Another app is the *Factor Game*. Through the game, students demonstrate understanding of prime and composite numbers. The game addresses the need for the students, from various grade levels, to identify factors and multiples of a given number. Students select a number and all its factors. In the easiest level (Figure 3.2 (a)), numbers 1 to 30 are placed on a 5×6 grid arranged in order while in the 1-60 level (Figure 3.2 (b)), randomly selected numbers from 1 to 60, not in order, are placed on the 6×7 grid. Students may also select a time limit. The time limit can push students to strategize in selecting the numbers to get a higher score within the available time.



(a)



(b)

Figure 3.2 Screenshots of the *Factors Game*

In preparation for the performance task, a series of informal activities and formative assessments may be performed. Students may initially explore each game option in the app in a form of free play. At this point, the idea is to engage the students’ interest and motivate them to reach as many levels as possible. Teachers may monitor their students by asking them to send screenshots or a written record of their progress. The students’ outputs may enable teachers to see some learning patterns and provide timely feedback and interventions when necessary.

Students can also perform more structured activities. As a first activity, students can perform a “Shading Activity” using the divisibility rule. Given a grid of numbers, students are asked to shade all numbers divisible by 3 as in Figure 3.3 (a). In the end, the shaded numbers will form a familiar letter of the alphabet, object or an animal, as in Figure 3.3 (b).

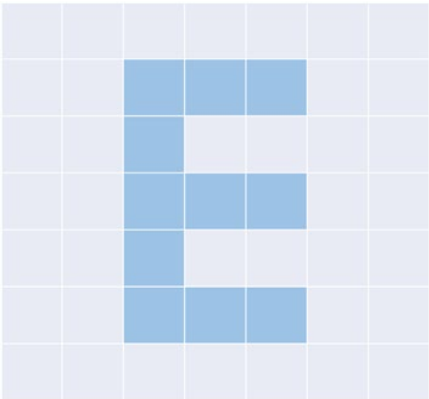
25	11	56	77	37	19	49
71	91	42	18	36	95	83
38	85	24	73	75	32	28
28	59	57	27	48	13	40
16	10	84	29	63	80	34
22	76	87	20	30	82	65
31	61	92	55	25	26	52

25	11	56	77	37	19	49
71	91	42	18	36	95	83
38	85	24	73	75	32	28
28	59	57	27	48	13	40
16	10	84	29	63	80	34
22	76	87	20	30	82	65
31	61	92	55	25	26	52

(a)
(b)

Figure 3.3 Shading activity

A sample performance task related to this, is one where the teacher asks the students to design their own puzzles. A student creates a puzzle by filling-up the grid (Figure 3.4 (a)) with two-digit numbers so that the numbers in the shaded squares are divisible by a number of their choice and the numbers in all other squares are not.



(a)

123 <u>3</u>	456 <u>3</u>
72 <u>1</u> 8	2 <u>7</u> 90
3 <u>9</u> 78	<u>6</u> 246
62 <u>2</u> 8	13 <u>0</u> 5

(b)

Figure 3.4 (a) A sample grid (b). Fill in the blanks activity

A variation to this task requires students to create a puzzle by designing their own image, for example, an arrow, animal, or any object so that when students shade the correct squares with numbers divisible by their chosen number, the students will end up with the image. Furthermore, students may also use grids of different sizes, with a mixture of two-digit and three-digit numbers. In addition, in this task, teachers may require students to come up with three or more grids where the images would follow a story. At the end of the task, students write their reflection. Sample guide questions are i) How did you choose the numbers to be placed on the grid? ii) Any difficulties encountered? Explain these challenges. iii) What did you learn from this task?

Another activity is the “Fill in the Blanks Activity” where students fill in the blanks with numbers that will satisfy the given conditions on divisibility. For example, students fill in the blanks so that the 4-digit number is divisible by 9 as in Figure 3.4 (b). At the end of the activity, teachers may ask if there are other solutions and ask students to explain. A sample performance task related to this is requiring students to answer more difficult non-routine problems such as those appearing in national

competitions. Sample problems are: i) If the nine-digit number A1234567B is divisible by 45, determine all possible values of A and B (2019 Metrobank-MTAP-DepEd Math Challenge - Grade 6 National Finals), ii) How many ways are there to arrange four 3's and two 5's into a six-digit number divisible by 11? (19th Philippine Mathematical Olympiad - National Oral Stage). At the end of the task, students explain the steps they followed to solve the problem. Although this performance task does not involve a real-life situation that is central to the GRASPS model, the task is aligned with other alternatives indicated by the official guidelines [10] because it provides students an opportunity to integrate their knowledge of divisibility in an environment of inquiry and problem solving.

3.2 Algebra

Addition and subtraction of integers and polynomial expressions are fundamental algebraic skills that students need to master. Different models have been utilized to represent the concepts underlying integer operations [13, 14]. However, there seems to be no app that covers polynomial expressions. *AlgeOps* is an interactive visual mathematical app [15] which makes use of pictorial and symbolic representations of integers and polynomials (see Figure 3.5 for screenshots from the app). The app incorporates both the neutralization and number line models [16] which offer a more holistic representation of integers and eventually, strengthens instruction on integer and polynomial operations.

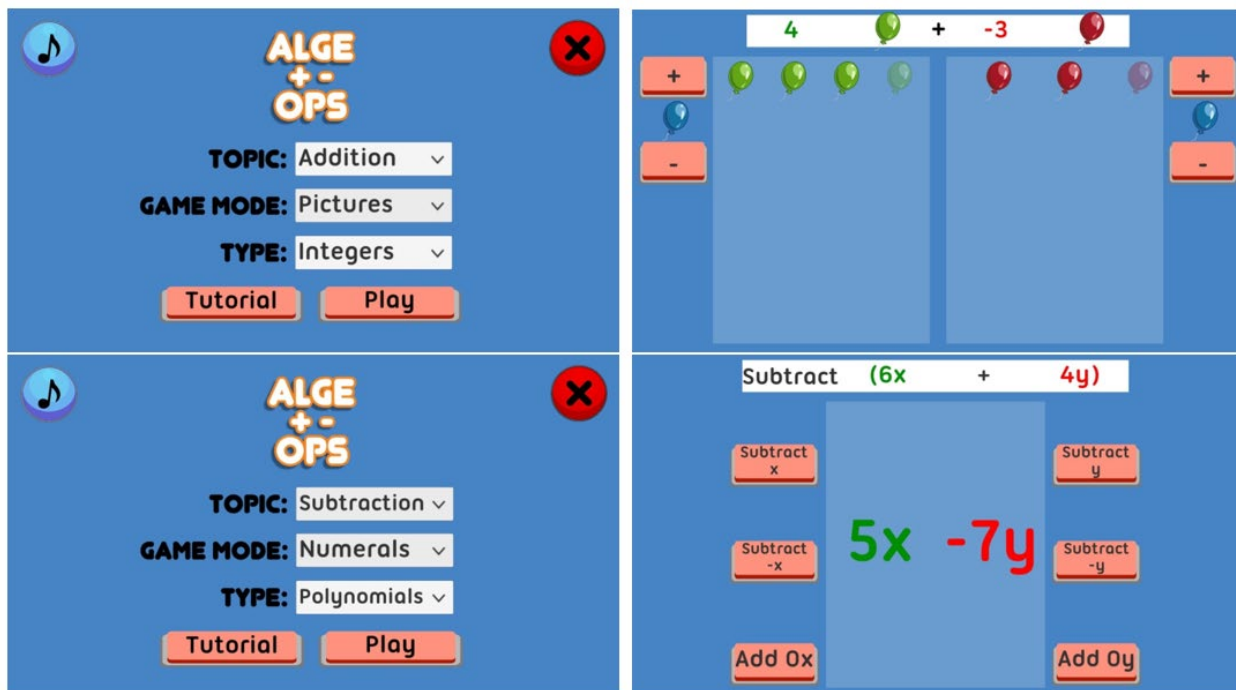


Figure 3.5 Screenshots of the different modes of an updated version of *AlgeOps*

To facilitate the operations, students use the different buttons found on the left and right panels of the screen to construct the expressions. For example, Figure 3.6 (a) illustrates the addition problem of $(-2) + 5$. It is represented by two red balloons and five green balloons. These visuals are displayed when the (-) left button is clicked twice and the (+) right button is clicked five times. Moreover, the

three green balloons representing the correct answer (Figure 3.6 (b)) remain after the animation of the neutralization method.

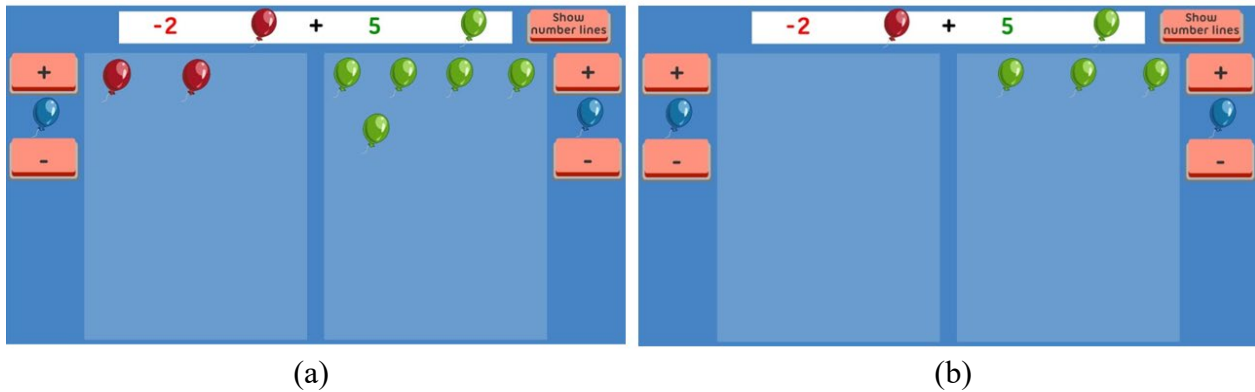


Figure 3.6 A representation of the addition problem: $(-2) + 5 = 3$

Performance tasks are usually open-ended and often yield more than one correct answer. This characteristic is manifested in the following task:

Students are to construct different situations where the sum of the balloons is the same as the given diagram. The diagram shows two green balloons plus two green balloons (or $2x + 2x$) and three green apples plus two red apples (or $3x - 2y$). The diagram (Figure 3.7) can be unique to each student since it is generated by the device used by the student.

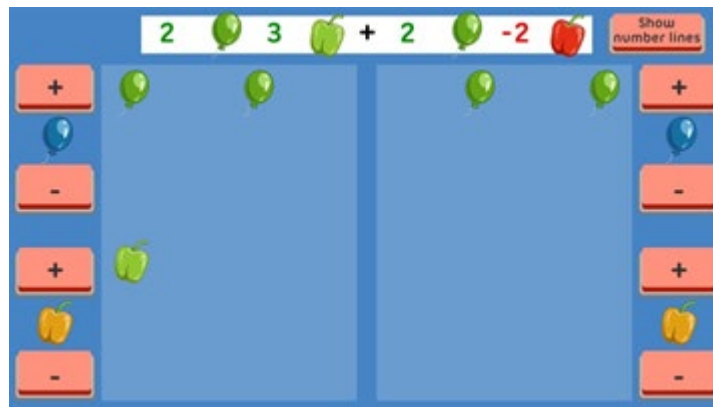


Figure 3.7 Addition of polynomials

Indeed, the task involves providing different values of a , b , c , and d that satisfy the equation $ax + by + cx + dy = 4x + y$. As such there will be numerous responses that students can provide satisfying the equation. Additional constraints on the values of a , b , c , and d can be added to make it more challenging. For example, the values must range from -20 to 20 and the values need not be integers. In consideration of guidelines for assessment, this performance task can involve competencies in another subject area. For example, the student's output requiring creative storylines and visual representations of the polynomials can be used competencies in writing and the arts.

The use of *AlgeOps* in designing the performance task provides a digital medium that replaces the usual pen and paper format of assessment. The generation of unique exercises to students illustrates

its efficiency and the animation of the neutralization method provides a dynamic representation of the addition and subtraction of integers and polynomials.

3.3 Geometry

One of the advantages afforded by technology in performance tasks in geometry is the dynamic geometry environment [9]. As will be discussed here, a digital geometry environment in assessment can develop mathematical and technological competencies and connect mathematical ideas to everyday experiences and to other disciplines [17]. It can also promote mathematical and technological competencies, such as developing and applying new mathematical knowledge through problem solving; developing mathematical reasoning, developing visualization skills to assist in processing information, making connections and solving problems, and generalizing and exploring properties of objects [7]. In addition, the feedback provided by a dynamic geometry environment supports interaction during assessment [7, 18].

One example of a performance task for Geometry in Grade 7 employing the GRASPS model is a real-life application in which students gain experience and discover the significance of triangle centers such as the centroid, incenter, circumcenter and orthocenter. This can be facilitated with a *Geogebra* applet “Bisectors of Triangles” that allows the student to explore the various centers (Figure 3.8).

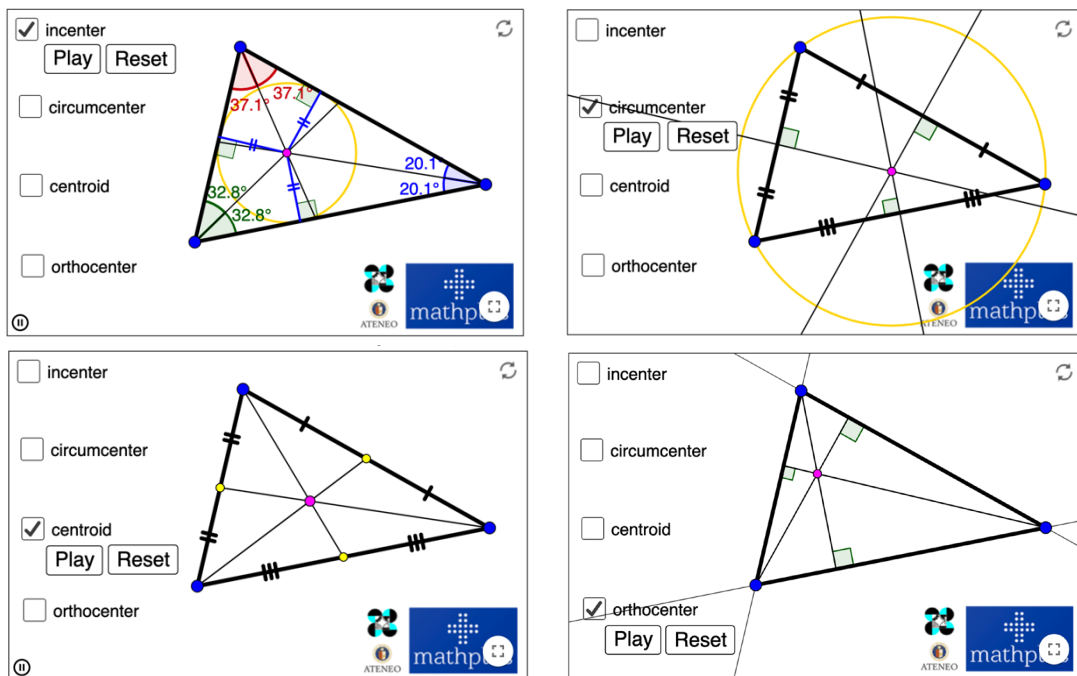


Figure 3.8 Screenshots from the *Geogebra* applet: Bisectors of Triangles

The GOAL of the performance task is for the students (their ROLE as a research team) to present and discuss a proposal to the Chief Executive Officer (AUDIENCE) of a local amusement park, *Enchanted Kingdom*, regarding the best possible location of a snack stand in the park so that it will be strategic to the three most popular rides in the park (SITUATION). The final output (PRODUCT) is a multimedia presentation (using a presentation software, video demo, poster) that showcases i) the location of the snack stand such that it is central to the three main popular rides in the park and how

this was determined, ii) maps and pictures to support the recommendation, and iii) the advantages and disadvantages of the location, supported by sound geometric principles. The multimedia presentation will be one session (in an online environment, this will be one synchronous session) where the students will present the proposal to the teacher.

The performance task is divided into four parts.

1. Students will work in groups. To get students to start thinking about this task, some questions posed could be: “Have you ever visited the amusement park?”, “What was your experience in getting a snack in between rides?”, “Where would be a location that would be "central" to most customers?”. The location of the snack stall is critical in order to attract more potential customers. Using a Google Form, students then conduct a survey among classmates and friends of the top three favorite/popular rides in the park.
2. In order to be guided with regards to the strategic location of the stall in the park, explorations on the definition and properties of the various four (4) centers of a triangle will be carried out using the *Geogebra* app “Bisectors of Triangles” (see Figure 3.8) which will be given to the students. This is a very important part of the task where students explore the various centers by using the “drag feature” of the app. By being able to drag the vertices of a triangle, students develop visualization skills, which are important in processing information. For example, some key concepts and properties that can be explored for the incenter using the *Geogebra* app are: the incenter is the center of the triangle’s inscribed circle; the incenter is always inside the triangle, the perpendicular segments from the center to sides of the triangle serve as radii of the inscribed circle. The incenter can be verified for various types of triangles.
3. For the discussion/analysis part, students will consider the four different types of circle centers when solving this task. They will also need to clearly establish the assumptions that are necessary in order to come to a solution, i.e., how did you define “best site of the stand,” etc.

In the proposal analysis, the student justifies the appropriate use of the triangle center depending on its application. (i) Is it important that the center be equidistant from the three vertices or sides of the triangle? ii) Should it be that the center, when connected by a line segment to each location (vertex) creates three portions that are equal in area? iii) Should the circle be circumscribed about or inscribed in the circle? iv) Is this choice realistic in a given situation?

Through the dynamic geometry software, the students can employ trial and error techniques, interpret the behavior of the triangle and its parts to make inferences about it, and test the validity of their conjectures. The students’ conjectures can be communicated through email or Messenger app for quick feedback as they go through the task and seek confirmation of their work.

4. Students will then find evidence of the distance between the food stand and the three rides using Google Earth (see Figure 3.9) or some other application.

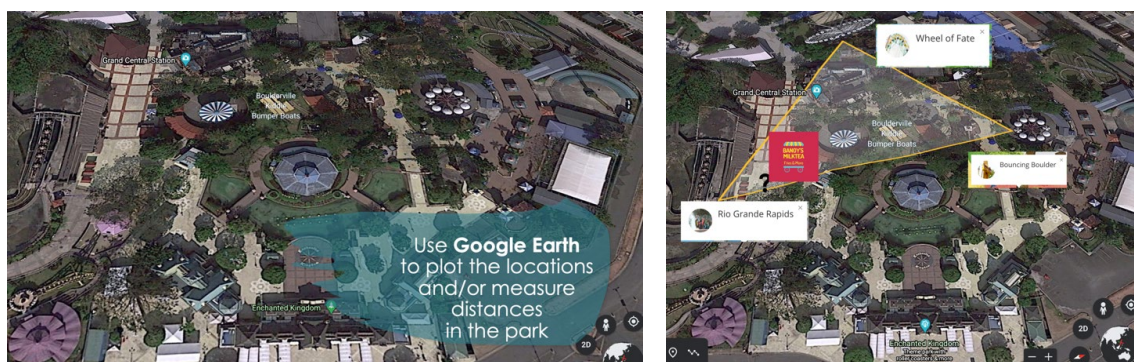


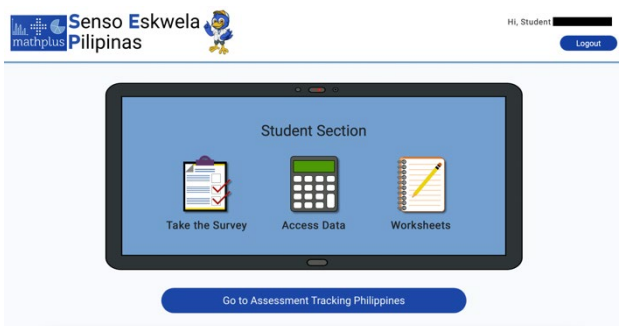
Figure 3.9 Screenshots from the *Geogebra* applet: Bisectors of Triangles

For the STANDARDS part of the GRASPS framework, the following criteria may be used: i) data/assumptions used to arrive at the location of the stand (survey, geometric explorations using the app), ii) geometric principles used to arrive at location, iii) accuracy of the geometric solution, iv) supporting figures/screenshots from app/tables, and v) presentation (video, PowerPoint, poster) and delivery.

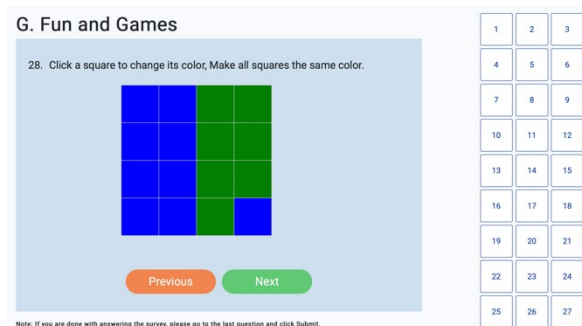
3.4 Statistics

The role of context in learning and teaching statistics has consistently been emphasized in the literature. For instance, according to Cobb and Moore [19, p. 801], “Statistics requires a different *kind* of thinking, because data are not just numbers, they are numbers with a context.” Moreover, the Pre-K-12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II) [20, p. 11] indicates that assessments “must require students to use statistical reasoning with context and variability at all stages of the statistical problem-solving process.” In this section, in light of the role of context in Statistics Education, the design of performance tasks using data from the web-based application *Senso Eskwela Pilipinas* (SEP) is discussed. The SEP app provides teachers and students not only contextualized but also authentic and relatable data. The benefits afforded for the exploration and transformation of data with digital technology reveals fresh approaches to analyzing statistics [21].

SEP is “an online platform for gathering, storing, and accessing readily available and relatable data for learning and teaching statistics” [22, p. 174]. It is a web application patterned after *CensusAtSchool*, which was first established in the UK in 2000 with the “dual thrust to enliven data handling activities within the classroom while also educating children about the principles and processes involved in conducting a census” [23, p.1]. In SEP (Figure 3.10 (a)), students anonymously log in to the platform (freely available at mathplusresources.com) to answer a 31-item survey (Figure 3.10 (b)) involving questions on different topics such as food, hobbies, and interactive tasks. The students’ responses become part of the SEP database, to which they have access. Students and teachers can retrieve the data of their own class/es or random samples from the entire database.



(a)



(b)

Figure 3.10 Screenshots from the SEP platform: (a) The landing page for student accounts, (b) A question from the SEP survey

It is anticipated that students' use of data from the SEP database in performance tasks can increase their engagement and interest as “the sense of belonging that participating pupils feel, purely because they know their own responses will become part of a database of responses from their peers, is fascinating and motivating for them” [24, p. 175]. Another clear advantage is the ready access that students have to the data from the SEP database. In situations in which the actual data collection process is difficult or is not integral to the objectives of the task, readily available data will prove to be convenient.

The performance task *From SEP to Infographic* discussed below recognizes the aforementioned role of context and adapts the GRASPS model. It addresses the following Grade 7 Statistics competencies in the Philippine K-12 curriculum [12, pp. 229-230]: “poses real-life problems that can be solved by Statistics, gathers statistical data, organizes data in a frequency distribution table, uses appropriate graphs to represent organized data: pie chart, bar graph, line graph, histogram, and ogive, draws conclusions from graphic and tabular data and measures of central tendency and variability, draws conclusions from graphic and tabular data (and measures of central tendency and variability).”

The task involves the creation of an *information graphics* or *infographics*, which has become more common in recent years. These infographics, which are visual representations of information, are mainly used to present complex data in a form that is more understandable and, at times, more impactful. This performance task allows learners to choose from the varied data available in the SEP database and use these data to create their own infographics. The GOAL is for the learners to retrieve data from the SEP database and organize, process, and analyze these data so that they can create an infographic on their chosen topic.

The students are given the following setting and instructions for the performance task: “In this task, you will play the ROLE of a researcher and writer for a social media page that mainly posts interesting information (e.g., their hobbies, activities, interests) about the Filipino youth. The administrator (or admin) of the social media page is asking the page’s researchers and writers to submit new infographics (PRODUCT) that can be posted on the page. Naturally, the admin would like to receive submissions that have the potential for increased engagement from the page’s followers (AUDIENCE). Moreover, the infographic should be based on real data, so the admin has advised the use of the SEP database.

As one of the researchers and writers, you are now faced with the following tasks (SITUATION):

1. Determine a suitable topic of the infographics based on the available data in the SEP database.
2. Process and organize the data into graphs or charts.
3. Analyze the data and the graphs/charts.
4. Design and create an infographic, with the graphs/charts integrated in a creative or meaningful way, that presents clear, concise, and interesting information on the chosen topic.”

The students are also given the following criteria for grading (STANDARDS): i) topic of the infographic, ii) correctness of graphs or charts, iii) organization of the infographic, iv) design and creativity.

The performance task using the SEP platform offers a number of advantages. First, since the SEP database contains a large amount of readily available and varied data, students are given more space to think creatively in deciding the topic and direction of their performance task. This aspect is enhanced further by the fact that the students also answered the SEP survey themselves. Because the data is relatable to them, the students may easily develop a curiosity towards how their responses compare to others. Moreover, the Access Data feature (Figure 3.11) of the SEP platform offers the students flexibility and allows them to make their own decision/s in terms of i) using their own class' data or a random sample from the database, ii) which questions in the SEP survey they want to work with, iii) which grade level/s they should focus on, and iv) how small or large a sample size they

should use. These allow them to ask themselves questions like “Do we use data only for Grade 7, or Grades 7 to 10? Do we do a comparison between grade levels? Do we try to relate the data from two SEP questions?”

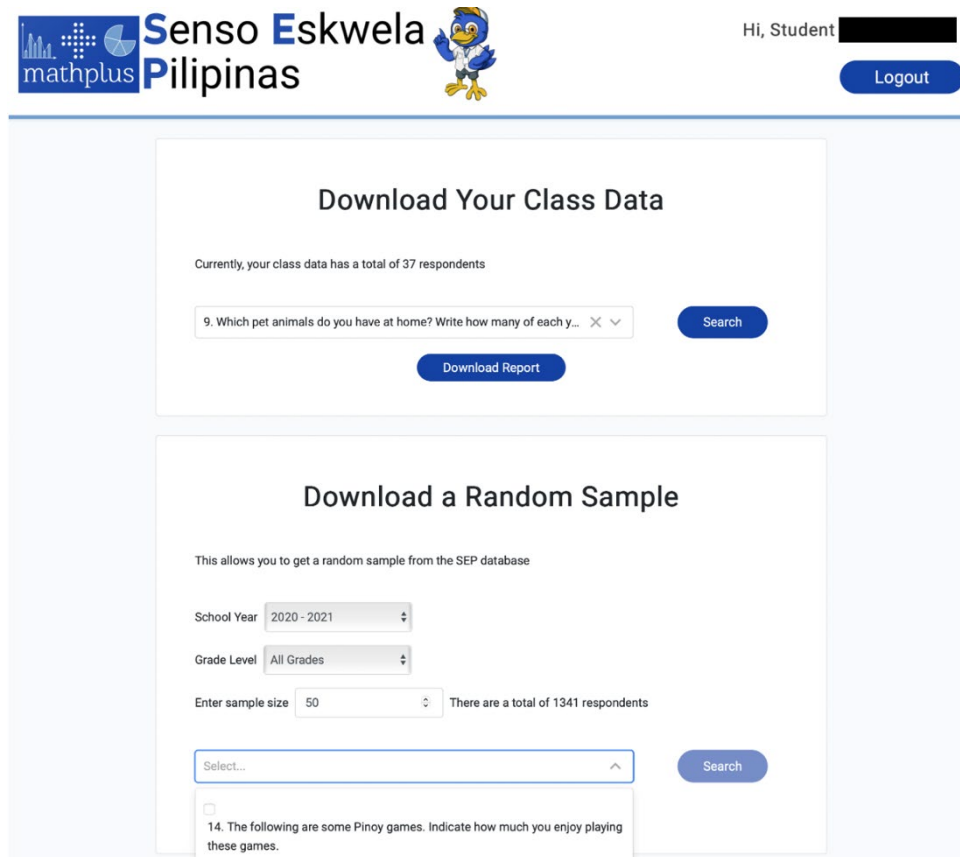


Figure 3.11 The Access Data feature of the SEP platform

Given sufficient time, students will also be able to create multiple infographics, from which they can choose their best work for submission. With the SEP database, this is possible for them without having to repeat the data collection process. Finally, accessing and processing data from the SEP database mirror steps in doing statistical analyses using data from actual databases. Students get raw data from the SEP database, then organize and process them using the skills that they have learned. This results in a more authentic context for the performance task.

Moreover, conforming to one of the thrusts of the Department of Education’s Order No. 31 s.2020 [1], the performance task can integrate competencies in other subject areas. For example, the students may be asked to accompany their infographic with a feature article in English or in Filipino. The article may serve as an assessment for their written communication classes. The design aspect of the infographic may also be used as an assessment for their arts class. Lastly, their output may also be used as an assessment for their Understanding Culture, Society and Politics class if the possible topics of the infographic are restricted appropriately.

The SEP database naturally leads to a number of performance tasks. The infographics task is suitable for Grade 7, but more tasks can be designed to assess more advanced statistical concepts. For example, in *Hypothesis Testing Using SEP Data*, which is designed for Grade 11 classes, students play the role of researchers for a non-profit organization that promotes the health and welfare of

Filipino youth. After performing data analysis and hypothesis tests, they are to prepare a written report for the officers of the organization to help in their planning of programs and campaigns. The complete details of this performance task are available in the SEP website (<https://mathplusresources.wordpress.com/senso-eskwela-pilipinas-for-grades-1-11-statistics/>).

Finally, the use of SEP in performance tasks offers a possible avenue for doing effective online assessment. In addition to the elements mentioned above, these performance tasks can be easily implemented in an online setting. Teachers can easily communicate the instructions to students and receive students' submissions online (e.g., via learning management systems, email, or official/school-authorized messaging platforms). In the same way, students can conveniently retrieve their data sets from the online SEP database. Moreover, depending on the involved topic/s, students may opt to use freely available online tools (e.g., a spreadsheet application) to organize and process their data.

4. Conclusion and Future Direction

Designing mathematics performance tasks for summative assessments must be adapted to the changes brought by the shift to a blended learning modality of education. The summative assessments in the form of performance tasks described in this article were designed to address the urgent educational difficulties driven by the Covid-19 pandemic and the need of the teachers to create forms of assessment while in a blended learning modality. The design of these tasks considered alignment with curriculum goals so that integrity of assessments is still met.

This study presented the design of mathematics performance tasks that creatively make use of mathematical apps to support student learning. The sample tasks address the need of making assessments in four key content areas: Numbers, Algebra, Geometry and Statistics. This set of tasks can provide teachers in basic elementary, junior and senior high school with creative ideas on designing performance tasks in the future. The designs of all performance tasks discussed in this paper considered the creation of an environment of inquiry and the integration with other subject areas whenever applicable. Further, the task designs for Geometry and Statistics were guided by the GRASPS framework by Wiggins and McTighe [11].

While these apps were designed to help students gain more insight and develop stronger mathematical skills, this paper described how the apps can also be integrated into performance tasks that teachers can assign their students. The performance tasks were presented in the webinar for Philippine teachers *Amplifying Development of Assessment and Performance Task Skills with Mathplus Resources*. Although the performance tasks and apps were designed for Filipino learners, these can assess learning outcomes that are present in many other national curricula as well. As a goal, the authors affirm the recommendations of [6] for an "open assessment repository (OAR) or an education observatory" that addresses the need for challenging but engaging technology-assisted performance tasks. The tasks designed by the authors can form part of this repository.

Acknowledgements The authors would like to thank the Department of Science and Technology-Philippine Council for Industry, Energy, and Emerging Technology Research and Development (DOST-PCIEERD) and the University Research Council (URC), Ateneo de Manila University for the support of the development of the performance tasks and interactive resources discussed in this paper. Deep gratitude is extended to our app developers Mr. Victor Antonio M. Ortega, Mr. Nigel Benedict D. Cargo, Mr. Jose Teodoro P. Lacson, and Sparksoft Solutions Inc. We also thank Mr. Amiel Damian F. Justiniani for the apps' art assets. Mr. Marc Pison and the Logik team for the *SEP*.

We are indebted to Mr. Restituto I. Rodelas, Education Program Supervisor of Department of Education-School Divisions Office Mandaluyong City, for his assistance.

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The loci of virtual points constructed with elementary models of planetary orbits

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Abstract

The orbits of the planets around the Sun are ellipses, close to circles. A simple model is given by concentric circles centered at the Sun. The radius of the orbit and the linear velocity of the planet may be found in numerical form in the literature. The motion of a planet on its orbit is then described by a trigonometric parametrization. Using such parametrizations for two planets, various mobile "virtual points" such as their midpoint or their center of gravity, are defined and their trajectories are studied. A good opportunity to experience different kinds of animations with software, depending on the affordances of the package. By that way, classical plane curves appear in a dynamical way. It appears quickly that the level of accuracy of the involved numerical data influences strongly the shape of the constructed curves. Despite the fact that loci and envelopes are often related, and automated commands exist for both in the same packages, their respective appearances may be different, the one very rich and the other much apparently less. The plane curves which appear as loci of some of the virtual points have been known for a long time, but separately. We have here an opportunity to see them as members of a larger family. A byproduct of the activities is an incitement to study classical plane curves and their properties, in a dynamical framework and in relation with real world. Despite the physical non-accuracy of the models, this is a typical STEAM approach.

1 Introduction

A few months ago, the news were filled with announcements about three space probes launched in July 2020 towards Mars and which arrived to Mars surface (the American rover called Perseverance, by NASA) or were inserted in a martian orbit (a Chinese probe, and Hope, a martian orbiter from the United Arab Emirates) in February 2021. The various national space agencies had a wonderful opportunity to attract public attention to space science and technology. Numerous websites have been developed and are freely accessible to a general audience. There are good reasons for mathematics educators to be part of this atmosphere, showing applications of mathematics to real world. Maybe not a daily concern, but the same world which appears in news.

According to the 1st Kepler’s law ([11], p. 127), the orbit of a planet around the Sun is an ellipse, with the Sun at one of the foci. As the foci are very close, and in order to make the example as simple as possible, we consider an approximation of the orbits as coplanar concentric circles, centered at the Sun. It will be enough to know radius (i.e. distance to the Sun, actually a mean distance) and velocity. Table 1 gives some data, not the velocity. Kepler’s 2nd law is illustrated by Figure 1, taken from [11] p. 129: the areas of the shaded sectors, covered by the radius in equal times (i.e. it takes equal times to travel distances AB , CD and EF), are equal. In our very simple models, we will consider motion with constant angular velocity on circular

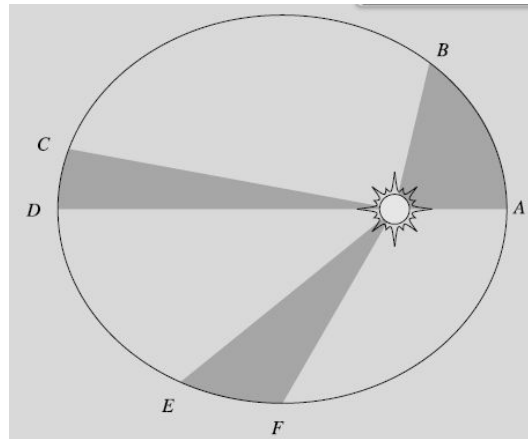


Figure 1: Kepler’s 2nd law of planetary motion

orbits. We compute the velocities according to the year length of the planet, with Earth year equal to 1. Note that Table 1 displays only the eight objects, officially called planets by the international astronomical organization. According to 3rd Kepler’s law, the orbital velocity is a function of the distance to the Sun.

Planet	Distance to the Sun (km)	Period (1=terrestrial year)
Mercury	57.91 10^6 km	0.2408
Venus	108.2 10^6 km	0.6152
Earth	149.6 10^6 km	1
Mars	227.9 10^6 km	1.8808
Jupiter	778.5 10^6 km	11.862
Saturn	1.434 10^9 km	29.457
Uranus	2.871 10^9 km	84.018
Neptune	4.495 10^9 km	164.78

Table 1: Some orbital data

Because of the huge differences between the distances and the hardware constraints (we mean mostly the size of the screen and the number of available pixels), we will consider examples with Earth and Mars only. The same activities can be done with the pair Venus-Earth, they will produce the same family of curves. Nte that in order to make the first examples easy, we use approximations less precise than in Table 1.

For the simulations and the animations, we use GeoGebra, a freely downloadable software, whose main characteristic is devoted to Dynamic Geometry. For some uses, including automated determination of loci and envelopes, it can be supplemented with the package GeoGebra-Discovery, also freely downloadable. A general analysis of the automated methods for loci and envelopes is given in [13]. We will also use the Computer Algebra System Maple for its specific animated affordances.

The goal of this paper is to present mathematical situations with a STEAM¹ approach where plane curves, either algebraic or not, are presented and some of their properties explored using technology. We see this kind of study as an opportunity to explore also classical families of plane curves. Nothing beyond that.

2 The midpoint Earth-Mars

Because of Mars ubiquity in the news during last year, we consider now the pair Earth-Mars and their midpoint. We choose the distance Sun-Earth as equal to 1, which is consistent with the general definition of the so-called *astronomical unit* ([11], p.24), the distance Sun-Mars is then almost equal to 1.5. We describe the respective orbital motions of the planets by the following parametrizations:

$$\begin{aligned} \text{Earth: } (x, y) &= (\cos t, \sin t) \\ \text{Mars: } (x, y) &= \left(r \cos \frac{t}{h}, r \sin \frac{t}{h} \right) \end{aligned}$$

where $1/h$ encodes the angular velocity of a planet on its orbit and r denotes the ratio between the radii of the orbits. For Mars, we will take $h = 1.881$ or a coarser approximation. The t -interval on which to consider these parametrizations has to be defined later, either experimentally after some trials with the software, or by computation of the number of needed orbits, let's say for the Earth, in order for the pair of planets to return to the initial position. For integer values of the parameters, a LCM of two integers has to be computed; otherwise, the computation is harder and an experimental approach will be often preferred. This general setting allows to construct a general applet, useful for different pairs Earth-other planet, and will be translated by GeoGebra into a slider bar.

Of course, when proposing to students such an activity, an educator should make clear that this midpoint has no physical meaning. Figure 3 illustrates this. In July 2020, the NASA launched a rover called *Perseverance* which landed on Mars in February 2021. The trajectory was not a segment of line, and the trajectory in space was roughly the union of arcs of ellipses. Figure 3(b) shows the midpoint on the trajectory; it is easy to see that this point has no relation with a midpoint of a segment Earth-Mars. The simplified model that we will use in the activities contributes to this understanding.

¹STEAM = acronym for Science, Technology, Engineering, Arts and Mathematics.

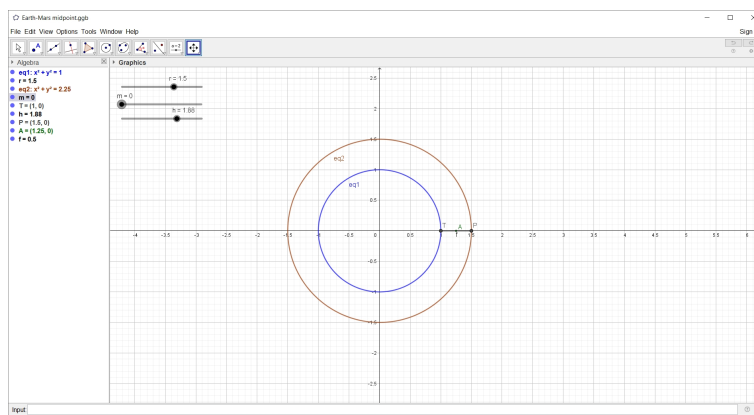
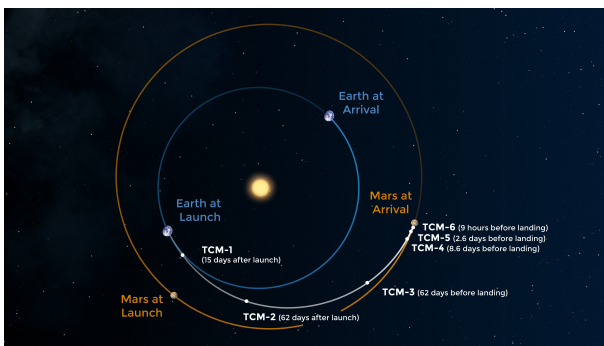
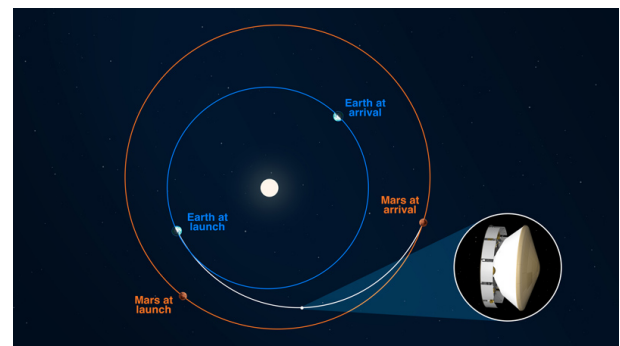


Figure 2: A snapshot of the general applet for the midpoint of two planets



(a) The trajectory - an arc of ellipse



(b) At midpoint of the space trip

Figure 3: Perseverance’s trajectory from Earth to Mars (Credit: NASA/JPL-Caltech)

2.1 First example: $r = 1.5$, $h = 1.9$

The midpoint of the pair Earth-Mars is described by the parametrization

$$\begin{cases} x = \frac{1}{2} \left(\cos t + 1.5 \cos \frac{t}{1.9} \right) \\ y = \frac{1}{2} \left(\sin t + 1.5 \sin \frac{t}{1.9} \right) \end{cases} \quad (1)$$

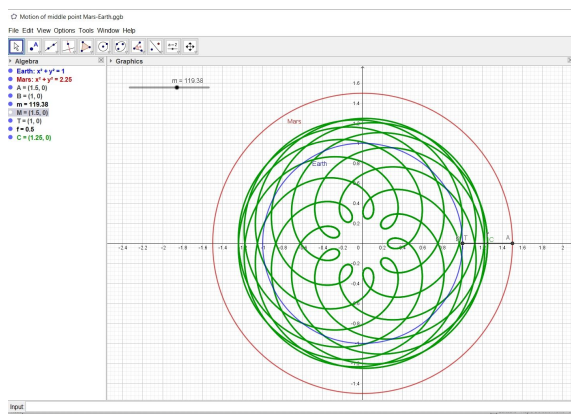
In Figure 4, the orbit of the Earth (point T) is represented in blue and the orbit of Mars (point M) in brown. The midpoint is denoted by A and is plotted in dark green. Note that in the applet, a slider for a parameter m instead of t is used, as t has a special meaning in GeoGebra. There are two ways to use the applet from which we have here a snapshot:

- Automated command: use the Locus command, either typed in the command line or using a specific button. The syntax is as follows:

`Locus(<Point creating the Locus line>,<slider>)`

. Here we need **Locus(A,m)**.

- Dynamics: Use **Animation On** for the slider m . The curve is created dynamically.



(a) Animated construction

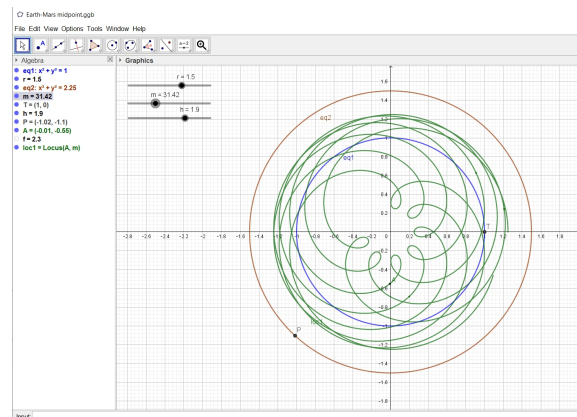

 (b) Automated construction with **Locus** command

 Figure 4: The locus of the midpoint Earth-Mars - first attempts with $h = 1.9$

In both cases, if a defining interval for the parameter is "too short", then the obtained curve is incomplete. Either experiments and successive trials or computations, depending on the computational skills of the user, lead to a suitable interval. The first option may be time consuming. Figure 4(a) has been obtained within 10 minutes with **Animation On** for the slider and **Trace On** for the point A . Figure 4(b) shows an incomplete attempt with the **Locus** command: the given interval for the parameter m is $[0, 30\pi]$ and is too small.

Remark 1 Figure 4 displays both the algebraic and the graphical windows of the GeoGebra session. The algebraic window is easily written, and the order of commands can be checked with the windows displaying **Object Properties**, or better, with the windows displaying the **Construction Protocol**, accessible from the main menu.

The same activity can be performed using the animate command of Maple. In the following code, the animations to simulate the moving of Earth and of Mars on their orbits are defined separately, then displayed together. The resulting plot is quite trivial, as it creates dynamically two concentric circles. This may be used as a preliminary control, before plotting the animation for the midpoint, which comes afterwards.

```
restart;
with(plots):with(plottools): setoptions = (thickness = 2):
earth := <cos(t), sin(t)>;
mars := <1.5*cos(t/1.9), 1.5*sin(t/1.9)>;
plotearth := plot([earth[1], earth[2], t = 0 .. 2*Pi], color = blue);
plotmars := plot([mars[1], mars[2], t = 0 .. 4*Pi], color = brown);
display(plotearth, plotmars);
animearth := plots[animate](plot, [[earth[1], earth[2], t = 0 .. A],
    color = navy], A = 0 .. 2*Pi);
animmars := plots[animate](plot, [[mars[1], mars[2], t = 0 .. A],
    color = red], A = 0 .. 4*Pi);
animmidpt := plots[animate](plot, [[1/2*(earth[1] + mars[1]), 1/2*(earth[2] + mars[2]),
    t = 0 .. A], color = green], A = 0 .. 28*Pi);
display(animearth, animmars, animmidpt);
```

At start, the plot is empty; it is made alive by a left click, then by clicking on the suitable button in the command row shown in Figure 5. The animation speed is controlled from the command row to. Eventually, to improve the visualization, the segment TM can be added to the animation, using the **line** command..

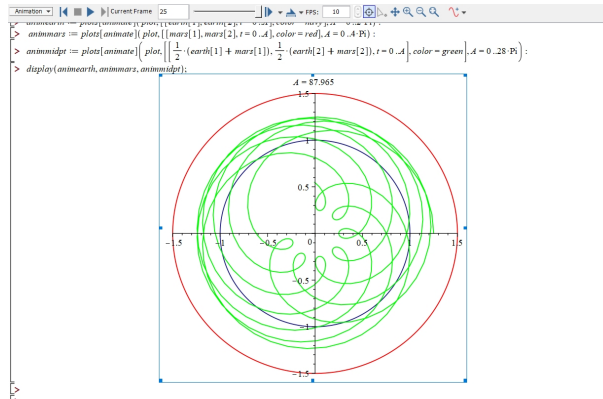


Figure 5: Snapshot of a Maple animation

2.2 Second example: $r = 1.5$, $h = 1.85$

We modify slightly the approximation of the parameter h encoding the period of Mars on its orbit, and take here $h = 1.85$ and $h = 1.86$. Snapshots of the resulting geometric loci of the midpoint of the two planets are displayed in Figure 6. The small modifications of the value of

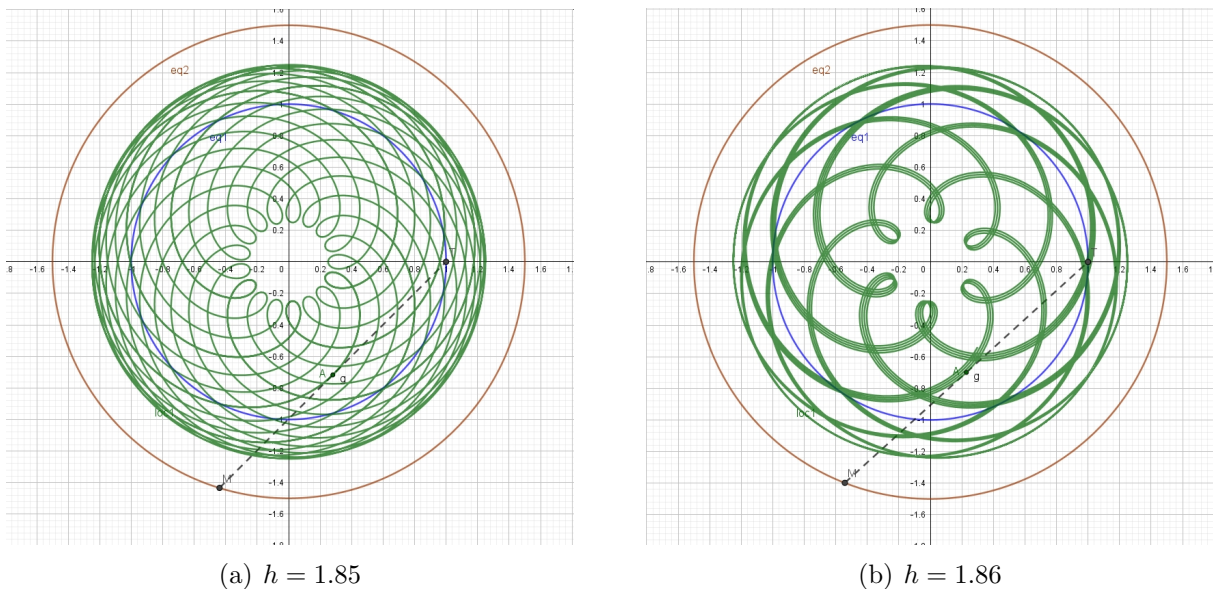


Figure 6: The locus of the midpoint Earth-Mars: the influence of the approximations

the parameter h result in very different geometric loci. The existence of rotational and axial symmetries has been preserved, but their nature has changed.

2.3 Generalisations.

A GeoGebra applet is available to explore the influence of the different parameters and of changing the rounding for numerical values. A screen snapshot is displayed in Figure 7. Actually, the ratio of the radii of the orbits is much closer to 1.881. The resulting geometric locus is still more complicated than Figure 6(a). Note that rounding has to be changed to 3 digits in order to use efficiently the slider and fix $h = 1.881$; to change the increment in the **Object Properties** windows is necessary, but not enough. Pay attention to the 3 sliders on the screen. Their range

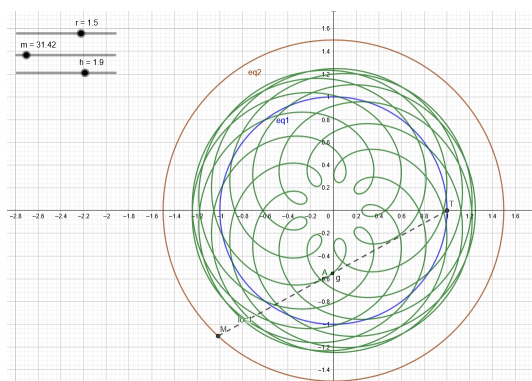


Figure 7: An applet to generalize the previous study

can be modified, most of the possible values cannot be considered as good approximations of the celestial parameters, but enable a broader exploration of the family of curves.

Further generalization is provided as follows:

- The ratio r of the radii of the orbits can be changed. For example for Venus, we have $h = 0.6152$ and $r = 0.723$ (see Table 1).
- The masses of the two planets may be taken into account, if interested in the center of gravity of the pair of planets.

A complete GeoGebra applet is available for such an exploration. Once again, nice plane periodic curves are created. Note that this applet allows any value for each parameter, which is not true in the real world, first because we use a very simplified model with circular orbits, and further according to Kepler's laws, which establish a relation between the distance to the Sun and the velocity on the orbit. Once again with the simplified model, Figure 8 shows the locus of the center of gravity of the pair Earth-Mars, with a rounding of 4 digits.

Remark 2 *Different values of the parameters yield different closed curves with rotational symmetries. This is clear already from the trigonometric parametrization given for the curves and the midpoint. When an algebraic curve is given, an important question is whether it has a rational parametrization or not [15]. It is well known that rational parametrizations are available for a circle. Here above, it was more efficient to use trigonometric parametrizations, not only to have a more "linear" behaviour of the animations, an issue mentioned and studied in [4, 7], but in order to reflect the motion of each planet on its orbit, with constant velocity. This will be useful in Section 3.*

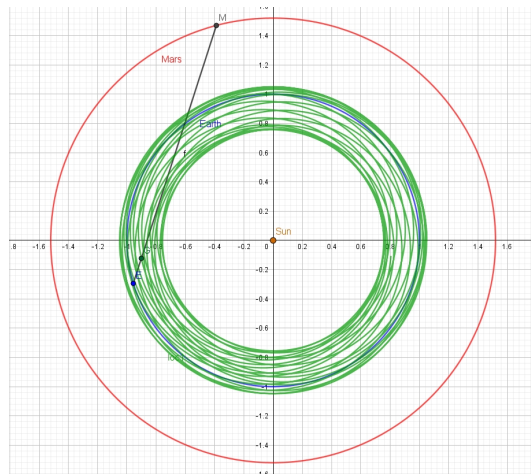


Figure 8: The trajectory of the center of gravity Earth-Mars

3 Identification of classical curves

The parametric presentations that are in use in the previous sections have a too specific form not to be already known among the plane periodic curves. A quick websearch reveals that they are special cases of the family of *epitrochoids* (from the Greek "trochos"=wheel). These curves are defined by the following equations (see [18], p.233-234, and the MacTutor website:

$$\begin{cases} x = (a + b) \cos t + c \cos \left(\frac{a}{b} + 1\right) t \\ y = (a + b) \sin t + c \sin \left(\frac{a}{b} + 1\right) t \end{cases} \quad (2)$$

where a, b, c are real parameters. With these notations, let C_1 be a circle of radius a and C_2 a circle of radius b rolling outside C_1 . A point P at a distance c from the center of C_2 rolls together with C_2 . Its locus is an *epitrochoid*.

With the same simple model of coplanar concentric circular orbits as above, another description of epitrochoids is as the apparent trajectory of a planet when seen from another planet. Derivation of equations in this setting is beyond the scope of this paper. The obtained animations explain why, in certain circumstances, a planet seems to travel in reverse direction (what is called *retrograde motion*). Figure 9 shows an illustration by Kepler of Mars motion on its orbit, viewed from Earth (see for example [10] p. 86).

For special values of the parameters, well-known plane curves are obtained. For example, if $a = b$, i.e. C_1 and C_2 have the same radius, the corresponding curve is a Pascal Limaçon, for other values it is a nephroid, etc.. The illustrations in Figure 10 have been obtained with the following Maple code with $p = 1$:

```
x := 1/2*(p*cos (t) +k*cos(t/k));
y := 1/2*(p*sin(t) + k*sin(t/k));
for k from 2 to 6 do
    plot([x, y, t = 0 .. 20*Pi]);
end do
```

With $p = 3$, Figure 11 is obtained:

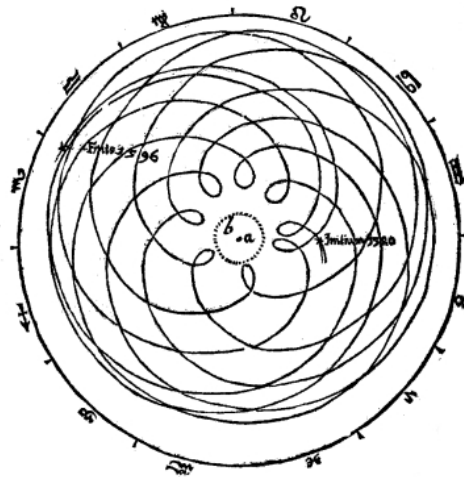


Figure 9: Kepler's drawing of Mars orbit viewed from Earth

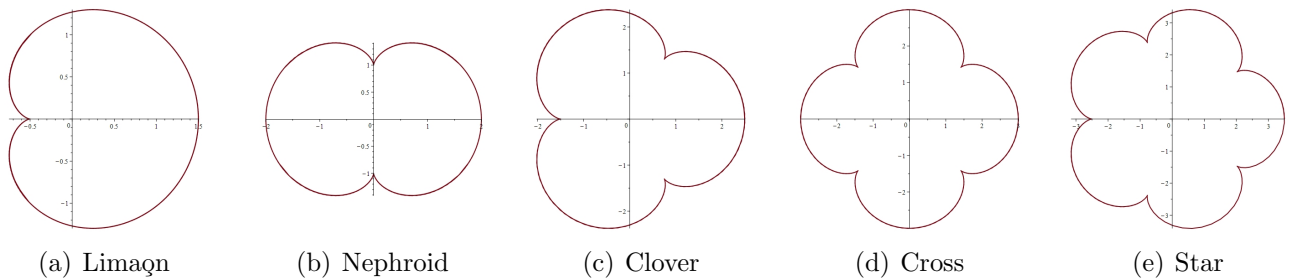


Figure 10: Special epitrochoids

3.1 Envelopes

A detail appearing in Figure 6 has not been mentioned until now: the segment TM is there, dotted. It may help to visualize the coordinated motion of the 3 points T, M, A . Another feature may be worth of an exploration: does the family of segments TM have an envelope?

A first exploration may be performed using the DGS. Figure 12 shows two trials with different values of r and h .

With the current settings, i.e. when everything depends on the parametric presentation of the points T and M , we cannot use GeoGebra's command **Envelope**. It has been developed on another basis. Four different definitions of envelopes are given and their similarities and differences analyzed in [3]. Here we consider one of their definitions, the only one considered by Berger [2]3(sections 9.6.7 and 14.6.1). Some recent developments and examples can be found in [6].

Definition 3 Let C_m be a family of plane curves given by an equation of the form $F(x, y, m) = 0$, where x, y are real variables and m is a real parameter. An envelope of this family, if it exists, is the solution set of the system of equations:

$$\begin{cases} F(x, y, m) = 0 \\ \frac{\partial F}{\partial m} F(x, y, m) = 0 \end{cases} \quad (3)$$

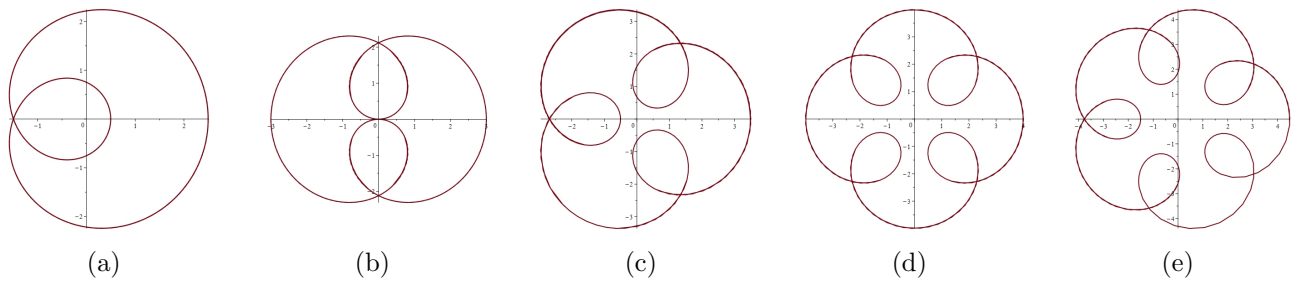


Figure 11: Other epitrochoids

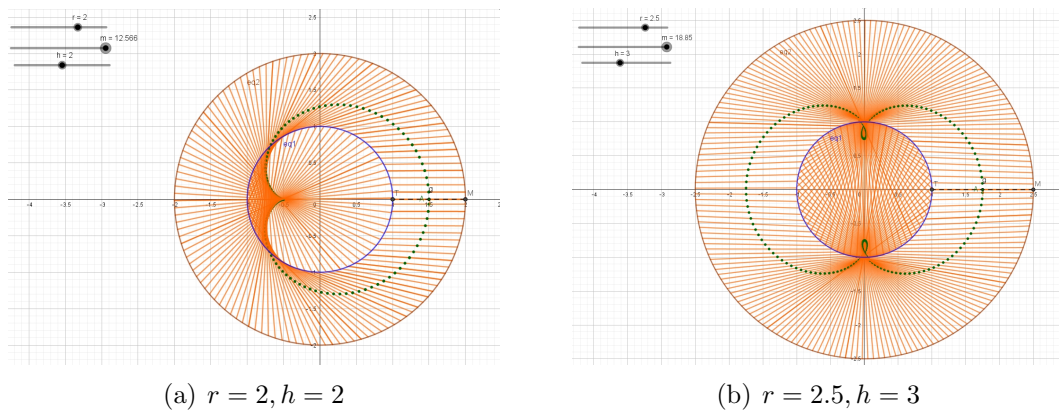


Figure 12: Experiments to determine envelopes graphically

Actually we will look for an envelope of the family of lines TM . It is easily shown that the family has no fixed point, so an envelope should exist. First we compute an equation for the lines; the role of the parameter m in Definition 3 will be played by the parameter t in the parametric presentation of the moving planets. Working with Maple, here is the source code of a suitable session:

```
restart;
with(plots);with(LinearAlgebra);
T := <cos(t), sin(t)>; # Earth
M := <r*cos(t/h), r*sin(t/h)>; #Mars
mat := Matrix(2, 2, [[x - T[1], M[1] - T[1]], [y - T[2], M[2] - T[2]]]);
line := Determinant(mat); # implicit equation of the line TM
dline := diff(line, t);
solve({dline = 0, line = 0}, {x, y}); #the solutions determine the desired envelope
env:=simplify(%);
```

An implicit equation for the line TM is

$$x \left(r \sin \frac{t}{h} - \sin t \right) + y \left(\cos t - r \cos \frac{t}{h} \right) + r \left(\cos t \cos \frac{t}{h} + \sin t \sin \frac{t}{h} \right) = 0$$

and, after simplification:

$$x \left(r \sin \frac{t}{h} - \sin t \right) + y \left(\cos t - r \cos \frac{t}{h} \right) + r \sin \left(\frac{h+1}{h} t \right) = 0. \quad (4)$$

We denote by $F(x, y, t)$ the polynomial in left hand side. Solving now Equations (3) we obtain the output of last command row above::

$$\begin{cases} x = \frac{(h \cos(\frac{t}{h}) - \cos(t))(-\sin(\frac{t}{h}) \sin(t) + \cos(t) \cos(\frac{t}{h}) - 1)}{\cos(t)(h+r) \cos(\frac{t}{h}) + \sin(t)(hr+1) \sin(\frac{t}{h}) - h - r} \\ y = \frac{(\cos(t) \cos(\frac{t}{h}) hr - \cos(t)^2 + h + 1) \sin(\frac{t}{h}) + \sin(t) (\cos(\frac{t}{h})^2 hr - \cos(t) \cos(\frac{t}{h}) - r(h+1))}{\sin(t)(hr+1) \sin(\frac{t}{h}) + (\cos(t) \cos(\frac{t}{h}) - 1)(h+r)} \end{cases} \quad (5)$$

The first output displayed by the CAS is quite complicated, but **simplify** commands are helpful. Nevertheless, the final display with general parameters is still heavy and trying to identify the curve using catalogs of curves and websearch is unilluminating, even hopeless. Moreover, in order to try and implicitize the parametric presentation, methods such as substituting rational expressions for sine and cosine may be considered, but even for the CAS it can be a heavy task. Substituting specific values for the parameters, the plot command yields a plot of the curve, and the result is somehow surprising (an example is given in Figure 13). Graphical superposition of the curves obtained with an animation as in Figure 12 and 13 showed differences. We have here still an open question.

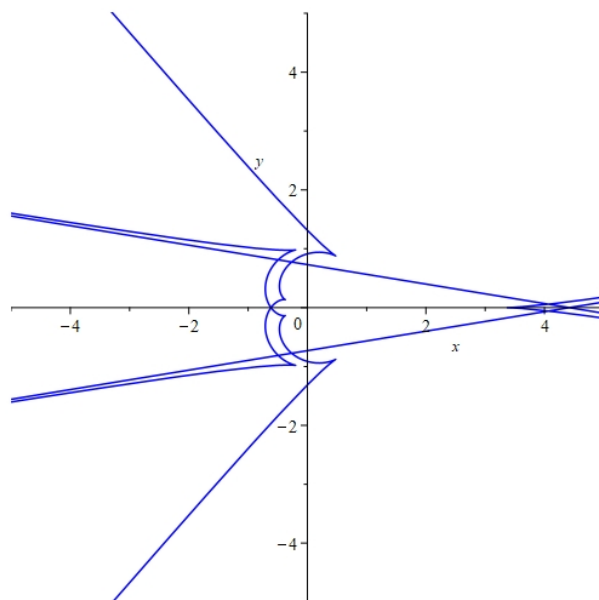


Figure 13: The envelope of lines TM for $r = 1.5$ and $h = 1.9$

And, after all, the literature, either printed or electronic, devoted to the epitrochoids, show only integer values for the parameters and do not display any implicit equation, which can mean that these curves are not algebraic. We invite the reader to check.

4 Discussion

This is not the first time that we analyze the different contributions of a CAS and of a DGS for exploring the same question. Generally, the roles are complementary, and with the help of both kinds of software together, letting them having a dialog, a precise insight into the problem is obtained [6, 7].

Here a first exploration has been done with the DGS, using the following features:

- The numerous options for the animation (interval, speed, oscillating, etc.) may be changed in the **Object Properties** window, which has to be opened. Sometimes, numerous trials are needed.
- The animated construction and the plot of the locus are driven by different commands.
- Zoom-in and zoom-out are easy to perform with the mouse. This is important to discover properties of the curves in neighborhoods of singular points. As an example, see the study of the relations between the cusps of a given curve and of its offsets [6]. A strong zooming with the DGS provided conviction, and an algebraic proof was not needed. Here with non integer values of the parameters, zooming is strongly needed to distinguish arcs through multiple points, as can be seen in Figure 8.
- Modifications (size, proportions, colors, etc.) may be performed quickly with the mouse, for example with a right-click on the object.

Exploration with the CAS is slightly different.

- After a command line to define a parametric curve, an **animate** command has to be entered. Its output is not immediately visible.
- A left-click on the graphical window is necessary, and it switches automatically to the row of graphical buttons.
- Here too, the relevant values for the parameters (number of frames, speed, etc.), in order to obtain a significant graphical output have to be looked for experimentally, using the buttons.
- Other modifications of the output may require changes in the written commands.

The animation speed is quite different with the two software packages². A complete plot for the midpoint of Earth-Mars with $h = 1.881$ (see Figure ??) has been obtained with Maple within a few seconds; with GeoGebra it took much longer. Interesting conjectures and results can be obtained with a single software only, but joint work (we should say collaboration between packages) provides a richer insight. We do not always have a strong dialog between the two kinds of software, as in [7], but the collaboration man-and-machine in this triangular setting is efficient and enriches the study.

The starting point of the study is STEAM oriented, namely using a scientific model from an item in the news. Students may have prior interest in the domain, without having a strong knowledge. The present topic offers an opportunity to collaborate between educators, between man and machine, of course between students. The study output is multiple, and among the "rewards" we have:

- Acquisition of new mathematical knowledge: classical curves (epitrochoids), which are not part of the regular curriculum, have been discovered and studied. Epitrochoids are members of a larger family of curves, which involves roses, epicycloids, etc. Activities as

²We wish to mention that we worked with both packages on the same computer.

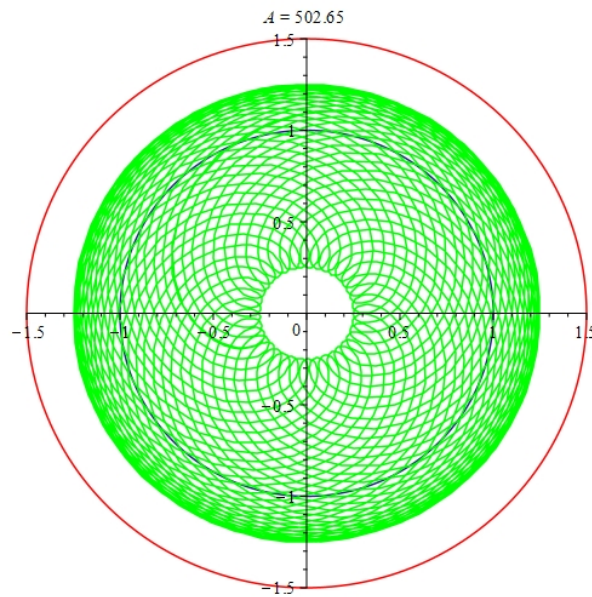


Figure 14: The locus of the midpoint Earth-Mars with $h = 1.881$

in this work may be a nice incitement to explore other situations and to broaden horizons. The literature describes generally the epitrochoids for integer values of the parameters, and our experimentations showed also more general settings.

- Emphasis on the importance of the data precision (in space, contrary to most classrooms, nothing is measured by integers) and of rounding. We considered non integer ratios of radii of orbits, and of orbital angular velocity, approximations and rounding became an important issue. We could discover that different precisions in the approximation yield very different output. This is probably a crucial product of this work: students do not always believe that mastering errors is important, and they believe that the answers provided by a numerical calculator are always accurate. Asking them which answer is true among the cases that we studied with different rounding should lead at least to some questioning.
- Emphasis on the differences between results obtained by a graphical method and results obtained by an algebraic method. There may be a difference between what you see and what actually exists. Zooming appeared as an important feature. More general studies are available in [12, 8, 19].
- Development on new technological skills, which are part of the new mathematical knowledge [1].
- Emphasis on multidisciplinary tasks, whence development of STEAM skills.

In the present work, the geometric locus of the midpoint of two planets (within a very simplified model of motion) has been explored using the dynamical features of the DGS. An automated method could be applied to find this locus for various values of the parameters. Then sliders allow to modify interactively the values of the parameters and to observe the

consequent changes in the obtained loci. Algebraic work was not strongly needed, but could add another layer to the study. We did not elaborate on this topic, as our discovery of the different loci for positive integer values of the parameters showed classical plane curves, namely the epitrochoids. Anyway, the different websites devoted to the curves that we discovered provide parametric presentations and polar equations, but no implicit equation. This is a good enough reason for the educator not to push in that direction, and purely algebraic methods have not been applied.

The fact that we started from a concrete question with planets had two goals: to attract students to explore mathematical situations linked to topics from their everyday life. We have here a good example of the importance of teaching mathematics in relation with the cultural background of the students. In [5, 9] various situations accessible at an early stage of the curriculum have been proposed. Here we propose a more advanced environment, where classical mathematics, sometimes already known in Antiquity, model very modern situations. Some websearch will enrich more the new developed knowledge. In 21st century, STEAM is the name of the game; around curves, models of planetary motion, usage of software, we propose a truly STEAM oriented activity.

WE devoted a brief section to envelopes. This, because envelopes are a classical topic with numerous applications, and they provides also new constructions of classical curves, as in [4, 6]. Moreover, the notions are related, and automated commands exist in the same technological environments for loci and envelopes. Nevertheless, it happens, and we have here an example, that one topic provides richer results than the other.

Finally, we wish to emphasize that among the components of the STEAM approach, we focused on STEM and did not address issues involving Arts. No need to say that the curves which have been discovered provide patterns for rosette-like design, generally more complicated than the traditional rosettes on European monuments. The reader's imagination will do the rest.

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The Potential of Outdoor Mathematics in a Digital Context

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Abstract: *In this paper, we present the MathCityMap system which aims at bringing the idea of mathematics trails into an educational context. To facilitate the preparation and conduct from both, teacher's and students' perspective, the system provides two components: a web portal to create outdoor tasks and a smartphone app to guide students to discover mathematics outdoors. In this paper, we focus on the benefits of the math trail idea for educational purposes and, in particular, its enrichment through digital tools in the context of mobile learning. By taking learning theories and empirical findings into consideration, the potentials of the system are analyzed and evaluated. Hereby, the focus is on different aspects of math trails and, in particular, math trails with MathCityMap that differ from "classic" mathematics teaching, i.e. the autonomous learning, the potential for modelling tasks, the use of mobile learning, the role of embodiment and their empirical impact on students' performance and motivation.*

1. Theoretical Background of Math Trails

Learning and the learning of mathematics can take place in different learning environments. For sustainable learning, [8] describe that a learning environment should offer enough space for own discoveries and still introduce new discoveries stepwise. In addition, it should communicate the required knowledge directly when needed and the task difficulty should not be too high.

In such a learning environment, mathematics does not have to take place exclusively in the classroom. Due to the demand for relating mathematics education to reality, i.e. contextualized tasks with relevance to the living world, a current trend emerges to do mathematics outside – often found under the catchword "outdoor mathematics" [10]. So-called mathematical trails (also known as "math trails") offer a possibility to meet this demand and to consciously perceive and apply mathematics in the environment. When running a math trail, objects in the environment become the centre of mathematical tasks through appropriate questions. In this way, it is possible to transfer tasks that are known from the textbook to everyday objects, places and concrete situations that are familiar to the students. A math trail is thus a mathematical path along a map with several fixed places where mathematics can be experienced. At these places, there are tasks that can only be solved on site because to solve the problem, the own data have to be collect exactly at these places. In order to decide which data must be collected, the task solver must have a mathematical model for this situation [15]. This can be, for example, the model of direct proportionality when counting paving stones in a limited area, the gradient triangle when calculating the slope of a wheelchair ramp, or a suitable geometric body when determining the weight of a stone.

The idea of the math trail is not new, but already several decades old. However, its original intention was not focused on a school setting but was to popularize mathematics in society. The math trail created by [4] in Melbourne in the 1980s was intended as a vacation activity for the whole family. Accordingly, the tasks were set in such a way that simple basic arithmetic was sufficient to discover or just discuss mathematical relationships and phenomena in the environment.

Over the past several decades, there have been scattered reports of positive experiences using math trails in schools [17]. Hereby, using math trails in school seems legitimate in many

respects. Emotions, interest and personal significance have been shown to correlate positively with (mathematical) performance [18].

Already at first glance, math trails differ from the “classic” mathematics teaching and learning inside the classroom. When walking a math trail, students work in small groups on various tasks. The groups are independent of each other and also quite independent of the teacher. Because of this open form of learning, math trails are reminiscent of the “learning at station” method, in which students work on tasks independently. [3] emphasizes, in particular, the autonomous learning at one's own pace and the observing, rather passive role of the teacher during the processing as advantages of this method. It is true that the observing role of the teacher in math trail can only be realized to a limited extent due to the spatial distance of the tasks – in contrast to learning at station in the classroom. The feature of the Digital Classroom of MathCityMap (see section 2), which allows the teacher to observe the students' entries, walking paths and hint retrievals and to analyze them after they have been completed, provides the teacher with diagnostic possibilities. In addition, students can chat with the teacher if necessary and ask for individual hints. Nevertheless, this request for assistance and, in particular, the development of the solution lies with the students, whereby autonomous learning takes place.

Not only methodically, but also in terms of content, the math trails require independent decisions from the students. Unlike in the textbook, the tasks do not specify data about the object and students have to decide independently which data should be collected based on a mathematical model and which data can be collected at all [10; 12]. Through a math trail, an introduction to mathematical modelling is usefully possible.

Another benefit of working on math trail tasks is the own (physical) activity on site. Studies show that one's own physical activity has a positive effect on cognitive learning [19], which is understood in terms of "embodied mathematics" as an important basis for grasping mathematical concepts [16]. The embodiment view emphasizes that mathematical concepts and terms cannot be meaningfully thought about and grasped without corresponding bodily experiences. For example, the circumference and diameter of a cylinder can be internalized in a contrasted manner if these quantities have been actively measured and thus the difference is also evident in the active measurement. Especially the connection between these enactive actions on the one hand and iconic representations (e.g. a sketch of the object) and the symbolic representation (e.g. formula of an object's volume) on the other hand, is valuable for learning mathematics [15].

Nevertheless, these theoretical benefits have not led to a widespread use of math trails in mathematics education. The main reason for this is probably the high effort required to create a math trail. In particular, the development of the tasks and the compilation of the map that contains the tasks should be mentioned here. More recently, there has been increased experimentation with electronic maps (e.g. Google Maps) and QR codes for task delivery to cell phones. But even this did not lead to the desired minimization of the workload in creating math trails. By combining a web-based database and a mobile app, the MathCityMap system has brought the idea of math trails into a digital and didactic context [15].

2. The Digital Components of the MathCityMap System

The MathCityMap system consists of two main technical components. First, there is a web portal (www.mathcitymap.eu; Figure 2.1 left) that acts as an international database and community portal. On the other hand, there is a corresponding smartphone and tablet app (Figure 2.1 right),

which loads a selected math trail from the portal and makes it available to users (e.g. students) while they are walking along the trail.



Figure 2.1 The MathCityMap Web Portal (left) and Smartphone App (right)

Task and Trail Creation in the Web Portal. The web portal offers the possibility to view tasks to get ideas and create your own tasks. When creating tasks, the portal allows to position the task pin on the map by mouse click and to upload a photo of the task's object. If geotagging is activated when the picture is taken, the system automatically takes over the positioning of the task pin. During the task creation, it is possible to choose from different answer formats. As solution format, the system allows exact values, intervals, multiple choice, fill-in-the-blanks, set, vector as well as GPS coordinates as answer. Thus, combinatorial tasks with an exact solution as well as measurement and modelling tasks, where small deviations should not lead to a wrong result, can be equally realized. In addition, stepped hints and one possible sample solution must be provided for each task.

To make it easier for the user to create tasks, there is a catalogue of so-called "generic tasks" – selected tasks that can be created as if by magic with the help of templates (Task Wizard) with just a few clicks. The intention here is to make frequently found task objects, such as the slope of a ramp or the speed of an escalator, transferable to new locations with as little effort as possible.

Users can combine their self-created tasks with public tasks to form a math trail. Once the trail has been created, it is assigned a unique code. The trail can now be downloaded using the MathCityMap app via code. Afterwards it is available for the actual math trail walk.

The MathCityMap App. In the app's section "Add Trails", students can add a private trail by entering the trail code from the web portal. Once the download of a trail is complete, no further internet connection is required to run a trail with the basic functions described in this section. With the help of a map and your own location (Figure 2.2 first row left), the app guides students along the tasks of the trail. By clicking on the respective task pin, the task text and the picture of the object are displayed (Figure 2.2 first row middle).

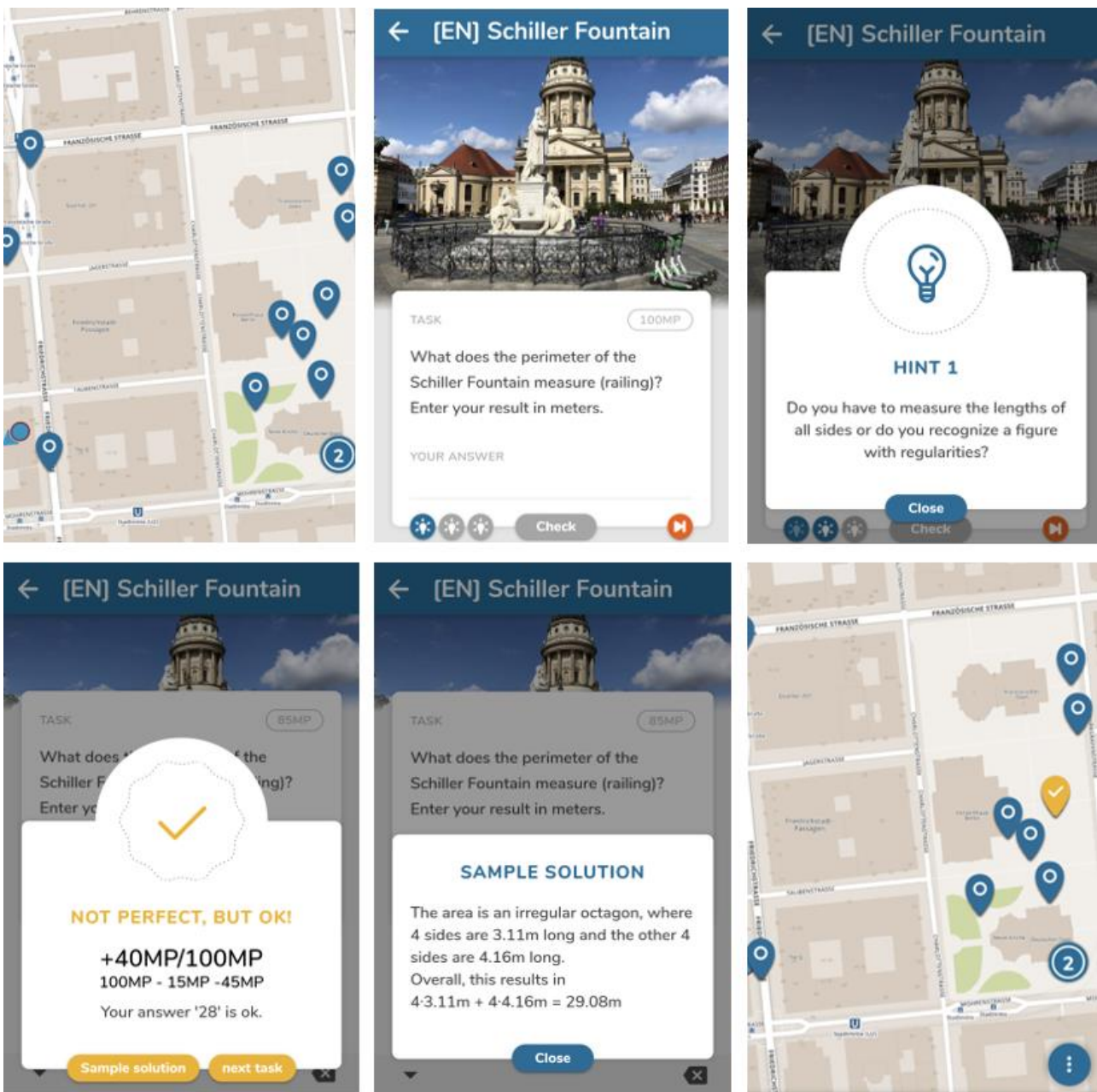


Figure 2.2 Screenshots in the MathCityMap App
 from left to right - First Row: Map (left), Task Formulation (middle), Hints (Right) – Second Row: Answer Validation (left), Sample Solution (middle), Map with Validation (right)

The app upgrades the paper version of a math trail with a stepped help system by displaying hints (Figure 2.2 first row right), as well as automated and direct feedback after solutions have been entered (Figure 2.2 second row left). Task creators can add points as gamification elements when creating a trail. Then students receive up to 100 points per task. From the second wrong solution entry, 15 points are deducted to prevent guessing. The use of hints, on the other hand, does not lead to a point deduction. With the "interval" answer format, further points may be deducted, depending on the quality of the solution (in the example of Figure 2.2 the deviation is about 45 points). When setting the solution interval, the collected measured values and possible deviations are taken into

account. This results in a green interval for very good solutions, a yellow interval for acceptable solutions and a red interval for wrong solutions. After entering a correct or acceptable solution or giving up on the task, it is possible to view the sample solution (Figure 2.2 second row middle). Depending on the range of the solution interval in which a task was solved, the app gives corresponding feedback and displays the tasks that have already been solved on the overview map (Figure 2.2 second row right). Blue pins represent tasks that have not yet been worked on. Grey pins represent tasks that have been skipped and can be worked on again later.

MathCityMap sample tasks. After presenting the basic technical components, this paragraph presents sample MathCityMap tasks get an idea about the actual use of outdoor mathematics. One of the main goals of outdoor mathematics is to foster the application of mathematics and mathematical modelling competencies.

Hereby, we understand modeling as the ability to work out relevant questions from the environment, to transfer them to mathematics, to work on them mathematically and finally to validate and interpret them on the basis of the given real situation. Furthermore, the ability to choose from different models and to evaluate them is also part of the modeling ability [5]. In particular, the aspect that different models can be chosen to address a real-world problem is emphasized by MathCityMap tasks. At this point, it should be noted that MathCityMap tasks are mainly questions that are particularly suitable for providing an introduction to modeling and should therefore be distinguished from complex and extensive modeling tasks. The fact that a math trail is usually composed of ten different tasks and that each task should be completed in a time frame of up to 15 minutes makes it clear that MathCityMap tasks cannot require every single modeling step in its full complexity. Therefore, in the following we deal with individual steps in the modeling cycle and present suitable examples from the MathCityMap project. Most of the MathCityMap tasks focus on simplifying and mathematizing the real situation into an adequate mathematical model, which corresponds to steps 2 and 3 of the seven-step modeling cycle (Figure 2.3 left) according to [5]. In simplifying, important information is separated from unimportant information taken from the real situation. Mathematization involves the translation of the simplified real situation into mathematical models. This is illustrated by the sample task (see Figure 2.3 right).

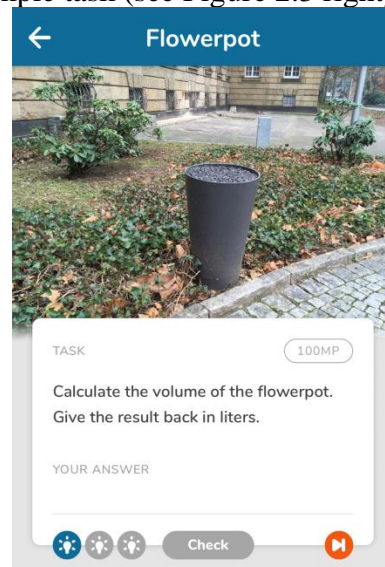
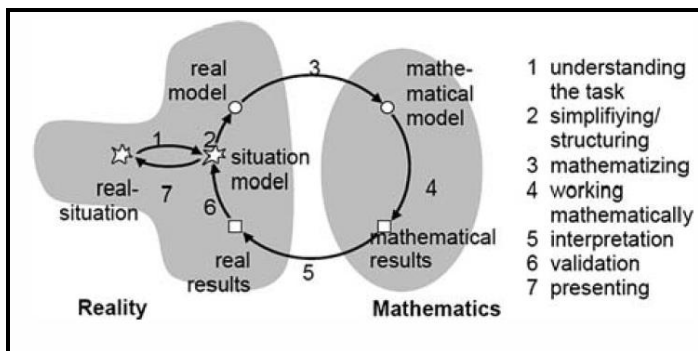


Figure 2.3 The Modelling Cycle according to [5] (left) and the Sample Task “Flowerpot” (right)

For the flower pot, which can be described fairly accurately by a truncated cone, three different geometric approaches could be observed among 9th grade students solving the task outdoors. This can probably be attributed to the fact that the formula of the volume of a truncated cone is not familiar or present to every group. Despite students not being able to solve the task, the following modelling processes were performed (see Figure 2.4).

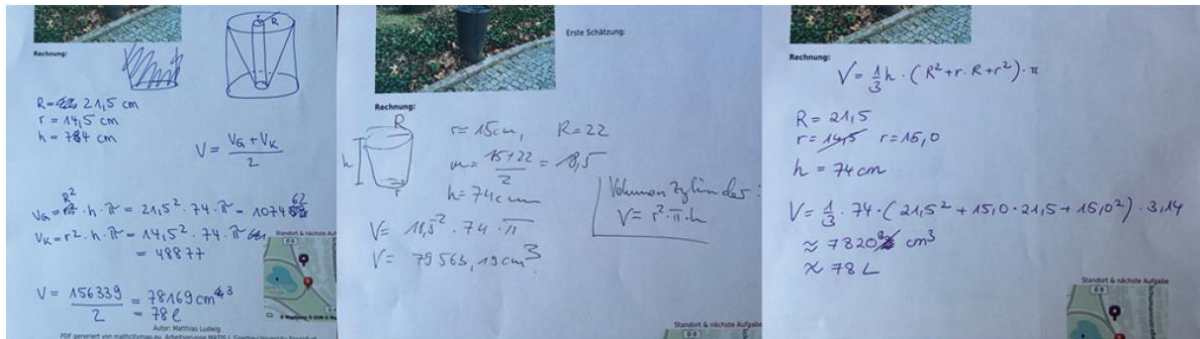


Figure 2.4 Three different Student Solutions on the Sample Task “Flowerpot”

Here, each solution emphasizes a different mathematical model and the students have worked with it mathematically in different ways. The first solution variant (Figure 2.4 left) approximates the result by the mean value of the volume of a cylinder with the large radius (R) and a cylinder with the small radius (r), thus:

$$V = \frac{R^2 + r^2}{2} \cdot h \cdot \pi$$

The second possible solution (Figure 2.4 middle) approximates the result by a middle cylinder. For this, the students take the mean value of the small (r) and large radius (R) as the radius, resulting in the following mathematical model:

$$V = \left(\frac{R+r}{2} \right)^2 \cdot h \cdot \pi$$

The third student solution (Figure 2.4 right) is based on knowledge of the formula of the volume of a truncated cone with:

$$V = \frac{R^2 + Rr + r^2}{3} \cdot h \cdot \pi$$

It seems particularly interesting here that the real results of all three approaches hardly differ. On the one hand, this can be explained by the carefully collected measured values of all three groups of students. On the other hand, the shape of the truncated cone shown here does not differ that much from an exact cylinder, so that the models used only lead to minor deviations.

We conclude: Each solution deals with different mathematical models. For each approach, the students have to create their own real model and then they mathematize it by adding variables they have to measure. The students hereby have to think about which data they have to measure – a difference to modelling tasks in the classroom.

Other typical sample task in the MathCityMap system – also with respect to the idea of generic tasks at frequently found objects – are the determination of the slope of a handrail, the weight of a stone or the age of a tree (see Figure 2.5).

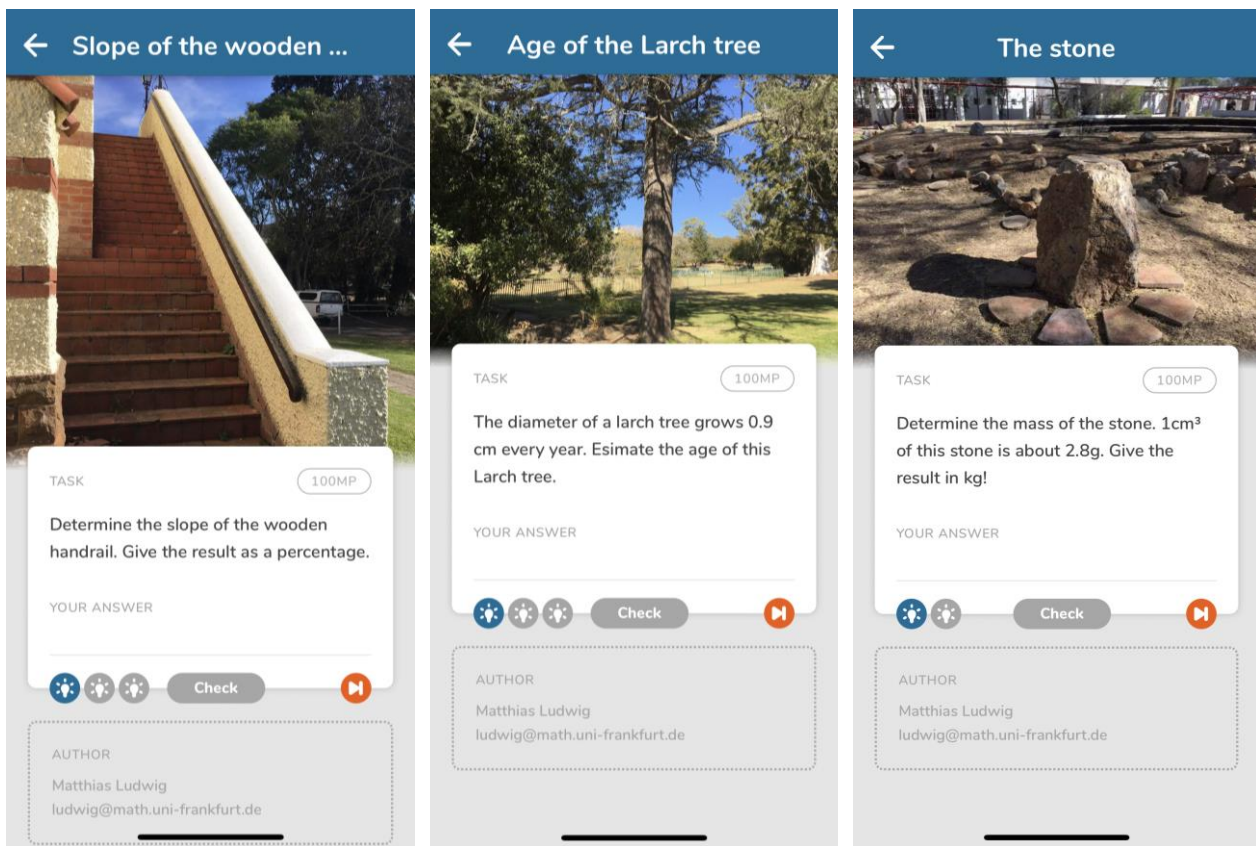


Figure 2.5 Three Generic Tasks in the MathCityMap App

The Digital Classroom. A challenge when conducting a MathCityMap math trail with a school class is to keep track of what is happening. Similar to the learning at station method, when starting a MathCityMap math trail, students are sensibly divided into small groups to work at different stations in the area of the math trail. Subsequently, they work on task after task. This makes it impossible for the teacher to keep an eye on all the learners and, if necessary, to provide support. This is where the digital classroom of MathCityMap comes into play. The digital classroom is a temporary pedagogical digital environment that allows the teacher to communicate with the learners via smartphone during a math trail, while at the same time tracking their learning status and determining their position.

To create a digital classroom, the teacher goes to the MathCityMap web portal and selects the desired trail, he/she would like to use in the digital classroom. Since the digital classroom is intended to be used only temporarily, a start and end time must be defined. The system now generates an access code for the digital classroom that has just been created. This code is used to download the trail onto the smartphone. The participating students are informed about the conditions of use and have to enter a "player name" and the group members. While the digital classroom is active, the teacher has access to a special interface in the web portal. In this interface, three main functions for class management and diagnosis are available during the math trail.

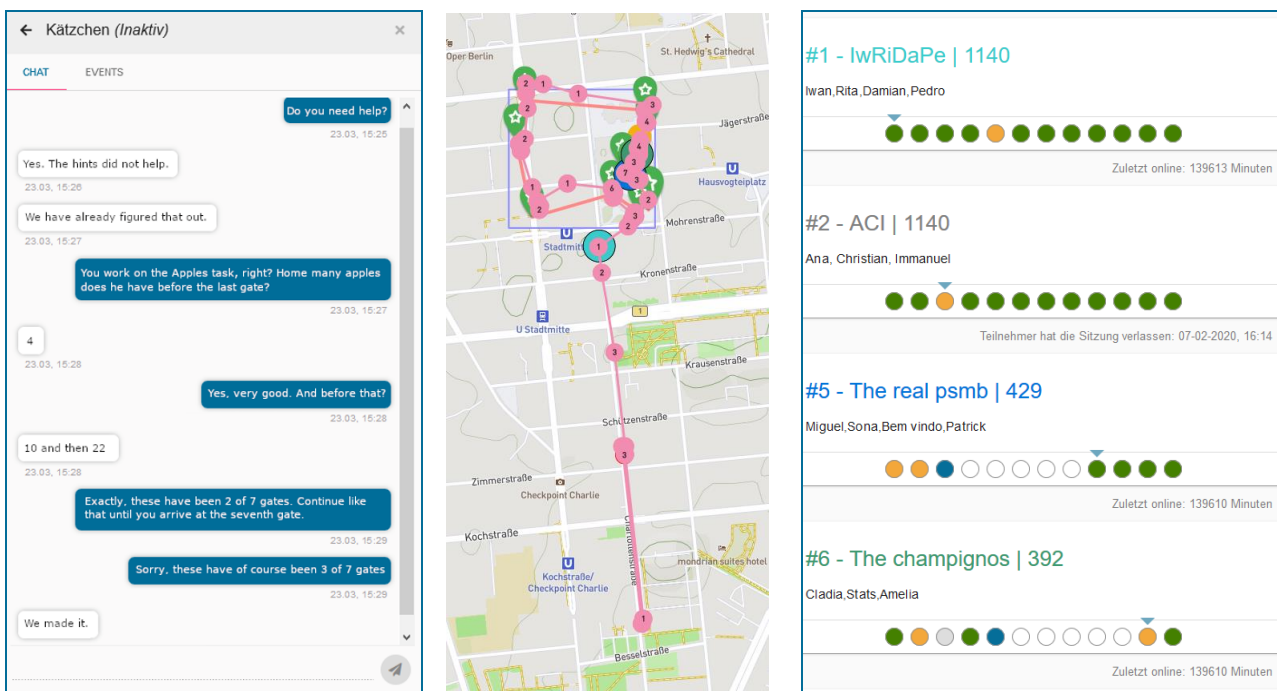


Figure 2.6 Chat (left), Walking Paths (middle) and E-Portfolio (right)

The chat (see Figure 2.6 left) as a communication channel allows the teacher to give instructions or differentiated help to all or selected students digitally in real time. In addition, students can request help when problems arise or have part of their solution (e.g. measured values) validated by the teacher.

The walking path tool (see Figure 2.6 middle) shows the teacher on which tasks the learners are currently active as well as their previous walked path during the math trail. For example, if several small groups are jammed at one station, the teacher can recognize this without being there and react to it via a broadcast message. In this way, teachers retain pedagogical control over their learning group without being physically present at all stations.

The e-portfolio (see Figure 2.6 right) is an additional evaluation and diagnostic tool. It contains information about the progress along the math trail for each group. This includes, for example, the number of tasks completed so far, the hints used, and the answers entered. In addition, learners can transmit their answers and calculation methods to the teacher in the form of text, images and voice message, using the smartphone's camera and microphone. The information obtained through the e-portfolio can be used for diagnosis and incorporated into further lesson planning.

MathCityMap in the context of mobile learning. As presented in the introductory section, running a math trail with MathCityMap is supported by the use of digital media. From the student's perspective, this is done in particular through the use of the smartphone app. The use of mobile technologies in an educational context is called "mobile learning" (m-learning). Mobile learning describes learning that is enabled by mobile technologies at any time and any place, or learning that is supported by the use of mobile technologies in a profitable way [15]. Especially the second aspect is considered by connecting math trails with mobile technologies and mobile learning is implemented with MathCityMap. The added value of using a smartphone app compared to math trails with paper and pencil is seen - from a student perspective - in the following aspects:

1. Navigation support: The app shows your own location and the task location. In order to train the use of maps, the app does not provide a leading navigation, it is only supported by the position display. After completing a task, the app automatically shows the position of the next task in the trail and thus organizes the trail sequence.
2. Retrieval of hints: Unlike an analog math trail, the app not only presents the task and task object, but also holds hints entered by the author or teacher if needed. These are graded, i.e. up to three hints are stored, which increase in specificity.
3. Validation of the solution: The solution is validated automatically and immediately after input. The students thus learn directly on the spot whether their result is correct, acceptable or incorrect and can revise incorrect considerations again if necessary. Furthermore, the sample solution is available to them after correct answer or after giving up the task. By additional playful elements, like points or group sequences, math trails can be embedded further into an optional competition character.

The Math Trail Community. One reason why the math trail idea has not spread as much as one could have expected in the last century was that no (international) community could develop in which tasks could be exchanged or experiences could be shared. The previous math trail projects (e.g. Niagara Falls Math Trail and Canadian Math Trail by E. Muller) were therefore isolated solutions for individual areas or regions, or relatively old-fashioned and unknown websites that were not interactive and therefore did not contribute to the exchange among users [20]. It was important from the beginning of the MathCityMap project to create the possibility that on the one hand the web portal can be offered in different languages and that on the other hand the contact among the users can be established.

In the context of these demands, the MathCityMap project defines different roles for the users with different tasks. There is the classic simple user who can create tasks and combine them to a private trail. The users have the possibility to contact other users directly via the community portal.

In order to publish tasks, so that every user can view and use them, the tasks have to be put into a review process. Over the years, quite a large number of reviewers have been established in various languages who review MathCityMap task voluntarily. Currently, the system has more than 27.000 tasks of which nearly the half are public. These tasks have been created by more than 8.700 users on all continents in more than 40 countries around the world.

In addition to the reviewers, there are also translators who are responsible for ensuring that the latest new developments are translated into the respective languages.

Each member of the community receives various awards for reaching goals in certain categories (see Figure 2.7).

- Taskmanager (number of tasks created)
- Pathfinder (number of trails created),
- Teacher 4.0 (number of Digital Classrooms conducted),
- Influencer (number of followers),
- Consultant (number of reviews conducted) and
- Apptastic (number of downloaded own trails).

All badges



Figure 2.7 The Badges the MathCityMap Community

We do not know exactly which of the factors (e.g. community website, the MathCityMap idea itself, outdoor learning, mobile learning) has led to the fact that the MathCityMap community has been developing almost exponentially for the last six years. Every year, the users create as many tasks as at large of all previous years. The only exception was 2020, the first year of Corona pandemic. For 2021, we expect an increase of more than 15.000 tasks and 4.000 users. It is encouraging that the users have created 10 % more tasks per user since the Corona crisis. Before Corona, we were averaging 2.76 tasks per user, now we are averaging 3.06. This development can be seen in Figure 2.8.

Users and Tasks

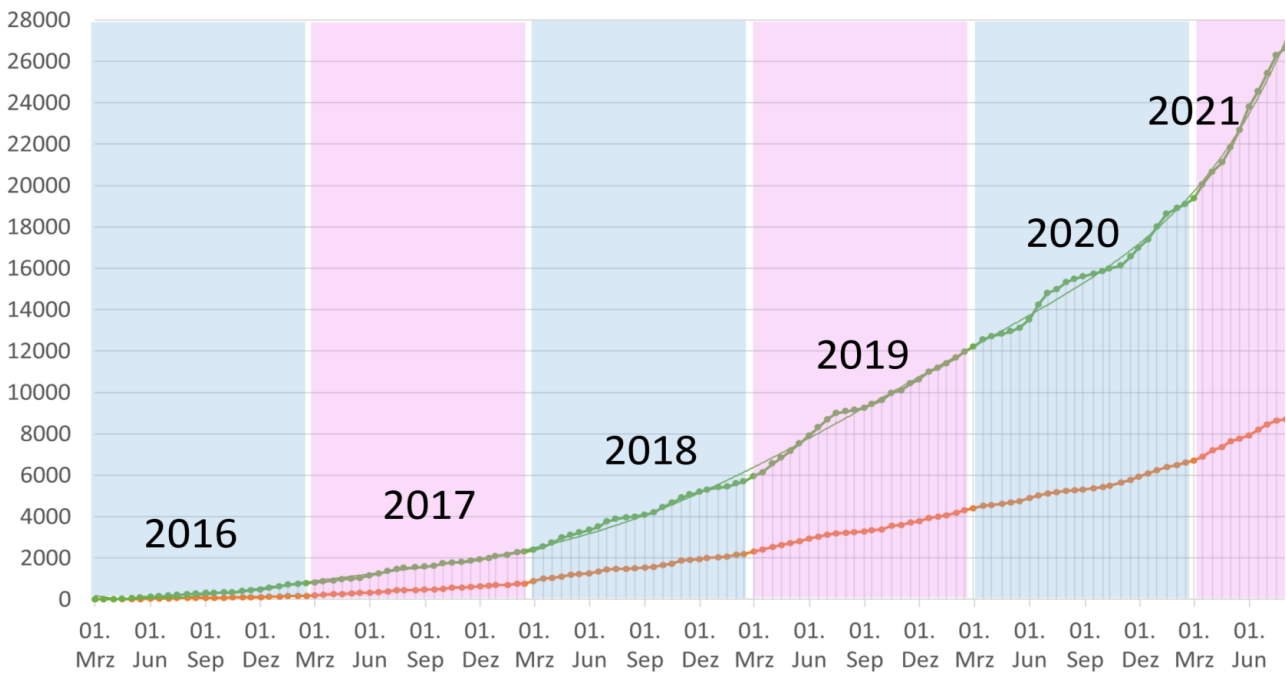


Figure 2.8 The Exponential Development of the Number of Tasks and Users

3. Research Interest on Mobile Math Trails

As stated in the introductory part, MathCityMap has been created for the use and digital enrichment of math trails in the educational context. In order to state the relevance of math trails in mathematics teaching and learning, considerations on both levels – theoretical and empirical – are necessary. Especially the digital enrichment of the math trail idea through the digital components of MathCityMap are of interest for the educational purpose. In order to legitimate the use of math trails on the one hand and the digital enrichment through MathCityMap on the other hand, we focus on the following research question:

In what way do math trails and, in particular, their digital enrichment support the teaching and learning of mathematics?

In order to answer the question, several research findings from the educational context are taken into consideration. Hereby, the focus is on different aspects of math trails and, in particular, math trails with MathCityMap that differ from “classic” mathematics teaching. More specifically, we narrow the research question to the aspects of performance, motivation, and mathematical skills using modeling as an example. Already at this point, it should be pointed out that this compilation of potential benefits certainly cannot claim to be exhaustive.

4. Empirical Findings on Math Trails

Math trails in the context of performance. [6] was the first to investigate the impact of the MathCityMap system on mathematical performance. For this, an empirical study was conducted in Indonesia with over 500 students. The experimental group tried math trails with the MathCityMap app, while the control group had normal math classes [21]. The Indonesian study found no significant difference between the control and experimental group regarding the pretest on mathematics ($p = .35$ for a two-sided t-test). However, the post-test has a significant difference between the two groups ($p < .000$, $d = 1.2$) [21].

Also [20] investigates the influence of math trails with MathCityMap on mathematical performance. For this purpose, an empirical study was conducted in 2017 with 235 German ninth grade high school students on the topic of cylinders. After a grouping test, treatment and control groups were formed. For the treatment group, math trails with cylinder tasks were created and run in two 90-minute sessions. In a comparison test on cylinder tasks, there was a strong significant difference between treatment and control group ($p < .01$) with a medium effect ($d=0.5$) in favor of the treatment group. This showed an increase in performance especially after the second run [20]. Notably, long-term learning through math trails was demonstrated in the context of the study with a smaller sample, which is consistent with the findings of previous studies on long-lasting memories of learning in outdoor situations [7].

Math trails in the context of motivation. In addition to the pre- and post-test, [6] used the Self-Determination Index in both experimental and control group to specify the influence of MathCityMap on the students’ motivation. The values of the experimental group were significantly higher than those of the control group. The follow-up math trail activity and the survey one year later showed that increased motivation is also a long-term effect.

Additionally, the previously described German study examined the influence of gamification elements (game elements) on students' motivation while completing a math trail. In total, 196 ninth grade students were divided into three different gamification groups [9]:

- G0: No Gamification - The app only provides feedback on the correctness of the answer.
- G1: Points gamification - There are up to 100 points for each task. Each incorrect answer is penalized with a point deduction. The first incorrect entry has no consequences yet. Answers within the acceptable solution range are rewarded with partial points depending on their proximity to the correct solution.
- G2: Local Leaderboard Gamification - In addition to points, groups can see which group is ahead or behind them in the score.

After completing a trail with one of the above gamification variations, students completed an intrinsic motivation questionnaire. Missed entries and the number of tasks solved were also recorded using the app.

The results show that gamification variants G1 and G2 increase motivation, but not significantly. Nevertheless, they significantly change the groups' solving behavior. On the one hand, the number of incorrect entries - and thus the guessing behavior in particular - was reduced. In particular, the leaderboard increased the number of solved tasks [9].

Math trails in the context of modelling. Also when it comes to mathematical skills, outdoor mathematics shows potential to enrich the teaching and learning of mathematics. As stated in section 2, through the necessity of collecting data from the real environment under consideration of simplification, a strong connection to mathematical modelling [5] can be assumed. In the context of a comparative case study [14], the differences in indoor and outdoor modelling settings were taken into consideration. Through the nature of outdoor mathematics, the modelling process was mainly limited to the “Structuring and Simplifying”, “Mathematizing” and “Validating” steps [5].

Two groups of students solved similar modelling tasks – one group being outside at the real object’s location and one group solving the task inside the classroom with a picture including an object of reference. Both group’s solving processes involve the “Structuring and Simplifying” and “Mathematizing” modelling steps. The basic chosen models are often similar in the indoor and outdoor context.

Still, the discussion which model should be chosen is more intense in the group of students being outdoors. While doing so, they take different perspectives and discuss which data and knowledge are needed. Through the possibility of measuring at the real object, the students try to be as precise as possible and do not take any estimations. Still, it is their aim to work effectively. The students inside the classroom have fewer possibilities to choose a model because of the limited data that can be collected. Through the picture, they only have one unchangeable perspective of the object. In contrast, the students indoors have more intense discussions on the assumptions and estimations that they have to make in order to collect the necessary data for the mathematical model [14]. These findings are exemplary presented in Figure 4.1 which gives an overview on the different modelling steps and the duration for the outdoor and the indoor group. The activities in “Simplifying/Structuring” and “Mathematizing” are divided into subcategories. Still the choice of colour (deep and light blue and red) should symbolize their relationship.

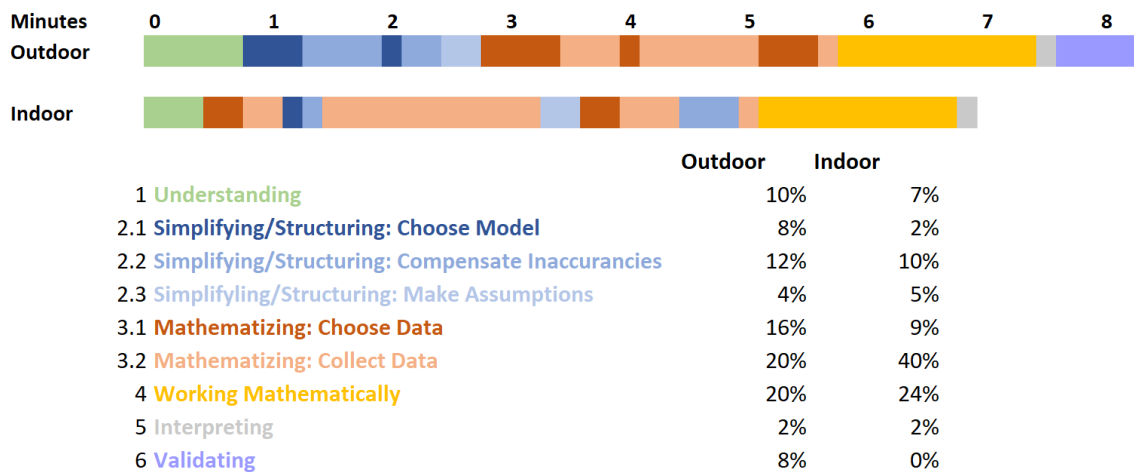


Figure 4.1 Modelling Activity Diagram [1; 14]

5. Discussion and Limitations

On the one hand being an open form of learning, on the other hand through strongly emphasizing embodiment and enactive actions, we have assigned math trails in the introductory part a special role alongside or in mathematics learning and teaching. The results from the various empirical studies confirm the special significance of math trails hypothesized from the theory. In addition, the empirical results show that this special role indeed seems to have an impact on the learning of mathematics.

First, the impact on (initially short-term) learning performance could be shown. In both studies presented by [20] and [6], the experimental groups that were outside with MathCityMap performed better. With reference to the theoretical advantages of mobile math trails (own decisions, open-ended work, embodiment and enactivity, and digital enrichment), the theoretical and empirical considerations seem to fit together. Nonetheless, it remains unclear at this point which influences lead to this increase in performance – going outside, the smartphone, or even a mixture of both. This remains to be investigated on a qualitative level.

Furthermore, circumstantial evidence for long-term learning success emerges in [20]. Based on a subsample of his study, he hypothesizes that math trails with MathCityMap lead to long-lasting memories of what is learned. This hypothesis will be investigated by [2] in the quantitative longitudinal study “MEMORI”.

In terms of motivation, the MathCityMap system was found to be intrinsically motivating. As with performance, this may be due on the one hand to going outside, but also to the use of the smartphone. Especially the latter shows relevance in the study of [9]. The supplementary use of digital gamification elements shows that these playful elements have an additional influence on motivation and prevent undesired behavior such as guessing.

To confirm the quantitative results also on a qualitative level, we have listed the results on the actual solution process when modelling in the classroom and outside. Here we find different emphases regarding the steps structuring and mathematizing from the modeling cycle of [5]. Validating also has a different meaning in the context outside and is particularly stimulated by the digital component of the app. Modelling outside the home thus seems to be a profitable enrichment of "classical" mathematics instruction in the classroom. In the qualitative study "MAP - Modelling,

Arguing and Problem Solving in Outdoor Mathematics" we will investigate the modelling competence in more detail and furthermore the competences arguing and problem solving in outdoor mathematics tasks [11].

The limitations and outlooks show that there is still much work to be done concerning the special learning form of math trails. Both, theoretically and empirically, math trails show advantages that can be used to enrich normal mathematics instruction. Thus, in addition to these research activities, our practice-oriented goal is to convince teachers to use the tool regularly. In the strategic partnership MaSCE³ (Math Trails in School, Curriculum and Educational Environments of Europe), we realize this by creating theme-based trails – trails that fit a specific topic from the curriculum (e.g., slope or statistics) and that can be created on objects that can be easily found in numerous locations (e.g., stair railings or city maps) [13]. Ultimately, through this curricular adaptation, we hope to see an even greater emphasis on the benefits of math trails – and an ever-growing international community of teachers and instructors willing to use them.

Acknowledgements MathCityMap is part of the MaSCE³ project (Math Trails in School, Curriculum and Educational Environments of Europe; 2019-1-DE03-KA201-060118) which is co-funded by Erasmus+.

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Contributed Papers

A Tragedy at the National Gallery of Art: Problem-based learning approach to online teaching of mathematics.

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Abstract: *This paper examines an experience of mathematics preservice teachers of using problem-based learning (PBL) approach in an online course about teaching applied mathematics. The study results showed that PBL approach to teaching mathematics supports collaborative/cooperative work and increases PSTs' engagement in and out of class. The structure of the course that involved PBL experiences from students' perspective followed by analysis of PBL approach from teacher's perspective provided PSTs with sufficient background to develop their own PBL-based instructional units. The study results also showed that while PSTs' teams were successful in developing curriculum-based problematic situations with real-world connections, they were challenged to justify the use of technology in these experiences. PSTs also had difficulty with development of authentic assessments within PBL approach.*

1. Introduction

In March of 2020 the pandemic demanded that teachers design a 21st century learning solution that is scalable, adaptable, engaging, and meaningful. We were forced to untether our thinking and create a new world for learning. This transformation required educators at all levels to move quickly to create fundamentally different virtual environments that provided learning experiences that had never before been offered. Transforming the traditional in-person classroom to a virtual environment while maintaining student-centered teaching became one of the most difficult challenges for all educators during the COVID-19 pandemic. This paper examines experiences of mathematics preservice teachers' (PSTs) in an online problem-based learning (PBL) course about teaching and learning applied mathematics designed to address the challenges of the online environment, e.g. truly engaging students in learning mathematics in and out of class, providing space and time for collaborative/cooperative work, and rethinking formative assessment. The course was designed to meet the following goals:

- i. Apply the philosophy, theory, and rationale behind PBL through the experience of a variety of mathematics modeling and applications PBL activities.
- ii. Use a variety of resources, including the use of technology to facilitate and create PBL units that demonstrate mathematics modeling and applications.
- iii. Use authentic assessments, techniques, and tools in planning and evaluating PBL lessons that reflect the needs of the students.
- iv. Illustrate how to employ a PBL in mathematics courses through the creation of a collaborative and engaging environment.
- v. Evaluate the effectiveness of a PBL lesson by using a set of criteria to determine if the lesson meets the educational needs of the students.

The purpose of this paper is to discuss my success with these goals. To do so, I address the following research question: Did PBL-based design of online course support PSTs' engagement in mathematics problem-solving, collaborative/cooperative work, and development of their own PBL-based instructional units for secondary mathematics classrooms?

2. Background

Studies show that PBL has been an effective approach to teaching and learning mathematics in the in-person settings. In [1] a meta-analysis of 217 studies showed that the implementation of PBL into secondary school mathematics courses significantly enhanced learners' higher-order thinking skills in mathematics. In higher education, the implementation of PBL in mathematics courses led to increased student achievement [2], interest, engagement, enthusiasm, and sense of ownership [3], and improved reasoning skills [4]. Research also shows that PBL is an effective approach to teacher education. Specifically, empirical evidence suggests PBL can deepen PSTs' mathematics content knowledge [5], and prepare PSTs for implementing PBL in their own classrooms [6]. While there is overwhelming evidence that PBL is an effective approach to teaching and learning mathematics and mathematics education courses in the in-person settings, very few studies examine the PBL approach in online mathematics or mathematics education courses. In [7] a study that compared academic achievement of undergraduate students in online and in-person mathematics PBL course, students in the online course performed significantly better than in the in-person course. Moreover, the online environment supported the use of social media for student discussion in a hybrid mathematics PBL course, which led to increased student interaction and development of the mathematics learning community [8]. However, studies are needed to examine how online PBL-based courses can support preparing mathematics PSTs to implement PBL in their future classrooms.

According to [9] students learn best when their teachers maintain a high-level of cognitive demand throughout the lessons, which can be achieved through PBL approach. Therefore, in order to provide an active learning environment with high cognitive demand tasks, I decided to use a problem-based learning (PBL) approach as suggested in [10]. In this approach each unit starts with a problematic situation that serves as the organizing center and context for learning. The problematic situation is usually ill-structured and messy, that students perceive as important and relevant. The PBL process steers students through the complex tasks of brainstorming ideas, identifying useful knowledge, asking appropriate questions, and crafting a strategy for finding answers [11]. However, PBL requires a high level of interaction and collaboration which is not naturally supported by the virtual environment. Thus, I needed to redesign a course about teaching applied mathematics for mathematics PSTs that would transform PBL approach used in the in-person settings to a virtual environment.

3. Course design and implementation

In order to create a highly active learning environment where PSTs could discuss and share ideas, and collaboratively solve problems, I analyzed high-impact influences on student learning [12]. John Hattie [12] examined and synthesized more than 1,600 meta-analyses comprising more than 95,000 studies about factors affecting student learning and ranked them according to their effect size (Cohen's d). Among teaching strategies with the largest effect sizes that could be enhanced by open source instructional technology and effectively used in an online PBL environment I selected Cognitive Task Analysis (CTA, $d = 1.29$), classroom discussion ($d = 0.82$), and providing feedback strategies ($d = 0.62$). I knew I could support these strategies in an online environment by using Nearpod (<https://nearpod.com/>), a multimedia student engagement platform for designing interactive lessons using various multimedia content, e.g. PhET simulations, interactive videos, VR field trips, Desmos graphing calculator, etc. Nearpod also includes a variety of activities that could be used for formative assessment, e.g. Collaborate board, Draw-it, different types of questions, and FlipGrid video. Live Nearpod presentations could be used in a synchronous mode of teaching enabling

teachers to control student progression through the lesson. An asynchronous mode can provide self-paced Nearpod presentations so students can experience the same interactive features at their own pace. All my classes were conducted live, so it was very important for me to have an asynchronous option for students who might have difficulties accessing the Internet during classes. I also wanted to be able to integrate various resources in my lessons, and Nearpod enabled me to do that within one platform, so students would only need to join a single Nearpod lesson to have access to all resources that I had planned for them.

The goal of the redesigned online course was to explore and experience the ways in which applied mathematicians approach practical applications, from understanding the underlying problem, creating a model, analyzing the model using mathematical techniques and digital technology, and interpreting the findings in terms of the original problem. In order to provide PSTs with first-hand PBL experience, I started each unit of the course with a problematic situation on a topic aligned with 7 – 12th grades mathematics curriculum. The course outline is shown in Table 1.

Table 1. Outline for the course *Teaching applied mathematics*

<i>Week</i>	<i>Focus</i>	<i>Topics</i>
Week 1	Introduction	Introductory problem: A greenhouse extension
Week 2	PBL philosophy and theory	What is PBL? Why use PBL?
Week 3		Teacher role in PBL.
Week 4		Student role in PBL.
Week 5	Middle school	Problem 1. Crime scene investigation: Classroom experience from students' perspective.
Week 6	PBL model	Developing a PBL.
Week 7		Criteria for good applied mathematics PBL.
Week 8	Algebra, geometry	Problem 2. Moving a ladder: Classroom experience from students' perspective
Week 9		Evaluating PBL in the classroom.
Week 10	Precalculus, calculus	Problem 3. Designing a detector: Classroom experience from students' perspective.
Week 11		Brainstorming PBL scenarios for the final project.
Week 12	Assessment in PBL	Alternative assessment options.
Week 13	Gains and pains of PBL	Teachers, students, parents.
Week 14	Student presentations	Individual/group presentations of problems. Peer evaluation.
Week 15	Final project	Reflection on experience. Course evaluation.

3.1. PBL design

Designing a problem scenario that results in the targeted learning outcomes is critical. You want to make sure that the problem scenario encourages students to take charge of their learning, that it emphasizes critical and creative thinking, problem-solving, collaboration, and self-directed learning. The following criteria were used to evaluate problematic situations [:

- Does the problem approximate reality? Will your students perceive the problem scenario as realistic? Avoid creating a scenario that is unbelievable. Avoid designing a problem that is so global and generic that students will throw up their hands in frustration and dismay. If students think that the scenario could never happen in a million they won't be bought into solving it.

- Is the problem curriculum-based? You want to ensure that the problem addresses the target knowledge and skills. Avoid getting carried away writing a problem that you think is intriguing and thrilling, when in reality the problem doesn't match teaching objectives.
- Is the problem engaging? How well will the problem stimulate and sustain curiosity, or the need to explain or understand, or the desire for order, or the need to resolve the situation, or the need to improve a condition? The problem must sustain the motivation to continue interaction and learning from beginning to end.
- Does the problem allow inquiry? The problem presentation should further inquiry and analysis. Is it sufficiently open-ended, ill-structured, and messy?

The problem design is an iterative process, it takes time and energy to write a good problem. The *Tragedy* problem scenario went through multiple iterations before it took its final form shared in this paper as an example of a problem that could be used at the beginning of a middle school unit to teach proportionality and similarity in scale drawings. Here is the excerpt from the problem scenario [13]:

A Tragedy at the National Gallery of Art

As a team of forensic specialists from Lloyds of London, you have been asked to investigate a theft of a priceless Picasso painting that took place at the National Gallery of Art in Washington, D.C. The painting, titled The Tragedy was housed in the East Building, Upper Level, Gallery 99. The theft of this painting marks the first successful removal of a painting from the National Gallery of Art. The National Gallery had The Tragedy insured by Lloyd's of London for \$15,000,000, but of course, the painting is priceless.

Neither the audible alarm nor the police alarm had been activated. The wall, which had just hours earlier housed several of Picasso's greatest works, was now missing The Tragedy, one of the most prized paintings of Picasso's "Blue Period". To the officers' surprise, there was no one in the gallery. Black paint was found on the floor beneath the area where the Picasso had hung, and spatters of black paint were found throughout the gallery. In addition, footprints from the paint were found leading from the missing painting to the gallery exit door and outside.

Digital images were taken of the paint spatters and footprints. After examining digital images of the trace evidence left at the scene, your team should be prepared to present quantitative evidence that can lead to the warrant for the arrest of the suspected thief. Lloyd's of London and the National Gallery of Art are relying heavily on your team of forensic specialists to determine who stole the Picasso.

The problem then provides a list of suspects identified by the police, digital images, background on the painting, and requirements for the dossier to be submitted by the forensic team (you can see the full text of the problem at <http://bit.ly/3uWcCMg>). Considering the fact that this problem was developed for an online environment, instead of a 'real' crime scene I used digital images 'taken at the crime scene' that were 'sent' to the forensics lab, thus creating an opportunity for students to conduct an investigation using online resources only. This problem becomes the context for student learning about proportionality and similarity in scale drawings. The whole unit for this topic could take up to ten days with the plan shown in Table 2.

Table 2. Teaching plan for the middle school unit on similarity in scale drawings.

Day 1	Introduction of the CSI scenario. Whole class development of learning needs using problem-solving grid.
Day 2	Review of ratios and proportions. Introduction of similar figures.
Day 3	Practice solving similarity problems, including problems with proportions in the human body.

Day 4	Learning to use GeoGebra software. Practice constructing similar figures and exploring their properties in GeoGebra.
Day 5	Practice solving similarity word problems dealing with indirect measurements. Group work with digital evidence – qualitative analysis.
Day 6	Introduction to scale drawings. Scale factor. Practice drawing objects to scale using ratios and proportions.
Day 7	Practice finding distances on the map or scale drawings given scale factor.
Day 8	Practice finding distances on the map or scale drawings given scale factor.
Day 9	Group presentations of evidence, results, and conclusions
Day 10	Peer review: Presentation of rebuttal analysis from the “defense attorneys”.

3.2. PBL implementation

This problem places students in an active role of forensic specialists to investigate the theft of Picasso’s masterpiece, *The Tragedy*, from the National Gallery of Art in Washington, DC. In order to provide authentic experience to PSTs, I modeled the first day lesson the way it would be taught to middle school students. The Nearpod slides for this lesson are shown on Figure 1. (You can access this lesson at join.nearpod.com with the code UZGN9 that is valid through September 4, 2022).

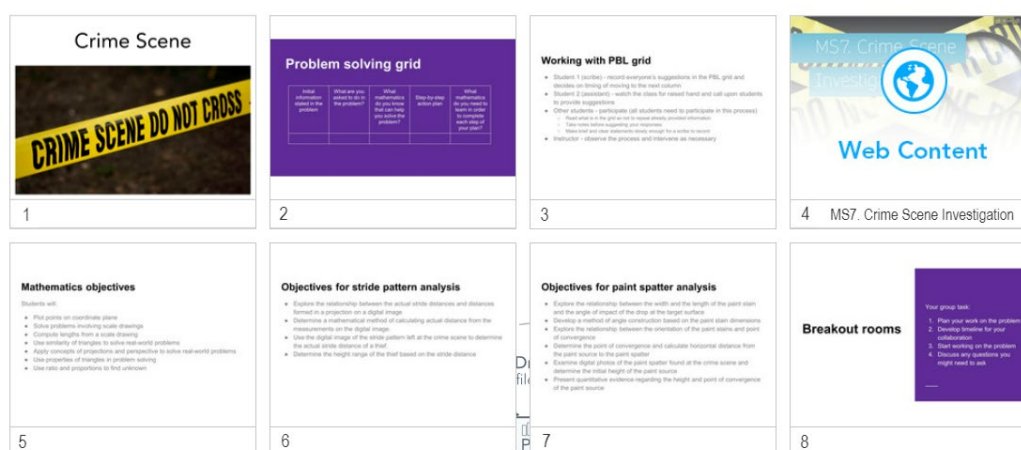


Figure 1. Nearpod slides for Day 1

In this lesson we first discussed the rules of working with the problem-solving grid, a graphic organizer for logical brainstorming (Figure 1, slides 2-3). This graphic organizer is used to help students unpack the problem, e.g. to identify the information given in the problem and translate it into mathematics language, then formulate problem questions in mathematics terms, determine what mathematics they know and develop a plan on how they intend to approach the problem. The final step leads to identifying learning needs for solving the problem that become learning objectives for the whole unit. The process of working with the problem-solving grid as a whole class involves assigning one of the students to be a scribe, a person who facilitates whole class discussion and records information provided by other students. The whole class starts working with the first column until the scribe decides to move to the next column. Students can still add information to the previous column, but they cannot move forward until the scribe decides to do so. One very important aspect of this whole class activity is recording all student ideas without judgment or evaluation.

After the rules were clarified, Nearpod took PSTs to a Microsoft Sway (Figure 1, slide 4) that included a full problem statement, digital images from the crime scene, link to editable Google Document with a blank problem-solving grid, as well as some additional information about the actual painting. At this point PSTs were given quiet time to read the problem and then each of them had an opportunity to use a ‘raise hand’ feature of Zoom in order to add information to the grid. Working

with the problem-solving grid took about 30 minutes with adult learners, but I would have allocated one full class period for unpacking the problem if I were working with middle school students.

There were 18 PSTs in this course and every one of them contributed to the grid. They were truly interested in the plot and wanted to share their ideas about solving the problem. The Google Document that everyone could see replaced the white board we would have used in the physical classroom. At the same time, PSTs' ideas were saved in a document that was easily accessible to all of them. This is just one example of how Google Apps can be used as tools for collaborative problem solving, inquiry-based learning, and student discourse in online courses.

In a PBL classroom, a problematic situation is used as a context for learning the topic. Thus, it is critical that the learning needs that students identify through their work with the problem-solving grid match the unit teaching objectives. I consider it a successful problem design if students' learning needs and my teaching objectives match at 80% or more. In my previous experience teaching middle and high school students, it took about 1-2 months to get to this matching level. After the introduction of the problem on the first day of a unit, the problem becomes a student group project for the length of the unit. When a problem is engaging, students want to learn mathematics as they immediately see how they can use what they learn in class to resolve a given problematic situation, and I think that is the major advantage of the problem-based learning.

With PSTs we continued the lesson to discuss pedagogical aspects of the task, we compared their learning needs with my teaching objectives (Figure 1, slides 5-7), and discussed how the unit could continue in a middle school mathematics classroom. In the lessons that followed I continued using the Nearpod platform to embed inquiry activities, problem-solving, technology tasks to demonstrate how this platform could be used throughout the unit to maintain a high level of engagement and to provide students with collaborative activities and feedback. Nearpod has over 8,000 premade lessons available online, so teachers can find lessons that will meet their teaching needs for most of the topics and modify these lessons as needed. As an example, we reviewed a Nearpod lesson that I modified from the available premade lesson to demonstrate a possible way to introduce a Day 6 topic of scale drawings (selected slides are shown on Figure 2).

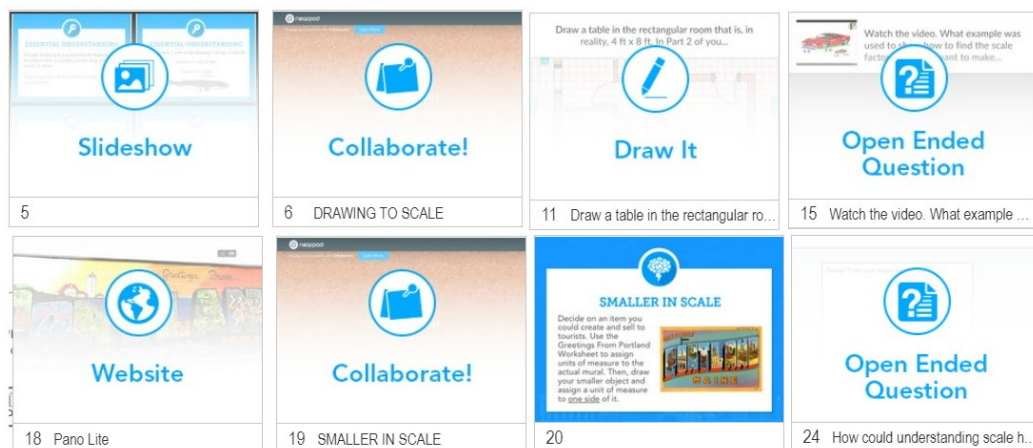


Figure 2. Selected slides from Day 6 Nearpod lesson

This lesson uses interactive features such as slideshow, video, and VR to present material to students in an engaging way. Nearpod's Collaborate board serves as a place for sharing ideas, and student drawings and answers to open-ended questions provide the teacher and the students with ongoing feedback. At the end of the lesson, students are asked to reflect on their learning and how what they learned could help them in their crime scene investigation. All student work is

automatically saved in Nearpod reports, providing the teacher with another opportunity to review student learning and make necessary adjustments to the next lesson. (You can access this lesson at join.nearpod.com with the code ZEGCN that is be valid through September 4, 2022).

This lesson also helped me engage PSTs in a discussion of pedagogical aspects of teaching applied mathematics through a PBL approach. Throughout the course for each problematic situation that I introduced to PSTs, we continued discussions about PBL philosophy and theory, design of problems, models for PBL implementation, issues of assessment, as well as issues of using a PBL approach in teaching mathematics in an online environment.

3.3. Examples of PSTs' solutions of the *Tragedy* problem

In order to have a full PBL experience, PSTs working in groups of 3 were required to solve each unit problem as part of their out-of-class assignments. For the *Tragedy*, the final product had to include the dossier with quantitative evidence that could lead to the search warrant of the suspected thief (thieves). These were the requirements for the dossier content:

1. Summary of your findings based on your analysis of the digital evidence left at the crime scene that can be understood by the judge who is not a mathematics professor.
2. Appendix that includes:
 - a. Step-by-step explanation of your analysis.
 - b. GeoGebra files with completed constructions and calculations.
 - c. Explanations of how you used GeoGebra with illustrations from the software.

When problems are ill-structured and messy, you can expect very different approaches and solutions from different students. Team A presented a flowchart showing their process of solving the problem (Figure 3).

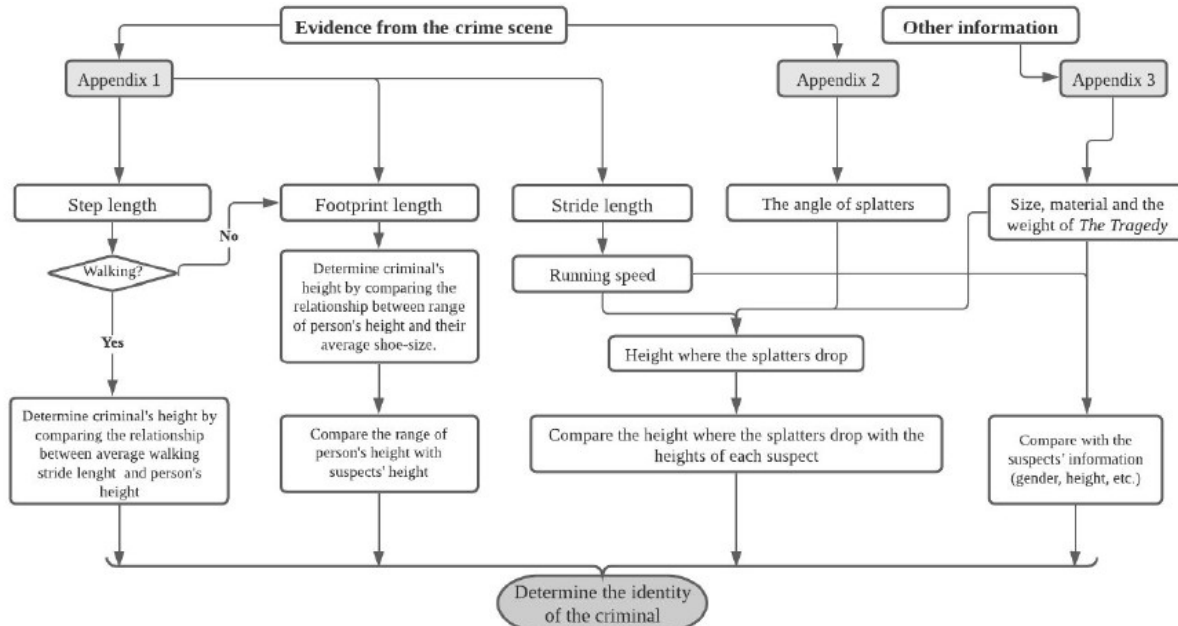


Figure 3. The problem-solving process used by Team A to identify the suspect.

Following this plan, team A produced mathematical analysis to estimate the size of the footprint, the stride and the speed of the criminal, the height where the paint spatter came from, and the weight of the framed painting. Using these data, they were able to identify a suspect most likely responsible for the theft.

Team B used GeoGebra to perform an analysis of the provided image of multiple footprints that were found on the floor of the gallery (Figure 4).

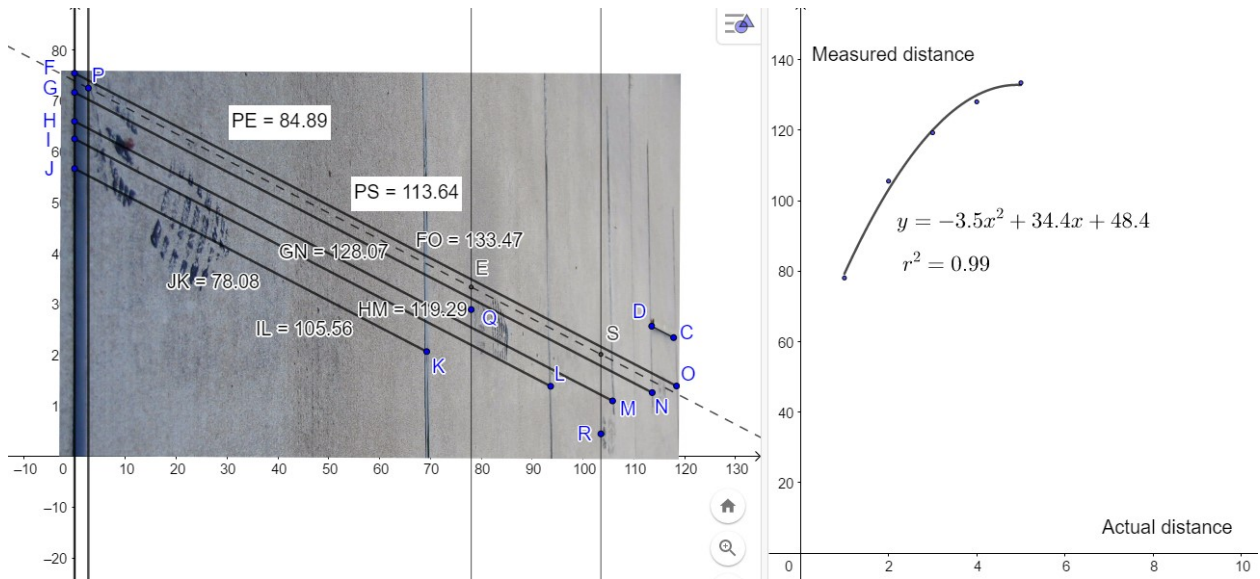


Figure 4. GeoGebra analysis of the footprints image by Team B. The black tape is placed at 1-meter intervals.

Here PSTs measured distances between the black tape on the image and then plotted the measured distance against the actual distance. They then used regression analysis and found a quadratic function to model this relationship. Using this function, students found the distance between the base of one footprint to the base of the next footprint, which led them to an estimation of the height of the suspect.

In order to understand the paint spatter, Team C conducted an experiment with spattering water onto a paper to find a spatter pattern similar to that in the digital evidence (Figure 5).

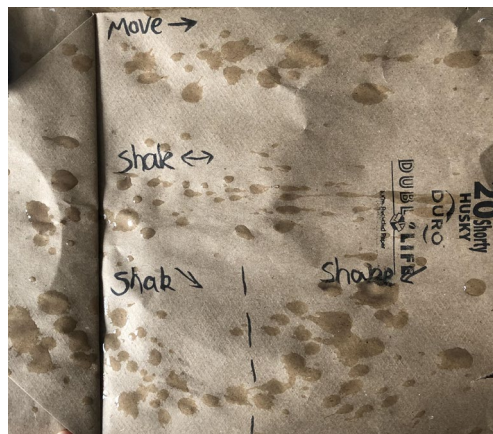


Figure 5. Water spatter experiment by Team C to understand paint spatter evidence

Team D used physics to model the shape of paint drops on the floor (Figure 6) and properties of right triangles to determine the height from which the paint was spattered (Figure 7) in order to estimate the height of the suspect.

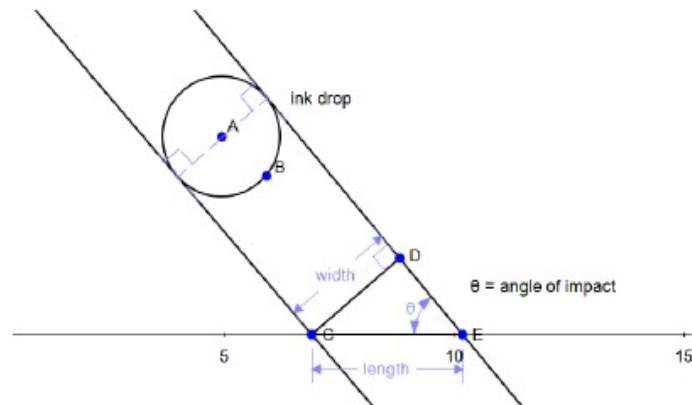


Figure 6. Physical model used by Team D

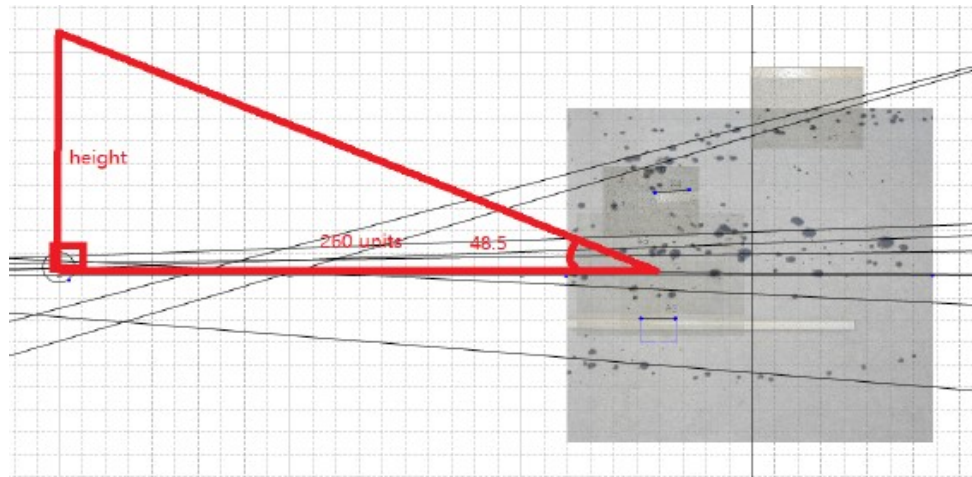


Figure 7. Paint spatter analysis by Team D

When time came to share their solutions, I wanted PSTs to engage in peer review in a meaningful way. Therefore, each of them was assigned to a suspect identified by a peer as a ‘defense attorney’. The defense attorney had the following task:

Your job is to help your client, and therefore, you will be looking for flaws in the quantitative evidence provided by the forensics team. After a careful review of the documents provided to you by the forensic team, submit a summary of your comments and recommendations for the judge based on your analysis of these documents.

This task created another problematic situation for PSTs that engaged them in much deeper analysis of the work developed by their peers. Here is an excerpt from one of the peer reviews:

Your Honor,

I respectfully ask that you disregard the forensic report’s findings based on the following reasons:

1. First and foremost, there is a grave inconsistency in this report. The scientists affirm they have calculated a step length of 1.07 m. Later, they affirm the stride length of 2.86 m. This can mean one of two things. Either the suspect has a left step of 1.07 m

and a right step of 1.79 m, or the scientists miscalculated. If they made an error here, who is to say where else there are hidden mistakes? Note that Frothingan (2018) suggests the stride is usually 2 times the step length (meaning, the left and right steps are roughly the same size).

2. The report includes the calculations for only one footprint. In case an average was calculated (like in the case of the paint spatter), the scientists could have a stronger case.
3. This report simply dismisses the women suspects, affirming they are not strong enough to carry a heavy painting. This sort of stereotypical profiling is often deceptive. What if the women suspects practice CrossFit? Further, the report assumes that the footprints were left by the thief by chance, when everyone knows any woman could be smart enough to plant evidence to point to other suspects.
4. The calculations of the paint spatter assume that the suspect was running and dropped paint on the floor by accident. However, it could be that a suspect was actually dispersing paint deliberately with horizontal and vertical initial velocity.

Given the reasons above, please consider releasing my client from preventive custody. The prosecutor's case is very weak. (a member of Team B)

4. Analysis of PST-generated PBL units

As part of the course final project, PSTs engaged in designing their own problematic situations. The scenarios covered a wide range of topics, such as planning a school concert during pandemic (Team F) to address middle school mathematics standards, park design competition (Team C) and building accessible entryways to public schools (Team B) to address high school geometry standards, designing a soccer shooting machine (Team A) and planning an indoor intelligent traffic flow for social distancing (Team E) to address high school algebra standards, and Coca-Cola bottle design competition for Olympic games (Team D) to address calculus topics. Each team submitted written PBL package that included teacher's materials and student package. Teacher's materials had to include instructional objectives and research-based rationale for the PBL-based unit, emphasizing PBL design in relation to meeting instructional objectives, justification of the topic relevance and value for students, and explanation of connections of mathematics knowledge and skills with real-world problems or natural or social science applications. PSTs also had to justify essentiality and affordances of technology integrated into the PBL experience. The teacher's materials also included unit teaching plan and assessment strategies. Student package had to include PBL scenario with suggested reading and Internet resources, expectations for the final product, and detailed assessment instrument of the final product.

The team's PBL packages were assessed using 4-point rubric that measured quality of PBL package in five different domains: research-based rationale for PBL (RBR), analysis of real-world connections of mathematics knowledge in the PBL unit (RWC), technological essentiality and affordances (TEA), pedagogical approach to support and assess student PBL experience (PA), and quality of scenario/problematic situation according to PBL evaluation criteria described above in section 3.1 (QPS). Table 2 represents results of this analysis for the six team projects. These results indicate that the most difficult aspect of designing PBL experience for this group of PSTs was meaningful integration of technology and providing research-based rationale for the use of technology by students. For example, Team A included requirement for the students to use GeoGebra in order to find a mathematical model for the trajectory of a soccer ball, but they did not include any explanation of how students will be using the software and they did not justify why technology was

essential for problem-solving. Team E just made a generic statement that students will use technology to “develop the plan of the situation, create a model, and perform calculations aligned with the social distancing guidelines”. Team F included requirement for students to use GeoGebra “to sketch the school gymnasium, create a location of each audience seat, and calculate the distance between each seat” but did not explain how GeoGebra features could be used for this task.

Table 2. Descriptive statistics for PSTs’ written PBL packages

	RBR	RWC	TEA	PA	QPS	Group Mean	SD
Team A	4.0	4.0	1.0	2.0	4.0	3.0	1.26
Team B	4.0	4.0	3.0	4.0	4.0	3.8	0.40
Team C	4.0	4.0	3.0	3.0	4.0	3.6	0.49
Team D	4.0	4.0	4.0	4.0	4.0	4.0	0.00
Team E	2.0	4.0	2.0	2.0	4.0	2.8	0.98
Team F	3.0	4.0	2.0	3.0	4.0	3.2	0.75
Class mean	3.5	4.0	2.5	3.0	4.0	3.4	0.58

The pedagogical aspects of integrating PBL into curriculum-based instructions were also challenging for majority of PSTs. Specifically, development of authentic assessment that was aligned with unit instructional objectives, and provided opportunities to assess individual students within groups was the most challenging task for the same teams. Team F failed to explain the expectations for the student final project and all three teams (A, E and F) developed rubrics that did not align with unit objectives and were mostly measuring completion and organization of the student project rather than their learning of mathematics.

However, the results also demonstrate that all teams of PSTs were able to develop an engaging problem-based scenario for the selected instructional unit, that represented good approximation of real-world problem, provided opportunities for student inquiry, and was sufficiently open-ended and ill-structured. All problematic situations were curriculum-based and the teacher’s materials included thorough synthesis and analysis of the connections between mathematics knowledge and skills and real-world applications.

5. Conclusions

While the focus of the course was mostly on pedagogical aspects of teaching, my experience with PSTs demonstrated that they got deeply engaged in problem solving and actively participated in class discussions and activities, despite the physical separation of remote learning. The final projects developed by PSTs suggest that course design and structure supported their pedagogical content knowledge for using PBL approach in teaching mathematics online. These findings are also supported by PSTs’ comments from the course evaluations:

- *Experiencing the PBL approach first hand and reading about the PBL approach in scholarly journals that specifically discuss the pedagogy behind the different decisions made in using a PBL approach helped to prepare for using PBL in the classroom.*
- *It was great to learn about problem-based learning. It was also interesting to work on PBL as students and also create our own PBL as teachers. It was motivating.*
- *PBL approach encourages students to go beyond the rote memorization aspect of learning. It motivates students to learn more. Students can answer the following question: why am I taking this course?*

- *PBL makes me understand that students could learn by themselves, and teachers could play more roles not only the "teacher".*

At the same time, the study identified challenges PSTs had in integrating PBL into teaching mathematics that related to use of instructional technology and assessment of individual students within collaborative settings of PBL approach. Future studies are needed to examine how PSTs could be supported in online pedagogy courses to develop skills necessary to effectively integrate technology and design assessment tools appropriate for PBL classrooms.

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Effectiveness of Problem-Based Learning Strategy in Improving Academic Performance of Grade 10 Students in Mathematics

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Abstract: *Evidence of poor performance in mathematics by secondary school students highlights the fact that the most desired technological, scientific and business application for mathematics cannot be sustained. This makes it paramount to seek a strategy for teaching mathematics that aims at improving its understanding and performance by students practically. Problem-Based Learning as a method of teaching may be used to accomplish the instructional roles of learning basic facts, concepts, and procedures, as well as goals for problem-solving. Thus, the aim of the study reported here was to ascertain the extent to which the Problem-Based Learning (PBL) strategy would affect the achievement and retention of students when teaching Mathematics concepts as applied in the online learning in the Philippine secondary schools. The subject of this research was the twenty Grade 10 students of Calbeg National High School who were diagnosed to have difficulty in solving probability problems during the third quarterly examination of this School Year 2019-2020. All the needed data for the study were drawn from the questionnaires taken from the Learner's Material of Grade 10 and other reference books in Mathematics. Frequency and t-test were used to treat the academic performance of the Grade 10 students in the pretest and posttest.*

1. Introduction

The development of students' Mathematics understanding is generally accepted to be one of the major goals of K-12 mathematics teaching. Mathematics is a branch of mathematics in which symbols (usually letters) represent unknown numbers in mathematical equations [10]. Over the years, there have been students who have not performed well on math testing as they have in the classroom. Students' experiences with Mathematics begin with the acquisition of knowledge about procedures or operations that are used in dealing with Mathematics situations. Topics in Mathematics are taught for its usefulness in other branches of mathematics, and in the generalization of scientific truth, its power and verification of results more simply and satisfactorily; and its practical value in trades and industries [1]. Mathematics provides a conceptual foundation for the understanding of other concepts that students encounter in the school mathematics curriculum. The importance of this area of mathematics has been underlined by the increasing attention the teaching and learning of Mathematics have received over the past decade from teachers and researchers alike. Children's understanding of Mathematics concepts begins in the early years of their school life and continues throughout their mathematics learning experiences in high school and beyond. Mathematics also provides an effective way of expressing complicated relations and as a good instrument for mental training. Mathematics was chosen because it inculcates the power of analysis and provides a good instrument for mental learning [1].

The ubiquity of the subject matter of Mathematics in the K-12 mathematics curriculum further attests to its critical role in helping students develop an appreciation

of links that exist among other topics in mathematics. Indeed, this issue has been given considerable attention to the agenda of major curricular documents [5].

The present-day teaching and learning of mathematics are far from being satisfactory. [8] observed that mathematics is one of the most poorly taught, widely hated and poorly understood subjects in our schools. [6] had blamed this on several factors, which range from the students' perception that mathematics is difficult, shortage of qualified mathematics teachers and lack of mathematics laboratory. The persistent poor performance of secondary school students in mathematics examinations both in teacher-made tests and external examination is now a global issue. The students' low performance in mathematics is an indication of low mastery of the subject. This poor result calls for serious concern and this has made researchers in mathematics education to consider several factors. One of the factors, as examined in this study, is the appropriate method of teaching.

[2] noted that very little work has been done on how achievement in mathematics can be improved by focusing attention on the students from which efforts at improvement should emanate. Others, according to Ali have suggested the improvement of the cognitive demand levels of the secondary school curriculum in mathematics. Many non-professional and in-experienced teachers present topics in mathematics to the student in such ways that the students find it difficult to grasp some mathematics concepts [3]. According to [4] many teachers cling to traditional methods in which answers to the previous day's home works are first given, then the teacher-directed explanations are used to present materials for the new lesson. The powers of thinking and understanding are thus not developed in the students. One of the many strategies that have the potentials to put students at the center of their learning is through Problem-Based Teaching. Today, it is recognized that every person must be empowered to suggest possible explanations, to propose ways to test personal or class, to collect and interpret data obtained, to communicate the process and results to others.

The use of Problem-Based Learning in the mathematics classroom with the aid of technology may enhance the quality of mathematics teaching and learn in the Philippines. Problem-Based Learning as a method of teaching may be used to accomplish the instructional roles of learning basic facts, concepts, and procedures, as well as goals for problem-solving. The purpose of this study is to shed light on this issue. Thus, the aim of the study reported here was to ascertain the extent to which the Problem-Based Learning (PBL) strategy and the use of technology to employ it would affect the achievement and retention of students when teaching Mathematics concepts in the Philippine secondary schools.

2. Conceptual Framework

With Problem-Based Learning (PBL), learning begins with a problem to solve, and the problem is posed in such a way that the students need to gain new knowledge before they can solve the problem [12]. PBL as an instructional strategy based on constructivism is the concept that students construct their understanding by relating the concrete experience to existing knowledge where the process of collaboration and reflection are involved. PBL is generally based on ideas that originated earlier and nurtured by different researchers like Dewey, Bruner, Piaget, Ausubel, Novak, and Hanesian [9].

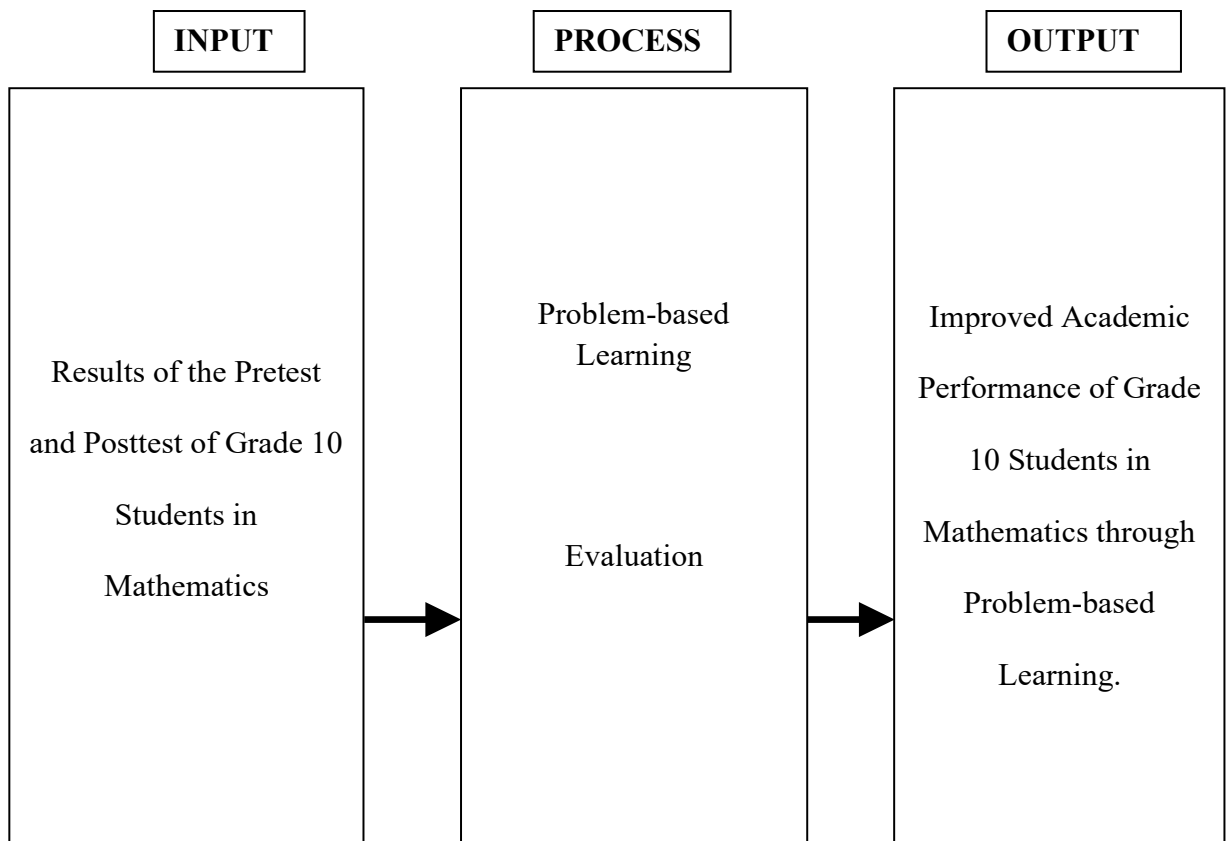
In Problem-Based Teaching, the teacher acts just as facilitator, rather than a primary source of information or dispenser of knowledge. [12] argued that within Problem-Based Learning environments, teachers' instructional abilities are more critical than in the traditional teacher-centered classrooms. Beyond presenting mathematical knowledge through the use of a PowerPoint presentation to the students,

teachers in Problem-Based Learning environments must engage students in marshaling information and using their knowledge in applied sand real settings.

In the Problem-Based Learning model, the students turn from passive listeners of information receivers to active, free self-learner and problem solvers. It also shifts the emphasis of educational programs from teaching to learning. It enables the students to learn new knowledge by facing the problems to be solved instead of feeling boredom. Problem-Based Learning affects positively certain other attributes such as problem-solving, information acquisition, and information sharing with others, group works, and communication, etc. Again problem-solving is a deliberate and serious act, involves the use of some novel method, and with the aid of technology, higher thinking and systematic planned steps for the acquisition set goals. The basic and foremost aim of this learning model is the acquisition of such information based on [7].

The teacher-researcher used the Independent–Dependent variable. The independent variable was the use of Problem-Based Learning. The dependent variable was the performance of the Grade 10 students in Mathematics

Figure 1.1 The Paradigm of Study



3. Statement of the Problem

This study focused on the improvement of students' academic performance in Mathematics using problem-based learning to Grade 10 students of Calbeg National High School, Municipality of Malasiqui, Schools Division Office of Pangasinan I, during the School Year 2019-2020.

Specifically, this study aimed to answer the following research questions:

1. What is the academic performance of the Grade 10 students in the pretest and posttest?
2. Is there a significant difference between the academic performance of the Grade 10 students in the pretest and posttest?

4. Hypothesis

There is no significant difference between the levels of performance of the Grade 10 students in the pretest and posttest.

5. Scope and Delimitation

The respondents of this study handled by the teacher-researcher were the Grade 10 students which consists of twenty students out of the eighty-one students. The respondents were identified having a difficulty in Mathematics as identified during their third quarterly examination. It is a heterogeneous group of students composed of five male students and fifteen female students.

6. Research Design

The teacher-researcher used the experimental research design. An experimental research design was employed to improve the Grade 10 students' academic performance in Mathematics using problem-based teaching applied in technology. The teacher presented the lesson thru powerpoint presentation and video lesson and the students will access using their gadgets. The teacher-researcher used one group pretest-posttest design in his study. Pretest and posttest questions were drawn on the topic during the third quarter. It focuses mostly in solving probability.

The researcher used the experimental method of research since it is the only method of research that can truly test hypothesis concerning cause-and-effect relationships. It represents the most valid approach to the solution of educational problems, both practical and theoretical.

Also, according to [7] when properly applied, the experimental research design is the best type for testing hypotheses about cause and effect relationships. It contends the experimental research determines the impact of an intervention on an outcome for participants in a study.

According to [10] the experimental research design is a controlled procedure that sees the manipulation of an independent variable (IV) to observe or measure its effect on a dependent variable. It contends the experimental research determines the impact of the intervention on an outcome for participants in a study. Experimental design allows one to make causal inferences about relationships among variables.

7. Sources of Data

The study was conducted in Calbeg National High School, Municipality of Malasiqui, Schools Division Office of Pangasinan I. The research had spanned from March to May. The respondents of this study handled by the teacher-researcher were the Grade 10 students which consists of twenty students out of the eighty-one students. The respondents were identified having a difficulty in Mathematics. It is a heterogeneous group of students composed of five

male students and fifteen female students. These students were selected based on their scores on the said examination.

8. Instrumentation and Data Collection

The teacher-researcher asked permission to conduct the study from the office of the School Principal. The intervention used is Problem-Based Learning which was employed using technology platform like video lesson and PowerPoint Presentation after the conduct of pretest followed by a posttest. Pretest and Posttest was sent to the students thru the created group chats.

9. Tools for Data Analysis

Responses of the respondents were tallied, tabulated and analyzed both descriptively and inferentially, to answer the problems raised in this study;

Problem 1 dealt with the extent of the performance of Grade 10 students before and after the use of Problem-Based Learning in-class activities and performance presented in the PowerPoint. This was treated using frequency and percentages.

Problem 2 dealt with the significant difference between the extent of performance of Grade 10 students in their Mathematics class activities and lessons before and after the use of Problem-Based Learning. This was treated using a t-test.

10. Results and Discussions

Table 3.1 presents the academic performance of the Grade 10 students in the pretest and posttest. The table discloses that the performance of the Grade 10 students has a mean of 13.1 in the pretest and 17.75 in the posttest.

Table 3.1 *Academic Performance of the Grade 10 students during the Pretest and Posttest*

Students	Pretest	Posttest
1	11	17
2	18	25
3	11	15
4	11	15
5	12	15
6	14	16
7	9	19
8	10	16
9	19	20
10	14	15
11	10	18
12	14	24
13	16	17
14	11	16
15	13	17
16	18	20
17	15	16
18	8	17
19	11	17
20	17	20
Mean	13.1	17.75

The students' low performance in Mathematics is an indication of low mastery of the subject. This poor result calls for serious concern and this has made researchers in mathematics education to consider several factors. One of the factors, as examined in this study, is that of the appropriate method of teaching.

This suggests that the Problem-Based Learning (PBL) students have developed competence in essential skills of numeracy and a deeper understanding of the content knowledge of learned materials compared to when no intervention is being employed yet.

Significant Difference between the Academic Performances of Grade 10 Students in the Pretest and Posttest

Table 3.2 presents the significant difference between the academic performance of the Grade 10 students in the pretest and posttest.

Table 3.2. *Significant Difference Between the Academic Performance of the Grade 10 students in the Pretest and Posttest*

Academic performance	N	Mean	t	t-crit at 0.05	
Pretest	20	13.1			
Posttest	20	17.75		- 6.94	-1.729

The most important achievement of a teacher is to help his/her students along the road to independent learning and the aid of technology. Presenting the students with a problem, allow them to take risks, to adopt new understandings, to apply knowledge, to work in context and to enjoy the thrill of being discoverers. [11] In those classrooms in which the Problem-Based Learning strategy is used for the instructional process, the students take much more responsibility for their learning. They have become independent and long- life students and can continue to learn in their whole life.

It appears from Table 2 that the t-computed value of -6.94 is beyond the t-critical value of -1.729 at 0.05 level of significance with 19 degrees of freedom (df), The hypothesis which states that there is no significant difference between the academic performance of the Grade 10 Students in the Pretest and Posttest is therefore rejected.

This means that students taught with problem-based learning have achieved better as compared when there is no intervention yet. Students made significant progress in learning outcomes by being stimulated with realistic questions. They gradually became more involved in the instruction and feel the learning was easy and fun. The findings revealed the efficacy of the use of PBL (Problem-Based Learning) in enhancing students’ achievement in Mathematics. Thus, students in groups can find ways to develop real-life problem-solving capabilities and develop the competencies to become self-guided students.

The findings corroborate that of [8] from Pakistan, [10] from South Africa as well as [11] in Nigeria, who all attested that students’ learning outcomes were observed to be better than those with ordinary learning approaches.

11. Conclusions

Based on the findings the following conclusions were drawn:

1. At the implementation of Problem-Based Learning (PBL), the instructional activities became more interesting to students. The learning method changed from static to dynamic. It is evident from the findings of this study that the use of a Problem-Based Learning strategy could provide a good way for students to learn Mathematics.
2. In the process of PBL, students were found to obtain key knowledge, skills, and competences by cooperating with others to collect information, share opinions, and select problem-solving plans.
3. Problem-Based Learning enhances students’ academic performance in Mathematics. A student who is exposed to this type of strategy is more likely to possess a meaningful

in-depth knowledge of the content area. Students were able to organize their thoughts in an orderly manner that is essential for problem-solving and the acquisition of basic practical skills in mathematics.

4. PBL should, therefore, be used as an additional teaching strategy to other traditional methods of teaching mathematics. This could help in improving students' performance in the subject.
5. Another remarkable finding related to the retention period is that students in PBL remember more of the acquired knowledge.

12. Recommendations

Based on the conclusions, the following recommendations are

1. If this method, proposed by this study, is adopted in Mathematics teaching and learning, it will boost the performance of students in skills acquisition, problem-solving ability, and development of the right type of attitude toward mathematics as a subject.
2. Mathematics teacher trainees should be trained on the use of a Problem-Based Learning strategy with the aid of technology;
3. Teachers of mathematics should use Problem-Based learning to improve the academic achievements of the students, and seminars and workshops should be organized for Mathematics teachers in elementary and secondary schools to employ Problem-Based Learning in the classrooms.

Acknowledgements: The researcher would like to extend his deepest sincerest gratitude to all the people who helped there in any manner, who have share the effort and knowledge in order to make this research a reality. To the **Grade 10** students of Calbeg National High School, for being the researcher's respondents; To my **family** especially my father and siblings, friends, and co-teachers, for their moral encouragement, financial assistance as well as spiritual support in every path the researcher takes. Above all, the researcher offers his greatest appreciation to **Our Almighty Father**, for giving the wisdom, strength, and knowledge in exploring things.

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Curved Patterns in the Graphs of PPTs

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Abstract: In a graph of all Primitive Pythagorean Triples (PPTs), with legs up to length 10,000, conic section-like patterns can be observed. We show that they are indeed parabolic curves, which follow in a natural way from the mathematics of the subject matter. This work lives at the intersection of Arithmetic, H.S. Algebra, and Analytical Geometry. It is easily accessible by students. Computer programs for Dynamic Geometry (*Sketchpad*, *GeoGebra*) and *Mathematica* (or *Maple*) were used to build the graphs.

1. Introduction

A *Pythagorean Triple* (PT) is a set of 3 positive integers (a,b,c) , which satisfy the Pythagorean Equation $a^2+b^2 = c^2$. The numbers a , b , c are associated with the sides of a right triangle ΔABC [5].

For example, $(3,4,5)$ is a Pythagorean Triple, since, by the Pythagorean Theorem, the right triangle ΔABC with sides of length 3, 4, and 5, satisfies $3^2+4^2 = 5^2$.

Given a PT (a,b,c) , if the integers a , b , c are relatively prime (no common factors other than 1, so $GDC\{a,b,c\} = 1$), then call the triple (a,b,c) a *Primitive Pythagorean Triple* (PPT).

For example $(3,4,5)$ is a PPT, but the PT $(6,8,10)$ is not a PPT, since all the terms are even.

We wish to analyzing a graph of PPTs, which is construct as follows. Given a PT (a,b,c) , $a < b < c$, with associated right triangle ΔABC , there is a way to graph it which allows us to compare different PTs. Choose the ordered pair (a,b) and graph it in the 1st quadrant of the xy -plane, Figure 1.1.

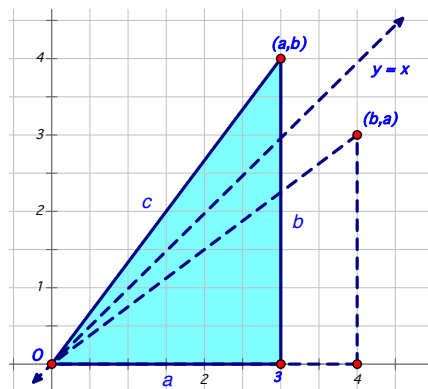


Figure 1.1 The Graph of a PPT

Then the right triangle ΔABC with sides a,b,c is congruent to the triangle formed by the origin O , the point $(a,0)$, and the point (a,b) . Thus the point (a,b) corresponds to ΔABC .

Another congruent copy of ΔABC is provided by the graph of the point (b,a) . The point (b,a) is the reflection of (a,b) about the line $y = x$.

We assumed that $a < b < c$, when referring to the PT (a,b,c) , but occasionally it will happen, by way of an argument, that we have the PT (b,a,c) , with $a < b < c$, as happened in the triangle figures just above. The line $y = x$ divides the first quadrant into two regions, one which contains the points (a,b) , $a < b$, and one which contains the points (b,a) , $a < b$. Since the points (a,b) and (b,a) are symmetric about the line $y = x$, we call (b,a) the *reflected point* of (a,b) .

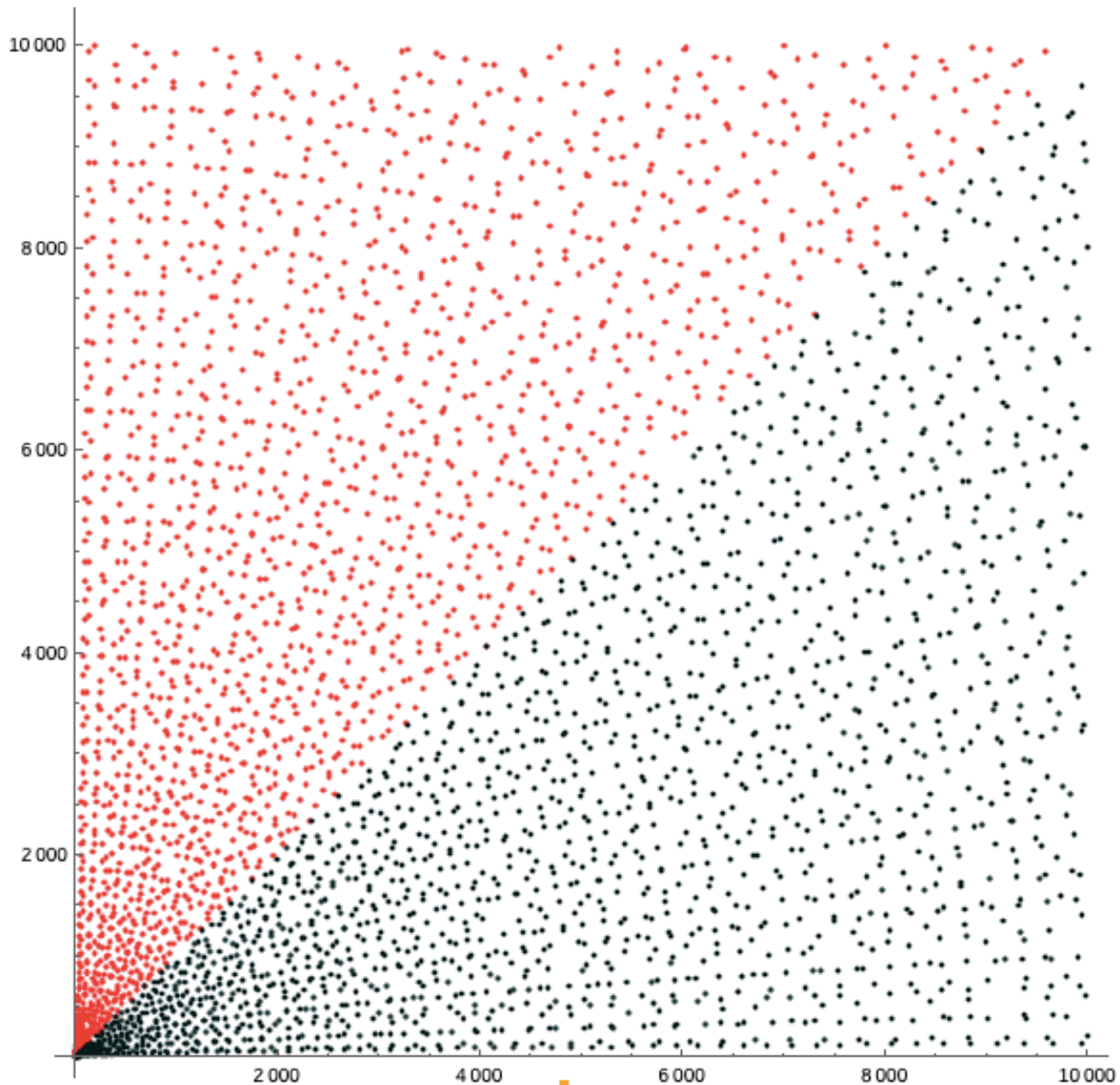


Figure 1.2 The Main Graph of all PPTs with $0 < a < b < 10,000$

Observe that if (a,b,c) is a PPT, then one of a, b must be odd, and the other must be even, while c is always odd. Also, it cannot happen that $a = b$, or that a or $b = 1$ [5].

If we list all PPTs with $0 < a < b < 10,000$, and graph them and the reflected points on *Mathematica* [9], using the graphing method above, the amazing graph shown in Figure 1.2 is the result [6]. An identical graph was constructed by R. Knott [3]. Both graphs were constructed using *Mathematica*, but using different codes.

The red region is the set of points (a,b) , where $a < b$, and the black region is the set of reflected points (b,a) . Thus the two regions are symmetric about the line $y = x$. We call Figure 1.2 the Main Graph.

2. Analysis of the Main Graph

In order to analyze this graph, consider the following set of the first 18 PPTs ordered by the size of the short leg a .

3, 4, 5	5, 12, 13	7, 24, 25	8, 15, 17	9, 40, 41	11, 60, 61
12, 35, 37	13, 84, 85	15, 112, 113	16, 63, 65	17, 144, 145	19, 180, 181
20, 21, 29	20, 99, 101	21, 220, 221	23, 264, 265	24, 143, 145	25, 312, 313

Those PPTs which are in **bold** print are meant to attract your attention. These particular 12 PPTs all have terms “ a ” which are odd numbers, and they also have the form $(a, b, b+1)$, that is b and c are consecutive numbers, $c = b+1$.

It is given that these particular triples all satisfy the Pythagorean equation $a^2 + b^2 = c^2$, and the terms are obviously relatively prime, since consecutive numbers cannot have any common divisors. They also satisfy the equivalent equation:

$$b = (a^2 - 1)/2 \tag{2.1}$$

This means that a^2 (and thus a) must be an odd integer, so that b is then an even integer.

So if an arbitrary odd positive integer “ a ” is given, then the triple $(a, b, b+1)$ is a PPT, whenever b satisfies equation (2.1). Thus these PPTs all satisfy the form $(a, (a^2-1)/2, (a^2+1)/2)$, for “ a ” an odd positive integer. This determines a one-to-one correspondence between the odd positive integers “ a ” and those PPTs which have the form $(a, b, b+1)$.

Note that if any two integers are relatively prime, then listing a third number with them makes a relatively prime list of three numbers.

The above formulas for $(a, b, b+1)$ above are not a new result, they are well known. According to Proclus (410-485 AD), these PPTs were known to the Pythagoreans (570-495 BCE), and perhaps before [1]. However, this does not seem to prevent the result from being ‘rediscovered’ occasionally.

Using *Sketchpad* [7], graphs of these PPTs in the xy -plane are obtained by graphing the set of points $(a, (a^2 - 1)/2)$, and the reflected points $((a^2 - 1)/2, a)$, for “ a ” an odd positive integer.

For example, the odd integer 3 determines the points $(3,4)$ and $(4,3)$, and the odd integers 5, 7, 9, ..., 25, determine the points corresponding to the PPTs given in bold in the list above.

A graph of these points and their reflected points in the xy -plane in the range $0 < a, b < 120$ is given in Figure 2.1.

It suggests parabolic shaped curves about the positive x and y axes. The red points have the form $(a, (a^2 - 1)/2)$, so they satisfy the parabola equation $y = (x^2 - 1)/2$, and the reflected points have the form $((a^2 - 1)/2, a)$, so they satisfy the reflected parabola equation $x = (y^2 - 1)/2$.

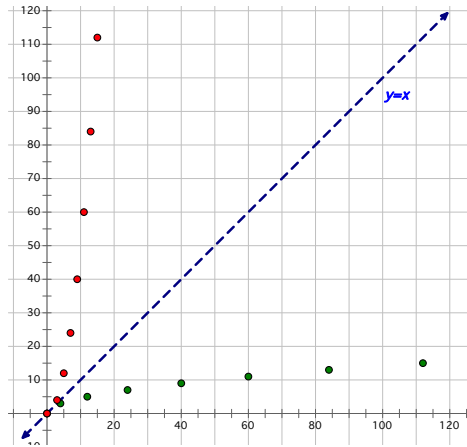


Figure 2.1 The Graph of PPTs for $d = 1$

The first parabola opens about the positive y -axis, with focus at O , and vertex $(0, -1/2)$. We denote this set of PPTs by $d = 1$. The second parabola opens about the positive x -axis, has focus at O , and vertex $(-1/2, 0)$, Figure 2.2. We will give a notation for it below.

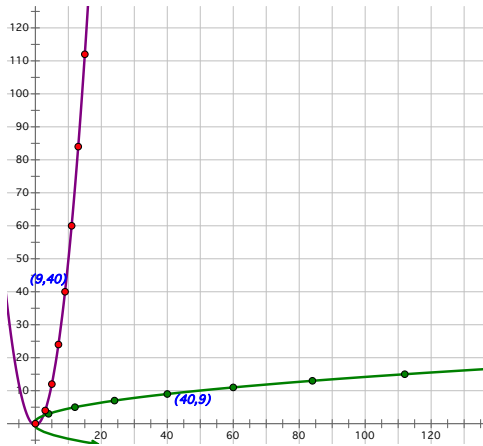


Figure 2.2 Graph of Parabolas for $d = 1$

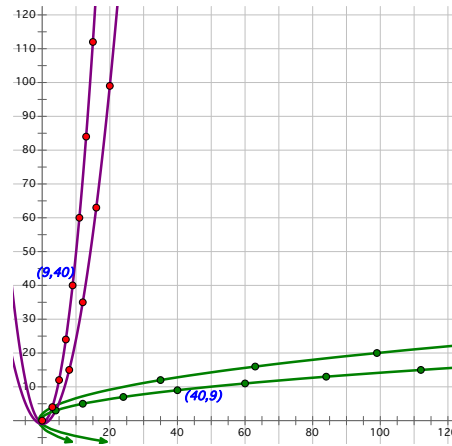


Figure 2.3 Graph of Parabolas for $d = 1, 2$

Look again at the list of the first 18 PPTs above, and notice that 5 of the PPTs have the form $(a, b, b+2)$. Denote the set of all such PPTs by $d = 2$, since the ‘difference’ $c - b = 2$. The set of PPTs which satisfy $d = 2$ also satisfy the equivalent equation:

$$b = (a^2 - 4)/4 \tag{2.2}$$

This means that a^2 , and thus a , must be even, and b must be odd. The numbers a, b, c are relatively prime, since the odd terms b, c , are 2 units apart, so they cannot have any common divisors. This form is not new, Proclus attributed this result to Plato (429-347 BCE) [1].

From the list of $d = 2$ PPTs above, note that “ a ” is even and $a = 0 \pmod{4}$. But if “ a ” is even, and not a multiple of 4, then $a = 2 \pmod{4}$, so the triple is a PT, but not a PPT, hence it is not on the list. The “ \pmod ” notation is short-hand from modular arithmetic. If the integer $z > 1$, and x, y , are integers, then the equation $x = y \pmod{z}$, means $x - y = zk$, for some integer k , or equivalently, $x = zk + y$, for some integer k .

Thus, if $a = 0 \pmod{4}$, then $a = 4k$, for $k > 0$, and $(a, b, b+2) = (4k, 4k^2 - 1, 4k^2 + 1)$, a PPT. But if $a = 2 \pmod{4}$, then $a = 4k + 2$, and $b = 4k^2 + 4k$, so $b+2 = 4k^2 + 4k + 2$. This PT is not a PPT, as all terms are even. So every other even number “ a ” determines a PPT. The curves which are for these PPTs with $d=2$, satisfy the parabola equations $y = (x^2 - 4)/4$, and $x = (y^2 - 4)/4$. They have vertex points at $(0, -1)$ and $(-1, 0)$, resp., and both have focus at the origin O , Figure 2.3.

In general, the parabolas for arbitrary values of $d > 0$ are given by the equations:

$$y = (x^2 - d^2)/2d \tag{2.3}$$

$$x = (y^2 - d^2)/2d \tag{2.4}$$

where “ a ” and “ d ” are either both odd or both even positive integers, and $d|a$. However we find that a lot of the values of “ d ” determine PTs, but not PPTs, see below.

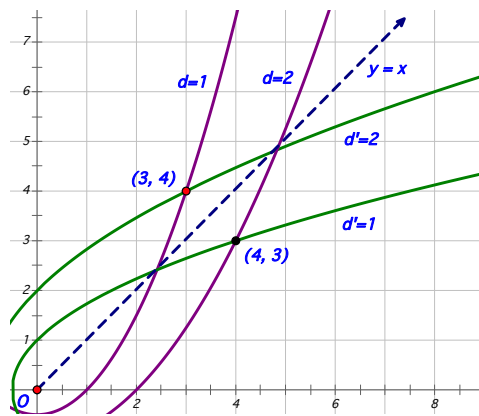


Figure 2.4 Graph of Parabolas for $d = 1, d' = 2$, and $d' = 1, d = 2$

There is another d -value associated with the PPT (a,b,c) , which is determined by the form (b,a,c) , namely $d' = c - a$. The point (a,b) (and (b,a)) will occur at the intersection of parabolas for d and d' . For example, the graph above, Figure 2.4, shows the point $(3,4)$ at the intersection of the parabolas labeled $d=1$ and $d'=2$, and the point $(4,3)$ at the intersection of the parabolas labeled $d'=1$, and $d=2$.

Any integer point (s,t) which is on a parabola $y = (x^2 - d^2)/2d$, equation (2.3), for some d will satisfy $t = (s^2 - d^2)/2d$, and (s,t,u) is a PT for that d , with $u = (x^2 + d^2)/2d$. Each integer point on the graph of the PPTs above is on a parabola for some value of d .

Note that the value $d = 3$ does not determine any PPTs, for all such PTs have a common value of 3 in their coordinates. The same thing happens for the d values 4, 5, 6, 7, 10, 11, and 12. The next PPTs, after $d = 1$ or 2, occur when $d = 8$ or 9, and these values have mixed results similar to those we found for $d = 2$.

3. Results and Conclusions

The list of d -values which determine PPTs begins as follows: 1, 2, 8, 9, 18, 25, 32, 49, 50, 72, 81, 98, ..., [2]. We call these numbers the *allowable values* of d . This list is in fact the OEIS sequence A096033. The graph of some of the representative points and parabolic curves for some of these d and d' values is shown in Figure 3.1.

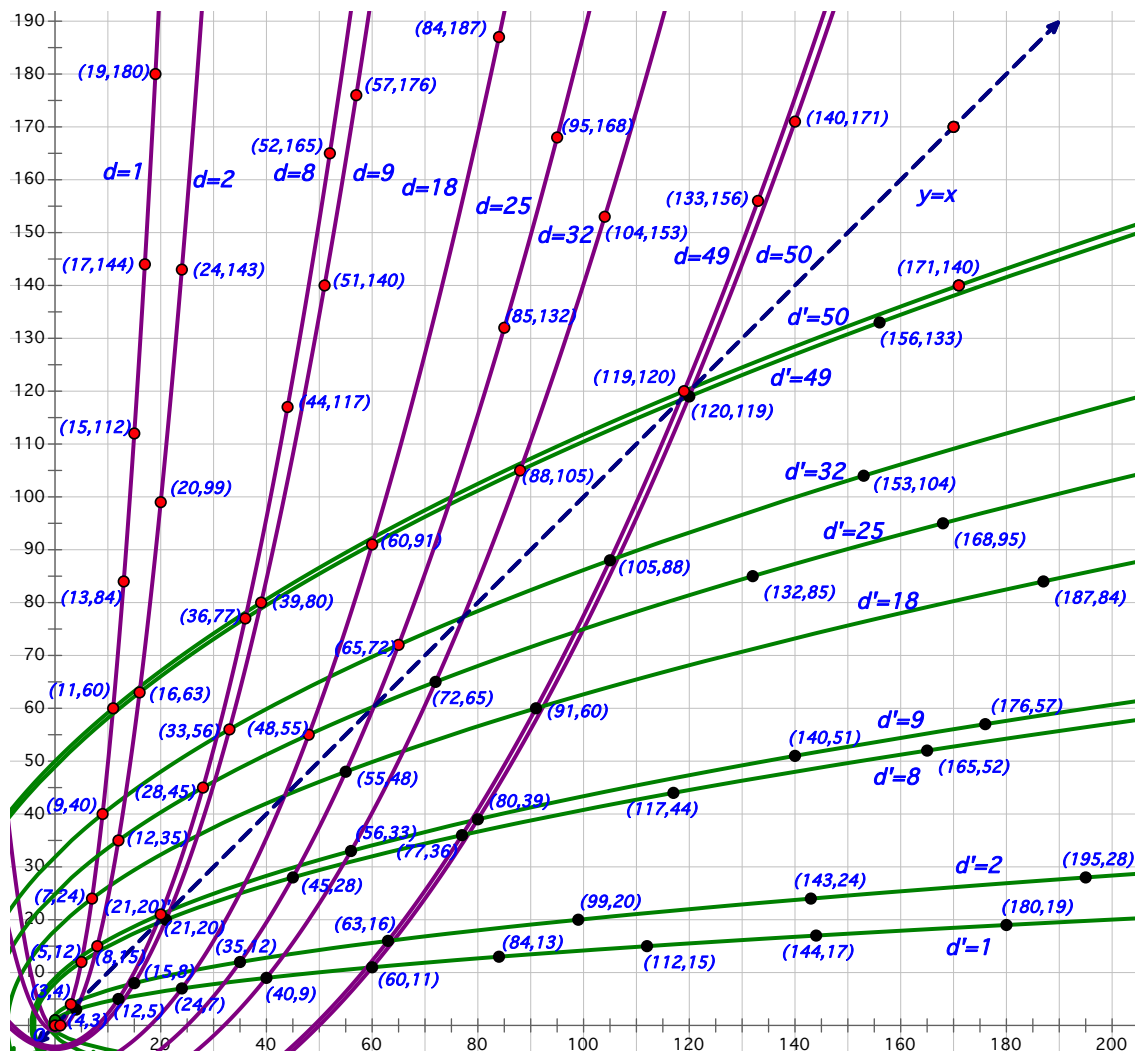


Figure 3.1 Representative Points and Parabolic Curves for Beginning d and d' Values

Comparing this graph with the Main Graph it becomes clear where some of the curved patterns originate. This is only a very small part of the lower left corner of the Main Graph, Figure 1.2, but the general picture is starting to become apparent.

These results are stated in the following Proposition on parabolas about positive x,y -axes with allowable d values.

Proposition 3.1 If (a,b,c) is a PPT with $a < b < c$, and values d, d' , such that $d < d'$, then (a,b) is a point at the intersection of the parabolas with equations $y = (x^2-d^2)/2d$, and $x = (y^2-d'^2)/2d'$, and (b,a) is a point at the intersection of the reflection of those parabolas about $y = x$, which have equations $x = (y^2-d^2)/2d$, and $y = (x^2-d'^2)/2d'$.

For the Main Graph, Figure 1.2, notice that there also appear to be parabolic curve patterns which open about the negative x,y -axes. These curves are mentioned in [3]. The equations for these curves are formed in a different manner from those above.

First consider the parabolic curves which open about the negative y -axis.

An example is the parabola with equation:

$$y = -x^2/(2 \cdot 3^2) + 3^2/2 \tag{2.5}$$

shown here in orange, Figure 3.2. This parabola has vertex point $(0, 3^2/2)$, and focus at O . It also contains the point $(3, 4)$.

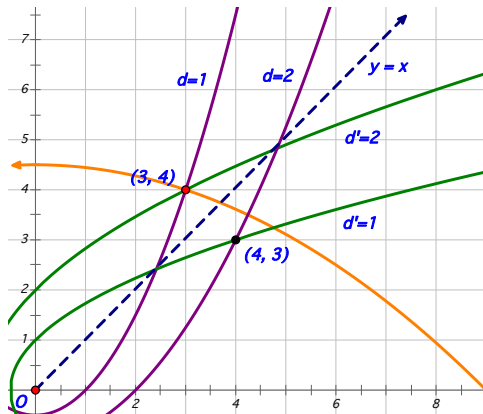


Figure 3.2 The Graph of Equation (2.5) in Orange

When studying the graphs in Figure 3.1 and Figure 3.2, notice that certain sets of points seem to be in a parabolic shaped pattern which opens downward. For example, the 4 points $(20,99)$, $(60,91)$, $(140,51)$, and $(180,19)$ appear to form a parabolic curve which opens about the negative y -axis. Note that the 1st coordinates of these 3 points are all multiples of 10. The LH point $(20,99)$ is from the triple $(20,99,101)$, and is on the parabola labeled $d = 2$. The equation for this parabola is:

$$y = -x^2/(2 \cdot 10)^2 + 10^2 \tag{2.6}$$

the vertex point is $(0, 10^2)$, and the focus point is $(0, -10^2)$.

For another example, consider the 3 points $(28, 45)$, $(56, 33)$, and $(84, 13)$. They also appear to be on a parabola which opens about the negative y -axis. The 1st coordinates here are all multiples of 28. The LH point $(28, 45)$ is from the triple $(28, 45, 53)$, and is on the parabola labeled $d = 8$. The equation for this parabola is:

$$y = -x^2/(2 \cdot 7)^2 + 7^2 \tag{2.7}$$

the vertex point is $(0, 7^2)$, and the focus point is $(0, -7^2)$.

These examples provide a pattern for the equations of the parabolas in the Main Graph which open about the negative y -axis. Given a series of points with *even* 1st coordinates all multiples of, say $2 \cdot n^2$, the general equation of these parabolas is:

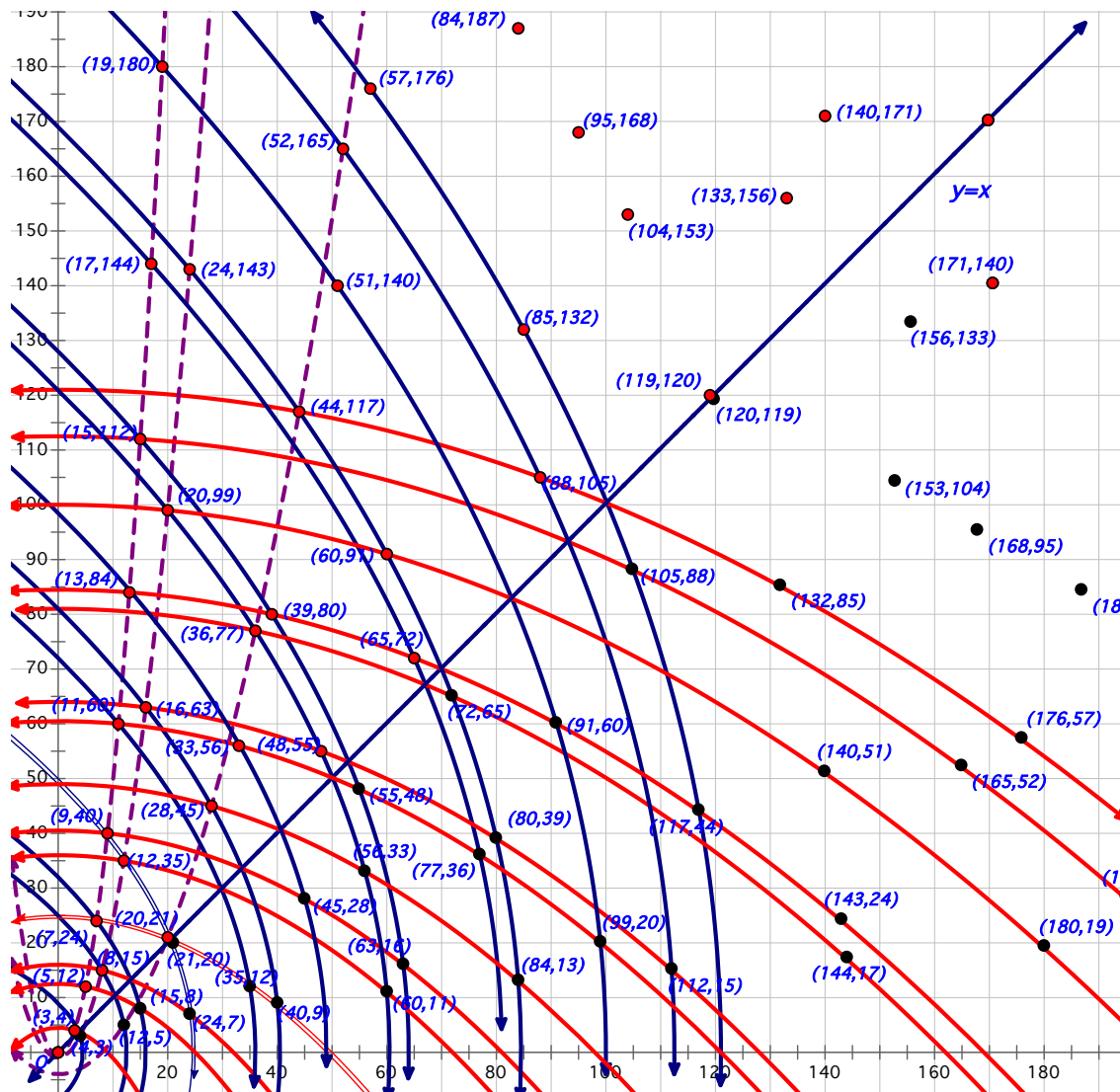


Figure 3.3 Examples of Graphs which Open about the Negative y -axis

$$y = -x^2/(2 \cdot n)^2 + n^2 \tag{2.8}$$

where $n = a/m$, for “ a ” the 1st coordinate in the LH point in the series, and $m = 2$, if this point is on the $d = 2$ parabola, and $m = 4$, if it’s on the $d = 8$ parabola. Thus every point from a PPT on the $d = 2$ and $d = 8$ parabolas (the middle and RH red dashed lines in the graph below) determines a parabola in this set. These curves are shown in orange in Figure 3.3.

If “ a ” is *odd*, as in the example with the point $(3,4)$ above, then the equation is derived as in the following example. The points $(13,84)$, $(39,80)$, $(65,72)$, $(91,60)$, $(117,44)$, and $(143,24)$, appear to be on a parabolic curve which opens about the negative y -axis. The 1st coordinates of these points are all multiples of 13, the LH point $(13,84)$ is on the parabola labeled $d = 1$, and it’s from the triple $(13,84,85)$. The equation for this parabola is:

$$y = -x^2/(2 \cdot 13^2) + 13^2/2 \tag{2.9}$$

It has vertex point is $(0,13^2/2)$, and the focus point is $(0,-13^2/2)$.

For another example, consider the points $(11,60)$, $(33,56)$, $(55,48)$, $(77,36)$, and $(99,20)$. They also appear to be on a parabola which opens about the negative y -axis. All of the 1st coordinates of the points are multiples of 11, the LH point $(11,60)$ is from the triple $(11,60,61)$, and is on the parabola labeled $d = 1$. The equation for this parabola is:

$$y = -x^2/(2 \cdot 11^2) + 11^2/2 \tag{2.10}$$

the vertex point is $(0,11^2/2)$, and the focus point is $(0,-11^2/2)$.

The general form for the equations of the parabolas which open about the negative y -axis, when “ a ” is odd, is:

$$y = -x^2/(2 \cdot a^2) + a^2/2 \tag{2.11}$$

where “ a ” is the value of the x -coordinate of the first LH point (a,b) on the parabola labeled $d = 1$, in the series of points being considered (the LH red dashed line) Figure 3.3.

Thus every point (a,b) , from a PPT point (a,b,c) , on the $d = 1$ labeled parabola determines a parabola in this set. These curves are shown in orange in the graph, Figure 3.3.

When these 2 sets of parabolas, for “ a ” even or “ a ” odd, are reflected about the line $y = x$, we have the corresponding parabolas for the d' -values. These parabolas open about the negative x -axis, and are shown in dark blue in the graph above. Not all of the curves which exist in this range are shown here.

These results are stated in the following Proposition on parabolas about negative x,y -axes.

Proposition 3.2 Let $(a_1,b_1), \dots, (a_k,b_k)$, be a finite sequence of points in the Main Graph such that the first coordinates a_1, \dots, a_k form an increasing sequence, and a_1 divides all of the other a_i , $i = 2, \dots, k$. If a_1 is odd, then (a_1,b_1) is on the parabola labeled $d = 1$, and the equation of the parabola which contains the points $(a_1,b_1), \dots, (a_k,b_k)$ is then $y = -x^2/(2 \cdot a_1^2) + a_1^2/2$. If a_1 is even, then (a_1,b_1) is on the parabola labeled $d = 2$ or 8 . The equation of the parabola which contains the points $(a_1,b_1), \dots, (a_k,b_k)$, is then $y = -x^2/(2 \cdot (a_1/m))^2 + (a_1/m)^2$, where $m = 2$, when $d = 2$, and

$m = 4$, when $d = 8$. The reflection of these parabolas about the line $y = x$ determines parabolas which open about the negative x-axis. In the case for $d = 1$, the equation of the reflected parabola is $x = -y^2/(2 \cdot a_1^2) + a_1^2/2$, and in the cases for $d = 2$ or 8 , the equation of the reflected parabola is $x = -y^2/(2 \cdot (a_1/m)^2) + (a_1/m)^2$, where $m = 2$, when $d = 2$, and $m = 4$, when $d = 8$.

That every parabola shown which opens about the negative y -axis has a point (a, b) from a PPT (a, b, c) on the parabolas labeled $d = 1, 2$, or 8 , follows from the observation that these parabolas are the closest ones to the y -axis for even or odd values of d .

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Using “Turtle Geometry” in the XXIst Century for teaching mathematics? Classic ideas and new software and hardware

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Abstract: In “Turtle geometry” there is a graphic cursor (the “turtle”) that obeys elementary orders related to the position and orientation of the turtle itself: move forward, move backward, turn right and turn left. That is, graphics are not based on a classic Cartesian reference system. Turtle Geometry is best known for its incorporation into Logo language. As monitors had no graphical capabilities in the 1960s, the movements ordered to the turtle from the first Logo versions were carried out by electromechanical devices connected by cable to the computers. After a great success in the '80s and '90s Logo has fallen into disuse, but there are also very modern visual computer languages based on the use of “blocks” for programming such as Scratch and Snap! Moreover, nowadays there are affordable programmable robots that use Turtle Geometry. Summarizing, wonderful powerful software and hardware that use Turtle Geometry, appropriate for teaching mathematics are available in 2021.

1. Introduction

The author has taught mathematics with ICT in teacher training programs since the late 1980s using computer languages including implementations of “Turtle Geometry”. He has also developed implementations of Turtle Geometry for different computer languages (e.g. *Maple*¹).

In Turtle Geometry (also known as “Turtle Graphics”) there is a graphic cursor (the “turtle”) that obeys elementary orders related to the position and orientation of the turtle itself. They are:

- move forward,
- move backward,
- turn right, and
- turn left.

That is, graphics are not based on a classic Cartesian reference system, and, consequently, the list of commands required to draw a certain pattern neither depends on where the pattern is to be allocated nor on its leaning. Another advantage with respect to working with Cartesian coordinates is that the trigonometric calculations regarding positioning are performed internally.

The Turtle Geometry applies constructionist ideas [1] and its range of possibilities is absolutely impressive, as can be seen in the seminal work [2].

Turtle Geometry is best known for its incorporation into *Logo* language [3]. For example, the image of Figure 1.1 is generated in *Logo* by typing FORWARD 200 RIGHT 90 FORWARD 100. Many regular and repetitive geometric designs (for instance some fractals) are very easy to program in *Logo*. A simple example can be found in Figure 1.2.

¹ All product names, trademarks and registered trademarks are property of their respective owners

Seymour Papert co-founded the MIT Artificial Intelligence Laboratory with Marvin Minsky in the early 1960s and *Logo* was introduced in 1967. It was a very powerful language that incorporated the latest trends: it was procedural and recursive, it handled lists, etc.



Figure 1.1 A *Logo* very simple drawing.

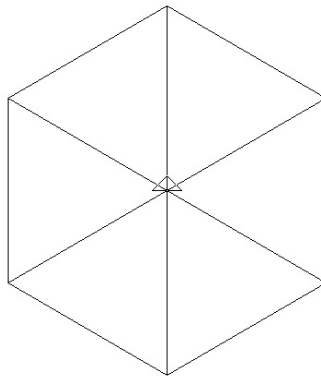


Figure 1.2 Drawing a regular hexagon and its three diagonals (that is, its six radii) in *Logo* is very simple nesting two REPEAT iterative loops (this way six equilateral triangles are drawn sharing a common vertex): REPEAT 6 [REPEAT 6 [FORWARD 200 RIGHT 120] RIGHT 60]

Meanwhile, “turtle robots” or “tortoises” were autonomous robotic creatures introduced by Grey Walter in the late 1940s [4,5]. As monitors had no graphical capabilities in the 1960s, the movements ordered to the turtle from the first *Logo* versions were carried out by electromechanical devices connected by cable to the computers [6,7].

After a great success in the '80s and '90s *Logo* has fallen into disuse, although there are exceptions like [8].

Nevertheless, there are nowadays very friendly and powerful free versions available, such as *FMSLogo* [9] (used for drawing Figures 1.1 and 1.2) and *UCB Logo (Berkeley Logo)* [10]. A

detailed comprehensive list of *Logo* language implementations, mentioning more than 250 dialects, can be found in [11].

2. Present situation (software)

Apart from *Logo*, there are very modern visual languages based on the use of “blocks” for programming that base their computer graphics upon Turtle Geometry, like:

- *Scratch* [12], a project of the Lifelong Kindergarten Group at the MIT Media Lab (Figure 2.1), and
- *Snap!* [13,14], presented by the University of California at Berkeley (Figure 2.2). Although less well known than *Scratch*, it has some advantages like the existence of a REPORT block that allows to easily implement recursive procedures.

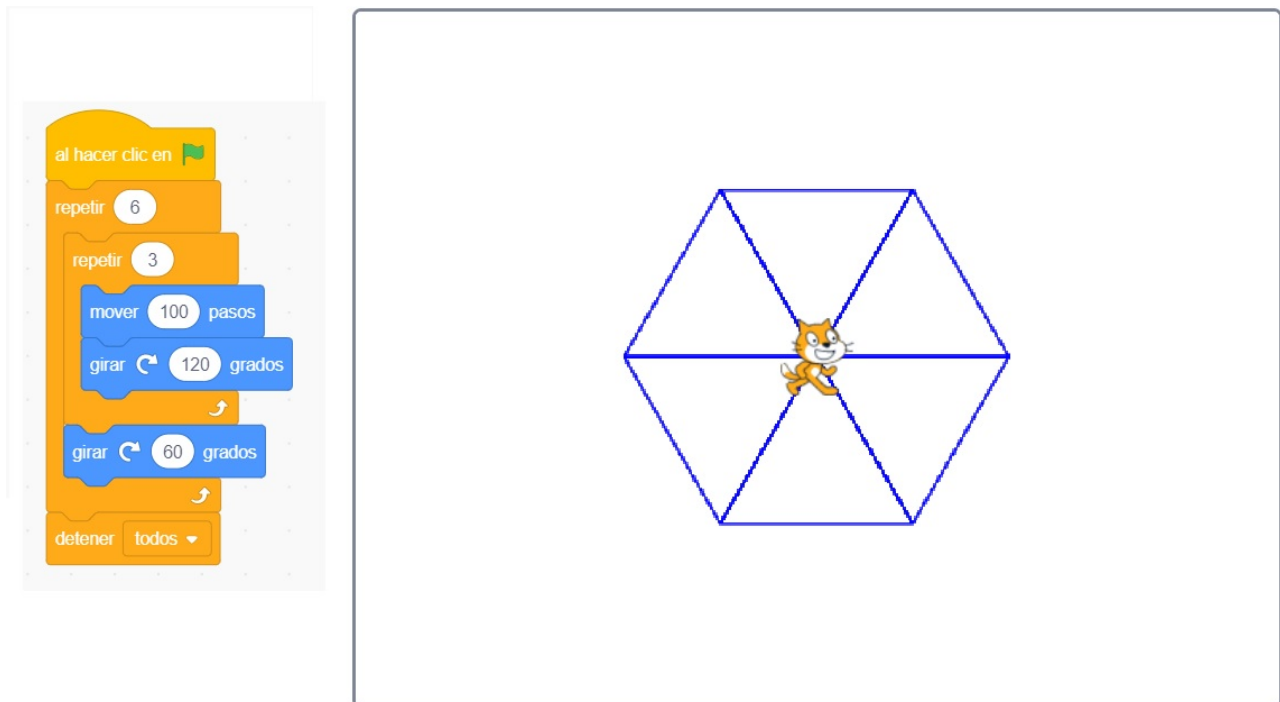


Figure 2.1 Drawing the regular hexagon and its three diagonals of Figure 1.2 with *Scratch 3* (in Spanish).

The use of blocks (that are grouped in “categories”, with different colors in *Scratch* and *Snap!*) has advantages for beginners with respect to traditional programming. For instance:

- the end user doesn’t have to remember the names of the commands as he/she only has to look for them in the corresponding category (that is intuitively chosen) and drag and drop them in the right position in the “programming window”,
- the colors help to identify the different types of commands,
- loops are clearly visible, without having to “indent” the code (Figure 2.1),
- ...

An expert programmer types faster than dragging and dropping blocks, but that is not the goal of these computer languages.



Figure 2.2 Drawing a row of square tiles with *Snap!*

A sophisticated example of the use of Turtle Geometry to draw fractal trees related to the spread of virus by this author (intended to visually explain it to children and to raise their awareness) can be found in [15] (*Scratch 3* version, see Figure 2.3) and [16] (*Maple* version, implemented on the Turtle Graphics implementation [17]).

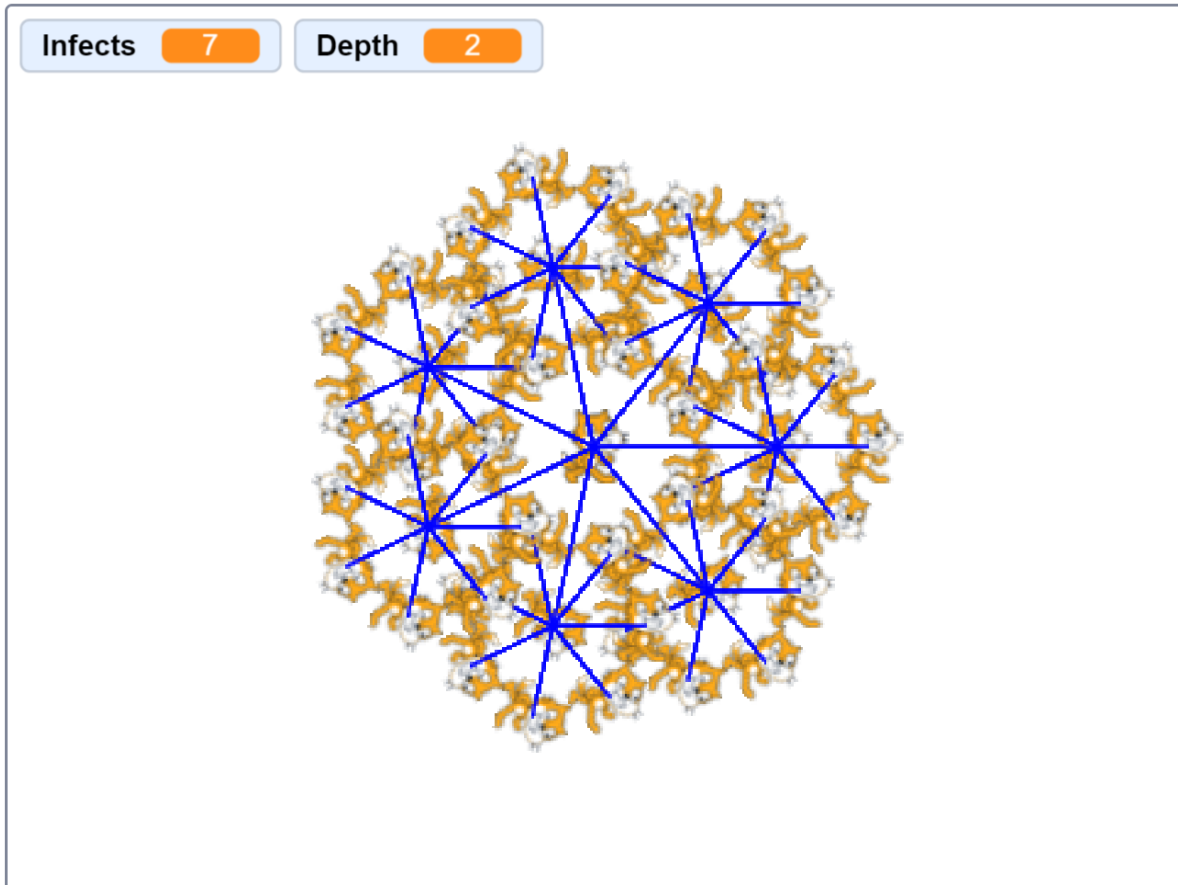


Figure 2.3 A figure illustrating virus propagation drawn with *Scratch 3*.

3. Present situation (hardware)

Nowadays, with the downsizing and cheapening of electronic chips, affordable programmable robots that use Turtle Geometry are available.

Already back in 1979 *Big Trak* was introduced. It was a six wheels programmable tank that moved using a reduced version of Turtle Geometry [18].

Much more recently a programmable car named *Pro-Bot* (Figure 3.1), incorporating a very close to *Logo's* implementation of Turtle Geometry was introduced. It has a display showing the program, it has connectivity with computers, it incorporates sensors, etc. [19]. It is very well suited for Primary and Secondary Education.

There are other similar programmable robots with a much simplified version of Turtle Graphics incorporating keyboards without letters or numbers. Four arrows (↑, ←, →, ↓) indicate Forward, Left, Right and Backwards, respectively, but they have no numerical input. Movements are restricted to 1 step forward or backwards and turns are restricted to 90 degrees clockwise or counterclockwise. For instance, a path like that of Figure 1.1 can be programmed typing:

↑↑↑↑→↑↑.

Therefore they are well suited for Early Childhood Education and can be used by children that still can't read and write but can count. Examples are *Bee-Bot/Blue-Bot* [20] and *Code & Go Mouse* (Figure 3.2).

These hardware somehow take us back to the early times of the mechanical turtles.



Figure 3.1 The keyboard and screen of *Pro-Bot*.



Figure 3.2 The keyboard of *Code & Go Mouse*.

4. The formal attitude with respect to the use of technology in the mathematics class

The CEMAT (Comité Español de Matemáticas –Spanish Mathematics Committee) has recently published a comprehensive report about the Primary and Secondary School mathematics curricula [21]. The need to connect teaching mathematics with programming as a positive synergic experience in problem solving is underlined ([21], p. 5). Although neither specific computer languages nor computer systems are explicitly mentioned, the use of different technologies such as graphic calculators, spreadsheets, dynamic geometry systems (DGS) and computer algebra systems (CAS) is recommended ([21], p. 15). In the more informal interview to some of its authors [22], the use of *Scratch* or *Snap!* in the mathematics class at Primary School level is specifically recommended

According to my experience, the authorities and the experts in Spain recommend the use of ICT but its use is not generalized and, when used, they are many times underused (for instance some Spanish high school textbooks propose to use the DGS *GeoGebra* to plot functions and perform some computations, but neither propose to use its dynamic geometric capabilities nor its computer algebra ones). My experience, after speaking with colleagues from different countries, is that the situation is not very different elsewhere.

5. The goal of this work. Our experience

I have thought for a long time that it is a pity to forget about using Turtle Geometry for mathematics teaching just because it is supposed to be outdated old stuff. This work gives a brief introduction to Turtle Geometry and a panoramic view of its present possibilities. The goal of this work is to try to convince the reader that modern Turtle Geometry implementations deserve a place in the set of useful pieces of software for mathematics teaching.

We have studied different aspects of Turtle Geometry and mathematics teaching:

- Regarding the effect on skills acquisition, we carried out an experience along three consecutive academic years about the effect of working with the Turtle Geometry (using Scratch 2) in the learning of geometric concepts of future Primary School teachers [23]. It

took place with students of the Universidad Complutense de Madrid and the experimental study carried out showed an improvement on both the academic performance as well as the student satisfaction. This was an extensive study, involving several teachers and 366 students.

- Another study about the attitude of 27 students of a Master's Degree in Secondary Teacher Training regarding their preferences: using computers versus using programmable robots (*Scratch 3* versus *Pro-Bot*) in their future work as teachers was detailed in [24]. The future teachers were really enthusiastic when working with the robots, in contrast with what we could qualify as normal interest in *Scratch*. But, curiously, they showed in the polls a clear preference for teaching using traditional hardware (possibly due to a lack of self confidence in the use of the new hardware). [25]. There are many articles about the use of robots in STEM, like the well-known [26,27], but we haven't found a similar one, about device comparisons.
- A possible use of Turtle Geometry to visualize abstract processes has been already mentioned in Section 3 (virus propagation). Spanish versions of the tale and video about a cat that propagates a virus (illustrated with images generated by *Scratch 3*) are available from the Instituto de Matemática Interdisciplinar (IMI) web page [28]. The English version of the tale (illustrated with images generated by *Maple*) can be found in *Mapleprimes* web page [29]. The good reception of these tales gave rise to the already mentioned articles [15,16].

Summarizing our experience with Turtle Geometry, it offers a very appropriate and comfortable environment for certain specific tasks (teaching some geometry concepts, illustrating fractals-related issues, etc.) on different kinds of devices.

6. Conclusions

There are fashions in the world of mathematics teaching too. For instance, Euclid's *The Elements* was the standard textbook at the Victorian period in England, a didactic approach opposite to that of "modern mathematics". But there are no unanimous opinions on these topics: consider, for example, the recent attacks against the reform-based curriculum in the Netherlands described in [30].

Moreover, some ideas that are considered "modern" are older than expected. For instance, "recreational mathematics" are probably best known thanks to Martin Gardner's works, like those in the "Mathematical Games" column in *Scientific American* [31], but there are also very interesting earlier works such as [32] (an extensive collection of problems, many of them reprinted from magazines and newspapers of the time). Recreational mathematics could have a much active role in the curriculum, as they provide a wide variety of strategies for solving mathematical problems. Many of them can be solved and/or checked using mathematical and computational techniques in fruitful synergy.

Turtle Geometry is suffering an unjust purgatory of oblivion, despite the arrival of these mechanical devices that somehow close the circle sending us back to the tortoises of the origins of *Logo*. Wonderful powerful software and hardware that use Turtle Geometry, and are appropriate for teaching certain mathematical issues, are available in 2021.

In my opinion, Turtle Geometry adapts very well to theories for mathematics education such as Realistic Mathematics Education [33], which six principles have been recently reformulated as: activity principle, level principle, intertwinement principle, interactivity principle and guidance principle.

Summarizing, as final conclusion of this work: despite the success of the (many times underused) DGS *GeoGebra*, the scarce use of CAS and the oblivion of Turtle Geometry, I believe there is place for the three of them in mathematics teaching. There are nowadays very interesting implementations of Turtle Geometry on different hardware that worth been used for mathematics teaching. I hope I have convinced the reader that Turtle Geometry shouldn't be obliterated. I would like to finish with a paragraph of the 1868 book *Cuentos del día (Tales of the day)* by the Spanish writer Ventura Ruiz Aguilera:

*From the old, we will keep all the sacred, everything
beautiful and everything useful and applicable to the
construction I propose... Everything else, out! Old,
just for being old, does not deserve even a tear.*

Acknowledgements Partially funded by the research project PGC2018-096509-B-100 (Government of Spain).

I would also like to thank the anonymous reviewers for their most valuable comments and suggestions.

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On the analysis of singularity structure in learning

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Abstract

The existence of singularities often affects the learning dynamics in neural network and caused plateau phenomena. Using Mathematica, we observed near singular regions by examining the evolution of the parameter and the dynamics of learning on the training loss surface. Our result is to investigate that the type of dynamics of learning changes when the overlap and the elimination singularity is approached from a distance by changing the initial values of the statistical model, and to clarify the plateau phenomenon observed near singular regions.

1 Introduction

In a hierarchical structure model which is a neural network, a set of true parameters consists of not a union of several manifolds. Watanabe[1], [2] investigated that the analytic set of parameters contains singularities by using algebraic geometry and Bayesian statistics.

Let the statistical model be a three-layer neural network, plateau phenomena were observed in singular regions where two hidden neurons can be rewritten with only one hidden neuron and pose a serious problem in neural networks[3], [4], [5]. Amari[5] showed that a subset of critical points corresponding to the global minimum of a smaller network can be local minima or saddles of the larger network. Amari[3] discussed the learning dynamics near the overlap singularity and the elimination singularity close to them. Also, Amari[4] introduced coordinate transformation of parameters of the statistical model and fixed variables moving quickly and searched trajectories of learning of variables moving slowly. Moreover, he[5] calculated stability and dynamics of learning near singular regions. Guo[6] classified dynamics of learning near the overlap singularity and the elimination singularity into five patterns.

Currently, the dynamics of learning near the elimination singularities far away from the overlap singularities still remains unknown. By continuously changing the initial value of learning, we investigate that the type of dynamics according to Guo's classification of overlap and elimination singularity when approaching from a distance changes.

2 Definition

Definition 1 (Input, noise, training data, test data) We assume that an \mathbb{R}^1 -valued random variable X that follows a probability density function $p(x)$ is input and that an \mathbb{R}^1 -valued random variable Z that follows a normal distribution, the average and standard derivation of which are $(0, \sigma)$, is noise. For $\theta_0 = (w_{11}^*, w_{12}^*, w_{21}^*, w_{22}^*, w_{31}^*, w_{32}^*) \in \mathbb{R}^6$, an \mathbb{R}^1 -valued random variable Y is determined by the training data or test data as follows[1], [2]:

$$Y := f(x, \theta_0) + Z = w_{31}^* \tanh(w_{11}^*x + w_{21}^*) + w_{32}^* \tanh(w_{12}^*x + w_{22}^*) + Z.$$

Definition 2 (Function approximation model) For parameters $\theta = (\mathbf{w}_1, \mathbf{w}_2, w_{31}, w_{32}) \in \mathbb{R}^6$, and \mathbb{R}^1 -valued function $f(x, \theta)$, an \mathbb{R}^1 -valued random variable Y is determined as a function approximation model as follows[1], [2]:

$$Y := f(x, \theta) + Z = w_{31}\phi(\mathbf{x}, \mathbf{w}_1) + w_{32}\phi(\mathbf{x}, \mathbf{w}_2) + Z = w_{31} \tanh(\mathbf{w}_1^T \mathbf{x}) + w_{32} \tanh(\mathbf{w}_2^T \mathbf{x}) + Z \\ = w_{31} \tanh(w_{11}x + w_{21}) + w_{32} \tanh(w_{12}x + w_{22}) + Z.$$

where $\mathbf{w}_1 = (w_{11}, w_{21})$, $\mathbf{w}_2 = (w_{12}, w_{22})$, $\mathbf{x} = (x, 1)$.

Definition 3 (Statistical model, true density function) A conditional probability density that follows function approximation model Y and is referred to as a statistical model is defined as follows[1], [2]:

$$p(y|x, \theta) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{|y - f(x, \theta)|^2}{2\sigma^2}\right).$$

A conditional probability density that follows output Y and is referred to as a true density function is defined as follows[1], [2]:

$$q(y|x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{|y - f(x, \theta_0)|^2}{2\sigma^2}\right).$$

Definition 4 (Overlap singularity, elimination singularity) An overlap singularity is defined as the special region in the parameter space in which w_i satisfies[3]

$$R_0 := \{\theta \in \mathbb{R}^6 | \mathbf{w}_1 = \mathbf{w}_2\}.$$

The elimination singularity is defined as the special region in the parameter space in which w_i satisfies[3]

$$R_1 := \{\theta \in \mathbb{R}^6 | w_{31} = 0\} \cup \{\theta \in \mathbb{R}^6 | w_{32} = 0\}.$$

We recall the following coordinate transformation from the parameter $\theta = (\mathbf{w}_1, \mathbf{w}_2, w_{31}, w_{32})$ to the parameter $\xi = (\mathbf{a}, b, \mathbf{v}, w)$ [3]:

$$\mathbf{a} = \mathbf{w}_2 - \mathbf{w}_1, \quad b = \frac{w_{31} - w_{32}}{w_{31} + w_{32}}, \quad \mathbf{v} = \frac{w_{31}\mathbf{w}_1 + w_{32}\mathbf{w}_2}{w_{31} + w_{32}}, \quad w = w_{31} + w_{32}.$$

Using coordinate ξ , the coordinate θ is as follows :

$$\mathbf{w}_1 = \mathbf{v} + \frac{1}{2}\mathbf{a}(b - 1), \quad \mathbf{w}_2 = \mathbf{v} + \frac{1}{2}\mathbf{a}(b + 1), \quad w_{31} = \frac{1}{2}w(1 + b), \quad w_{32} = \frac{1}{2}w(1 - b).$$

For $y = f(x, \theta_0) + Z$, we define the loss function

$$l(y, \mathbf{x}, \theta) := \frac{1}{2}(y - f(\mathbf{x}, \theta))^2.$$

Then, for a learning rate η , parameter θ , which is modified by the stochastic gradient descent algorithm, is as follows:

$$\theta(t+1) - \theta(t) := -\eta \frac{\partial l(y_t, \mathbf{x}_t, \theta_t)}{\partial \theta}.$$

Definition 5 (Learning equation of coordinate θ) For coordinate $\theta = (\mathbf{w}_1, \mathbf{w}_2, w_{31}, w_{32})$, the learning equation is defined as follows:

$$\dot{\theta}(t) := -\eta \left\langle \frac{\partial l(y, \mathbf{x}, \theta)}{\partial \theta} \right\rangle = \int -\eta \frac{\partial l(y, \mathbf{x}, \theta)}{\partial \theta} q(y|\mathbf{x}) dy d\mathbf{x}.$$

For loss $e(y, \mathbf{x}, \xi) := y - f(\mathbf{x}, \xi)$, negative gradients of the loss function $l(\xi)$ hold as follows[1]:

$$l_{\mathbf{v}}(\xi) = w \left\langle e(y, \mathbf{x}, \xi) \frac{\partial \phi(\mathbf{x}, \mathbf{v})}{\partial \mathbf{v}} \right\rangle + \frac{1}{8} w (1 - z^2) Q(\mathbf{v}, \mathbf{a}) + O(\mathbf{a}^3),$$

$$l_w(\xi) = \langle e(y, \mathbf{x}, \xi) \phi(\mathbf{x}, \mathbf{v}) \rangle + \frac{1}{8} (1 - z^2) \left\langle e(y, \mathbf{x}, \xi) \mathbf{a}^T \frac{\partial^2 \phi(\mathbf{x}, \mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}^T} \mathbf{a} \right\rangle + O(\mathbf{a}^3),$$

$$l_{\mathbf{a}}(\xi) = \frac{1}{4} w (1 - z^2) \left\langle e(y, \mathbf{x}, \xi) \mathbf{a} \frac{\partial^2 \phi(\mathbf{x}, \mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}^T} \right\rangle + \frac{1}{24} w z (1 - z^2) \left\langle e(y, \mathbf{x}, \xi) \frac{\partial D(x, \mathbf{v}, \mathbf{a})}{\partial \mathbf{a}} \right\rangle + O(\mathbf{a}^3),$$

$$l_b(\xi) = -\frac{1}{4} w z \left\langle e(y, \mathbf{x}, \xi) \mathbf{a}^T \frac{\partial^2 \phi(\mathbf{x}, \mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}^T} \mathbf{a} \right\rangle + O(\mathbf{a}^3).$$

where $Q(v, \mathbf{a}) := \left\langle e(y, \mathbf{x}, \xi) \frac{\partial}{\partial \mathbf{v}} \left(\mathbf{a}^T \frac{\partial^2 \phi(\mathbf{x}, \mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}^T} \mathbf{a} \right) \right\rangle$, $D(\mathbf{x}, \mathbf{v}, \mathbf{a}) := \sum_{i,j,k} \frac{\partial^3 \phi(\mathbf{x}, \mathbf{v})}{\partial v_i \partial v_j \partial v_k} a_i a_j a_k$.

Note that $l_{\mathbf{v}}, l_w$ is of order $O(1)$. By taking into account the fact that $\mathbf{a} \approx \mathbf{0}$, we see that the time evolution of (\mathbf{v}, w) is fast and converges to the partial equilibrium states that satisfies $l_{\mathbf{v}}(\xi) = l_w(\xi) = 0$ quickly.

On the other hand, note that $l_{\mathbf{a}}$ and l_b is of order $O(\mathbf{a})$ and $O(\mathbf{a}^2)$. By taking into account the fact that $\mathbf{a} \approx \mathbf{0}$, we see that the time evolution of (\mathbf{a}, b) is slow[4].

Definition 6 (Learning equation of coordinate ξ) For the coordinate $\xi = (\mathbf{a}, b, \mathbf{v}, w)$, the learning equation is defined as follows:

$$\dot{\xi} := -\eta \frac{\partial \xi}{\partial \theta^T} \left(\frac{\partial \xi}{\partial \theta^T} \right)^T \left\langle \frac{\partial l(y, \mathbf{x}, \xi)}{\partial \xi} \right\rangle.$$

Then, the learning equations hold as follows[3]:

$$\begin{aligned} \dot{\mathbf{v}} &= \frac{b^2 + 1}{2} l_{\mathbf{v}} + \frac{b^2 + 1}{2w^2} \mathbf{a} \mathbf{a}^T l_{\mathbf{v}} + \frac{b}{w} \mathbf{a} l_w - b l_{\mathbf{a}} - \frac{b^2 + 1}{w^2} \mathbf{a} l_b, & \dot{w} &= \frac{b}{w} \mathbf{a}^T l_{\mathbf{v}} + 2l_w - \frac{2b}{w} l_b, \\ \dot{\mathbf{a}} &= -b l_{\mathbf{v}} + 2l_{\mathbf{a}}, & \dot{b} &= -\frac{b^2 + 1}{w^2} \mathbf{a}^T l_{\mathbf{v}} - \frac{2b}{w} l_w + \frac{2(b^2 + 1)}{w^2} l_b. \end{aligned}$$

Then, we fix (\mathbf{v}, w) to its best approximation (\mathbf{v}^*, w^*) . We examined the evolution of parameter (\mathbf{a}, b) . For $\xi^* = (\mathbf{v}^*, w^*, \mathbf{0}, b)$, we defined

$$H(\mathbf{v}^*, w^*) := \frac{1}{4} w^* \left\langle e(y, \mathbf{x}, \xi) \frac{\partial^2 \phi(\mathbf{x}, \mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}^T} \right\rangle \Big|_{\xi=\xi^*}.$$

For loss function $l(y, \mathbf{x}, \xi)$, $\left\langle \frac{\partial^2 l(y, \mathbf{x}, \xi)}{\partial \xi \partial \xi^T} \right\rangle \Big|_{\xi=\xi^*} = (1 - b^2) H(\mathbf{v}^*, w^*)$ holds.

Theorem 1 (Stability of learning near singular regions) *For stability of learning near singular regions, it holds as follows: When the true density function is in a singular region, the entire critical line of R_0 is stable. When the true density function is not in a singular region, the stability of the entire critical line of R_0 is divided into the following three cases according to the eigenvalue of $H(\mathbf{v}^*, w^*)$ [5].*

- (1) both positive and negative eigenvalues: all points on the critical line of R_0 are unstable.
- (2) negative definite: the part $b^2 < 1$ is stable, whereas the part $b^2 > 1$ is unstable in R_0 .
- (3) positive definite: the part $b^2 < 1$ is stable, whereas the part $b^2 > 1$ is unstable in R_0 .

We assume $\tilde{\xi} = (\mathbf{v}^*, w^*, \mathbf{a}, b)$. The gradient of loss function $l(\tilde{\xi})$ holds as follows[3]:

$$\begin{aligned} l_{\mathbf{v}}(\tilde{\xi}) &= \frac{1}{8} w^* (1 - z^2) Q(\mathbf{v}^*, \mathbf{a}) + O(\mathbf{a}^3), \quad l_w(\tilde{\xi}) = \frac{1}{2} \frac{1 - z^2}{w^*} \mathbf{a}^T H(\mathbf{v}^*, w^*) \mathbf{a} + O(\mathbf{a}^3), \\ l_{\mathbf{a}}(\tilde{\xi}) &= (1 - z^2) H(\mathbf{v}^*, w^*) \mathbf{a} + \frac{1}{24} w^* z (1 - z^2) \left\langle e(y, \mathbf{x}, \xi) \frac{\partial D(x, \mathbf{v}, \mathbf{a})}{\partial \mathbf{a}} \right\rangle \Big|_{\xi=\tilde{\xi}} + O(\mathbf{a}^3), \\ l_b(\tilde{\xi}) &= -b \mathbf{a}^T H(\mathbf{v}^*, w^*) \mathbf{a} + O(\mathbf{a}^3). \end{aligned}$$

Note that $l_{\mathbf{a}}(\tilde{\xi})$ is of order $O(\mathbf{a})$ and $l_b(\tilde{\xi})$, $l_{\mathbf{v}}(\tilde{\xi})$, $l_w(\tilde{\xi})$ is of order $O(\mathbf{a}^2)$. Neglecting higher terms in the above equations and taking into account the fact that $\mathbf{a} \approx \mathbf{0}$, the learning equation near R_0 holds as follows[3]:

$$\dot{\mathbf{a}} = 2(1 - b^2) H(\mathbf{v}^*, w^*) \mathbf{a}, \quad \dot{b} = -\frac{b(1 - b^2)}{w^{*2}} \mathbf{a}^T H(\mathbf{v}^*, w^*) \mathbf{a} - \frac{2b(b^2 + 1)}{w^{*2}} \mathbf{a}^T H(\mathbf{v}^*, w^*) \mathbf{a}.$$

Theorem 2 (Dynamics of learning near singular regions) *An energy function $h(\mathbf{a}) := \frac{1}{2} \mathbf{a}^T \mathbf{a}$ of the dynamics of learning near singular regions, it holds as follows[3]:*

- (1) In the neighborhood of R_0 , we obtain the equation $\dot{h} = \mathbf{a}^T \dot{\mathbf{a}} = \frac{2w^{*2}(b^2-1)}{b(b^2+3)} \dot{b}$ and the dynamics of the learning equations are given by

$$h(\mathbf{a}) = \frac{2w^{*2}}{3} \log \frac{(b^2 + 3)^2}{|b|} + C.$$

- (2) In the neighborhood of $R_0 \cap R_1$, we obtain the equation $\dot{h} = \frac{w^{*2}(b^2-1)}{b(b^2+1)} \dot{b}$ and the dynamics of the learning equations are given by

$$h(\mathbf{a}) = w^{*2} \log \left(|b| + \frac{1}{|b|} \right) + C.$$

Definition 7 (Classification of dynamics of learning near singular regions) *The dynamics of learning near a singularity is classified into following five patterns by changing an initial value of the statistical model[6].*

- (1) *Overlap singularity: The learning process is significantly affected by overlap singularity.*
- (2) *Cross elimination singularity: The learning process crosses the elimination and reaches the global optimum after training.*
- (3) *Fast convergence: The learning process converges to the global minimum fast.*
- (4) *Near elimination singularity: When the parameters of the statistical model are near the elimination singularity in the training, the learning process is significantly affected by elimination singularity.*
- (5) *Output weight 0: After training, output weight w_i becomes nearly equal to 0.*

3 Construction of a neural network as the statistical model using Mathematica.

Using Mathematica, variables $F1$, $F2$, constants $elem0$, $elem1$, $elem2$, $elem3$ calculate as follows:

```
F1[a_] := NetInsertSharedArrays[NetChain[LinearLayer[1, "Weights" -> a, "Biases" -> None], "Linear1"],
F2[b_] := NetInsertSharedArrays[NetChain[LinearLayer[1, "Weights" -> b, "Biases" -> None], "Linear2"],
elem0 := ElementwiseLayer[# * (1/2)&], elem1 := ElementwiseLayer[# * (-1)&],
elem2[v_] := ElementwiseLayer[# * (v)&], elem3[w_] := ElementwiseLayer[# * (w)&].
```

First, to express the condition $w_{11}x = (v + \frac{1}{2}(b - 1)a)x$, we input the following:

```
net11[a_, b_, v_] := NetGraph[elem0, elem1, F1[a], F2[b], elem2[v], TotalLayer[],
NetPort["Input"] -> 1, 1 -> 3 -> 4, 3 -> 2, 4, 2, 5 -> 6]
```

and $net11$ output on the left-hand side of Figure 1.

To express the condition $w_{31} \tanh(x) = \frac{1}{2}w(b + 1) \tanh(x)$, we input the following:

```
net12[a_, b_, w_] := NetGraph[Tanh, elem0, elem3[w], F2[b], TotalLayer[],
NetPort["Input"] -> 1, 1 -> 2 -> 3 -> 4, 3, 4 -> 5]
```

and $net12$ output on the light-hand side of Figure 1.

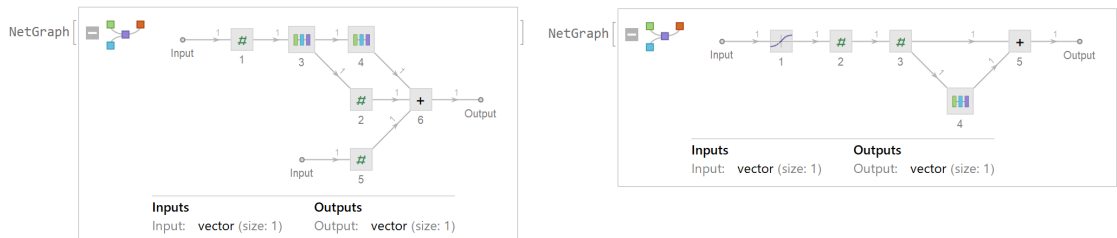


Figure 1: $net11$, $net12$

Similarly to express the condition $w_{12}x = (v + \frac{1}{2}(b+1)a)x$, $w_{32} \tanh(x) = \frac{1}{2}w(-b+1) \tanh(x)$, similarly we input in the same way. *net21*, *net22* output on the Figure 2.

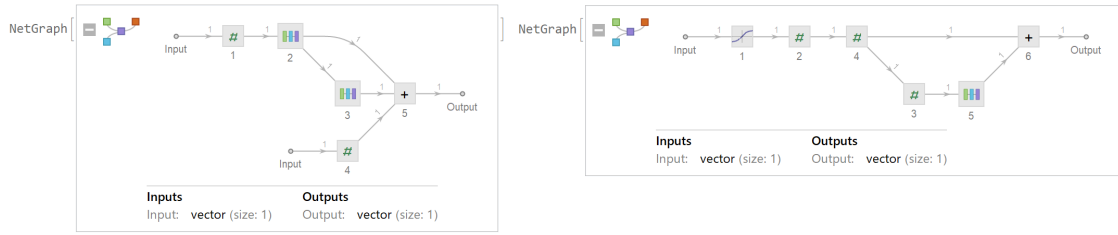


Figure 2: *net21*, *net22*

Next to express the condition $w_{31} \tanh(w_{11}x) = \frac{1}{2}w(b+1) \tanh[(v + \frac{1}{2}(b-1)a)x]$, we input the following:

$$net1[a_-, b_-, v_-, w_-] := NetGraph[net11[a, b, v], net12[a, b, w], NetPort["Input"] -> 1, 1 -> 2]$$

and *net1* output on the left-hand side of Figure 3.

Similarly, to express the condition $w_{32} \tanh(w_{12}x) = \frac{1}{2}w(-b+1) \tanh[(v + \frac{1}{2}(b+1)a)x]$, we defined *net2*.

Finally, to express the condition $w_{31} \tanh(w_{11}x) + w_{32} \tanh(w_{12}x)$, we input the following:

$$parameterNet[a_-, b_-, v_-, w_-] := NetGraph[net1[a, b, v, w], net2[a, b, v, w], TotalLayer[], NetPort["Input"] -> 1, NetPort["Input"] -> 2, 1, 2 -> 3 -> NetPort["Output1"], "Input" -> enc]$$

and *parameterNet* output on the light-hand side of Figure 3.



Figure 3: *net1*, *parameterNet*

Let us define the loss function as a log density ratio function. We input the following:

$$gaussianLikelihood[y_-, \mu_-] := PDF[NormalDistribution[\mu, 1], y]$$

$$trainingNet[a_-, b_-, v_-, w_-] := NetGraph[< | "params" -> parameterNet[a, b, v, w], "lhood" ->$$

$$ThreadingLayer[gaussianLikelihood], "neglog" -> ElementwiseLayer[-Log[#]&] >,$$

$$NetPort["Output"], NetPort["params"], "Output1"] -> "lhood", "lhood" -> "neglog" -> NetPort["Loss"]]$$

and *trainingNet* output on the light side of Figure 4.

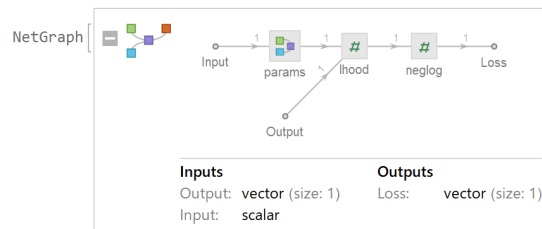


Figure 4: *trainingNet(log density ratio)*

For training data and test data, we input the following:

$$G[a_-, b_-] := \text{Mean}[\text{trainingNet}[a, b, v0, w0][\langle \text{"Input"} - \rangle \text{dataX}, \text{"Output"} - \rangle \text{enc}[\text{dataY}]] \langle \rangle]$$

$$H[a_-, b_-] := \text{Mean}[\text{trainingNet}[a, b, v0, w0][\langle \text{"Input"} - \rangle \text{testX}, \text{"Output"} - \rangle \text{enc}[\text{testY}]] \langle \rangle]$$

and defined training loss function G and validation loss function H .

4 Dynamics of learning near singular regions

4.1 Framework of dynamics of learning

Example 1 (Training data, true density function) For input X on $-3 \leq x \leq 3$ and noise Z of $\sigma = 0.05$, let the training data (are listed in Appendix.) be

$$0.25 \tanh(0.2x) + 0.25 \tanh(0.4x) + Z,$$

and the true density function be

$$q(y|x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{|y - (0.25 \tanh(0.2x) + 0.25 \tanh(0.4x))|^2}{2\sigma^2}\right).$$

For $a = 0.2, b = 0, v = 0.3, w = 0.5$, we consider that the dynamics of learning evolving under the influence of a critical line classified into five cases by changing the initial values of the statistical model for the case in which the true distribution near the singular regions is realizable by the statistical model. Let us define the loss function as the log density ratio function, and input the following:

```
results1[a_-, b_-] := NetTrain[trainingNet[a, b, v, w], < | "Input" - > dataX, "Output" - > enc[dataY]] >,
{"Weights", "TrainedNet", "RoundLossList"}, LossFunction - > "Loss", Method - >
"ADAM", "LearningRate" - > 0.1, BatchSize - > 30, MaxTrainingRounds - > { },
TrainingProgressFunction - > appendToLog
```

In addition, the neural network was trained.

4.2 Dynamics of overlap singularity and cross elimination singularity

Let the initial values of the statistical model be $a = 0.15, b = -2.0, -1.8, -1.5, -1.3, v = 0.3, w = 0.5$. The neural network was trained 140 times. We construct an array of parameters of a, b under the influence of the critical line. The evolutions of parameters of a and the evolutions of parameters of b are shown on the left-hand and middle-hand sides respectively, of Figure 5, and the evolution of parameters of a, b is shown on the right-hand side of Figure 5.

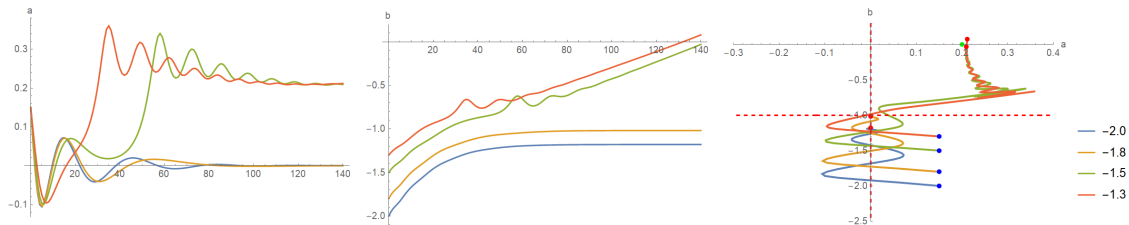


Figure 5: Evolution of parameters of a, b ($a = 0.15, b = -2.0, -1.8, -1.5, -1.3$)

The neural network was trained 140 times. We construct an array of the training loss, and the evolution of the training loss and the dynamics of learning of the training loss surface are shown on the left-hand and right-hand sides, respectively, of Figure 6.

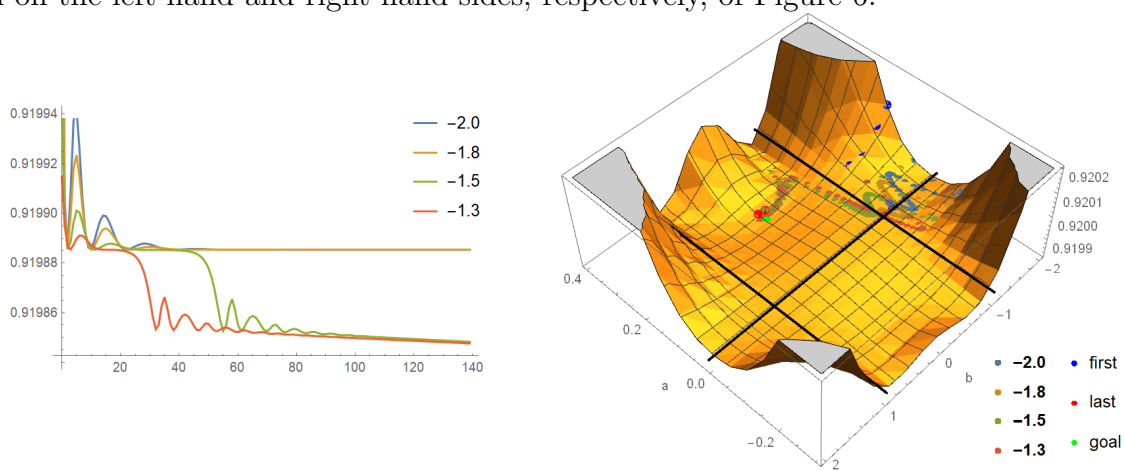


Figure 6: Evolution of the training loss and the dynamics of the training loss surface ($a = 0.15, b = -2.0, -1.8, -1.5, -1.3$)

We generalize the parameter of b ($-2.2 \leq b \leq 2.2$). The evolution of the parameters of a, b and the dynamics of learning of the training loss surface are shown on the left-hand and right-hand sides, respectively, of Figure 7.

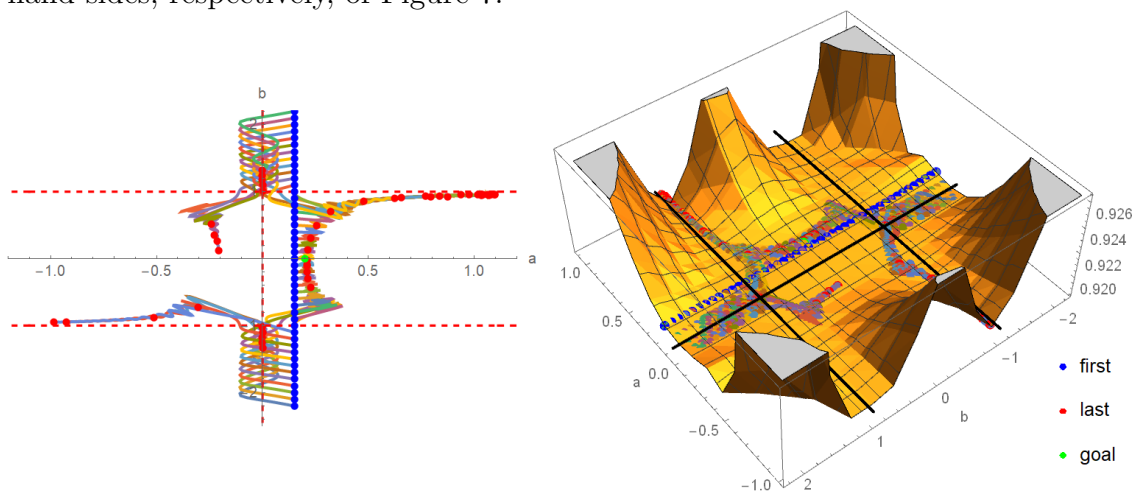


Figure 7: Evolution of the parameters of a, b and the dynamics of the training loss surface ($-2.2 \leq b \leq 2.2$)

- Result 1**
- (1) We find that plateau phenomena were observed on critical line $a = 0$ and that the dynamics of learning do not reach the true distribution in case $b = -2.0, -1.8$.
 - (2) We find that plateau phenomena were observed when crossing critical line $b = -1$ and that the dynamics of learning reach the true distribution in case $b = -1.5 - 1.3$.
 - (3) As the parameter b evolves to 0, the dynamics of learning change from overlap singularity to cross elimination singularity and from cross elimination singularity to fast convergence.

4.3 Dynamics of near elimination singularity and output weight 0

Let the initial values of the statistical model be $a = 0.5, 0.6, 0.7, 1.2, b = 0.75, v = 0.3, w = 0.5$. The neural network was trained 100 times. We construct an array of parameters of a, b under the influence of critical line. The evolutions of parameters of a and the evolutions of parameters of b are shown on the left-hand and middle-hand sides respectively, of Figure 8, and the evolution of parameters of a, b is shown on the right-hand side of Figure 8.

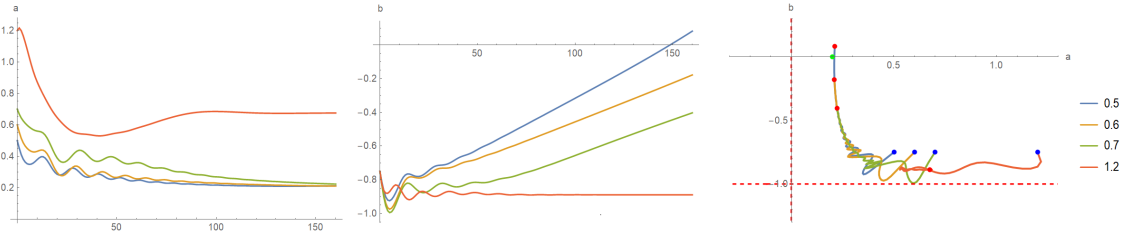


Figure 8: Evolutions of parameters of a, b ($a = 0.5, 0.6, 0.7, 1.2, b = 0.75$)

The neural network was trained 100 times. We construct an array of the training loss, and the evolution of the training loss and the dynamics of learning of the training loss surface are shown on the left-hand and right-hand sides, respectively, of Figure 9.

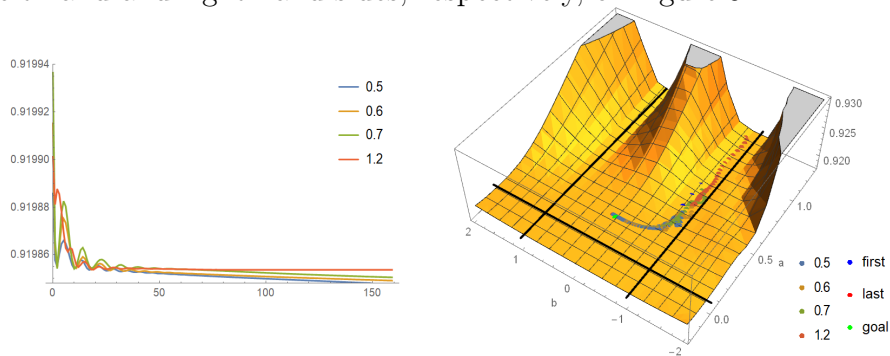


Figure 9: Evolution of the training loss and the dynamics of the training loss surface ($a = 0.5, 0.6, 0.7, 1.2, b = 0.75$)

We generalize the parameter of a ($0 \leq a \leq 2.2$). The evolution of the parameters of a, b and the dynamics of learning of the training loss surface are shown on the left-hand and right-hand sides, respectively, of Figure 10.

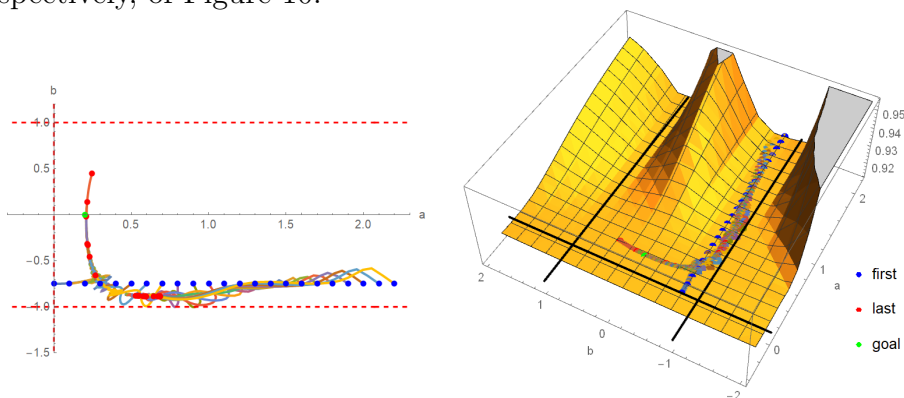


Figure 10: Evolution of the parameters of a, b and the dynamics of the training loss surface ($0 \leq a \leq 2.2$)

Result 2 (1) *We find that plateau phenomena were observed on critical line $b = -1$ and that the dynamics of learning do not reach the true distribution in case $a = 1.2$.*

(2) *We find that plateau phenomena were observed approaching critical line $b = -1$ and that the dynamics of learning reach the true distribution in case $a = 0.6, 0.7$. Moreover, we find that the dynamics of learning reach the true distribution more quickly in case $a = 0.5$.*

(3) *As the parameter of a evolves to 0, the dynamics of learning change from output weight 0 to near elimination singularity and from near elimination singularity to fast convergence.*

5 Conclusion

Firstly we constructed the neural network as the statistical model using Mathematica. Secondly, we observed plateau phenomena near singular regions by examining the evolution of the parameter and the dynamics of learning on the training loss surface. Finally we investigated that the type of dynamics of learning changes when the overlap and the elimination singularity is approached from a distance by changing the initial values of the statistical model.

The purpose of our research is not to study the current state of the art in neural networks but to make some concepts in phenomena in neural networks correspond (explain) to learning in educational activities. Specifically, it is to clarify what state the phenomenon of plateau and over-fitting and over-generalization are in educational activities.

The results can also be the foundation to investigate the singular learning dynamics in educational activities.

References

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6 Appendix

6.1 Training data of Example1

For input x_s , and output y_s as follows:

$$x_s = \{2.467685732795669, 1.6313896711002975, 1.7039693114471142, -2.353539095169551, 2.5106926463104458, -2.9742536653063, 1.4778884503387921, -1.7619315572659175, 0.8575206146347014, -1.9522402751318726, -2.9186556422433796, 2.3433244821789305, -2.3174593747595598, 0.24745360478229195, -0.43473282858294837, 2.0777962243403962, -0.7489587340884398, 0.40283200240701333, 1.4393667305075848, 2.6884952319559243, 0.4233060018195829, 1.3133371734415373, -1.8687861826912897, 2.641499809476027, 1.3536619131864676, 1.4261447937286373, -1.5373889449365947, 2.5833410435168336, -0.7634883775841974, -1.418229957030034\},$$

$$y_s = \{0.25047549494898375, 0.14195709642758433, 0.2416763776071971, -0.34055590890961035, 0.3082658314034902, -0.4292549244954509, 0.15038776701404105, -0.23410034295044008, 0.1674469014375939, -0.26548937037643955, -0.3321460817933551, 0.2720167181157782, -0.2892455062624837, 0.0520546848151971, -0.0009290519327547, 0.2940081059525326, -0.14421683321295234, 0.08562704853302514, 0.25724997978192643, 0.2668005655536598, 0.043918697553646746, 0.19753643159405437, -0.2627853499983649, 0.25989101875041354, 0.1395086144673041, 0.17062611740258554, -0.18466386529707135, 0.3690548490941195, -0.16241114605952112, -0.14051890248769974\},$$

Then, we defined training data as follows:

$$\{2.46769 \rightarrow 0.250475, 1.63139 \rightarrow 0.141957, 1.70397 \rightarrow 0.241676, -2.35354 \rightarrow -0.340556, 2.51069 \rightarrow 0.308266, -2.97425 \rightarrow -0.429255, 1.47789 \rightarrow 0.150388, -1.76193 \rightarrow -0.2341, 0.857521 \rightarrow 0.167447, -1.95224 \rightarrow -0.265489, -2.91866 \rightarrow -0.332146, 2.34332 \rightarrow 0.272017, -2.31746 \rightarrow -0.289246, 0.247454 \rightarrow 0.0520547, -0.434733 \rightarrow -0.000929052, 2.0778 \rightarrow 0.294008, -0.748959 \rightarrow -0.144217, 0.402832 \rightarrow 0.085627, 1.43937 \rightarrow 0.25725, 2.6885 \rightarrow 0.266801, 0.423306 \rightarrow 0.0439187, 1.31334 \rightarrow 0.197536, -1.86879 \rightarrow -0.262785, 2.6415 \rightarrow 0.259891, 1.35366 \rightarrow 0.139509, 1.42614 \rightarrow 0.170626, -1.53739 \rightarrow -0.184664, 2.58334 \rightarrow 0.369055, -0.763488 \rightarrow -0.162411, -1.41823 \rightarrow -0.140519\}.$$

On a Chasles construction of Cartesian ovals

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Abstract

This article deals with Cartesian ovals and one of the many methods of their construction - the so-called Chasles construction. The paper first shows an alternative justification of why points constructed by the Chasles construction satisfy foci definition of Cartesian oval, and - as a part of this justification - shows why the oval has three foci and how to construct the third focus by Euclidean means (assuming the oval is defined by two foci). Finally, the special case in which one of the circles in the construction is replaced by a straight line is discussed. It is shown synthetically that in such a case the construction renders a conic, and moreover, it can be shown that the focal definition of a conic and the definition using a directrix line are equivalent (but proof is not included in the article).

1 Introduction

The foci definition of the Cartesian oval has the following form:

$$c_1|PF_1| + c_2|PF_2| = k \quad (1)$$

where F_1 and F_2 are given points (foci) and c_1 , c_2 and k are given constants. All points P , satisfying the equation, belong to the oval.

There are many ways to construct the points of an oval geometrically. The most elementary one is to choose the distance PF_2 arbitrarily and calculate PF_1 . If both distances are non-negative and satisfy the triangle inequality, the point P is constructed as the intersection of two circles. However, there are subtler and faster constructions, one of which is that of Chasles [1] (Michel Chasles, French mathematician, 1793-1880). His procedure follows (Figure 1):

Chasles construction

Let two circles $k_1(F_1, R_1)$ and $k_2(F_2, R_2)$ are given. Let us choose a fixed point X on the line F_1F_2 . Let an arbitrary line through the point X intersects k_1 at P_1 and k_2 at P_2 . Denote P the intersection of the lines P_1F_1 and P_2F_2 . Then the point P belongs to the oval whose equation is:

$$\frac{|XF_1|}{R_1}|PF_1| - \frac{|XF_2|}{R_2}|PF_2| = |XF_1| - |XF_2| \quad (2)$$

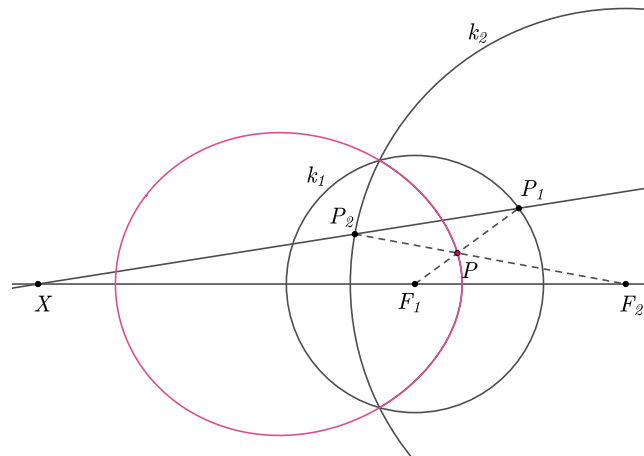


Figure 1: Chasles construction of oval

Since $|XF_1|/R_1$, $|XF_2|/R_2$ and $|XF_1| - |XF_2|$, are constants, this equation is equivalent to (1). This statement can be derived using Menelaus theorem (the line XP_1P_2 intersects the sides of the triangle F_1PF_2) or by spatial reasoning and subsequent projection onto a plane. These derivations are mentioned on the website [2], and we will not present them here.

Note: A line through X can intersect each circle at two points, so we can construct up to four points of P . It can be shown that two points belong to the oval with equation (2) and the remaining two points belong to the oval whose equation is similar to the equation (2), except that one of the mentioned three constants has the opposite sign. These two ovals are so called conjugate.

The article is divided into three parts. First, we give a new justification of why the points of P constructed using the Chasles construction belong to the oval. As a part of this justification, we show why a Cartesian oval has three foci and how to construct its third focus geometrically. In the second part, we show how to perform the Chasles construction of the oval, if two foci of the oval and its equation (1) are given. Finally, we consider the case where one of the circles is a straight line and show that in this case the construction depicts a conic. A by-product of this proof is a justification why the definition of a conic using a directrix line is equivalent to the definition using a foci definition.

2 Chasles construction and third focus of oval

We start from the construction described in the previous section. Our aim is to justify the following

Theorem 2.1 The points P constructed by Chasles' construction satisfy the equation (2).

Proof. Construct inverse points X_1, X_2 of the point X with respect to the circles k_1 and k_2 , respectively. Then construct the intersection E of the lines X_1P_1 and X_2P_2 and finally construct the intersection F_3 of the lines EP and F_1F_2 (Figure 2).

Due to the definition of inversion,

$$\frac{|XF_1|}{|P_1F_1|} = \frac{|P_1F_1|}{|X_1F_1|},$$

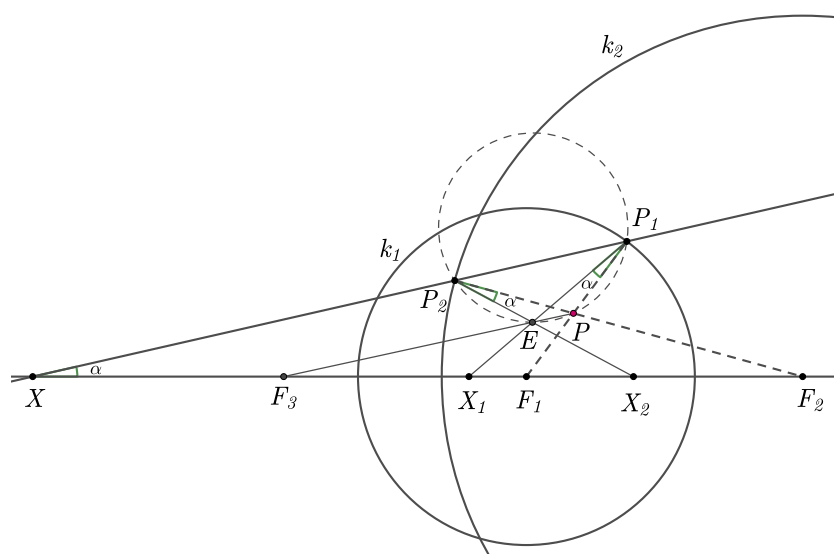


Figure 2: Chasles construction and the third focus

the triangles $X_1F_1P_1$ and P_1F_1X are similar. Therefore

$$\angle F_1P_1X_1 = \angle F_1XP_1 = \alpha$$

In the same way we derive the similarity of the triangles F_2P_2X and $F_2X_2P_2$ and the equality

$$\angle F_2P_2X_2 = \angle F_2XP_2 = \alpha$$

From the above equations follows that the quadrilateral P_2P_1PE is cyclic, therefore $\angle XP_2X_2 = \angle XP_2E = \angle P_1PE$. Hence, the triangles EPP_1 and X_2P_2X are similar. Since the triangles $F_2P_2X_2$ and F_2XP_2 are also similar, it holds:

$$\frac{|EP|}{|PP_1|} = \frac{|P_2X_2|}{|P_2X|} = \frac{|P_2F_2|}{|XF_2|} \quad (3)$$

In the same way, we arrive at equality

$$\frac{|EP|}{|PP_2|} = \frac{|P_1X_1|}{|P_1X|} = \frac{|P_1F_1|}{|XF_1|} \quad (4)$$

Therefore,

$$\frac{|PP_1|}{|PP_2|} = \frac{|F_1P_1| - |F_1P|}{|F_2P_2| - |F_2P|} = \frac{\frac{|P_1F_1|}{|XF_1|}}{\frac{|P_2F_2|}{|XF_2|}} = \frac{|P_1F_1|}{|XF_1|} \cdot \frac{|XF_2|}{|P_2F_2|}, \quad (5)$$

which, after easy rearrangement, gives the equation

$$\frac{|XF_1|}{|P_1F_1|} \cdot |PF_1| - \frac{|XF_2|}{|P_2F_2|} \cdot |PF_2| = |XF_1| - |XF_2| \quad (6)$$

identical to the equation (2).

Theorem 2.2 Point F_3 is the third focus of the oval.

Proof. To prove this statement, it is necessary to justify two facts:

1) The point F_3 is fixed, i.e. it does not depend on the position of the line XP_2P_1 . 2) Points P of the oval with foci F_1F_2 also lie on the (identical) ovals with foci F_3F_1 and F_3F_2

Due to the equality $\angle X_1EX_2 = \angle P_2EP_1 = \angle P_2PP_1$ and $\angle X_2X_1E = \angle F_1X_1P_1 = \angle F_1P_1X = \angle PP_1P_2$ the triangles X_2EX_1 and P_1PP_2 are similar. Since the ratio of of the sides $|PP_1|/|PP_2|$ is constant according to equation (5), the ratio of the sides $|EX_2|/|EX_1|$ is also constant. Let us denote this constant e :

$$\frac{|EX_2|}{|EX_1|} = e$$

Furthermore, the triangles F_3EX_1 and F_3X_2E are also similar since they share the angle at the vertex F_3 and the following equality holds:

$$\angle X_1EF_3 = \angle P_1EP = \angle P_1P_2P = \angle P_1P_2F_2 = \alpha + \angle P_2F_2X_2 = \angle P_2X_2F_3$$

Hence:

$$\frac{|F_3X_2|}{|F_3E|} = e \quad \text{and} \quad \frac{|F_3E|}{|F_3X_1|} = e \tag{7}$$

Multiplying these identities one gets

$$|F_3X_2| = e^2 \cdot |F_3X_1|$$

But

$$|X_2X_1| = |F_3X_2| - |F_3X_1| = |F_3X_1| \cdot (e^2 - 1)$$

Since the length of $|X_2X_1|$ is constant, the length of $|F_3X_1|$ is constant too and the point F_3 is fixed. (More precisely, it is the centre of the Apollonius circle for the triangle X_1EX_2 .) This completes the first part of the proof.

In the second part, it suffices to consider the foci of F_3F_1 . In the case of the foci of F_3F_2 the procedure is analogous. Let's consider an equation

$$\frac{|XF_2|}{|P_2F_2|} \cdot |F_3P| + |F_1P| = k \tag{8}$$

As $|XF_2|/|P_2F_2| = c$ is a constant, it is sufficient to show that k is also a constant. Putting the identity $|F_3P| = |F_3E| + |EP|$ into the equation (8) and using the relation (3), in the form $|XF_2|/|P_2F_2| \cdot |EP| = |PP_1|$, one gets

$$\frac{|XF_2|}{|P_2F_2|} \cdot |F_3E| + \frac{|XF_2|}{|P_2F_2|} \cdot |EP| + |F_1P| = \frac{|XF_2|}{|P_2F_2|} \cdot |F_3E| + |PP_1| + |F_1P| = \frac{|XF_2|}{|P_2F_2|} \cdot |F_3E| + |F_1P_1|$$

The right-hand side of the equation is constant if and only if $|F_3E|$ is a constant. Equation (7) implies

$$|F_3E| = e \cdot |F_3X_1|$$

Since we have proved that $|F_3X_1|$ is constant, $|F_3E|$ is also constant. The proof is complete.

3 Chasles construction of an oval given by focal equation

In this section we solve the following

Problem 3.1

Let an oval is given by two foci F_3 and F_2 and by the equation

$$c_3 \cdot |PF_3| + c_2 \cdot |PF_2| = k$$

where c_3, c_2 and k are constants. How to perform the Chasles construction of the oval?

Before we show the solution to this problem, we will return to the classical construction mentioned in the introduction. We will show that it is not unique, namely, that there are an infinite number of constructions giving the same oval. We choose one of them suitable for solving our problem.

Let the circles k_1, k_2 and the point X on the line of their centres be given. These objects, as we already know, determine the oval by the construction. Next, let us choose another point $Y \neq X$ on this line. How to modify the circles k_1 and k_2 to l_1 and l_2 , respectively, so that l_1, l_2 and Y determine the same oval? (Figure 3)

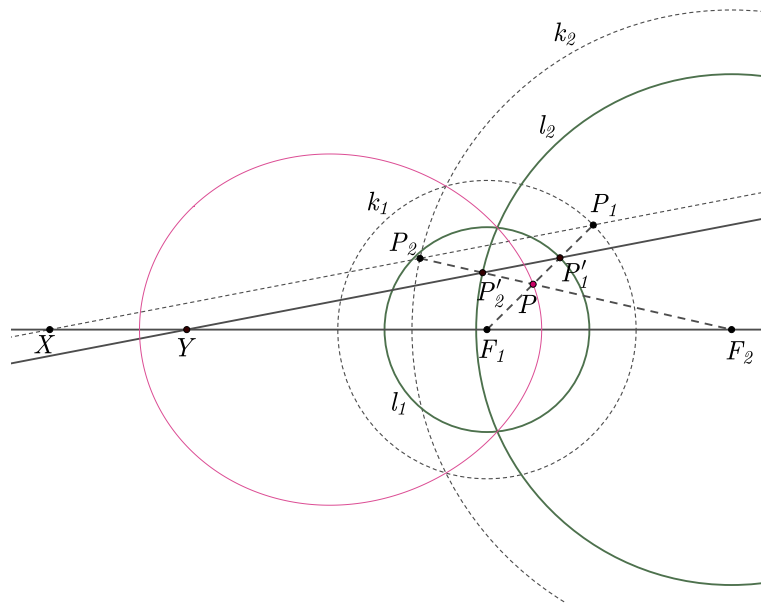


Figure 3: Chasles constructions of the same oval

Let the line XP_1P_2 determine the point P of the oval. Let us draw a parallel to this line through the point Y , which intersects the lines P_2F_2 and P_1F_1 at the points P'_2 and P'_1 , respectively. Then the intersection of the lines P'_2F_2 and P'_1F_1 coincides with the point P . It remains to justify that the points P'_2 and P'_1 move on the circles with centres F_2 and F_1 , respectively. From the similarity of the triangles XP_2F_2 and YP'_2F_2 follows:

$$|P'_2F_2| = |P_2F_2| \cdot \frac{|F_2Y|}{|F_2X|}$$

As the right-hand side of the equation is a constant, the point P'_2 moves on the circle l_2 centred at F_2 . Similarly, we prove that the point P'_1 moves on the circle l_1 centred at F_1 .

Also, the focal point F_3 , whose construction is described in the previous section, remains the same since (Figure 2 and Figure 3)

$$\angle P_1PF_3 = \angle XP_2X_2 = \angle XP_2F_2 - \alpha = \angle YP'_2F_2 - \alpha = \angle YP'_2Y_2 = \angle P'_1PF_3$$

where Y_2 is the inverse of the point Y with respect to the circle l_2 .

Now it is possible to choose a point Y in such a way that its inverse image Y_1 with respect the circle l_1 is identical to the inverse image Y_2 of Y with respect to the circle l_2 . Such a point always exists unless the point X - and Y - is not centre of similitude of the circles. In such case the solution is a conic and the third focus does not exist.

The proofs of these two propositions will be only sketched. Firstly, we show the existence of the point Y . Since Y_1 is inverse of Y in l_1 , it holds:

$$R_{l_1}^2 = |F_1Y| \cdot |F_1Y_1|$$

where R_{l_1} is radii of the circle. Dividing the equation by $|F_1Y|^2$ we get

$$\left(\frac{R_{l_1}}{|F_1Y|} \right)^2 = \frac{|F_1Y_1|}{|F_1Y|}$$

But the left hand side of the equation is constant for all points Y . Hence, the point Y_1 is the image of Y in homothety h_1 , with centre at F_1 and coefficient $(R_{l_1}/|F_1Y|)^2$. Similarly, we arrive at the conclusion that the point Y_2 is the image of Y in a homothety h_2 . Hence

$$h_1 \circ h_2^{(-1)}(Y_2) = Y_1$$

Now, composition of two homotheties $h_1 \circ h_2^{(-1)}$ is a third homothety h_3 if and only if the product of the coefficients of homotheties is not equal 1, in which case it is a translation. It is clear that the centre Z of the h_3 fulfils

$$h_3(Z) = Z$$

Therefore we can select Y in a way that $Z = Y_1 = Y_2$.

The second case (translation) occurs only if the point Y is the centre of similitude of the circles. In such case, the lines P_1F_1 and P_2F_2 intersect for the points P_1 and P_2 being antihomologous (for details see website [3]). They intersect in a centre of a circle tangent to the circles l_1 and l_2 . It is possible to show that the intersections belong to hyperbola with foci F_1, F_2 , if the tangent circle touches both circles externally or internally, and to ellipse, if the tangent circle touches one circle externally and second internally.

Let's return to the problem. The intersection E (Figure 2 with substitution of point labels Y, Y_1 and Y_2 for the labels X, X_1 and X_2) is identical to this common image, and since it lies on the line F_1F_2 , it must be the third focus F_3 .

Let us denote this point Y by X again, this time knowing that $F_3 = X_1 = X_2$. The updated Figure 2 then looks as follows (Figure 4):

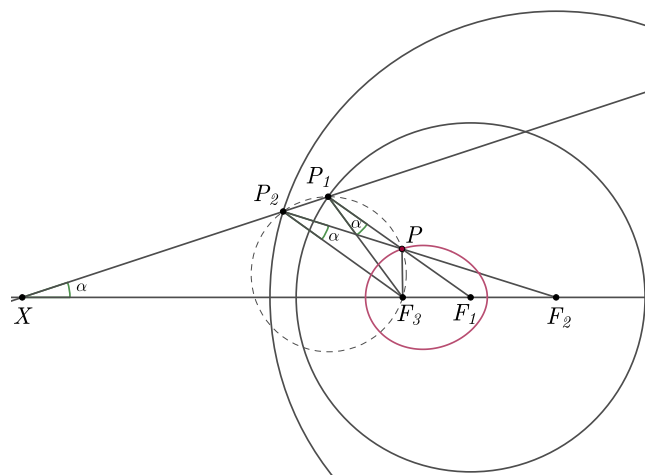


Figure 4: Chasles constructions with $F_3 = X_1 = X_2$

Denoting the radii of the circles $|P_1F_1| = R_1$ and $|P_2F_2| = R_2$, the new equations (3), (4), (6) and (8) are

$$\frac{|F_3P|}{|PP_1|} = \frac{|P_2F_3|}{|P_2X|} = \frac{R_2}{|XF_2|}$$

$$\frac{|F_3P|}{|PP_2|} = \frac{|P_1F_3|}{|P_1X|} = \frac{R_1}{|XF_1|} \tag{9}$$

$$\frac{|XF_1|}{R_1} \cdot |PF_1| - \frac{|XF_2|}{R_2} \cdot |PF_2| = |XF_1| - |XF_2| \tag{10}$$

$$\frac{|XF_2|}{R_2} \cdot |F_3P| + |F_1P| = R_1 \tag{11}$$

$$\frac{|XF_1|}{R_1} \cdot |F_3P| + |F_2P| = R_2 \tag{12}$$

Equations (10), (11) and (12) express an oval with foci F_1F_2 , F_3F_1 and F_3F_2 .

Solution of problem 3.1

In order to solve the problem 3.1, we will modify the oval equation to the form (12). Then we will proceed as follows:

- 1) Construct the circle $k_2(F_2, R_2)$.
- 2) Construct the inverse image X of the point F_3 with respect to circle k_2 .
- 3) Construct the Apollonius circle k_1 with the ratio distances of its points P_1 to the points F_3 and X equal to (9), $\frac{|P_1F_3|}{|P_1X|} = \frac{R_1}{|XF_1|}$. This ratio is given by equation (12). Denote the centre of this circle by F_1 .
- 4) The circles k_1 , k_2 and the point X determine the oval given by equation (12).

Negative constant in equation (12) does not change the above procedure - we work with absolute values of these constants. It may seem that absolute values of the two constants determine a total of four different equations (12) but the Chasles construction determines only two of them. It can be shown, however, that two of these four ovals are empty sets. Particularly, the oval of the equation (12) with $|XF_1|/R_1 > 0$ and $R_2 < 0$ is excluded immediately. One of the remaining three equations is always excluded due to the improper focal distance F_2F_3 .

4 Chasles construction for one of the circles replaced by a line

Now we prove that replacing one circle in the Chasles construction by a line, the set of points P is a conic. For completeness we describe the construction (Fig. 5) in detail.

Special case of the Chasles construction

Let a circle $k(F_2, R_2)$ and a line l be given and let the line p passes through F_2 and is perpendicular to the line l . Choose a point X on the line p . Let an arbitrary line passing through X intersects the line l at P_1 and the circle k at P_2 . Draw the perpendicular line p_1 to the line l at P_1 and denote the intersection of the lines P_2F_2 and p_1 as P .

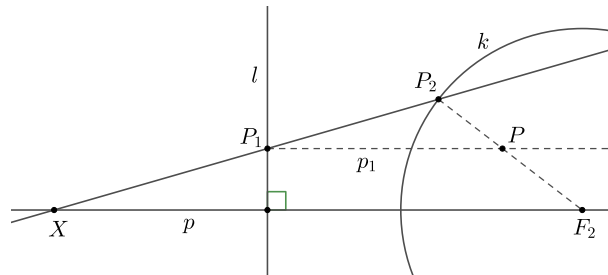


Figure 5: Special case of Chasles constructions

Theorem 4.1 The set of points P constructed in this way is a conic.

Theorem 4.2 The focus F_1 of this conic is constructed as follows (Figure 6):

- 1) Construct the axisymmetric image X_1 of the point X with respect to the line l .
- 2) Construct the inversion image X_2 of the point X with respect to the circle k .
- 3) Construct the intersection E of the lines P_1X_1 and P_2X_2 .
- 4) Denote F_1 the intersection of the lines EP and p .

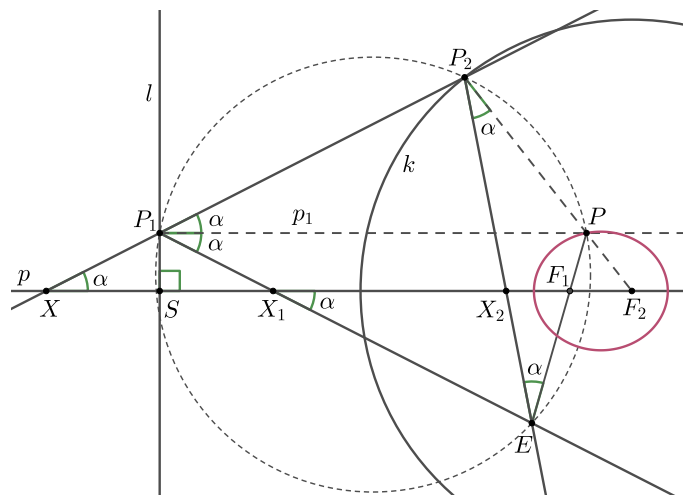


Figure 6: Construction of a focus of a conic

Proof of 4.1 and 4.2.

Let's start with a few obvious facts. It holds

$$\alpha = \angle F_2P_2X_2 = \angle F_2XP_2 = \angle PP_1P_2 = \angle PP_1E$$

Since $\angle PP_1E = \angle PP_2X_2 = \angle PP_2E$, the quadrilateral PP_2P_1E is cyclic and hence

$$\alpha = \angle PP_1P_2 = \angle PEP_2$$

Firstly, we will show that the point F_1 is fixed for any line XP_1P_2 . As the triangles X_1X_2E and $P_2X_2F_2$ are similar, we have

$$|X_1X_2| \cdot |X_2F_2| = |X_2E| \cdot |X_2P_2|.$$

The left side of this equation is a constant, therefore the product on the right side is also constant.

From the similarity of the triangles X_2F_1E and X_2P_2X follows

$$|X_2E| \cdot |X_2P_2| = |X_2X| \cdot |X_2F_1|.$$

The product on the left side is constant and the length of the line segment $|X_2X|$ is fixed. Therefore, the length of the line segment $|X_2F_1|$ is also fixed and the point F_1 is common to all lines XP_1P_2 .

Since $\angle PP_2E = \angle PEP_2$, the triangle P_2PE is isosceles.

Now we are at a crossroads: if the point X lies outside the circle k , the locus will be an ellipse (see below), if it lies inside the locus will be a hyperbola, finally if the point X lies at the intersection of p and k , the locus is parabola. We only hint why this proposition is true in the conclusion of the article, but before it, let's focus on the case of ellipse (Figure 6).

$$R_2 = |F_2P_2| = |PF_2| + |PE| = |F_2P| + |F_1P| + |F_1E|$$

Otherwise written:

$$R_2 - |F_1E| = |F_2P| + |F_1P| \tag{13}$$

This equation represents a focus definition of an ellipse if and only if $|F_1E|$ is constant. The similarity of the triangles F_1EX_2 and F_1X_1E implies

$$|F_1X_1| \cdot |F_1X_2| = |F_1E|^2.$$

The product on the left-hand side is constant and the proof is completed.

As in the case of the classical Chasel construction, the "special one" is not unique. Let S is the intersection of the lines l and p . Choose a point $Y \neq X$ on the line p , construct a circle k_2 with centre F_2 and radius $R'_2 = R_2 \cdot |YF_2|/|F_2X|$ and a line l_2 parallel to the line l such that the distance of intersection S_2 of lines p and l_2 to the point S is equal, $\overline{SS_2} = \overline{XY}$ (equality of oriented distances). Then the line l_2 , the circle k_2 and the point Y determine the same conic.

As in the previous case, the point Y can be chosen in such a way that the points Y_1 (axisymmetric point with respect to the line l_2) and Y_2 (the inverse image of the point Y with respect to the circle k_2) are identical, i.e. $Y_1 = Y_2 = F_1$. (To be exact: such a point always exists, unless the point Y does not lie on the circle k_2 , in which case the focus F_1 does not exist and the locus is a parabola.) By this transformation the relative position of the point Y and the

circle k_2 is preserved. It means that the points Y lie either inside of the circles k_2 in all of the configurations or outside in all of them. So we have the following configuration: points Y , F_1 , line l_2 (symmetrical axis of the points) and the circle k_2 with the property, that inverse image of Y is F_1 and vice versa. In this configuration the circle k_2 and the line l_2 are nonintersecting and it is possible to show: If the point Y lies outside of the circle, the locus is an ellipse If it lies inside, the locus is a hyperbola. We only sketch the proof. Let's begin the proof with the former case, Figure 7:

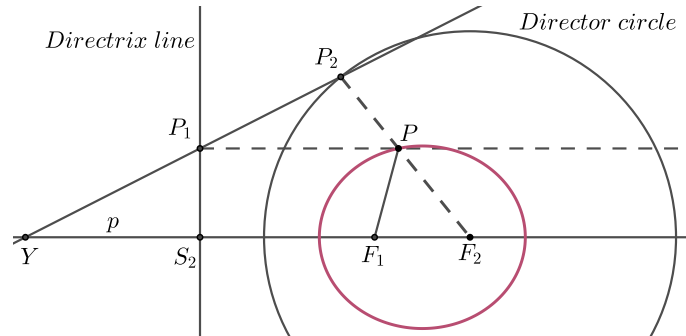


Figure 7: Directrix line and director circle in Chasles construction

We know, that $|EP| = |PP_2|$ and (in this configuration) $E = F_1$, so $|F_1P| = |PP_2|$. From the Figure 7 it is obvious that P lies always within the segment F_2P_2 . Hence

$$Constant = |F_2P_2| = |F_2P| + |PP_2| = |F_2P| + |F_1P|,$$

which is focal definition of ellipse.

The case of a hyperbola is analogous with the exception that the labels of the points Y and F_1 in the Figure 7 are interchanged (since the point Y lies inside of the circle). Again $|EP| = |PP_2|$, $E = F_1$ and $|F_1P| = |PP_2|$, but in this case, the point P lies outside of the segment F_2P_2 . Hence:

$$Constant = |F_2P_2| = |F_2P| - |PP_2| = |F_2P| - |F_1P|,$$

which is focal definition of the hyperbola. The exact proof why the point P lies in/out – side of the segment in respective cases is left to the reader. It is not hard to show that in this configuration the line l_2 is the directrix line of the conic.

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The Efficacy of an Integrative Performance Output in Senior High Subjects among STEM Learners

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ABSTRACT

In compliance to DepEd Order No. 12, s. 2020, teachers are compelled to collaboratively design and implement performance tasks that integrate two or more competencies within or across subject areas called “Integrative Performance Output”. Correlational research method was used in order to determine the significant relationship of the performances of the learners among subjects utilizing integrative performance task/output.

Research Paper was used as Integrative performance output for three integrated subjects. Teachers in the three (3) integrated subjects examined and combined their respective learning competencies to create an integrate performance output. Learners submitted their full research paper to their Practical research and Reading and writing teachers to assessed the overall content of the output while Chapter 4 was checked by the Statistics and Probability teacher.

Using F-test value of 4.81 the claim was rejected and concluded that there is a significant difference among the students’ performances in three (3) subjects; Practical Research 1, Statistics and Probability and Reading and Writing using an integrative performance task.

Likewise, using Pearson’s r of 0.89 and 0.82, it can be deduced that there is a high correlation between the students’ performance in Practical Research 1 and Reading and Writing and Statistics and Probability, respectively through the use of Integrative Performance Task/Output.

This implies that the utilization of an integrative performance output helped the students to improve their academic performance by making connections among concepts and experience so that information and skills can be applied to novel and complex challenges.

INTRODUCTION.

The Department of Education issued DepEd Order No. 31 s. 2020 “Interim Guidelines for Assessment and Grading in Light of the Basic Education Learning Continuity Plan,” summative assessment shall continue in the form of written works and performance tasks. Basically, learners are expected to do four (4) performance tasks per quarter per subject area. However, because of the implications brought about by the Online/Modular Distance Learning Modality that the Schools Division of Mandaluyong is currently implementing, learners are having difficulty in complying with this requirement. Hence, teachers ensure Academic ease in assessing student learning through an Integrative Performance Output.

Integrative Performance Output involves a series of integrated learning competencies from a variety of subjects applied to a variety of tasks. It can be as simple as breaking down an assignment into stages or having a group of related summative tasks that work together to increase capacity for additional learning and strengthen judgment abilities. The process begins with the three communicative modes as a reference point, and then moves to the selection of a theme followed by the creation of tasks around that theme, which corresponds to the three modes [5].

Teachers collaboratively design and implement performance tasks that integrate two or more competencies within or across subject areas. Observed that “complex learning is invariably slow learning, taking longer to grow than most modules last” [4]. Integrative assessment has the ability to lower the volume and stakes of summative assessments sooner in the learning process. This creates opportunities for students to learn from mistakes, to develop skills and understanding over time, and to generate dialogue between students and teachers about learning progress.

Several learning competencies in three (3) senior high subjects were combined namely, Statistics and Probability, Reading and Writing, and Practical Research 1. The mastery level of these combined learning competencies is deliberately assessed through a “Research Paper”. Research paper is the integrative performance output utilized for this study which covers different content standards and performance standards in three integrated learning areas. The study aims to determine the efficacy of using integrative performance output to improve learners’ academic performances in the three integrated subjects and help them overcome the burden of accomplishing different tasks in this Pandemic.

STATEMENT OF THE PROBLEM.

The study aims to determine the efficacy of an Integrative Performance Output in the academic performances of the STEM learners in three (3) SHS subjects. Specifically, it sought to answer the following questions;

1. What is the performance level of the learners in terms of their performance task in the three (3) combined subjects through the Integrative Performance Output?
2. Is there a significant difference among the performance of the learners in the combined subjects using an integrative performance output?
3. What is the significant relationship between the performance of the learners in Practical research 1 and Reading and Writing?
4. What is the significant relationship between the performance of the learners in Practical research 1 and Statistics and Probability?
5. What continuous improvement program can be recommended in order to sustain the efficacy of the study?

RESEARCH METHODOLOGY.

The researcher used a Quantitative research design which is an approach used to determine the level of performance of the learners after accomplishing the Integrative Performance Output of the three (3) subjects; Statistics and Probability, Reading and Writing and Practical Research 1. Correlational research method was used in order to determine the significant relationship of the performances of the learners among subjects utilizing integrative performance task/output.

The respondents for the study were chosen using the purposeful sampling technique, which is a non-probability sampling technique. The study included thirty-four (34) STEM students enrolled in the same topics for the School Year 2020-2021. They were picked from among other STEM sections because they had the lowest mean percentage grade in the third quarter. Subject teachers in the three (3) integrated disciplines evaluated the potential of merging their different learning competencies to create an integrated performance output as illustrated in figure 1.

Figure 1
Integrative Learning Competencies of Three (3) Senior High School Subjects

SHS SUBJECT	LEARNING COMPETENCIES	INTEGRATIVE PERFORMANCE OUTPUT
Practical Research 1	1. designs a research project related to daily life	RESEARCH PAPER
	2. selects relevant literature	
	3. synthesizes information from relevant literature	
	4. writes coherent review of literature	
	5. presents written review of literature	
	6. describes sampling procedure and sample	
Reading and Writing	1. Explains critical reading as looking for ways of thinking	RESEARCH PAPER
	2. Identifies claims explicitly or implicitly made in a written text a. Claim of fact, b. Claim of policy, and c. Claim of value	
	3. Formulates evaluative statements about a text read a. Formulates assertions about the content and properties of a text read, and b. Formulates meaningful counterclaims in response to claims made in a text read	
	4. Identifies the unique features of and requirements in composing texts that are useful across disciplines a. Book Review or Article Critique, b. Literature Review, c. Research Report, d. Project Proposal, and e. Position Paper	
	5. Identifies properties of a well-written text a. organization, b. coherence and cohesion, c. language use, and d. mechanics	
Statistics and Probability	1. distinguishes between parameter and statistic	RESEARCH PAPER
	2. identifies sampling distributions of statistics (sample mean).	

In examining the connections in curriculum, the evidence suggests that integrated assessment approaches are likely to improve the overall student experience by presenting curriculum as an integrated, holistic opportunity for students to engage with the overarching aims and intention of the programme. [3].

Research Paper was the Integrative performance output used for this study. Learners submitted their full research paper to practical research and reading and writing teachers. While, Chapter 4 of their research paper was submitted to their statistics and probability teacher. The Teachers in Practical research and Reading and Writing used rubrics in assessing the research paper of the learners based on their learning competencies and the statistics and probability teacher checked the statistical treatment of data used by the learners.

The scores of the students were recorded and tabulated to determine the level of performance of each learner in the three (3) subjects. Pearson Product-Moment correlation was used to determine the significant relationship between the performances of the learners in Practical Research 1 versus Reading and Writing and Practical Research 1 versus Statistics and Probability through an Integrative Performance Output. Likewise, F-test was used to determine the significant difference among three (3) subjects utilizing an integrative performance output.

RESULTS AND FINDINGS.

1. Performance Level of the learners in the three (3) combined subjects using the Integrative Performance Output

Table 1 illustrates the performance of students in the performance task through the integrative performance task/output utilize by the three (3) subjects.

Table 1
Performance Level of the Learners in the Three (3) Subjects through Integrative Performance Task/Output

Performance Task	Statistics & Probability		Reading & Writing		Practical Research 1	
	f	%	f	%	f	%
Outstanding (90-100)	18	52.94	27	79.41	26	76.47
Very Satisfactory (85-89)	16	47.06	7	20.59	8	23.53
Satisfactory (80-84)	0	0	0	0	0	0
Fairly Satisfactory (75-79)	0	0	0	0	0	0
Did Not Meet Expectation (Below 74)	0	0	0	0	0	0
TOTAL	34	100	34	100	34	100
OVERALL MEAN GRADE	91 OUTSTANDING		92 OUTSTANDING		93 OUTSTANDING	

Data showed that most of the learners performed “Outstanding” in the Integrative Performance Output in all the combined three (3) subjects. 18 or 52.94% performed “Outstanding” in Statistics and Probability, 27 or 79.41% of the learners in Reading and Writing performed “Outstanding” and 26 or 76.47% of learners performed “Outstanding” through an Integrative Performance Output (Research Paper).

Data showed that through an integrative performance output helped students in maintaining an excellent academic performance in the three subjects with the computed overall mean of 91% for Statistics and Probability, 92% for Reading and Writing and 93% for Practical Research 1.

This implies that using an integrative approach in assessing the students' performance greatly helped improve the overall student experience by presenting curriculum as an integrated, holistic opportunity for students to engage with the overarching aims and intention of the program. [3]

2. Difference in the Performance Level of the learners in the three (3) subjects using the Integrative Performance Output

Table 2 indicates the significant difference in the performance level of the learners on the three (3) subjects utilizing Integrative Performance Output.

Table 2
Difference in the Performance Level of the Learners in the Three (3) Subjects Using the Integrative Performance Output

Performance Output	Mean	F-Value		Remarks	Decision
		Computed	Tabular		
Practical Research 1	93	4.81	3.702	Reject	Significant
Statistics & Probability	91				
Reading & Writing	92				

Based on the results of Analysis of Variance (F-test), the computed value of 4.81 is greater than the tabular value of 3.702 using 0.05 level of significance, the claim was rejected and concluded that there is a significant difference among the students' performance in three (3) subjects; Practical Research 1, Statistics and Probability and Reading and Writing using an integrative performance task.

Data implies that an integrative performance task of two or more subjects affects the students' performance based on an integration of knowledge within and across learning areas, the provision of new information and a requirement to explain issues using a combination of knowledge from the learning competencies of the integrated subjects [1].

3. Relationship between the performance of the learners in Practical research 1 and Reading and Writing using the Integrative Performance Output.

Table 3 indicates the relationship between the performance of the learners in Practical research 1 and Reading and Writing using the Integrative Performance Output.

Table 3
Significant Relationship Between the Performance of the Learners in Practical Research 1 and Reading and Writing using the Integrative Performance Output.

Research Output	Mean	Pearson's r	Correlation	t-statistics		Remarks	Decision
				Comp	Tabular		
Practical Research 1	93	0.89	High Correlation	11.04	0.339	Reject	Significant
Reading & Writing	92						

Based on the results of Pearson's r of 0.89, it can be deduced that there is a high correlation between the students' performance in Practical Research 1 and Reading and Writing through the use of Integrative Performance Task/Output. Furthermore, since the computed t-statistics of 11.04 is greater than the tabular value of 0.339 using 0.05 level of significance, the claim was rejected and concludes that there is a significant relationship between the students' performance in Practical Research 1 and Reading and Writing using an integrative performance task.

Data implies that the utilization of an integrative assessment affects the students' performance since similar learning competencies were integrated in these two subjects. Thus, students applied their knowledge in Reading and Writing subject in terms of technical and academic writing towards their research paper.

4. Significant relationship between the performance of the learners in Practical research 1 and Statistics and Probability using the Integrative Performance Output.

Table 4 indicates the relationship between the performance of the learners in Practical research 1 and Statistics and Probability using the Integrative Performance Output.

Table 4
Significant Relationship Between the Performance of the Learners in Practical Research 1 and Statistics and Probability using the Integrative Performance Output.

Research Output	Mean	Pearson's r	Correlation	t-statistics		Remarks	Decision
				Comp	Tabular		
Practical Research 1	93	0.82	High Correlation	8.10	0.339	Reject	Significant
Statistics & Probability	91						

Based on the results of Pearson's r of 0.82, it can be deduced that there is a high correlation between the students' performance in Practical Research 1 and Statistics and Probability through the use of Integrative Performance Task/Output. Furthermore, since the computed t-statistics of

8.10 is greater than the tabular value of 0.339 using 0.05 level of significance, the claim was rejected and concludes that there is a significant relationship between the students' performance in Practical Research 1 and Statistics and Probability using an integrative performance task.

Data implies that the utilization of an integrative assessment affects the students' performance in terms of Chapter 4. It shows that the students were presented and computed statistical treatment of data for their Research paper using their knowledge in Statistics and Probability.

DISCUSSION.

This paper demonstrated that when performance task of two or more subjects or learning areas were combined, it helped to improve overall student performance among the combined subjects. There integration is when students were able to connect what they are learning in one subject area to related content in another subject area. John Dewey claimed that learning could be more meaningful if content areas are blended for curriculum and instruction. The use of unifying themes and real-life activities could lead to more relevant learning. The quality of learning outcomes is able to integrate information across disciplines instead of acquiring them in isolation.

The study showed a significant relationship of performance of the learners in two combined subjects. In the subject Reading and Writing, learners were taught of formulating evaluative statements about a text they read, formulating assertions about the content and properties of a text read and literature review. These learnings were incorporated and used by the learners in writing their review of related literature in the subject Practical Research 2. Likewise, in Practical Research 2, students were taught how to write chapters 1 to 5 of their research which the knowledge they got from Reading and Writing could be applied like identifying properties of a well-written text in terms of organization, coherence and cohesion, language use, and mechanics.

The integrated approach of a performance task is a vehicle that allows teachers to design lessons and activities that meet practical knowledge for the students. In addition, with this study, it is clear that integrative learning brought by an integrative performance task is important to prepare students to deal effectively both with the complexity of every learning area and the challenges facing the broader society today and in the future.

CONCLUSIONS.

In this paper, through providing curriculum as an integrated, holistic chance for students to connect with the goals and intention of easing up student tasks, an integrative approach to evaluate students' performance dramatically improved the overall student experience. The students' performance is influenced by the integrative performance output of two or more subjects, which

includes the integration of knowledge within and across learning areas, the creation of new products, and the requirement to explain issues using a combination of knowledge from the integrated subjects' learning competencies. The use of an integrative output is important in assisting students in applying their knowledge from the Reading and Writing subject in terms of technical and academic writing to their research paper in the Practical Research subject. Likewise, by applying their knowledge in Statistics and Probability, students were able to improve their performance in Chapter 4 in terms of presenting and computing statistical treatment of data.

Hence, in this pandemic, the integrative performance output contributed to the learners' academic performances in the three integrated disciplines and assisted them in overcoming the pressure of completing many activities.

Research Output

Integrative Strategic Testing and Reporting – (Project iSTAR)

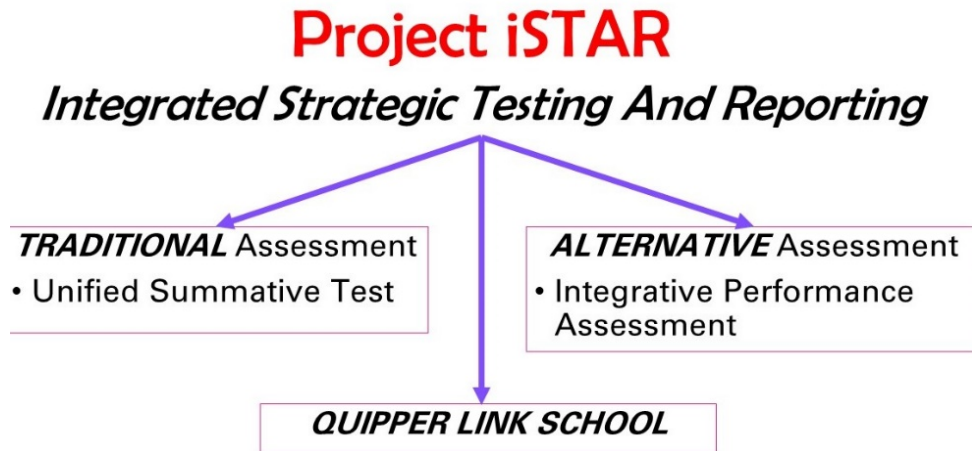
Background and Rationale:

Through DO 12, s. 2012, assessment should be holistic and authentic in capturing the attainment of the most essential learning competencies. Assessment is integral for understanding student learning and development at the same time variations of assessment strategies is necessary, with formative assessment taking priority to inform teaching and promote growth and mastery.

Teachers need to be creative and flexible in assessing student learning, while still adhering to the principles of quality assessment practice. With safety, health, and well-being foremost in mind, assessment decisions must be made in the best interest of all learners, ensuring that all assessment activities align with the most essential learning competencies, should be reliable, valid and transparent. Assessment activities are fair, inclusive, equitable, practical and manageable for both learners and teachers. These assessment tasks should give learners arrange of ways to demonstrate their learning and provide timely and accurate information as basis for feedback.

In compliance to this DepEd Order, the MPNAG SHS teachers introduce an Integrated Strategic Testing and Reporting for Mathematics and Research subjects or Project iSTAR.

Project iSTAR Framework



Objectives.

This project iSTAR aims to:

1. Develop a unified objective assessment (summative test) for SHS Mathematics and Research subjects.
2. Encourage teachers to develop an integrative performance task activity for students in Statistics and Probability, Practical Research 1, III and all other subjects that have something in common.
3. Provide scholarly reports on students' assessment in a form of quarterly summative test.
4. Use the summative test report for students' intervention and teacher's pedagogical improvement.

Acknowledgements: The author would like to thank his colleagues for their valuable comments and input.

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Integer Partitions using Generating functions and their Applications in Graph Theory

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Abstract

Integer partitions is one of the main subjects in Mathematics and which is taught at various levels in colleges and Universities. The subject was introduced by Leibniz and Euler. The chief aim of this paper is to give the ideas about Euler generating function which is the main tool to learn various properties of partitions of integers along with bijective function. Bijection is used to find out various identities and it always helps to prove results. It is argued that Mathematica software is important to learn and teach the subject. We give some applications of integer partitions in graphs theory.

1 Basic concepts of Integer Partitions

A partition of a positive integer n is defined as a way of writing n as the sum of positive integers. We denote the number of partitions of n by $P(n)$. An explicit formula for $P(n)$ valid for all positive integers n was discovered by Rademacher in 1937 [1], but since it is a complicated infinite series and is not needed for the purposes of this paper. The theory of integer partitions is a rich source of identities, bijections, and interrelations at the confluence of number theory, combinatorics, algebra, analysis, and the physical sciences. Let $p = (p_1 + p_2 + p_3 + \dots + p_k)$ where p is any partition denote a generic partition, with integer parts $p_1 \geq p_2 \geq \dots \geq p_k \geq 1$. It is very interesting to overview the concepts of Integer Partitions Function $P(n)$. The integer Partition function $P(n)$ is essentially the quantity of all integer partitions of whole number n . The total number of integer partitions is finite and n is always itself a partition. A **PartitionFunction** $P(n)$ of integer n , $n \geq 1$, is a non increasing sequence $\{n_1, n_2, n_3, n_4, \dots, n_k\}$ such that $P(n) = \sum_{j=1}^k n_j = n$, where n_j is the part of the partition $P(n)$ and $P(-n) = 0$, $P(0) = 1$ by default and $P(1) = 1$.

There are two common diagrammatic methods to represent partitions: as Ferrer diagrams [2], named after Norman Macleod Ferrers, and as Young diagrams [3], named after the British mathematician Alfred Young. Both have several possible conventions. In the Figure 1, partitions of the integer 5 are given with the corresponding Ferrer diagrams.

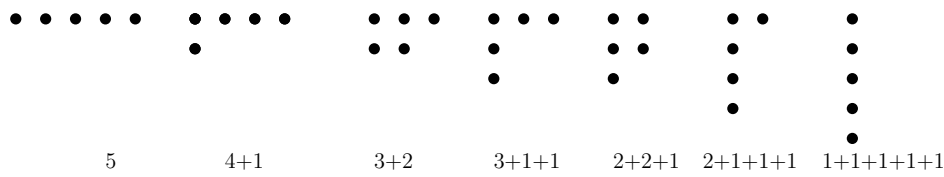


Figure 1: Ferrer diagram for integer 5

We can use *Mathematica* to find different sets about Integer Partitions.

In[1]: `IntegerPartitions[n]` gives a list of all possible ways to partition the integer n into smaller integers.

In[1]: `IntegerPartitions[5]`

Out[1]: `{{5}, {4, 1}, {3, 2}, {3, 1, 1}, {2, 2, 1}, {2, 1, 1, 1}, {1, 1, 1, 1, 1}}`

In[2]: `IntegerPartitions[n, k]` gives partitions into at most k integers.

In[2]: `IntegerPartitions[8, 3]`

Out[2]: `{8}, {7, 1}, {6, 2}, {6, 1, 1}, {5, 3}, {5, 2, 1}, {4, 4}, {4, 3, 1}, {4, 2, 2}, {3, 3, 2}}`

In[3]: `IntegerPartitions[n, {k}]` gives partitions into exactly k integers.

In[3]: `IntegerPartitions[8, {3}]`

Out[3]: `{{6, 1, 1}, {5, 2, 1}, {4, 3, 1}, {4, 2, 2}, {3, 3, 2}}`

1.1 Generating Function

There are many ways to formulate different properties in number theory, enumerative combinatorics and in discrete mathematics but one of the ways is generating function which helps to give a precise formula to various terms. Generating functions were first introduced in 1730 by Abraham de Moivre to solve the problem of common linear repetition. In order to generalize a formal power sequence to several asymmetric numbers, information about an infinite multidimensional array of numbers can be encoded. Generating functions work better when a discrete sequence is known [7]. Any problem which has a sequence in the result can be converted into a generating function. For this task, let us say an infinite sequence $\{e_0, e_1, e_2, e_3, \dots, e_r, \dots\}$ can be expressed as a function $f(x) = \sum_{i=0}^{\infty} e_i x^i$ or $f(x) = e_0 + e_1 x + e_2 x^2 + \dots + e_r x^r + \dots$, where e_r is number of ways to get r - objects. Any finite sequence can be used in the same manners as $\{e_0, e_1, e_2, e_3, \dots, e_r, 0, 0, 0\}$ is a finite sequence but can be expressed as infinite sequence. Every finite or infinite sequence can be expressed as a generating function which gives a polynomial. Some sequences with their generating functions are given in Table 1. Binomial theorem plays a huge role in the solving of any generating function which

Table 1: Some sequences with respective generating functions

Sequence	Series
$\{1, 1, 1, 1, 1, \dots, 1, \dots\}$	$1 + x + x^2 + x^3 + \dots$
$\{0, 1, 2, 3, \dots, n\}$	$1 + x + 2x^2 + 3x^3 + \dots + nx^n$
$\{0, 2, 4, 6, \dots\}$	$1 + 2x + 4x^2 + 6x^3 + \dots$
$\{1, 3, 5, 7, \dots\}$	$x + 3x^2 + 5x^3 + 7x^4 \dots$
$\left\{\binom{\beta}{0}, \binom{\beta}{1}, \binom{\beta}{2}, \binom{\beta}{3}, \dots, \binom{\beta}{n}\right\}$	$\binom{\beta}{0} + \binom{\beta}{1}x + \binom{\beta}{2}x^2 + \binom{\beta}{3}x^3 + \dots + \binom{\beta}{n}x^n$

is stated as

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}$$

which is the expansion of the function $(x + y)^n$ and $\binom{n}{i}$'s are the coefficients of x^i 's in it.

The generalized form of this theorem is stated as

$$(1 \pm x)^\alpha = 1 \pm \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 \pm \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} x^3 + \dots + (-1)^r \frac{\alpha(\alpha - 1)(\alpha - 2) \dots (\alpha - r + 1)}{r!} x^r \dots$$

where $y = 1$. Using this theorem we can have many identities, some of them are given in Table 2. The integer partitions function $P(n)$ can be expressed using generating function. In this paper, there

Table 2: Some generating functions and there expansions

Generating Function	Expansion
$(1 - x)^{-1}$	$1 + x + x^2 + x^3 + \dots$
$(1 - x)^{-2}$	$1 + 2x + 3x^2 + 4x^3 + \dots$
$(1 - x)^{-n}$	$1 + \binom{1+n-1}{1}x + \binom{2+n-1}{2}x^2 + \binom{3+n-1}{3}x^3 + \dots + \binom{r+n-1}{r}x^r + \dots$
$(1 - x^m)^n$	$\binom{n}{0} - \binom{n}{1}x^m + \binom{n}{2}x^{2m} - \binom{n}{3}x^{3m} + \dots + (-1)^r \binom{n}{r}x^{rm} + \dots$

is a frequent use of generating function for understanding some complicated results.

2 Generating Functions and $P(n)$

Generating functions plays very important role in calculating of different identities of integer partitions function $P(n)$. There is a very efficient and popular generating function to find sequence for integer partition function that is given in [10]. As we have discussed above the partitions of any integer can be partitioned into to the positive integers less or equal to that integer. Each partition has different parts of different sizes i.e 1's, 2's, and 3's so on. For instant, $5 = 3 + 2$ is partition of integer 5 with two parts of sizes 3 and 2. For the concept of generating function, we will use sizes of k parts, that how many 1's, 2's, 3's and so on are appeared in k parts of the partition n . The generating function for the $P(n)$ will be generated using the polynomial of all the sizes available in the integer partitions of n , the polynomials for different sizes of parts are given in Table 3. Using the polynomials

Table 3: Polynomial corresponding to sizes of parts

Sizes	Polynomial
Size 1	$x^0 + x^1 + x^2 + x^3 + \dots$
Size 2	$(x^2)^0 + (x^2)^1 + (x^2)^2 + (x^2)^3 + \dots$
Size 3	$(x^3)^0 + (x^3)^1 + (x^3)^2 + (x^3)^3 + \dots$
...	...
Size k	$(x^k)^0 + (x^k)^1 + (x^k)^2 + (x^k)^3 + \dots$

from Table 3, we can combine all the factors, as all the factors are distinct so we can multiply them to form a function. Such that

$$f(x) = (x^0 + x^1 + x^2 + x^3 + \dots)(x^0 + x^2 + x^4 + x^6 + \dots)(x^0 + x^3 + x^6 + x^9 + \dots) \dots (x^0 + x^k + x^{2k} + x^{3k} + \dots) \dots$$

Using instantly Table2, we will get

$$f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \frac{1}{1-x^k} \cdots$$

As we are working on integer partitions function $P(n)$, let us assume $f(x) = \sum_{i=0}^{\infty} P(i)x^i$, where $P(i)$ is total number of partitions of integer i . So that the generating function for integer partition function is defined as

$$\sum_{i=0}^{\infty} P(i)x^i = \prod_{i=1}^{\infty} \frac{1}{1-x^i} \tag{1}$$

In above the coefficients of x^i indicate the value of $P(i)$.

We can use *Mathematica* to see the expansion of this generating function as it will be hard to expand manually.

```
In[1]: Series[Product[1/(1-x^k), {k, 1, 10}], {x, 0, 10}]
```

```
Out[1]: 1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+30x^9+42x^10+O[x]^11
```

In which the coefficients of x^i indicates the number of partitions of integer i i.e $P(3) = 3, P(6) = 11, P(10) = 42$ and $P(19) = 490$.

3 Some Identities of $P(n)$

There are so many interesting identities available in literature[6]. Euler was the one of the amazing mathematicians who did different experiments on polynomials. His one of the interesting theorems is on pentagonal numbers, known as "Euler's Pentagonal number theorem". Pentagonal numbers are the total polygon numbers which are generating recursively from a pentagon, see Figure 2. The



Figure 2: Recursive Pentagonal shapes starting from $r = 1$ then $r = 2, r = 3, r = 4$ and $r = 5$ respectively

sequence we get from recursive pentagonal shapes is 1, 5, 12, 22, 35, 51, 70.... This sequence is known as "Pentagonal numbers $\xi(r)$ ". This sequence can be calculated using the formula

$$\xi(r) = \frac{r(3r - 1)}{2} \tag{2}$$

The generating function for this sequence is

$$\frac{x(2x + 1)}{(1 - x)^3} = x + 5x^2 + 12x^3 + 22x^4 + \dots \tag{3}$$

We can use software *Mathematica* [4] to find this sequence, the code is given as

```
In[2]: PolygonalNumber[5, Range[0, 15]]
```

Out [2] : {0, 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, 287}

There is another beautiful sequence named "triangular numbers" [11] which consists of the number of dots used in the formation of a triangle. the structure is given in the Figure 3. The sequence obtained from the shapes is 1, 3, 6, 10, 15, 21, 28, ..., which can be calculated using formula $t(r) = \frac{r(r+1)}{2}$. The

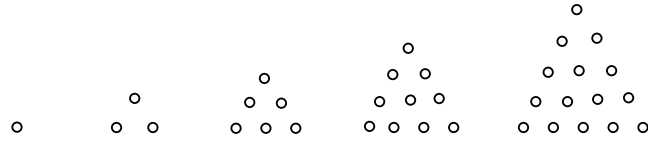


Figure 3: Recursive triangular shapes starting from $r = 1$ then $r = 2, r = 3, r = 4$ and $r = 5$ respectively

"Euler's Pentagonal number theorem" is stated as

Theorem 3.1 [6] Let $E = P(n|even\ number\ of\ distinct\ parts)$ and

$$O = P(n|odd\ number\ of\ distinct\ parts)$$

then

$$|E| - |O| = \begin{cases} (-1)^r & n = \frac{r(3r\pm 1)}{2} \\ 0 & otherwise \end{cases} \quad (4)$$

This theorem can be explained using triangular number, pentagonal numbers and ferrer diagrams. If we rearrange the pentagonal numbers given in Figure 2, we can write them in dots only, as shown in Figure 4. The pattern in the rearranged pentagonal shapes is related to the triangular shapes, so the

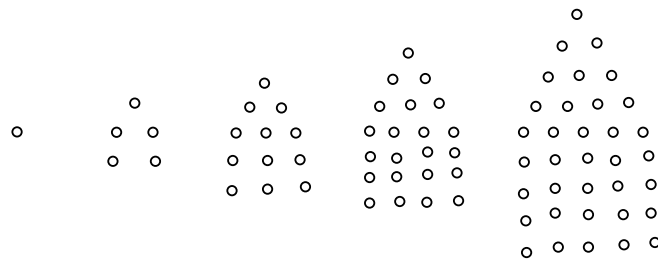


Figure 4: Rearranging Pentagonal Shapes from $r = 1$ then $r = 2, r = 3, r = 4$ and $r = 5$ respectively

formula can be modified as

$$\xi(r) = \frac{r(r+1)}{2} + r(r-1) = \frac{r(3r-1)}{2} \quad (5)$$

Now again arrange the pentagonal numbers from Figure 4 in form of ferrer diagrams by converting them into straight rows, shown in Figure 5. the partitions corresponding to the ferrer diagrams are all distinct. Now we need to establish bijection between the even number of distinct parts and odd number of distinct parts for the formation of Euler's Pentagonal theorem. Let us take different ferrer diagrams of distinct partitions, shown in Figure 6, and try to construct another corresponding ferrer diagram by removing the smallest part and place all the dots of that part at the right side of the parts

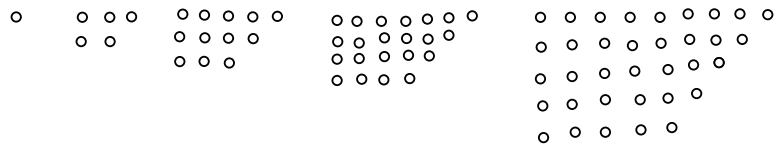


Figure 5: Rearranging Pentagonal Shapes from $r = 1$ then $r = 2, r = 3, r = 4$ and $r = 5$ respectively into Ferrer diagrams, the partitions corresponding to the ferrer diagrams are: $1, 3 + 2, 5 + 4 + 3, 7 + 6 + 5 + 4, 9 + 8 + 7 + 6 + 5$ respectively

from top to bottom. In the procedure, we see first two obtained diagrams are of again distinct parts, but the third diagram won't. Every ferrer diagram can be converted into other ferrer diagram of the same integer and there are two type of situations we yield. First will be in the form of first two examples in which parts of the obtained ferrer diagrams are less than or equal to the original diagram and the second will be in the form of third example in which the parts in the obtained diagram is greater than the original one. In the opposite direction, we will take a point from some large segments and create

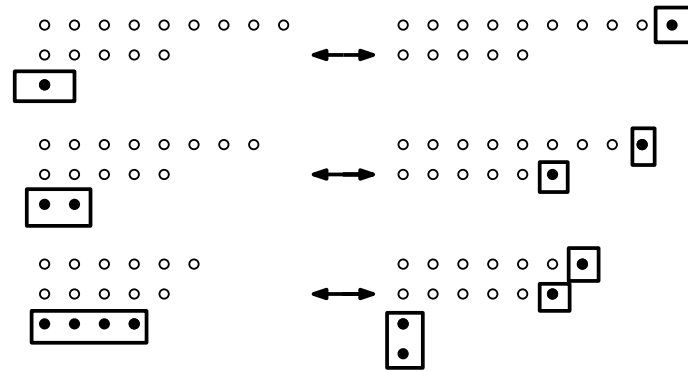


Figure 6: Three examples with transformations

a new small row. A well-defined number of points to move will be the number of rows separated by a single point, starting with the largest row. In other words, we will find the most accurate diagonal of the Ferrer graph. Figure 7. Now we need to decide which transformation is used and when? for

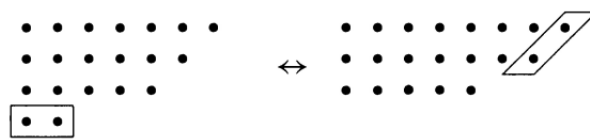


Figure 7: Inverse transformations

this, we check if the rightmost diagonal is shorter than the last row? If it is, then move it otherwise move the latter. The Ferrer graphs in the first case are the pentagons of size $r(3r - 1)/2$ dots that we considered in the beginning of this section. here we say $e(n)$ is added to make the theorem complete for this case, so we have

$$n = \frac{r(3r - 1)}{2} \tag{6}$$

However, there is a case where the above modification does not create a valid Ferrer graph, i.e., when the smallest row actually intersects the right diagonal in the lower right corner of the graph, and the row is the same length or point longer than the diagonal, Figure 8. In the second case, the Pentagon's

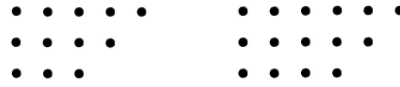


Figure 8: Inverse transformations

rectangular section contains an additional column, giving a total size of $r(3r + 1)/2$ dots. We say $e(n)$ is added to make the theorem complete for this case, so we have

$$n = \frac{r(3r + 1)}{2} \tag{7}$$

We have described a change that divides each partition n into an odd number of different parts and even number of distinct parts except these pentagonal partitions. Therefore using equations 6 and 7, we have

$$|E| - |O| = \begin{cases} (-1)^r & n = \frac{r(3r \pm 1)}{2} \\ 0 & otherwise \end{cases}$$

where $e(n)$ is 0 unless $n = \frac{r(3r \pm 1)}{2}$ for some integer r , in which case it shall be 1 if the number of parts is even and -1 if odd. This proves Euler's pentagonal number theorem.

4 Application of $P(n)$ in Graph theory

In the following, we give few results for the structural properties of graphs G_n , where G_n is defined in [9] using integer partitions, see Figure 9(a), graph G_3 is given for $n = 3$. Here we give some theorems.

Theorem 4.1 For the graph G_n , the total number of vertices $N(G_n)$ is given as

$$N(G_n) = \sum_{i=1}^n P(i), n \in \mathbb{Z}^+ \tag{8}$$

where $P(i)$ is the number of partitions of integer i .

Proof: We use mathematical induction to prove the theorem. It is trivial to see that $N(G_1) = P(1) = 1$ Suppose that it is true for $n = m$. We have to show that the statement is also true for $n = m + 1$. As

$$N(G_m) = \sum_{i=1}^m P(i), m \in \mathbb{Z}^+ \tag{9}$$

$$N(G_m) + P(m + 1) = P(1) + P(2) + P(3) + \dots + P(m) + P(m + 1). \tag{10}$$

By using 1 in (10), we get

$$\prod_{p=1}^1 \frac{1}{1-x^p} + \prod_{p=1}^2 \frac{1}{1-x^p} + \dots + \prod_{p=1}^m \frac{1}{1-x^p} + \prod_{p=1}^{m+1} \frac{1}{1-x^p} \tag{11}$$

$$= \sum_{i=1}^{m+1} \prod_{p=1}^i \frac{1}{1-x^p} = \sum_{i=1}^{m+1} P(i) \tag{12}$$

Now, according to the graph in Figure 9(b)

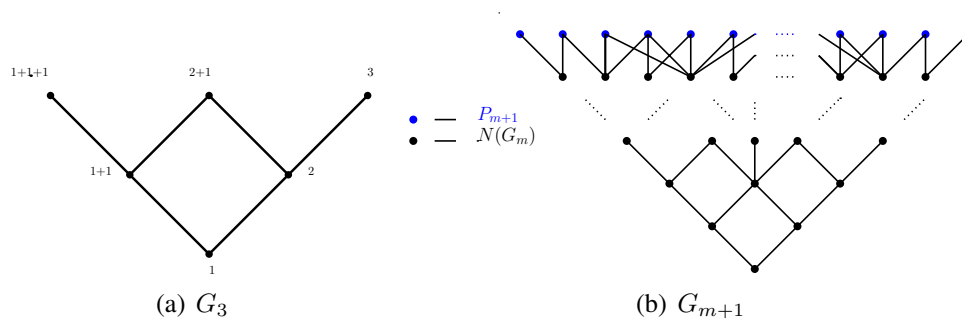


Figure 9: Integer based graphs

$$\implies N(G_{m+1}) = \sum_{i=1}^{m+1} P(i)$$

This completes the proof.

Here is another result in the form of theorem.

Theorem 4.2 Let $m(G_n)$ is the total number of edges of the graph G_n , then

$$m(G_n) = (n - 1)P(0) + \sum_{i=1}^{n-1} (n - i)P(i) \tag{13}$$

where $n \geq 1$ and $P(0) = 1$ by default.

Proof: Let $m(P(i))$ is the number of edges that can produce by $P(i)$, where $P(i)$ is partition of integer i , $1 \leq i \leq n$. As the number of 1's in all partition for the integer n is exactly same as $m(P(n - 1))$. Let $\psi(n)$ denotes the number of 1's in all partition of n ; $n \geq 2$, then

$$\psi(n) = \sum_{z=0}^{n-1} P(z) = m(P(n - 1))$$

Here $m(P(n - 1))$ denotes the number of edges that are added in G_{n-1} graph to produce G_n .

$$\implies \sum_{k=2}^n m(P(k - 1)) = \sum_{k=2}^n \sum_{z=0}^{k-1} P(z)$$

$$\implies m(G_n) = \sum_{k=2}^n m(P(k-1)) = \sum_{k=2}^n \sum_{z=0}^{k-1} P(z)$$

So to prove given result we have to show that

$$\sum_{k=2}^n \sum_{z=0}^{k-1} P(z) = (n-1)P(0) + \sum_{i=1}^{n-1} (n-i)P(i)$$

We use mathematical induction to prove this result. for $n = 2$,

$$\sum_{k=2}^2 \sum_{z=0}^{k-1} P(z) = (2-1)P(0) + \sum_{i=1}^{2-1} (2-i)P(i)$$

$$\sum_{z=0}^1 P(z) = P(0) + (2-1)P(1)$$

$$P(0) + P(1) = P(0) + P(1)$$

Now for induction step, let us assume that the result is true for $n = r$.

$$\implies \sum_{k=2}^r \sum_{z=0}^{k-1} P(z) = (r-1)P(0) + \sum_{i=1}^{r-1} (r-i)P(i)$$

We have to prove that it is also true for $n = r + 1$.

$$\sum_{k=2}^{r+1} \sum_{z=0}^{k-1} P(z) = (r+1-1)P(0) + \sum_{i=1}^{r+1-1} (r+1-i)P(i)$$

$$\sum_{k=2}^r \sum_{z=0}^{k-1} P(z) + \sum_{z=0}^{r+1-1} P(z) = (r-1)P(0) + P(0) + \sum_{i=1}^r (r-i)P(i) + \sum_{i=1}^r P(i)$$

$$\implies \sum_{k=2}^r \sum_{z=0}^{k-1} P(z) = (r-1)P(0) + \sum_{i=1}^{r-1} (r-i)P(i)$$

. This implies that

$$m(G_n) = (n-1)P(0) + \sum_{i=1}^{n-1} (n-i)P(i)$$

We give the relationship between the number of vertices and the number of edges.

Theorem 4.3 Let $N(G_n)$ and $m(G_n)$ be the number of vertices and edges respectively then

$$m(G_n) = m(G_{n-1}) + N(G_{n-1}) + P(0) \tag{14}$$

Proof: Using (8) & (13), $m(G_{n-1}) + N(G_{n-1}) + P(0) = (n-2)P(0) + \sum_{i=1}^{n-2} (n-1-i)P(i) + \sum_{i=1}^{n-1} P(i) + P(0)$

$$\implies (n-2+1)P(0) + \sum_{i=1}^{n-2} (n-i)P(i) - \sum_{i=1}^{n-2} P(i) + \sum_{i=1}^{n-1} P(i)$$

$$\begin{aligned} \implies (n-1)P(0) + \sum_{i=1}^{n-1} (n-i)P(i) - (n-(n-1))P(n-1) - \sum_{i=1}^{n-1} P(i) + P(n-1) + \sum_{i=1}^{n-1} P(i) \\ \implies (n-1)P(0) + \sum_{i=1}^{n-1} (n-i)P(i) - P(n-1) + P(n-1) \implies m(G_n) \end{aligned}$$

This completes the proof.

5 Conclusions

We have briefly reviewed the basic concept of integer partitions. The idea of generating function is the basic tool to generate partitions of any finite integer. Another important topic which is covered in the paper is the use of bijective techniques which helps to prove results in getting new results and particularly identities for partitions of integers. At the end of this paper, we give a few new results for the applications of the partitions and particularly $P(n)$ which is used to find out the structural properties of a family of graphs.

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Learning Congruence Through Ornaments And Tiling

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***Abstract:** In order to investigate movement in the plane and congruent transformations, instead of traditional logic development, we can conveniently use intuitive involvement in the geometric ideas of ornaments with the dynamic geometry software GeoGebra and 3D software Tinkercad. In this paper, we will explore selected examples of ornaments and segmented geometric patterns. We bring up some ideas about what a teacher of mathematic can find in geometric patterns and how he can fit them into his teaching curriculum. In this paper, we present a few proposals of methodologies for teaching symmetry in the plane with digital technologies, manipulative activities, and controlled exploration. We also present the results of the research of this method.*

1. Introduction

In math, finding patterns is extremely important. Patterns make tasks simpler because problems are easier to solve when they share patterns. Once we recognize a pattern, we can then use the same problem-solving solution wherever that pattern exists.

In coding – like in math - patterns are made from ideas. Mathematicians and computer programmers use patterns to express themselves and to make their work more efficient. For example, they might use loops to allow for the repetition of a sequence of code multiple times.

The highly symmetric ornaments from real-life provide interesting examples of geometric construction for use in the mathematics classroom. Symmetry is the basic concept of harmony in all cultures, found in churches, mosques, and minarets throughout the world. Understanding the mathematics behind the symmetry and construction of patterns is the key to then using these intricate patterns as motivational tools in the classroom.

Geometric methods needed to construct the tiling have the potential of introducing students to traditional techniques of straightedge and compass constructions as well as modern geometric design with computer software like GeoGebra. By comparing the traditional with the modern, each student may be challenged to use their individual creativity to design their own pattern.

2. Symmetry as a tool for improving picture perception and spatial ability

Spatial ability is one of the most widely studied domains of cognitive ability. Broadly defined, the spatial ability comprises the processes involved in perceiving, memorizing, and manipulating mental representations of visual scenes, including two-dimensional and three-dimensional objects and the relationships between them (see [8] and [10]).

Developing the spatial imagination of schoolchildren is an important task in the educational process, especially with regard to the practical application of acquired competencies in real life [8]. Some studies state that there are certain time periods that are particularly favorable for the development of spatial imagination. When these periods are missed, one loses the opportunity to develop his/her abilities to the level given by genetic predispositions (see [5], [10], and [11]). There are two periods in childhood, which are connected with the rapid and intensive evolution of spatial

abilities. The first of them is the period between 5th and 6th years of life and then, the second period occurs between 10–15 years, which is the second half of compulsory school education. While the first, pre-school period is the time of spontaneous manipulations with materials and tools, in the second, school plays the important role. This is the period of structured and knowledgeable growth, the period of evolution of the ability of synthesis and abstract deduction. It's the optimal period in which to develop student's spatial abilities and to stimulate and train their spatial imagination [3].

Investigation of ornaments and tessellations can help the students understand geometry terms like basic shapes, sides, vertices, and interior angles of a polygon. Tessellations introduce pupils to slightly advanced concepts like irregular shapes, their dimensions, surface areas of irregular polygons, and complementary shapes. Tessellations for kids are a way to make math easy. Because of repetition, patterns are usually easy to identify, but making a pattern – or even describing how it is made – requires a much deeper level of thinking.

3. Using IT technology in geometrical education

The use of digital technologies in solving and expressing problems allows us observe, explore, and formulate conjectures about possible suppositions, and verify those. While it is difficult to say which skills or concepts develop what aspect of computational thinking, it is not hard to see that these can lead to developing some of the useful habits. For instance, abstracting information and describing them as variables in a computer program helps develop one's mind to think of factors in the real world as mathematical variables.

Repetition in geometrical patterns is an opportunity for practicing for-loop control flow statement as well as block structure of the code. We'll show how the geometric ideas of ornaments could be explored in the classroom with the dynamic geometry software *GeoGebra* and 3D software *Tinkercad*. Also, with the aid of modern tools students may gain further appreciation to what was accomplished by artists and geometers of bygone centuries who possessed the simplest of tools.

Tinkercad codeblocks

*Tinkercad*¹ is free software that can handle the majority of situations that beginners would see in both modeling and 3D printing environments. An illustrative user interface with a limited toolbox makes *Tinkercad* more intuitive than *SketchUp*² and easier to learn which is an excellent asset in an educational setting. Challenges in individual or group development to design a structure that can be printed in 3D play a noticeable role in the motivation of the students. Research [2] founded, that there was a positive relationship between pupil's perception of *Tinkercad* and pupil's computational thinking skills.

Tinkercad Codeblocks coding graphical language is similar to *Scratch*³. Solid shapes can be stretched, reshaped, copied and pasted, grouped, and varying shapes can be connected to design new shape blocks. Blocks for basic generator shapes of an ornament give the readable structure of the code and push pupils to procedural thinking⁴. Moreover, we can have library objects as an

¹ <https://www.tinkercad.com>

² <https://www.sketchup.com/>

³ <https://scratch.mit.edu/>

⁴ Tenedorio-Carty, K. (2019). Code-Generate patterns in Tinkercad, Hour of Code 2019, <https://blog.tinkercad.com/hour-of-code-2019>, available on [Tinkercad for Education](#),

output of programs. Thus, if a student or teacher created a complex geometric object, anyone can reuse it in many other designs.

This is what we often do while designing a complex pattern – draw a part of a construction, copy it on a separate paper and add to it some elements, then copy it again and add something else. If one of the steps is wrong, we can always go back to the previous drawing and start again.

GeoGebra

Dynamic software GeoGebra⁵ is a freeware software designed for primary education. Thanks to a long history of sharing materials on common space you are free to reuse or edit any project from the other teachers, students, or GeoGebra enthusiasts. Together with [3] and [7], we can conclude that GeoGebra is basic tool for developing digital literacy and computation thinking in mathematical middle school education.

Benefits of using dynamic geometry tools in the classroom:

- Saving a lot of time with precise drawing.
- Deductions and creation of perceptions. Active cognition develops memory, understanding the context, constructive thinking, and motivation of the pupils.
- Explaining and examining all possible relative positions of given elements.

The geometrical patterns could be designed by using GeoGebra command [sequence](#)⁶ with arbitrary geometrical objects, such as polygon, spline, or conic. Command *sequence* returns list, that could be the new object for designing frieze, rosette, or wallpaper pattern. The code in *Algebra View* or in *GGBScript* is similar to the block structure in *Tinkercad* explained in the previous paragraph. As an example, we use the frieze on Figure 1. This symmetrical geometrical pattern is created by using the triangle *ABC* (name *Triangle1*) and vector *u* for translation.

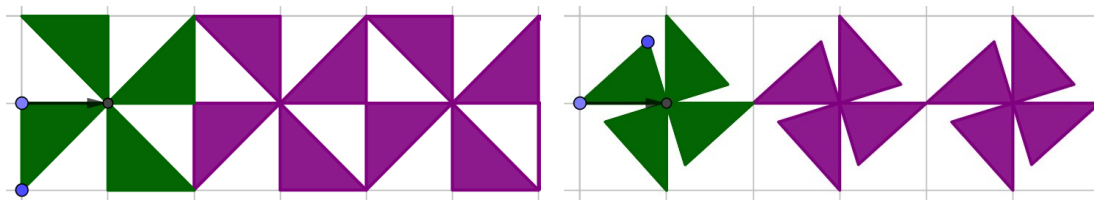


Figure 1 Two variants of frieze created from triangle by using the code

$$\text{Ornament} = \text{sequence}(\text{Rotate}, (\text{Triangle1}, 90^\circ, B), i, 1, 4)$$

$$\text{Frieze1} = \text{sequence}(\text{Translate}(\text{Ornament}, 2*j*u), j, 1, 3)$$

Moving with vertices of base triangle *ABC* helps to understand the geometry of the pattern. We could easily inspect the special position of the triangle (generator) with more global symmetries of the frieze.

⁵ <https://www.geogebra.org/>

⁶ https://wiki.geogebra.org/en/Sequence_Command

4. Geometrical patterns and symmetry groups

A mathematician while designing geometric patterns may use precise geometric constructions executed with a compass and ruler. He may also investigate shapes included in geometric patterns, proportions, and relations between particular elements of a pattern, symmetries global and local. In this case, we deal with classical constructive geometry and the tessellation theory [9].

Analysis of a wallpaper pattern – or even describing the wallpaper symmetrical groups – requires knowledge of abstract algebra, experience, and perfect visual perception. The best introduction is to start with a simple ornament and its symmetry. We can take a star with 8 vertices, a square, a long kite, a pentagon, or a photo of symmetrical object. For instance, rosettes in Figure 2 have rotation symmetry as well as mirror symmetry. Evidence of these invariants is necessary for further work.

Rosettes

Pattern with a central point, without any translations or glide reflections but, may have rotations about the central point or reflections in lines through the central point. Thus, rosettes fall into two major categories. These ornaments that contain only rotations form a cyclic group C_n and those that contain rotations and reflections are from dihedral group D_n . Index n denotes the number of symmetries.

Rosettes are typical in gothic and Islamic architecture. Figure 2 shows hexagram in Spanish Synagogue in Prague with D_6 symmetry and D_{12} rosette from the neo-gothic Catholic church in Prague. Is worth mentioning here that in our times, the gothic tracery is no longer developed while Islamic patterns are still used, and new patterns are still created [9].



Figure 2 left: hexagonal rosette in Spanish Synagogue, Prague (D_6 symmetry); right: dodecagonal rosette on Church of St. Ludmila, Prague's Vinohrady, 12 identical petals (D_{12} symmetry)

Frieze

The pattern on a belt that is repetitive in one direction has symmetries from Frieze group. Such patterns occur frequently in architecture and decorative art. The study of frieze patterns reveals that they can be classified into seven types according to their symmetries. The notation F_i^j give us reference to type of mirroring line (horizontal, vertical, glide reflection) and presence of reflection in point. Another notation used the IUCr symbol for more complex wallpaper groups. Frieze on Figure 1 has only 2-fold reflection ad translation, which means F_2 symmetry (UICr symbol $p2$).



Figure 3 Frieze group F_1^3 - absence of reflection in point, glide reflection ($pllg$)

Wallpaper pattern

A wallpaper pattern contains translational symmetries in two independent directions. These patterns are classified by the smallest angle of rotation that appears in the pattern. By a property known as the crystallographic restriction, the only angles that may appear in rotations for these patterns are 60° , 90° , 120° , 180° , or the pattern may have no rotational symmetry. If each of these patterns is considered to continue repeating in all directions then their symmetry groups are infinite. It can be shown that there are exactly seventeen symmetry groups for these types of patterns and they are referred to as the crystallographic or wallpaper groups [4].

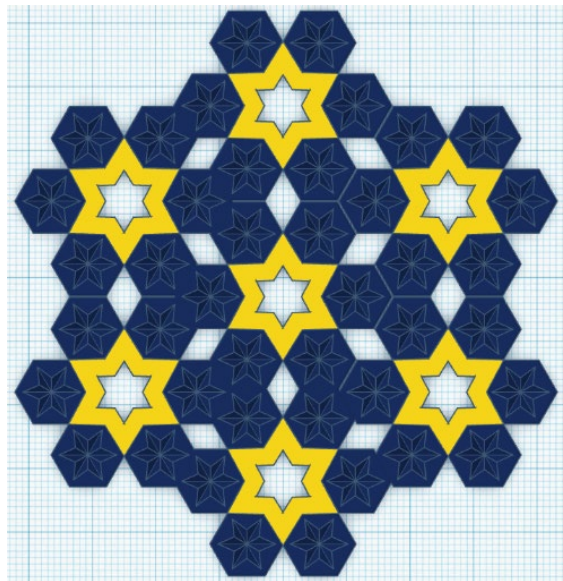


Figure 4 Fragment of the hexagonal tiling, pattern with symmetry group $p6m$, designed in *Tinkercad*

Hermann-Mauguin (also called IUCr) notation of the wallpaper group is given by appearance of the symmetry. It begins with either p or c , for a *primitive cell* or a *face-centered cell*. Digit n indicates the highest order of rotational symmetry: 1-fold (none), 2-fold (reflection in point) ... The next two symbols indicate symmetries relative to the main translation axis of the pattern. The symbols are either m , g , or 1 , for mirror, glide reflection, or none. Many groups include other symmetries implied by the given ones⁷.

One may notice that pattern in Figure 4 is the periodical repeating of one hexagonal ornament. Thus, to construct this pattern we need to construct the content of one tile and then use multiple copies of it. This way we can easily plan how a given space can be covered with this pattern. These polygons form a tessellation of the plane. Each of these polygons has the same symmetry lines as the pattern on it. This tessellation forms the hidden geometry of the pattern. As shown in [9] the same tessellation can be used to create a few different patterns and often the same geometric pattern can be created using two different tessellations.

A geometric pattern may have global as well as local symmetries. We recommend starting with local symmetries of a tile and then investigate special points on the pattern where multiple symmetry lines cross or points that can be treated as invariant points for rotations.

5. Education of geometrical pattern

Some basics in geometry can be best understood with the help of tessellations. For kids, it's simpler to grasp things when ideas are supported by pictures and models as examples. For example, it's after they try to tile a surface with random shapes that they shall understand the considerations for irregular shapes to achieve a uniform tessellation [12].

Investigation of tessellation and geometric pattern has an educational benefit in a geometric competence in every level of mathematical education:

- Geometric constructions – dividing angles and segments into equal parts, constructing perpendicular and parallel lines, constructing regular polygons and shields.
- Polygons and tessellations – tessellations with regular or symmetric polygons and shields.
- Symmetries – global and local symmetries of patterns and tessellations.
- Transformations – translations, rotations, mirror reflections, dilations, and iterations.
- Connection with technology – all examples discussed here can be developed using traditional drawing techniques with compasses and ruler, or they can be created with geometry software – *Tinkercad*, *Geometer's Sketchpad* or *GeoGebra*.
- Opportunities for students' projects – individual or collective.

The tasks are posed to the class along with a potential sequence of problems through which the class may be directed in solving the problem:

- How should construction of the tiling begin? Is there a particular tile that may produce the overall tiling? Could you describe a strategy for constructing a pattern similar to it?
- What are the symmetries of the tile?
- What is the minimal tile that is needed to be able to reproduce the entire pattern?
- How does the introduction of color affect the symmetry of the pattern?
- Does your tiling contain the same symmetries as the original tiling? Can you find a different tiling that contains the same symmetries as your tiling?

⁷ We recommend visual tools for symmetrical wallpaper patterns: <https://eschersket.ch/> or <https://math.hws.edu/eck/js/symmetry/wallpaper.html> (cit. 2021)

6. Experiment in the classroom

The ideas for teaching symmetry were tested in two classrooms at the elementary school. Pupils of the 4th and the 5th grade were involved (the 5th grade is considered as a transitional grade between primary and secondary education). The activities took place during several classes and were focused mainly on programming in *Tinkercad Codeblocks* and *GGBScript*. Education was focused on constructivist principles. The tasks of activities were achieved in the dimension of cognitive process, understanding, application, analysis, and evaluation. During research, we used the methodology of qualitative research.

Activity 1: Designing with Algorithm in Tinkercad

The simple *Codeblocks* scripts will enable your students to create a rosette that includes text or scrawl that they can easily modify to personalize their design⁸. Encourage students to modify and remix the *Codeblocks*.

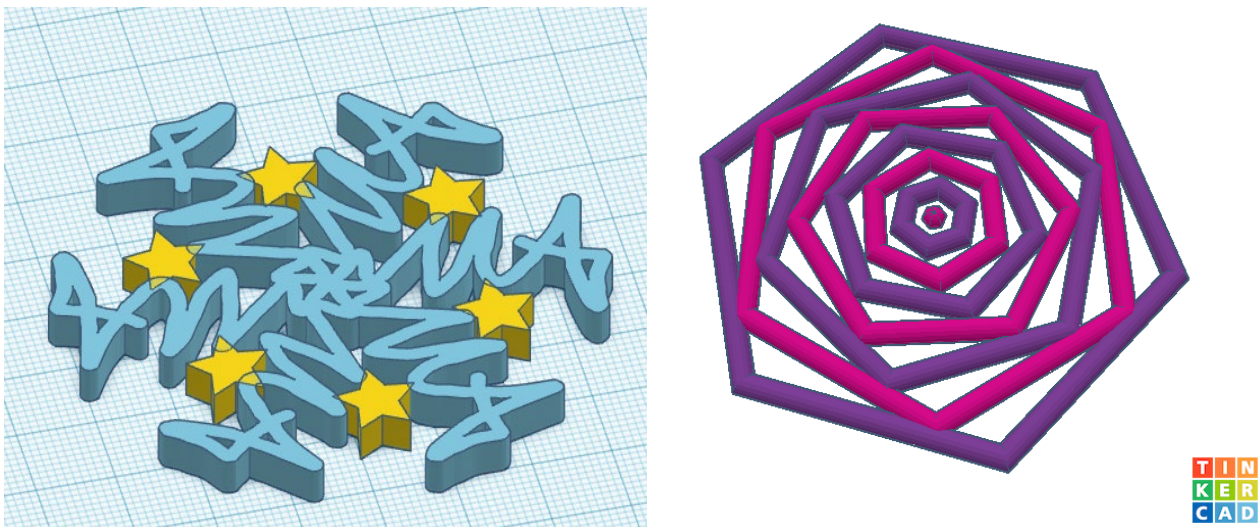


Figure 5 Rosettes with C_6 symmetry in *Tinkercad*, inspired by [Tinkercad for Education](#)

In this lesson pupils will:

- Discuss and debate an issue of human and computer interaction.
- Recognize, define, and solve computational problems in a rosette design in order to modify it.
- Explore the use of control structures to set parameters, allowing students to compute more design iterations quickly and efficiently.
- Communicate and collaborate clearly and express themselves using simple code.
- Make interesting, code-generated rosette or frieze designs that they can share through 3D printing.

⁸ <https://www.instructables.com/Designing-With-Algorithms-in-Tinkercad/>

Activity 2: Drawing symmetric shapes in GeoGebra

GeoGebra book “[Symetrie ornamentu](#)“ (Symmetry of Ornament)⁹ demonstrates finite symmetric groups C_4 , C_6 , D_4 and D_6 . During activity, students were supposed to draw in shapes in *GeoGebra* on a cartesian and isometric grid and experiment with position and shape of the basic objects. This activity significantly motivated the students as they had immediate feedback. Even weak students solved the problems with enthusiasm and search another symmetrical picture on the web independently on the teacher.

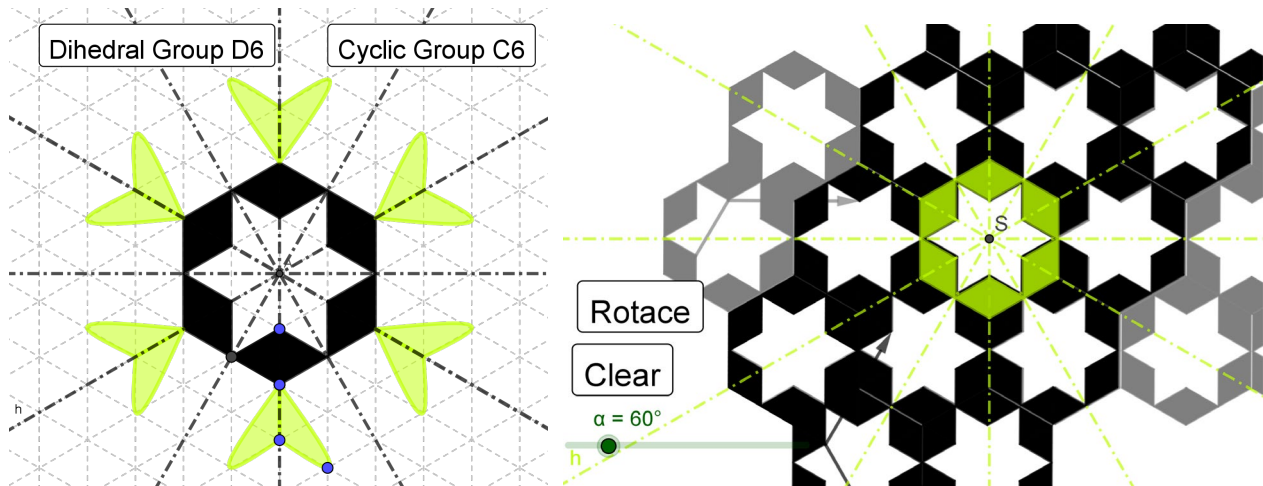


Figure 6 Ornament with 6-fold rotation and reflections in six lines (on the left). Regular hexagon (on the right), its symmetry and tilling with symmetry $p6m$.

Activity 3: Construct your own pattern with a restricted family of transformation

In this activity, we observed a significant difference between classes, especially in pace and skill. None of the students were able to determine how to construct the given wallpaper pattern without the translation.

Survey research

Research data were collected using information form, formative test, and pupil perception questionnaire about *Tinkercad* and *GeoGebra* software. The results are not truly comparable, because there were different forms of learning (present/online) and also different instructional models (transmission / constructivist).

Not surprisingly, the survey shows that rosettes are more readable for all students: 75% find the symmetry in line, 43% of the pupils find the rotation symmetry. The wallpaper pattern $p6m$ in Figure 6 were not fully recognized in the class without software.

In addition, when looking at students' perceptions of *Tinkercad*, it was determined that they were highly motivated for interest and appreciation and found *Tinkercad* to be generally useful and easy to use. Pupils appreciated the export to *.stl* file for 3D print, because “the end result is a physical object”.

⁹ <https://www.geogebra.org/m/pzb5mfku>, public GeoGebra book „Symetrie ornamentu“ (in Czech)

7. Conclusion

The role of CAD and dynamic geometric software in the teaching of geometry is discussed for a long time. Could they contribute to the development of reasoning skills and spatial ability? Will the dissemination and accessibility of these tools mean that students will be no longer consider elementary geometry as the most relevant for their future career? It's evident, that tablets and computers can't substitute the manipulatives activity and compass and ruler constructions. On the other hand, there are a lot of problems, which could be educated more effectively with software by using different solving strategies. Some of the foreseeing changes could be similar to those that the emergence and dissemination of pocket calculators already implied for mathematics education, not only about how certain problems have to be solved, but, more generally, concerning what type of techniques and problems should be considered as the true objective of mathematics education.

By introducing history, culture, and art into discussions in the classroom, students have an opportunity to see mathematics as a multidimensional embroidery woven across time and space [9]. It is indeed a stimulating exercise for students to attempt to understand the beautiful symmetry of geometrical pattern and learn the geometry required to reconstruct them.

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Socscistatistics Integration on Test and Evaluation in Statistics (SITES): Mastering Statistics Using Free Online Resource

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Abstract: *Statistics is kernel part of Mathematics designed to ease future research endeavors of students taking it. With its mastery, learners would be able to easily justify their research results in a manner that is not challenging to them at all. Traditionally, they are taught in a way that exhausting long computations take place so that each test statistic will be determined. Careful solving is necessary for them to achieve the correct values which will then affect the overall interpretation of the results. With such profound tasks, they tend to have very little confidence in achieving their understanding. This study is about shifting the mindset of learners in mastering statistics using free website called socscistatistics. This website can be easily accessed in any search engines and functions like Statistical Package for the Social Sciences (SPSS) but is more friendly to learners and start-up researchers. It employed the quasi-experimental design. To determine the overall performance of students in statistics, a pre-test was conducted. Scores obtained are subjected to 3 levels namely Poor Level (0% to 70%), Good Level (71% to 85%) and Satisfactory Level (86% to 100%). This test was done after the full discussions of statistical tools such as mean, median, mode, variance standard deviations, correlation analysis, relationship tests and difference tests. Results of the tests were used as a basis in implementing the intervention called SITES or Socscistatistics Integration on Tests and Evaluation in Statistics. A post-test was then given after 2 months of utilizing the mentioned website. Results indicated a higher performance in posttest. Results were tested at 0.05 alpha level of significance using a t-test which indicated that there was a significant difference between the means of pre-test and post-test scores of the respondents.*

1. Introduction

In the year 2012, the Department of Education of the Philippines implemented the K-12 Curriculum also known as the Enhanced Basic Education Act of 2013 (Republic Act 10533). According to the then President Benigno Aquino III, this will pave the way to most Filipino students while providing basic education that is at par with international standards. This act created the now called “Senior High School”.

In relation to this, Senior High School offers 5 tracks for which the students can choose from. Regardless of these tracks, students have “common” subjects which are called “Core and Applied Subjects”. Among these subjects is the Statistics and Probability which is a pre-requisite for the Research subjects to be taken later on by these students.

Statistics for a fact is simple. It uses simple operations from basic ones up to exponents and roots. Other than that, what makes it complicated is the thorough and step-by-step process for which the learners have to carefully do to arrive at the correct values they needed. With such delicate steps, several studies have been conducted to improve students’ performance in this particular subject.

A study conducted by [4] indicated that the rapid development and growth of technology can be of great help in enhancing student’s academic performance specifically on statistics.

They attempted to determine the relationship between Information Technology (IT) capabilities, learning experiences and performance of teaching and learning Mathematics. Developing the application called Multimedia Probability and Statistics System (MMPASS), they have found that students' learning experiences and performances of their academic achievement have been improved by using MMPASS.

Another study was designed to improve student performance in statistics through the use of flipped classroom. [7] found that those taught using the flipped classroom approach perceived the module significantly more interesting and the proportion of students who perceived the module to be difficult was roughly half that under the traditional teaching approach. Moreover, a study led by [2] revealed that peer-tutoring those students who have low performance in undergraduate statistics is another way of alleviating their statistics performance. In relation to this, [3] suggested that a mat word-problem a day improves the overall problem solving skills of students which is also another efficient way to deal with such research problem.

This research is conducted to determine the level of academic performance of the selected Grade 11 students of Highway Hills Integrated School in statistics. Upon identifying certain problems about this case, the researcher will also attempt to apply Socscistatistics Integration in Tests and Evaluation in Statistics (SITES) strategy to upscale the academic performance of the selected respondents.

2. Conceptual Framework

Statistics mastery is crucial in the future research endeavors of most students in senior high school. Without mastery on such subject, performance in research subjects will definitely be affected. Thus, resolving such problem is necessary.

This study is anchored to the Behaviorism Theory of Thorndike. According to [6] behaviorists believe external encouragement influences one's learning behavior, rewards and punishment can change one's learning performance. Researchers and scholars confirmed that Behaviorism has greatly promoted and effectively implemented in programmatic instruction, and has strongly promoted and widely applied in computing-assisted-instruction and the development of educational technology.

Also, the Theory of Diffusion of Innovations describes the steps of technology innovation, process and characteristics of accepting new technologies, as well as receivers' role in the receiving process of technological innovation. Everett M. Rogers, its proponent groups people, according to the degree of acceptance of innovation, into various stages: innovators, early adopters, early followers, late followers and laggards. Understanding the trainees' receiving capacities of new technologies can assist training institutions and trainers to design and deliver the training more effectively.

In this regard, this study uses the IPO (Input-Process-Output) framework which shows the Grade 11 students' statistics performance as the input, and the test scores before using the Socscistatistics Integration in Tests and Evaluation in Statistics (SITES) strategy. The process, was the administration of the pre-test, the implementation of the strategy for 2 months, the administration of post-test examination, and the computation for test for the difference between the pre-test and post-test scores. The output was difference between the test scores of Grade 11 students' statistics performance and after using the SITES. Below is the Figure1. which presents the Paradigm of the Study.

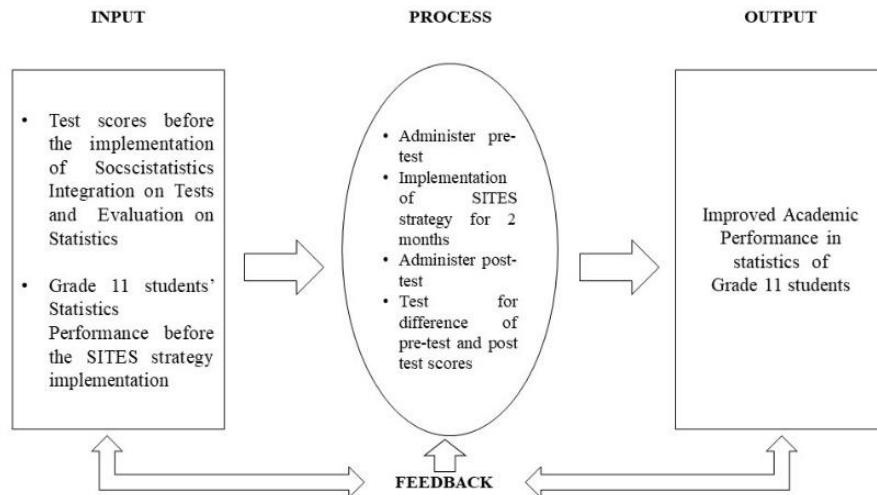


Figure 1. Conceptual Paradigm of the Study

3. Statement of the Problem

This research is conducted to determine the impact of Socscistatistics Integration in Tests and Evaluation in Statistics (SITES) in upscaling the performance on Statistics of selected Grade 11 students of Highway Hills Integrated School for the academic year 2020-2021.

Specifically, it aimed to answer the following questions:

1. What is the level of performance on Statistics of Grade 11 students before the implementation of Socscistatistics Integration in Tests and Evaluation in Statistics?
2. What is the level of performance on Statistics of Grade 11 students after the implementation of Socscistatistics Integration in Tests and Evaluation in Statistics?
3. Is there a significant difference between the results of pre-test and posttest after employing the SITES strategy?

4. Hypothesis

The hypothesis stated that there is no significant difference between the results of the pre-test and post-test of the Grade 11 students on their performance in Statistics through the use of Socscistatistics Integration in Tests and Evaluation in Statistics intervention at 0.05 alpha level of significance.

5. Scope and Limitations

This action research was limited to the one-hundred Grade 11 students of Highway Hills Integrated School for the school year 2020 - 2021. The respondents are composed of 55 males and 45 females. These students are given pre-test on Statistics after the discussion of descriptive statistics, and hypothesis testing and post-test after they have undergone the SITES intervention program for 2 months.

6. Research Design

The researcher made use of the following research design and method; quasi-experimental design, and one shot pretest and posttest design. The researcher selected to use quasi experimental because the study wanted to determine the level of performance in Statistics of the respondents and the effects of the intervention after it was utilized. The study measured the significant difference between the two assessments.

SITES is an intervention devised by the researcher to solve the said research problem. This intervention is done by utilizing the use of free site called socscistatistics to upscale the test scores of students on tests involving descriptive statistics, differences and relationship tests and interpreting results from hypothesis testing. This website functions just like an SPSS which when studied carefully on its usage, will allow easier interpretations on the part of students. Just like SPSS, its main functions include the requirement of just simple input of raw data and the rest of the work will be done by this site. Solving, interpreting and presenting results will be done and students will just have to arrange them in a manner expected by the statistics teacher.

A pretest was conducted on Statistics after the discussion of descriptive statistics, and hypothesis testing and post-test after they have undergone the SITES intervention program for 2 months. Discussions on how to place inputs in each of statistical tools and tests was done. Practice exercises were given to test their mastery on usage of the website. A monthly test to determine the state of students is then conducted to observe the impacts of the said intervention.

7. Sources of Data

The target-respondents were one-hundred (100) Grade 11 students who were enrolled in Highway Hills Integrated School during the school year 2020-2021. The respondents belong to the class of the researcher. There were 55 boys and 45 girls. The study was conducted from November 2020 to January 2021 before the start of daily discussions.

8. Instrumentation and Data Collection

This study made use of quasi-experimental design, and one shot pretest and posttest design to gather the needed data. The researcher used the Socscistatistics Integration in Tests and Evaluation in Statistics (SITES) strategy to improve the performance on Statistics of the respondents.

The data is gathered by recording a weekly evaluation of the students tests in statistics. A table graph was used to determine if there is a certain trend on the pre-test and post-test results. This action research was conducted at Highway Hills Integrated School using the selected Grade 11 as respondents.

The researcher administered a 50-item pre-test on statistics after their discussion on descriptive statistics and hypothesis testing to determine the level of performance in statistics of the respondents. Then, an action research proposal was given to the principal of the school to implement the intervention program. After which, the socscistatistics website was thoroughly discussed to the students in terms of the “how to” of its usage. This was done per topic in statistics and after each discussion, a practical exercise was given for the respondents to do. After making sure that the respondents have mastered the use of the site, a formative assessment was given.

A 50-item post-test composed of the same scope as the pre-test but of different variant was conducted after the implementation of the SITES Strategy. Both the two tests are designed by the researcher and are subjected to Cohen’s Kappa statistic to determine the inter-rater reliability. The supervisor in-charge in Mathematics was invited to rate the tests. It was found that the Kappa statistic is 0.89 justifying that there is a significant agreement between the raters signifying a satisfactory inter-rater reliability.

9. Tools for Data Analysis

To provide valid and accurate findings from the generated data, appropriate statistical tools were employed by the researcher.

For problem 1 and 2

To determine the Grade 11 students' level of performance in statistics the following formulae were used:

$$\text{Academic Performance Level in Statistics} = \frac{\text{Number of correct answers} \times 100\%}{\text{Number of questions}}$$

Academic Performance Level of the Grade 11 students was classified as follows:

Level	Level of Ability
Satisfactory	86%-100%
Good	71%-85%
Poor	70% below

Note that these cut off scores are solely based on the assigned point system by the researcher considering the passing score of 75% and taking into account that using three scales results to 33.33% per scale and adjusting it to a more appropriate scale which is not that low and at the same time not that high.

For problem 3

To determine the significant difference in the level of problem solving skills of the Grade 11 students before and after the assessment, a pre-test post-test T-test was used.

10. Results and Discussions

Level of Academic Performance of the Grade 11 Students in Statistics Before the Use of SITES Strategy

The researcher proponent administered a pre-test on statistics involving 100 Grade 11 students enrolled during the school year 2020-2021. The test aimed to determine the level of problem solving skills of the Grade 11 students. Based on the pre-test results, the students are classified into three (3) levels: satisfactory, good, and poor.

Table 3.1 reflects the results of the pre- test and the performance level in statistics of the Grade 11 students before the use of Socscistatistics Integration in Tests and Evaluation in Statistics (SITES) strategy.

Table 1
Result of the Pre-Test on Statistics

Problem Solving Ability	Frequency	Percentage
Poor	70	70%
Good	23	23%
Satisfactory	7	7%
Total	100	100%
Mean Score	22. 14	

Mean Percentage Score	44.28%
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It can be seen from table 1 that out of 100 students, only 7 of them have Very Satisfactory Level in terms of their performance level in statistics. Also, it can be observed that 23 out of 100 or 23% of the students are at Good Level. Moreover, 70 out of 100 or 70% of the students are under the Poor Level with respect to their performance level in statistics subject.

Moreover, it can be seen that there is a mean percentage score of 44.28 % (obtained from the mean score in pretest of 22.14) which is way below the passing rate of 75%.

These results are analogous in the longitudinal study conducted by [5] which revealed that there is a significant projecting influence on later mathematics achievement for self-perceived competence. Also tenacity, attitude, and positive attributional dimension of internality have an additional significant contribution.

Level of Academic Performance of the Grade 11 Students in Statistics After the Use of SITES Strategy

Table 2 presents the level of performance of the Grade 11 students in statistics subject after the implementation of Socscistatistics Integration in Tests and Evaluation in Statistics (SITES) strategy.

Table 2
Result of the Post-Test on Statistics

Problem Solving Ability	Frequency	Percentage
Poor	13	13%
Good	57	57%
Satisfactory	30	30%
Total	100	100%
Mean Score	40.15	
Mean Percentage Score	80.3%	

It is remarkable that after employing the SITES strategy, it can be observed that 13% of the students are at the Poor Level of academic performance in statistics signifying a great decline. Also, 57% of the students are at the Good Level of performance in statistics. In connection to this, 30% of the students became to the Satisfactory level in terms of their performance in statistics. Moreover, it can be seen that the mean percentage score is 80.3% (obtained from the mean score in pretest of 40.15) which has significantly amplified compared to the pre-test mean percentage score.

These results are similar on the publication of [1] which claimed that the future of mathematics education after the COVID-19 pandemic is the one with which teachers use technology to teach students rather than reverting back to the old ways of teaching process.

Difference Between the Pre-Test and Post Test Results in Problem Solving

Table 3 presents the difference between the pre-test and post-test results in level of performance in statistics of the 100 selected grade 11 students of Highway Hills Integrated School after the implementation of the Socscistatistics Integration in Tests and Evaluation in Statistics (SITES) strategy.

After the execution of the strategy, the scores on the pre-test and post-test results of 100 students were compared and analyzed to determine if there is a significant difference. T-test for two dependent means was used having 0.05 alpha level of significance and 99 degrees of freedom. Table 3 shows the significant findings of the inferential analysis.

Table 3
Difference Between the Pre-Test and Post Test Results in Statistics

Category	Mean Score	Degrees of Freedom	Computed t-value	Critical t-value	Interpretation
Pre-Test	22.14	99	5172.98	2.626	Significant
Post-Test	40.15				

Based on these results, the null hypothesis which states that there is no significant difference between the results of the pre-test and post-test of the Grade 11 students on their performance level in statistics through the use of Socscistatistics Integration in Tests and Evaluation in Statistics (SITES) strategy at 0.05 alpha level of significance is hereby rejected. The rejection of the null hypothesis is due to the fact that the computed t-value of 5172.98 is higher than the tabular value of 2.626.

Thus, there is a significant difference between the results of the pre-test and post-test of the Grade 11 students on their performance in Statistics through the use of Socscistatistics Integration in Tests and Evaluation in Statistics intervention at 0.05 alpha level of significance. From these results, it could be deduced that the use of proposed SITES strategy enhanced/improved the performance on statistics of the respondents. Hence, employing such strategy would eventually upsurge the performance level of the students.

This result is the same with [8] which signified that integrating digital technology in mathematics education uplift the level of performance of the learners. It has been suggested that teachers should have efforts in showing and guiding students on how to effectively use the digital tools rather than just merely giving them these and letting them explore on their own.

11. Conclusions

In view of the findings, the following conclusions were drawn.

1. There are 70 identified students under poor level of performance in statistics, 23 under good level and 7 under satisfactory level of students before the SITES intervention.
2. There are 13 identified students under poor level of performance in statistics, 57 good level students, and 30 satisfactory level students after the SITES intervention.

3. At 0.05 alpha level of significance, there is a significant difference between the results of the pre-test and post-test of the 100 selected Grade 11 students on their performance in Statistics through the use of Socscistatistics Integration in Tests and Evaluation in Statistics intervention.
4. Socscistatistics Integration in Tests and Evaluation in Statistics can enhance level of performance in statistics of students through time.

12. Recommendations

With the above conclusion, the following recommendations are offered:

1. Mathematics teachers shall use socscistatistics website in teaching statistics on a daily basis to improve the level of performance of the students.
2. The Socscistatistics Integration in Tests and Evaluation in Statistics (SITES) strategy shall be used as an intervention to improve the level of performance of the students under poor level.
3. The school shall constantly monitor the level of performance of the learners and provide technical assistance on improving it.
4. A similar study shall be conducted to ascertain the findings of this research.

Acknowledgements. The author would like to thank his family, specially Jen, Arjhen and Iah for always inspiring him to do better.

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Problem Based Learning approach to train pre-service teachers in mathematics education: an experience.

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Abstract

Societies around the world are focusing in motivating scientific vocations in children. STEM education is a priority in many Governments' education policies. Accordingly, actions should begin by improving pre-service teachers training, including the acquisition of skills in teaching methodologies and ICT tools, which will contribute to increase the motivation of students in STEM subjects. Among the most used methodologies for teaching STEM subjects, we consider the Problem-Based-Learning approach, because of its interdisciplinary vision and scientific working method, based on resolution of real situations. Aiming to improve STEM training in pre-service teachers, the authors conducted an experience in an initial-training mathematics teachers master course, where the pre-service teachers were asked to design an activity to be later addressed, with PBL methodology, by real students of a Madrid High-School. Considering the importance of improving teachers' training in ICT tools, it was required, the inclusion of GeoGebra in the design of the proposed activity. The experience is evaluated in terms of analysis of the problems suggested by the pre-service teachers, and of the results of the in-depth interviews conducted with several of the involved master's students, as well as with the teacher responsible for applying the problems in the High School. We appreciate that the inclusion, in the experience, of GeoGebra and other ICT tools within the PBL methodology approach, was a motivating element. Application of learning-by-doing method, led the pre-service teacher to know better about PBL methodology and GeoGebra, increasing their skills and possibilities to apply both tools in their professional future.

1. Introduction

Science, Technology, Engineering and Mathematics (STEM) education is a key approach for an increasingly complex society based on knowledge, that is why it has become a priority for most Governments. STEM education should be addressed from an early age. It is at school that children must be motivated to learn Science, Technology, Engineering and Mathematics to lead them to become responsible, creative, innovative citizens, and to make them aware of the world around them, that they can contribute to improve [15]. In order to assure the best learning process for our students, pre-service teachers must have also the best training. This involves training them in theoretical, as well as in methodology and technology [12].

The philosophy of Problem-Based Learning is to respond to a situation-problem in a real context from an interdisciplinary point of view. This makes it an ideal methodology to address the study of STEM subjects, because it combines this global and practical vision of what scientific work entails [16]. PBL is an excellent methodology to be used in mathematics education, increasing student motivation, giving theoretical mathematics concepts a real context, and the possibility of analysing their applications.

Based on previous studies we present an experience in training pre-service teachers of Secondary Education level in problem-based learning. In a way of *learning by doing*, the pre-service teachers themselves prepare a *Problem* to be applied in a real class, so that they will be capable to repeat it in the future with their own students. As part of their training process, pre-service teachers had to develop not only the *Problem* (including context, mathematical concepts needed as a base, and those to be achieved, a guide to solve it, and a solution), but also how to evaluate the task, hypothetical mistakes made by the students and how to help and guide them.

Information Technology and Communication (ITC) tools and GeoGebra, as well as the interdisciplinarity character of the proposal, were an important part of the learning and teaching process in this experience, both for pre-service teachers and for the students solving the problem.

We will start the article giving a brief context to the experience, and setting some theoretical remarks, both in what Problem Based Learning means as methodology to be used in training teachers, and in educational uses of GeoGebra and ICT in STEAM education. Then, in the next Section, we will explain the experience, the way we developed it, details of the process and ICT tools employed, and an example of a pre-service teachers proposed problem.

We finish, commenting the obtained results, in terms of the designed problems, and in the results of the in-depth interviews with several pre-service teachers, and with the school formal teacher.

2. Conceptual framework

1.1 Training pre-service teachers

In order to assure a good education to our students, the best training must be assured to pre-service teachers [11]. As a matter of fact, teachers are the final responsible of the teaching process, and so, teachers training should be a priority in every country [1]; [4]; [6].

In Spain, the requirement of reviewing the actual model of initial teacher training to make it suitable to the European one, has been established. An appropriate training in didactics and pedagogy, in ICT applied must complement theoretical knowledge of every subject.

In the case of secondary education in Spain (age 12-18), the model of pre-service teacher training is the “consecutive”. It is also applied in the surrounding countries as France, Italy, or Portugal, in it the training is centred in scientific knowledge acquired during the Bachelor degree obtained at the University [14], followed by a Master’s degree in education (in Spain, *Master of Teacher Training in Secondary Education*, is compulsory).

Nevertheless, there exist a gap between the training received at the master’s in education, and the classroom reality. During the master, a practising period of 180 hours is done at a high school, where a formal teacher tutorizes the students. It is useful for pre-service teachers, but insufficient.

In the case of mathematic learning, the problem is aggravated by the abstractness of the subject, which in many children is a lack of motivation and an obstacle to approaching the study of the subject successfully.

For all the above, it is important to give future teachers the opportunity to collaborate in teaching practices that require the use of ICT and active participation in the classroom, and not only attend theoretical classes in which these teaching methodologies are explained without reaching practice. ([2]; [3])

1.2 Problem Based Learning in secondary education students

Problem Based Learning (PBL) has proved to be a methodology specially issued to be used in the frame of STEM subjects [11].

The term PBL has been taking on different meanings and uses, which usually include [2]:

- Education takes place in the context of a situation or problem.
- Key content is learned from context and performance.
- It is part of a challenge or purpose shared by the students.

This is a methodology based in active learning, in which the students play the principal role, and the teachers acts as guide of the teaching and learning process. Knowledge is acquired in the process of solving a problem or project, as a group of scientists in a research context, or engineers working together to develop a project. PBL take the most when is combined with cooperative learning, and is a great way to motivate students, because of the real context of the problem to be solved, the challenge of self-employment in groups [7]

A problem posed in PBL for students from 12 to 17 years old, has the following common characteristics: real context, in which students feel involved; guide questions adapted to the age of the students and the educational stage, which helps students to identify the needs of the problem, the theoretical concepts to be applied, relevant data, and the search for accurate information to acquire them.

The objective of the methodology is not so much to solve the problem posed, as the process carried out for it. In fact, many times problems have open solution, or even several solutions. The acquisition of knowledge occurs when detecting the needs of the problem, performing the search for information to perform the solution process.

According to our experience in the use of this methodology with high school students, a student who approaches mathematical concepts through this system, not only assimilates them in a direct way as well (or better) than he would through a master class, but also develops other equally important skills ([5], [15], [16]):

- The student becomes more autonomous, since he has to face problems very close to real life , and has to "look for life" to be able to solve it using all kinds of "open" tools. Remember that a PBL does not have a single and guided solution but promotes the search for alternative solutions and the use of different tools. In addition, you must interact with real means: face problems that go beyond the merely mathematical, such as the organization of teamwork, decision-making, face technical problems (a file that is lost, a computer that does not work a colleague that does not come ...)
- The student increases his digital competence, especially in relation to the search for information on the web and the use of ICT tools for problem solving.
- The student becomes more collaborative. The problems presented are designed with such complexity that they require teamwork to be solved in a reasonable time.
- During the process of solving the problem, the student observes different paths and different possible solutions to it. He must evaluate and decide the best way to obtain what he considers the most adequate solution, developing the learning-to-learn competency.

- One of the deficits that students present historically in the learning of mathematics is that of the ability to face and solve problems, however simple they maybe. When facing these PBL more complex problems, whose resolution is not immediate, the student is taking an experience and confidence that will serve in the future to face simpler problems (the classic ones). A student who solves PBLs loses the fear of problems of immediate resolution, and even those apparently more complicated.

1.3 GeoGebra and ICT in Problem Based Learning

The PBL methodology supposes, as we have already established, a knowledge and skills in ICT of the students who face the resolution of the problem, and of the teachers who propose it. The search for information through Internet is fundamental in the resolution process; students will carry it out autonomously, guided to a greater or lesser extent according to their age by the questions included in the problem, and by the teacher himself. Looking for the necessary data and discarding the irrelevant ones, extracting the precise information, contrasting the materials obtained, means an important increase in digital competence for students.

In the other side, the best training pre-service teachers received in ICT, the better teaching and learning process of our scholars. That was an important point of including specialized mathematical software in the experience we show here: Master's students learned specialized in mathematical education ICT tools, making them capable to use its as teachers in the future.

Introducing the use of specific software to solve the problem posed is not necessary in PBL methodology, but it is highly recommended, since it expands the range of skills acquired in the learning process, in addition to acting as a motivating element.

There are many technology tools that can be included when considering a PBL to help in the resolution process. From the most basic such as scientific calculators, spreadsheets such as Excel, or the usual Word and Power Point (especially useful the use and learning of the Equation Editor that incorporate both applications), to more specific software such as the dynamic mathematics program GeoGebra, or Google Sketch-Up (free version).

In the case of the experience presented in this article, we have chosen to introduce GeoGebra as a common element in the proposals that the master's students had to present. Based on previous experiences [8],[9],[14], students had to include questions whose resolution went through the use of GeoGebra (functions, geometry, representation).

We consider that the use of freely available software GeoGebra is especially significant due its huge use in high schools and universities, and the millions of users all over the world [10]. On the other hand, it is important for us to transmit to pre-service teachers the spirit inherent in GeoGebra of sharing and contributing to improve for the good of the community, which we consider fundamental to be a good teacher.

It is advisable to distinguish at this point the use of ICT tools from the point of view of the student who solves the problem, and from the teacher who proposes it. In the case of the experience we show, the master's students play both roles, because, as we will see, the task they developed was to propose a problem to use the PBL methodology in the classroom, and to develop a hypothetical solution given by the students who solved the problem.

An important aspect to consider is the current situation arising from the COVID19 pandemic, which demands the use of collaborative remote work environments. In Spain there have been many students

of secondary education who have had part of their teaching remotely during the course 20-21, so that trying to bring active methodologies to distance learning is almost essential [3]. For this purpose, ICT tools for collaborative work such as MS Teams, Google Workspace, Nearpod, or the recent GeoGebra Classroom, are extremely useful.

3. An experience in training pre-service teachers in STEM education

3.1 Experience

The experience we present here was developed during the course 2020-2021 in the frame of the subject Didactic of Mathematics of the Master in Secondary Teachers Training at Rey Juan Carlos University in Madrid. This is an annual subject, with a total of 90 hours in weekly sessions. It is the third course we apply the activity, introducing new elements every year. In the actual course, lessons were conducted in remote, due to pandemic situation. MS Teams platform was used during the sessions to that purpose, but tasks were attached in Rey Juan Carlos University (Moodle) virtual classroom platform.

The number of students who participated in the activity was 33, working in group. There were seven groups of 4 students, one group of 3 students (in this case, there were partial time students who worked together in the activity out of the classroom), and one student with adapted evaluation who participated individually.

The task consisted in proposing a problem within the methodology PBL to be solved by real students aged 15 years (Grade 10) from *Montpellier School* in Madrid. This is a charter school, located in the metropolitan area, with around 1500 students from middle and working social class. This school introduced the project “1x1” (one student, one computer) in 2008 in Grades 7 to 10, and its teaching and learning project is based on cooperative learning, and active learning methodologies.

Formal teacher at Montpellier school sets the curriculum theme the problem must be related to, and the number of hours the students at school are able to dedicate to work in it. The proposed problem should contain the problem statement with the context, questions to guide its resolution, one solution of the problem, and hypothetical errors students from the high school were supposed to commit. This course we focussed our attention in GeoGebra, so that pre-service teachers had to include at least one question within the proposed problem to be performed using GeoGebra. They could ask the students at school to use GeoGebra also to present results (graphics, geometric representations, ...), if needed.

PBL methodology has been applied in this high school in mathematics learning for years, combined with cooperative learning. GeoGebra is used frequently in mathematics classes, every student has his own computer, as we mention above. A template is given to each group of children to guide them in the resolution of the problem. This template is provided to master’s students to be considered.

The activity took place on November 2020. Pre-service teachers participated in three training session before designing the problem, two in GeoGebra, and one in PBL. During two more classroom session, they work in groups developing the task. As we mentioned above, this master course was in remote learning, and students collaborated doing the task with MS Teams, supervised by the university teacher. After that, they worked autonomous in the activity for two weeks. Feedback was given in a first stage by the formal teacher from the high school, who analysed the proposed problems from a didactic point of view, making recommendations to pre-service teacher, to make them able to be applied with his students at the high school. Students prepared the final version during Christmas holydays, considering the reviews proposed by the teacher.

When the application of PBL in the high school with the proposed problems has concluded, formal teacher provides final feedback to pre-service teachers with the results of the experience. Students at Montpellier school filled a form at the end of the activity, with questions about the level of difficulty of the problem, understanding of the problem statement, questions aimed at reflect on the problem, and a final open question about their personal opinion about the experience.

3.2 Example of a proposed problem

As an example, we consider the following problem proposed by a group of students (we summarize the statement). The topic was trigonometry, and the problem was posed to students ages 15-16:

“Title: Repairing my sailboat. *You have inherited the sailboat from your grandfather, but it needs several major repairs. You must change the forestay, mast, and jib of the ship. But beware, there are measures that you will not be able to take, you do not have such a high ladder! That is why we provide you with a map of the boat, and we ask you to measure the elements (angles and lengths) that are closest to the deck of the boat*”.



Figure 1: Map of the boat

In order to repair the required ship parts, students must find: (1) Length of the forestay, (2) Mast height, (3) Dimensions of the jay (measurement of its sides) and its total surface (with the help of GeoGebra). In addition, students will be required to check the results by creating an scheme of the problem with GeoGebra, using the application's geometric tools.

The most complex part is the measurement of the angles of the jay to obtain the area of fabric necessary to repair it. To solve the problem, once aware that the angle between the mast and the ship deck is 90° , students must take this data into account and apply their knowledge of trigonometry, in addition to a new theorem (sine theorem). In addition, they should use their knowledge of the GeoGebra tool to find the area of the bow sail (jay). Figure 2 shows an example of solving the problem using GeoGebra. You can also see it in <https://www.geogebra.org/m/w4cnza5q>.

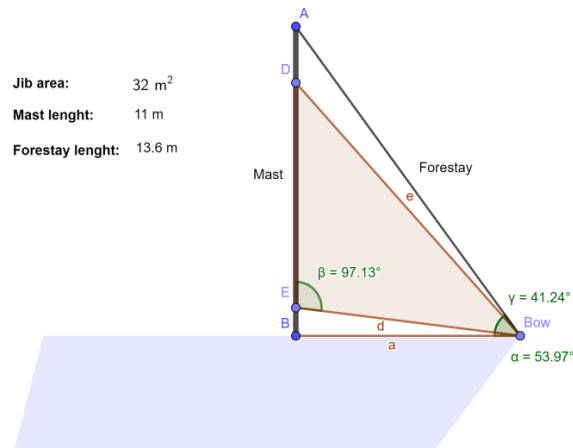


Figure 2: Proposed solution of the problem given by a group of students

3.3 Assessment

Every proposed problem is evaluated by the school formal teacher (70% of the task score), considering the point of view of its application in his classroom, in terms of complexity, quality of the problem as PBL, adequate level, duration, etc.; and by the teacher at University (30% of the task score), attending if the problem followed the given recommendations, analyzing the proposed solution, the inclusion of GeoGebra, and the statement of the problem and the given solution from the didactic point of view. Not every proposed problems were applied at school, formal teacher chose the ones who better fitted with the characteristics of his students (level of difficulty, extension, hours to be dedicated...).

4. Results

Once the set of all the videos was available, a qualitative analysis of them was carried out in a discussion group moderated by a professor from the Rey Juan Carlos University and integrated by 3 university professors from the area of Didactics of Mathematics, and 4 Grade 10 mathematics teachers. The discussion group held working meetings to analyze all the videos taking into account among other aspects if they conformed to the contents and learning standards for the stage, the curriculum approved in the legislation in force in Spain, and if they could be useful for the teachers of the subject [13].

Number of problems suitable to be used in the first stage (no revision needed)	1/9		
Number of problems with percentage of revision needed	20%-30% revision	30%-40% revision	More than 80% revision
	5	2	1
Number of problems suitable to be used in the second stage (after revision)	7/9		

Table 1: Number of problems suitable to be applied at high school in Grade 10

Most of the reviews suggested by the teacher of the high school, were related to the level of difficulty, knowledge necessary to solve the problem, necessary information, type of guide questions (writing, number), context of the problem.

To analyse the opinion of the students after the activity, and the view of the teacher about the experience, in-depth semi structured interviews were conducted with seven students from different groups, and with the teacher.

Some of the responses were:

Question: What advantages do you think using PBL has for teaching mathematics in secondary education?

"I think it is a great way to bring children closer to what they will find both in their studies at University, in the event that they take them, and in their day to day, both professional and personal"

"When a problem is posed for us, we usually have the tools to find ways to solve it, but not directly how to solve it; with the PBL methodology, we work in the search for those ways to solve it, so I think it is a great enhancer of critical thinking and the autonomy of students".

Question: What is your opinion of introducing this methodology as part of teacher training in STEM?

"The work process to develop our PBL proposal has allowed me to better understand which teaching situations are appropriate for each age group.

Question: Comment globally on your opinion about this activity (How do you value the possibility of carrying out with real students the proposed *Problems*? Has the feedback given by the teacher been important for you?).

"In my case, after all my years as a student, I think it is very interesting to know these methodologies and ICT tools, not so widespread and that can contribute a lot to the student".

"It has changed my perception of teaching mathematics after working on this project"

Teacher of the high school answered similar questions in another interview. Due the limitations of the article, we'll just transcribe the answer to questions 2 and 3:

Question: What is your opinion of introducing this methodology as part of teacher training in STEM?.

"PBL learning cannot be conceived without the use of ICT tools. If only for the mere fact of having to search for information on the Internet (inherent in any PBL type problem), the use of this type of technology is essential.

Beyond this, the environment that creates this method is ideal for the use of other ICT tools such as GeoGebra (especially in the areas of functions and geometry), Excel (in the management of tables, functions and statistics), Google Sketch-Up (for spatial geometry) or the usual Word and Power Point (especially useful the use and learning of the Equation Editor that incorporate both applications).

Question Comment globally on your opinion about this activity.

"I have a positive opinion, no doubt. Historically, in all educational centers, the teachers of the science area have been the spearhead, both in the use of new technologies in the educational field and new teaching-learning methods. This is because of the very essence of STEM subjects, based on the scientific method, rigorous research, and the use of ingenuity to solve problems. What better for any subject in this field than to put students in simulated situations similar to those they will encounter in the future?"

I believe that every teacher in general, but in this area, must carry in "his curriculum" a training in this method of teaching and a willingness to put it into practice when exercising his profession.

5. Concussions and future work

The type of activities such as the one we present here, represent a significant improvement in the training of future teachers. After the experience, the students of the Master's degree learned about the PBL methodology, increasingly present in teaching STEM subjects, and received training in a fundamental tool for teaching mathematics, GeoGebra. The method of learning by doing was applied, proposing themselves a problem as a task with which to apply the methodology. We bring them the option to do it with real students, thus bringing the real practice in the classroom to the teachers in training. Finally, the participation of the high school teacher, who applied his professional criteria to correct the proposed problems and give us the feedback of the application in the classroom, was invaluable, being highly appreciated by the students.

The success of the experience is also measured in terms of motivation. The students of the Master valued very positively the experience, as reflected in the responses of the interviews carried out to a sample of them. The common opinion is that this activity had allowed them to know and apply new methodologies and technology, permitting them to act as practicing teachers. In addition, five of the students continued with the research in PBL methodology, dedicating their Master thesis to this topic.

An 89% of the problems proposed in this course 20-21 by the pre-service teachers, were considered suitable to be applied at Grade 10 mathematical classroom by the discussion group who analyzed the problems, as we presented in the results. The level reached of some proposed problems was so good, that the teacher included them in the library of PBL class, becoming part of the project of teaching mathematics of the high school.

As proposals for improvement for later courses, we would like to introduce elements of GeoGebra Classroom to manage the resolution of the proposed problems, once modified with the indications of the teacher of the center. In this way, the students of the center could work on the resolution remotely, tutored by the corresponding group of pre-service teachers, who would be supervised both by the teacher of the center and that of the university. In this way, we could incorporate direct feedback to the activity, making it even more real.

The philosophy of sharing and collaborating in the educational community is present, so that another line of improvement in the future will be to publish the PBL made, together with the GeoGebra files and ICT tools used, so that they can be used and improved by other teachers, thus giving an unparalleled dimension to this type of experience.

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Research on Development of Teaching Material Put Emphasize on Retroductive Inference to Develop Statistical Thinking in Mathematics Education

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Abstract: *The purpose of this study was to develop teaching materials for 6th-grade elementary school students in Japan, focusing on retroductive inference, to develop their statistical thinking and explain its use. Recently, MEXT (Ministry of Education, Culture, Sports, Science and Technology) has made “Programming Education” compulsory at elementary schools in Japan. The purpose of “Programming Education” in Japan is to develop the programming thinking of students based on their computational thinking and enhance their academic ability in teaching and learning a particular subject, rather than learn coding. However, few studies have been conducted on the development of programming thinking based on computational thinking, for enhancing the academic ability of students. “Retroduction” is effective in promoting computational thinking. Also, the development of mathematical and computational thinking is interacted by one another. Therefore, we developed a teaching material to enable the students to engage in retroductive inference. A context and situation setting were provided to them to make retroductive inferences. The results indicated that it seems reasonable to assume that the teaching material promoted their computational thinking and developed their statistical thinking by creating phases for retroductive inferences.*

1. Introduction

The development of student’s mathematical thinking through mathematics lessons is essential. To do so, the teachers need to give due consideration, not only to mathematics teaching materials but also to the lessons, keeping in mind the teaching-learning contents and the mathematical processes.

Recently, MEXT (Ministry of Education, Culture, Sports, Science and Technology) has made “Programming Education” compulsory at elementary schools in Japan. The purpose of “Programming Education” is to develop the “programming thinking” of students, based on computational thinking, and enhance their academic ability in learning a particular subject, not learn to code [1]. It is necessary to develop their “programming thinking” based on their computational thinking and enhance their academic ability in mathematics education. However, surprisingly few studies (for example, [2] and [3]) have been conducted to develop the “programming thinking” of elementary school students based on their computational thinking and enhance their academic ability.

It suggests that the development of mathematical thinking and computational thinking are interacted by one another [3]. Therefore, it is necessary to discuss whether the use of computational thinking in finding solutions to problems has any role in the development of mathematical thinking [3]. Therefore, we developed teaching materials that could aid in the development of mathematical thinking, particularly statistical thinking, by promoting their computational thinking.

“Retroduction” is effective in promoting computational thinking [4]. Therefore, in this paper, we adopt “Retroduction” advocated by Charles Sanders Peirce.

The term “Retroduction” can be defined as “heuristic reasoning” and studying of facts for devising a theory, to explain the surprising fact observed [5, 6]. For example, students guess a relationship between times and water level is liner function while water is drained out from the bottom of a bottle full of water. However, they realize a relationship between these is a quadratic function if they carry out an experiment in which water is drained out from the bottom of a bottle full of water. In this experiment, the surprising fact observed by them is that the relationship between these is a quadratic function. If hypothesis the cause of what relationship between these is quadratic function is water pressure were true, what relationship between these is quadratic function would be a matter of course, hence, there is reason to suspect that hypothesis the cause of what relationship between these is quadratic function is water pressure is true.

We focused on “Retroduction” advocated by Charles Sanders Peirce and developed teaching materials to promote the computational thinking of the students by creating phases on which the students were asked to draw retroductive inference, to develop their statistical thinking.

The purpose of this study was to develop teaching materials and emphasize retroductive inference to develop the statistical thinking of students and discuss it.

2. Theory

It suggests that the development of mathematical and computational thinking is interacted by one another [3]. Therefore, it is necessary to discuss whether one should make full use of computational thinking in locating and finding the solution to problems to develop mathematical thinking [3]. In pursuance of the aim, we developed teaching materials that could enable the development of mathematical thinking, particularly statistical thinking, by promoting computational thinking of elementary school students.

According to Wing, computational thinking is analytical thinking and its essence lies in abstraction and automation [7, 8].

For example, students created programs by using Scratch developed by MIT (Massachusetts Institute of Technology), to construct an equilateral triangle (see Figure 1) by applying the facts that the lengths of the three sides of the triangle are equal and all interior angles are equal to 60° [4].

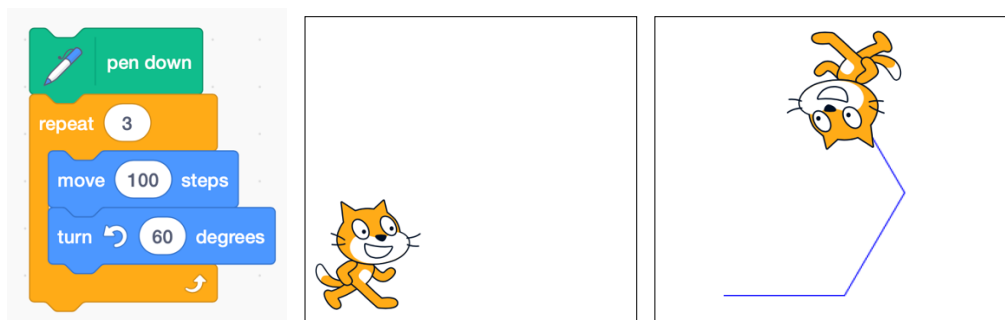


Figure 1 Program to Move A Character and A Movement of A Character

At this juncture, students extracted specific elements (i.e., sides and angles of equilateral triangles) from a variety of elements (i.e., vertices, areas of equilateral triangles, etc.), and created a program to move the character. This is a process of abstraction and automation.

However, if we look at this program (see Figure 1) which students created, we can see that it is not correct. Therefore, students considered improving the program, so that the character could be moved properly. Students repeated and refined the abstraction and automation by inferring why the character doesn’t move as expected in such a case (see Figure 1). This concerned drawing

inferences as to why things did or did not go as expected by observing the results, which promotes student's computational thinking.

Therefore, in this paper, we adopt "Retroduction" advocated by Charles Sanders Peirce to promote their computational thinking. The term "Retroduction" can be defined as heuristic reasoning and studying facts and devising a theory to explain the surprising fact observed [5, 6]. Drawing on the concept of Retroduction proposed by Pierce, the surprising fact observed by the students was that an equilateral triangle cannot be constructed using Scratch. Students formulated the following hypotheses: there is an error in setting the lengths of three sides; there is an error in setting interior angles, etc., to elucidate the reasons why they were unable to construct an equilateral triangle. Students found the right program by verifying the hypotheses they formulated, as "Retroduction" is effective in promoting computational thinking [4].

Based on the above example, it can be said that "Retroduction" works appropriately in promoting computational thinking. Therefore, we argue that "Retroduction" is a viewpoint to develop teaching material. Specifically, we created several phases which they inferred upon using retroduction.

3. Development of Teaching-Material

We developed teaching material for learning and teaching statistics in mathematics for 6th-grade elementary school students in Japan.

As mentioned in the preceding section, the development of a viewpoint on the development of teaching material in the creation of several phases of retroductive inference by elementary school students involves computational thinking. Therefore, it is necessary to provide a context and situation so that students draw retroductive inferences, for example, why the automatic vending machine in Japan rejects coins put in the coin slot.

The automatic vending machine in Japan accepts coins based on their weight, diameter, thickness, and quality of the material. The weight, diameter, and thickness of coins may be altered because of any flaw, soil, etc. Therefore, the automatic vending machine is expected to rarely reject coins dropped into the coin slot.

To start the lesson, the teacher presents a picture to show how the automatic vending machine rejects coins put in the coin slot. The teacher asks the students, "why the automatic vending machine would reject coins put in the coin slot?" In an attempt to investigate why the automatic vending machine rejects coins, the students will draw a flow chart (see Figure 2) to elucidate its structure.

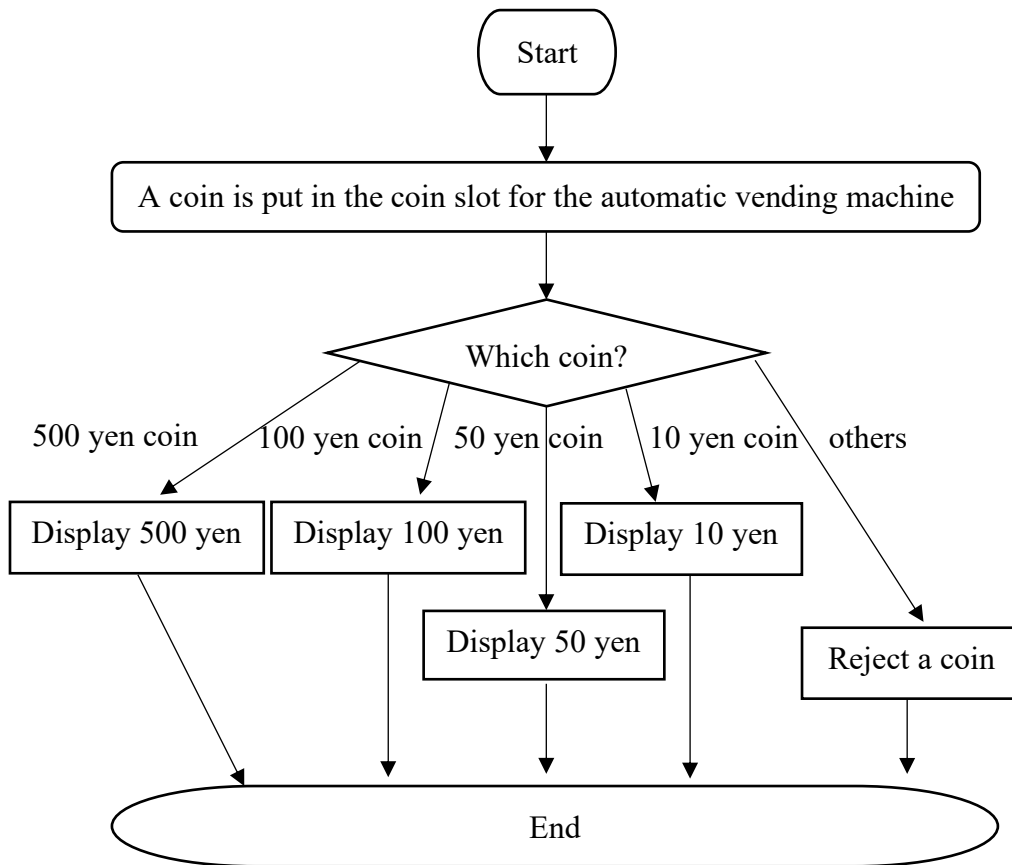


Figure 2 The processes of how the automatic vending machine recognizes coins

Students will consider how to judge coins in “Which coin?” (see Figure 2). With regard to this teaching material, we assumed that students would solve the problem by laying particular emphasis on the weight of the coins. The teacher prepares Scout™ Series Balanced-STX222JP of OHAUS (see Figure 3) if the students emphasized the weight of the coin for solving the problem.



Figure 3 Scout™ Series Balanced-STX222JP of OHAUS

Scout™ Series Balanced-STX222JP by OHAUS (see Figure 3) compares the weight of a sample against target limits and displays results as “Under”, “Accept” or “Over”.

In Japan, the coins have fixed weights, for example, a 10 yen coin weighs 4.5 grams, a 100 yen coin weighs 4.8 grams, and so on. Therefore, students are likely to consider that “Accept” will be displayed on the screen if the balance is set at 4.50 grams as “Over Limit” and a 10 yen coin can be put on the pan. However, this is not necessary as the weight of the coins might undergo changes depending on the flaw, soil, and so on (see Figure 4).

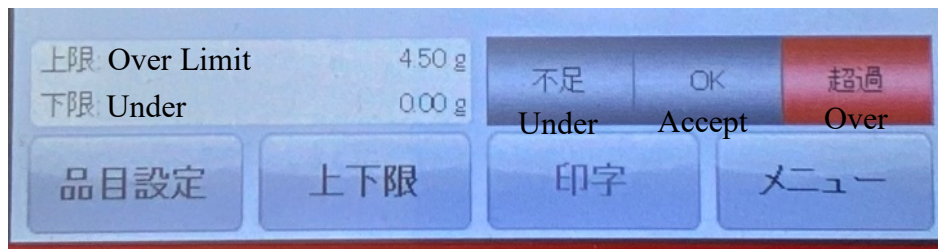


Figure 4 The Screen Display of Scout™ Series Balanced-STX222JP of OHAUS

At this time, students might think why “Over” is displayed on the screen of the balance when they set “Over Limit” as 4.50 grams and put a 10 yen coin on the pan. Students will hypothesize that the weight of a coin put in the coin slot is different from the original one and attempt to solve this problem by using a frequency table, a histogram, a dot plot, and the representative values.

4. Discussion

We discussed the teaching material developed in chapter 3 to promote computational thinking, by creating phases for retroductive inference by elementary school students and the development of statistical thinking by promoting computational thinking.

4.1 Promoting Computational Thinking

The essence of computational thinking is abstraction and automation [7, 8]. Therefore, it needs to be discussed whether repetition and refinement of abstraction and automation are promoted.

By using the teaching material, elementary school students were able to observe that an automatic vending machine in Japan would reject coins. Students hypothesized by doing trial and error to elucidate why this would occur. For example, something is wrong with the automatic vending machine, the coin put in the coin slot is counterfeit money, the weight of a coin put in the coin slot is different from the original one, and so on. At this time, students created a flow chart of the processes by which the automatic vending machine recognized the coins (see Figure 2), repetition and refinement of abstraction and automation through inferring why the automatic vending machine would reject coins to verify if their hypotheses were correct. This was a process of abstraction and automation.

For example, students extract a specific element (the weight of coins) from a variety (color, diameter, thickness of coins, etc.) and hypothesize that the weight of the coin that is put in the coin slot is different from the original one. This shows that students engage in abstract thought with regard to the rejection of the coins by the automatic vending machine.

Creation of the flow chart to depict the processes involved in recognition of the coins by the automatic vending machine (see Figure 2) was done to elucidate how the automatic vending machine judges coins to automate its programs.

From the above example, it can be assumed that setting a context and situation for retroductive inference promotes computational thinking in students.

4.2 Development of Statistical Thinking

To develop statistical thinking, it is important to solve problems and find new ones by employing statistical thinking in PPDAC (Problem, Plan, Data, Analysis, Conclusion) cycle [9]. Therefore, setting processes involving solving problems with statistical thinking through PPDAC cycle in mathematics lessons was to be explored.

The students observed the surprising fact that “Over” was displayed on the screen when they set “Over Limit” as 4.50 grams and they put a 10 yen coin on a balance (see Figure 4). Students hypothesized after trial and error that the weight of a 10 yen coin that was put in the automatic vending machine is not 4.50 grams to elucidate why “Over” is displayed on the screen.

The students investigated whether the automatic vending machine judges the 10 yen coins by their weight. This is the process of “Problem” step of PPDAC (Problem, Plan, Data, Analysis, Conclusion) cycle.

We assumed that the students solved this problem by following the steps described below.

Firstly, the students considered the data regarding weight (i.e., how many 10 yen coins should be collected ?; how to collect the weight data pertaining to the 10 yen coins? etc.) This comprised the process of “Plan” step of PPDAC (Problem, Plan, Data, Analysis, Conclusion) cycle.

Secondly, students collected the weight data pertaining to 10 yen coins and organized them in a table (see Table 1). The weight data of hundred 10 yen coins collected by the author is depicted below. This was the process of “Data” step of PPDAC (Problem, Plan, Data, Analysis, Conclusion) cycle.

Table 1 The Weight Data on A Hundred 10 yen Coins

4.42	4.43	4.45	4.45	4.45	4.45	4.45	4.45	4.45	4.46
4.47	4.47	4.47	4.47	4.47	4.47	4.47	4.48	4.48	4.48
4.48	4.48	4.48	4.48	4.48	4.48	4.49	4.49	4.49	4.49
4.49	4.49	4.49	4.49	4.49	4.49	4.49	4.50	4.50	4.50
4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50
4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.51	4.51	4.51
4.51	4.51	4.51	4.51	4.51	4.51	4.51	4.51	4.51	4.51
4.51	4.51	4.51	4.52	4.52	4.52	4.52	4.52	4.52	4.52
4.52	4.52	4.52	4.52	4.52	4.52	4.52	4.52	4.52	4.52
4.52	4.52	4.52	4.53	4.53	4.53	4.53	4.53	4.53	4.55

Thirdly, students illustrated the frequency distribution of the weight data of the hundred 10 yen coins with the help of a frequency table (Table 2), histogram (Figure. 6) and the representative values (Table 3) for these data were calculated and the variations in these data were illustrated with the help of a dot plot (Figure 5). We assumed that the students used computers to express in a

frequency table, a histogram, and a dot plot of the weight data on a hundred 10 yen coins and calculated the representative values for these data.

Table 2 A Frequency Table of The Weight Data on A Hundred 10 yen Coins

Interval		Frequency
or higher	less than	
4.42	~ 4.44	2
4.44	~ 4.46	7
4.46	~ 4.48	8
4.48	~ 4.50	20
4.50	~ 4.52	36
4.52	~ 4.54	26
4.54	~ 4.56	1
total		100

Table 3 Representative Values of The Weight Data on A Hundred 10 yen Coins

	(grams)
Mean	4.4975
Median	4.50
Mode	4.51

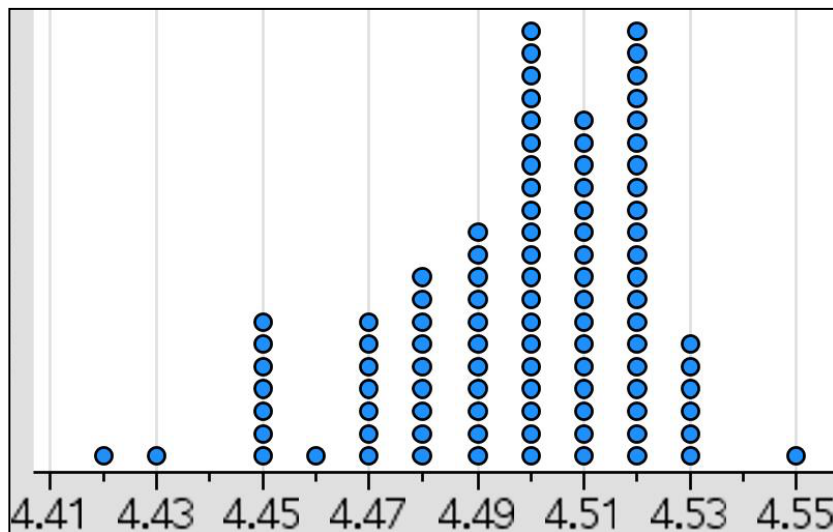


Figure 5 Dot Plot of The Weight Data on A Hundred 10 yen Coins

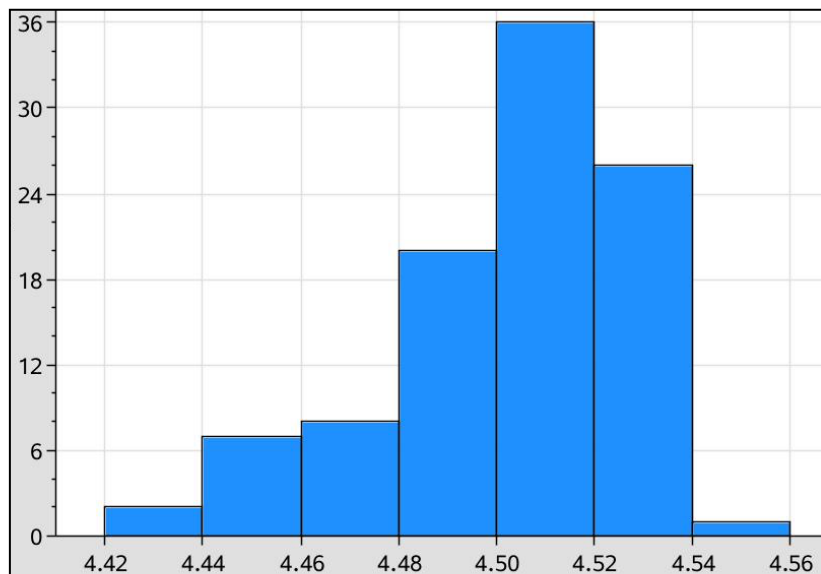


Figure 6 Histogram of The Weight Data on A Hundred 10 yen Coins

The process of “Analysis” step of PPDAC (Problem, Plan, Data, Analysis, Conclusion) cycle is described below:

Students found out what the frequency table, histogram, dot plot, and the calculated representative values represented, for example, they found what 10 yen coins weighing 4.53 grams should be ignored with the frequency table because a relative frequency of 10 yen coins weighing 4.54 grams or higher but less than 4.56 grams was 0.001, they found what 10 yen coins weighing 4.58 grams may be existed based on variations of the weight of 4.50 grams or less with the dot plot and so on.

Finally, students evaluated the graphs to determine values for their adequacy so that they could set the weight of the coins, in grams, for “Over Limit” and “Under Limit” to find out the number of 10 yen coins which would be adequate. The students concluded that the weight of “Over Limit” should be set as 4.53 grams, the weight of “Over Limit” should be set as 4.58 grams and the weight of “Under Limit” should be set as 4.42 grams, etc. During the processes of setting the weight of “Over Limit” and “Under Limit”, students found that setting them with representative values was not adequate.

Students must investigate not only the weight of the 10 yen coin but also the weight of 50, 100, and 500 yen coins. They should not only determine the weight of the coins but also their diameter, thickness, and quality of the material of coins. For example, if a 100 yen coin weighing 4.69 grams is put on balance, it could have been included in the “Accept” category, if they set the “Over Limit” as 4.70 grams. This was because the weight difference between a 10 and 100 yen coin was 0.30 grams (see Figure 7). The data collected by the author has been depicted in the dot plot. The process indicated above comprised the “Conclusion” step of PPDAC (Problem, Plan, Data, Analysis, Conclusion) cycle.

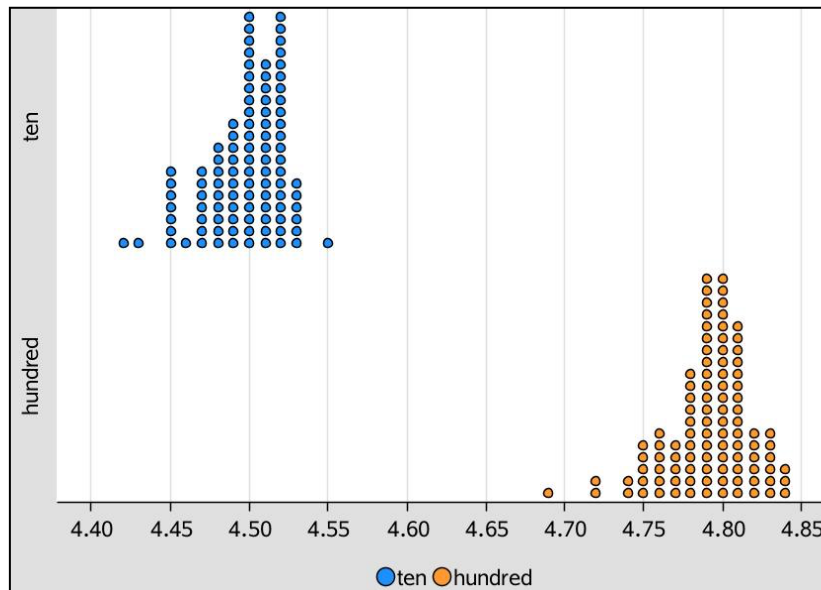


Figure 7 Dot Plot of The Weight Data on Hundred 10 yen Coins and Hundred 100 yen Coins

From the above process, it was expected that students set the problems, solve problems by employing statistical thinking and find new problems in the process during automation of programs, as to how the automatic vending machine judges the coins. Also, it was expected from the students that they validated their own judgment with the help of the frequency table, histogram, dot plot, and the representative values and reflect their essence. Therefore, it seemed reasonable to suppose that it is expected that promoting computational thinking develops statistical thinking.

5. Conclusion

To conclude, we developed teaching material to emphasize retroductive inference for 6th-grade elementary school students in Japan to promote their computational thinking and develop their statistical thinking and discussed it from the view of promoting their computational thinking by creating phases for retroductive inference by students. For example, a context and situation setting so that students inferred upon using retroduction why the automatic vending machine in Japan rejects coins put in the coin slot and development of their statistical thinking by promoting their computational thinking.

The utility of the teaching material in promoting student's computational thinking and development of their statistical thinking became evident.

One of the challenges before us was to use the teaching material for classroom practice and analyze effect of the teaching material on promoting student's computational thinking and development of their statistical thinking.

Acknowledgements: This work was supported by JSPS KAKENHI Grant Number JP19K14255.

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The article deals with the synthetic construction of specific planar curves which are defined by geometric motions and with the modeling of these constructions in GeoGebra dynamic system. We focus on the constructions of trajectories, envelopes, and centers of curvature and osculating circles of specific curves. We present a particular construction of all centers of curvature of these curves, i.e. the construction of the evolute of a curve. Our aim is to investigate the synthetic constructions without the use of coordinates and formulate the proofs of them. We newly model selected types of curves in GeoGebra dynamic system which is an additional output to our theoretical conclusions and also a significant teaching aid at the same time. The constructions are meant to be dynamic and the curves are formed gradually as the traces of points or curves. All examples presented in this article are intended to be used in the undergraduate courses on kinematic geometry (mandatory courses for secondary pre-service mathematics teachers who study teaching mathematics and descriptive geometry). The synthetic constructions together with their proofs demonstrated in GeoGebra dynamic system bring a new light into this area. It allows students to imagine the main idea of the proofs of the constructions and to investigate the properties of the curves more easily based on the pure geometry and visual aspects. Dynamic constructions, i.e. the possibility of changing positions of points and curves play the significant role here. GeoGebra offers also algebraic expressions which represent another tool for students how they can study and manipulate with those curves.

Keywords: kinematic geometry, trajectory, envelope, centrode, synthetic construction, descriptive geometry, centers of curvature, osculating circles

1 Introduction

This article addresses the study of specific planar curves which are defined by geometric motions. The properties of these curves are well known and were studied through ages, in the 16th century and 17th century mainly. Let us mention several famous names which are inextricably linked to this field of mathematics - Girard Desargues (1593-1662), René Descartes (1596-1650), Pierre de Fermat (1601-1665), Johannes Kepler (1571-1630), Isaac Newton (1643-1727), Blaise Pascal (1623-1662), François Viète (1540-1603) and many more [5]. In this article we broaden the study of planar curves with another aspects - the visualizations of all constructions in GeoGebra dynamic system. All examples and constructions modeled in GeoGebra are used in the undergraduate courses on kinematic geometry (mandatory courses for pre-service mathematics teachers who study teaching of mathematics and descriptive geometry, i.e. the prospective secondary school teachers). GeoGebra software helps students to investigate the properties of and the relationships between points and curves through the observations and experiments based on basic constructions and characteristics of planar objects.

Information and communication technologies (ICT) became an inseparable part of school instructions and learning processes [18]. If we focus on software in mathematics education, dynamic software systems (DGS) and computer algebra systems (CAS) have the great potential to be used in education process [19]. GeoGebra dynamic system became widespread all over the world among teachers and students. Because of its popularity GeoGebra is even further modifying and extending [10]. GeoGebra is so popular because it is very easy to use even for the absolute beginner and it is open-source software.

Much of literature summarize that the integration of dynamic software into the education process has a positive effect on better understanding, motivation of students, and transformation of the school instruction [14]. Many researchers claim that GeoGebra and its using at secondary schools and colleges support students' discovering new mathematical facts, their experimentation, and develop their own explanations [10].

I have a long-term teaching experience at the Faculty of Mathematics and Physics (Charles University, Czech Republic) where I have been teaching geometric courses (Euclidean, descriptive, computational, or kinematic geometry) over 10 years. Moreover, I also started teaching mathematics instructions in the Grammar school in the Czech Republic two years ago. At the faculty I work with pre-service mathematics teachers - students who study the specialization of teaching mathematics and descriptive geometry, i.e. the prospective secondary school teachers. The Grammar school where I also teach provides upper secondary education. This perfectly complements one another. I prepare university students for their future career in the secondary schools and I am also a secondary school teacher. I am aware of students' needs at university and for what they should be prepared when they enter secondary schools and start teaching.

As it has been already pointed out, I will show the synthetic constructions of selected planar curves defined by geometric motions and the use of GeoGebra for studying their properties. The constructions modeled in GeoGebra were created with the intention to be used in my university courses on kinematic geometry.

2 Preliminaries and Terminology

Kinematic geometry in the plane is a field of Euclidean geometry describing the motions of points, curves, and systems of points and curves without considering the forces that cause them to move. Kinematic geometry studies the determination of motions, the principles of planar mechanism, or the properties of curves given or determined by the motion.

We assume that the curves that we study are determined as trajectories of moving points or as envelopes of moving curves in the plane. Usually, the tasks in kinematic geometry are to determine a trajectory of a point or an envelope of a curve from the given inputs, to find special points of curves such as cusps or inflection points, to construct centers of curvature and osculating circles of curves or to demonstrate and to study the properties and the underlying geometry of mechanisms such as gears, screw systems, robots [8], [11], [16], [17], [20], [22].

Let us briefly introduce the basic terms related to kinematic geometry which will be then used in the constructions.

A movable plane Σ slides over a fixed plane Π in an arbitrary direction [9]. The final position of the plane Σ is obtained from the initial position by a continuous motion. A movable plane Σ slides over a fixed plane Π and it is never "turned over". The plane Σ is considered to be rigid during the motion and each of its points describes a *trajectory* (also called a *path*) in the plane Π and each of its curves describes an *envelope* in the plane Π . A trajectory of a point A will be denoted by τ_A and an envelope of a curve a by (a) . We also distinguish the individual discrete positions of the moving plane Σ and denote them by $\Sigma^1, \Sigma^2, \Sigma^3, \dots$, analogically the positions of a moving point A are denoted by A^1, A^2, A^3, \dots , and of a moving curve a by a^1, a^2, a^3, \dots .

The simplest motions in the plane are the identity, translations (the points of the plane move in the same direction and of the same distance) and rotations (the points of the plane rotate by an angle about a fixed point). The trajectory of a point in translation is a line, in rotation is a circle.

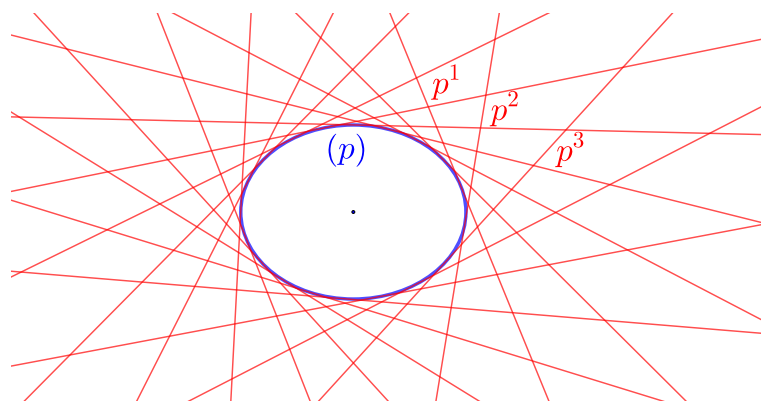


Figure 1: An ellipse (p) as an envelope of a moving line p .

To have a clear idea, an envelope of a moving curve a is a curve (a) which is tangent to

the each curve $a^1, a^2, a^3 \dots$ at some point, and these points of tangency together form the whole envelope. Two examples of envelopes can be seen in Fig. 1 and 2.

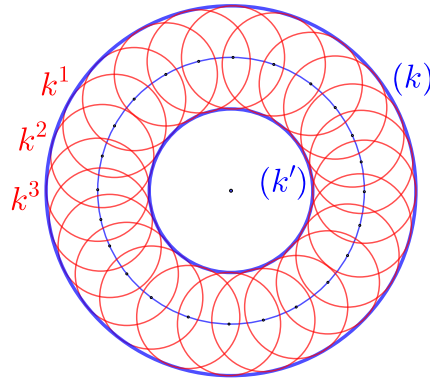


Figure 2: Two concentric circles (k) and (k') as an envelope with two branches of a moving circle k .

Very special cases of trajectories and envelopes can occur during the motion. For example, curves can even degenerate into points, let us mention a *point envelope*. We can imagine a simple example when a line a is rotating around a point (a) ; then the point (a) is the point envelope of the line a . I refer the reader to an interesting literature on curves where other definitions are provided and special curves are studied [4], [7]. Curves of kinematic geometry can be studied also using the coordinate system and algebraic expressions of curves, see [2].

2.1 Determination of Motions in the Plane

Now let us focus on more general examples of motions than identity, translation, or rotation. It can be proved that the motion is completely determined by the form of two trajectories of two distinct points, each on its trajectory, see Fig. 3. Another possibility is the determination by the form of two envelopes of two distinct curves, each tangent to its envelope, see Fig. 4. We can even combine it, it means that the motion is completely determined by a trajectory of a point and by an envelope of a curve. The concrete examples can be found in [15].

As has been already mentioned, when a movable plane Σ slides over a fixed plane Π it remains of the same orientation (it does not "turn over"). Then an arbitrary position Σ^i of a movable plane Σ can be moved to another position Σ^j of a plane Σ using one single translation or one single rotation. The proof can be found in [9]. This means that every two positions of a moving object of a movable plane are congruent and of the same orientation.

We can consider translations as rotation by a zero angle about a point at infinity. If the positions Σ^i and Σ^{i+1} are infinitely close we can construct the center of rotation S_i about which the position Σ^i is rotated to the positions Σ^{i+1} . This center of rotation is called

the *instantaneous center of the motion* at the instant i (for the translation it is a point at infinity) [23], [22].

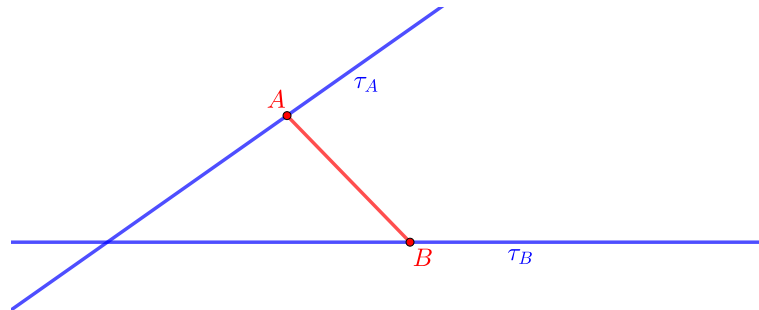


Figure 3: The determination of the motion by two trajectories τ_A and τ_B of two distinct points A and B .

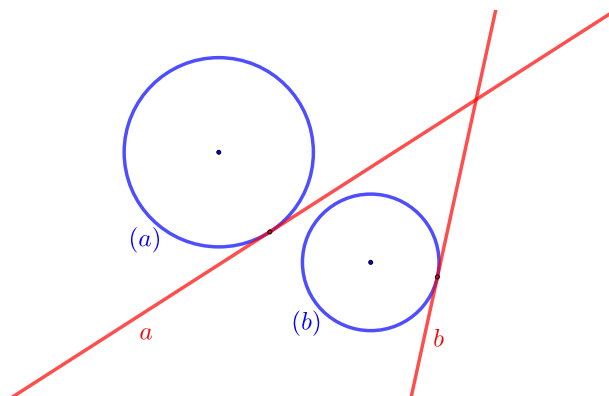


Figure 4: The determination of motions by two envelopes (a) and (b) of two curves a and b .

A locus of the instantaneous centers at every moment of the motion is a curve in the fixed plane Σ which is called the *fixed centrode* of the motion. The roles of the planes Σ and Π can be interchanged and the *inverse motion* is obtained. It means that what was moving in the original motion is fixed now and vice versa. A locus of the instantaneous centers at every moment of the inverse motion is a curve in the plane Σ which is called the *moving centrode* of the original motion. The fixed centrode of the original motion will be denoted by p and the moving centrode of the original motion by h , the i^{th} position of h will be denoted by h^i . The original and the inverse motion are given by each other. It can be proved that the fixed centrode of the original motion is the moving centrode of the inverse motion; analogically the moving centrode of the original motion is the fixed centrode of the inverse motion. It implies that with interchanging the roles of centrodes of the original motion we get the inverse motion. We defined the inverse motion so we could determine

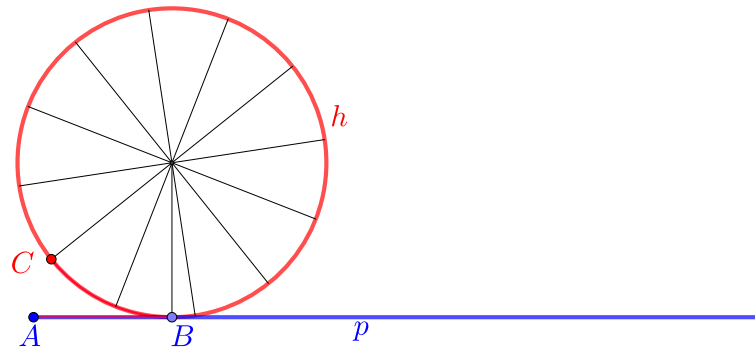


Figure 5: The determination of the motion by rolling the moving centrode h along the fixed centrode p .

the moving centrode. From this moment on, we consider only the original motion and we do not deal with the inverse motion.

Very important property of the instantaneous centers on the fixed centrode p is that these points are the intersections of the normals to the trajectories at their points in the particular position [12].

Considering the centrodes, we can add another possibility how to determine the motion. When a moving plane Σ slides over a fixed plane Π the moving centrode h in the moving plane Σ rolls (without sliding) along the fixed centrode p in the fixed plane Π . At each moment the centrodes are mutually tangent at the instantaneous center. Since the centrodes roll along each other without sliding it implies that the arc bounded by any two points on the fixed centrode p has the same length as the arc bounded by the corresponding points on the moving centrode h . See Fig. 5 where the motion is determined by the centrodes. For more detailed information regarding the basics of kinematic geometry see [3], [9], or [21].

3 Constructions using GeoGebra Dynamic Software

In this section we will show synthetic constructions of trajectories, envelopes, and centers of curvature and osculating circles of specific curves when some specific motion is given. The motions in the plane can be categorized according to the types of trajectories, envelopes or centrodes which determine them. The motions can be also sorted according to the types of curves which are created during the motion. We omit the categorization here and limit ourselves only on several examples. The categorization of motions and some other constructions are presented online and are accessible from [1].

All the constructions are newly modeled in GeoGebra dynamic software to demonstrate the motions in real time and to show the specific properties which can be hardly seen in

the static pictures. I am using these constructions in my university courses on kinematic geometry.

We can even draw the constructions by hand and GeoGebra software use as the control aid. I am also practicing hand-drawn constructions with my students.

I also refer the reader to other literature which deals with the constructions of trajectories, envelopes, evolutes, and many other types of curves using dynamic software systems [6], [13].

3.1 Construction of a Trajectory

Let us consider the motion is determined by two straight line trajectories τ_A and τ_B of two distinct points A and B , see Fig. 6. We can construct a circle determined by three points A , B , and O (the intersection of the trajectories τ_A and τ_B). The points A and B are considered in an arbitrary position. It is obvious that the instantaneous center S constructed for this concrete position also lies on this circle. Let us denote this circle h and its radius r . We construct the circle p with a center O and the radius $2r$. It is easy to prove that the circle p is the fixed centrode and the circle h is the moving centrode which is depicted in one concrete position.

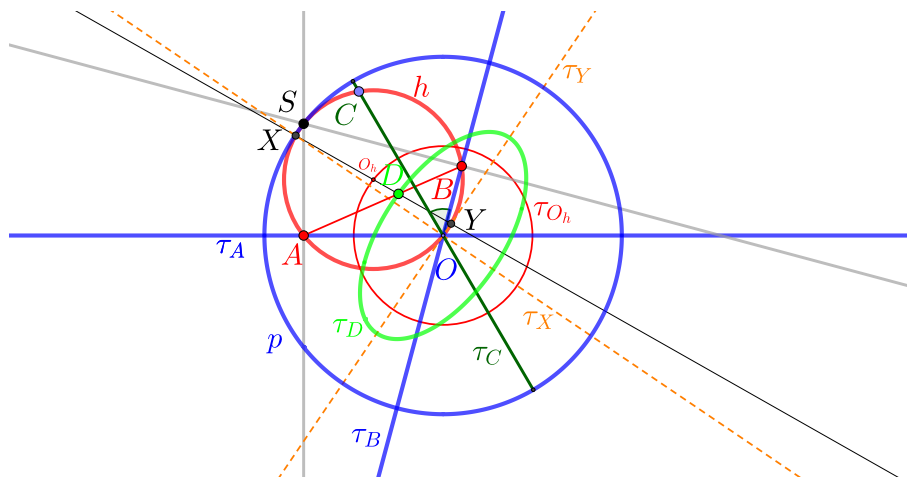


Figure 6: The trajectories of the elliptic motion.

The trajectories of points on the moving centrode h are segment lines - diameters of the fixed centrode p . It is clear that points A and B are moving on their trajectories τ_A and τ_B , if the point C is a point on the moving centrode h distinct from A and B then the length of an arc BC is the same in every position, thus the size of an angle $\angle BOC$ is constant. One ray τ_B of this angle is fixed, thus the second ray is fixed too. That is why the trajectory τ_C is the diameter of the fixed centrode p . The trajectory τ_{O_h} of the point O_h , the center of the moving centrode h , is the circle with the center O and the radius r . Finally, the trajectory τ_D of every other point D is an ellipse which is the reason for the name of this motion - the *elliptic motion*. A line DO_h intersects the moving centrode h in

points X and Y which have straight line perpendicular trajectories τ_X and τ_Y . Thus, the trajectory τ_D is an ellipse with the axes τ_X and τ_Y and the semi-major axis of the length DX and the semi-minor axis of the length DY . This is a known construction of an ellipse called - the *paper strip method* or the *trammel method* [15].

From these observations implies that the elliptic motion can be determined by two circles as centrodes i.e. the moving circle h rolls on the inside of the fixed circle p with double the radius. That is the elliptic motion is a special case of the *hypocycloidal motion* [15].

We can also construct trajectories synthetically point by point using the observations that the moving plane Σ is rigid, i.e. the distances between points in the moving plane Σ are preserved during the motion. We obtain the trajectory as the locus of all the positions of a moving point.

3.2 Constructions of Centers of Curvature and Osculating Circles

A suitable aid how to precisely describe the shape of the curves is to construct the centers of curvature and osculating circles. The center of curvature of a curve can be defined as the intersection point of two infinitely close normal lines to the curves [15]. The *evolute* of the curve is the locus of all its centers of curvature. The original curve is called the *involute* [15]. We can also say that the evolute is the envelope of the normals to the curve. The evolute of the involute is the original curve.

Let us consider the *cycloidal motion* is given, i.e. the moving circle h with the center O_h and the radius r_h rolls on the fixed straight line p (we can consider the straight line p as a circle with the center at infinity). The moving circle h is given in the initial position. We shall determine the trajectory τ_C of a point C of the moving circle h , that is a *cycloid*, and the center of curvature and the osculating circle in each point of the trajectory τ_C , see Fig. 7. To find all centers of curvature of the trajectory τ_C actually means to find the evolute of this trajectory.

At each moment of the motion, the fixed centrode p is the tangent line to the moving centrode h at the instantaneous center S , i.e the i^{th} position of h and the fixed centrode p are mutually tangent at the instantaneous center S_i . As has been already pointed out, the instantaneous center on the fixed centrode p is the intersections of the normals to the trajectories at their points in the particular position. Thus, we can construct the normal n_C to the trajectory τ_C at the point C , i.e. $n_C = SC$. The tangent line t_C to the trajectory τ_C at the point C is perpendicular to the n_C . We construct the point C' on the circle h opposite to the point C . The trajectory $\tau_{C'}$ of the point C' is a cycloid in the fixed plane Π which is congruent to the cycloid τ_C just moved by $\pi \cdot r_h$ in a direction of the fixed centrode p . The tangent line $t_{C'}$ to the trajectory $\tau_{C'}$ at the point C' is perpendicular to the normal $n_{C'}$ where $n_{C'} = SC'$. It is obvious that $n_C \parallel t_{C'}$. A segment line TS where T is the intersection of t_C and h determines the direction and the distance of translation which moves the tangent line $t_{C'}$ to the normal n_C . It implies that an envelope of n_C (i.e.

another types of curves and their synthetic constructions. My aim is to describe the constructions of special points of trajectories and envelopes such as cusps or inflection points.

5 Acknowledgments

The paper was supported by the project PROGRES Q17 Teacher preparation and teaching profession in the context of science and research and by Charles University Research Centre No. UNCE/HUM/024.

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Geometry of planetary motion

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Abstract

This contribution is inspired by the lecture “The Motion of Planets around the Sun” given by Richard Feynman, American physicist and Nobel Laureate, in 1964, namely by steps of Feynman’s geometrical proof of the law of ellipses, which was based on Isaac Newton’s approach to the problem. The contribution focuses on selected passages of this proof, interprets them using the program of dynamic mathematics GeoGebra and offers their use in the form of activities that can be implemented at different levels of mathematics curriculum of lower and upper secondary school. The activities presented in the paper are suitable for the implementation of the STEM approach to mathematics education as they combine the topics of the mathematics and physics curriculum, in addition against the backdrop of the captivating story of discovering the essence of the functioning of the universe.

1 Introduction

This study presents several specific educational activities which, at various levels of school mathematics and with significant use of the dynamic mathematics software GeoGebra [3], are focused on the selected properties of the ellipse with emphasis on the role of this curve as the trajectory of the motion of planets around the Sun.

One of the ambitions of this work is to show how the use of GeoGebra, with its respective geometric, algebraic and numerical capabilities, will enable educators to bring the subject of ellipses and the physical laws of planetary motion closer to pupils at different levels of school education and to make meaningful use of their school knowledge of mathematics and physics, such as geometry of the triangle, properties of an axial symmetry, definition of conic sections, Kepler’s laws of ellipses and of equal areas and Newton’s laws of motion and of gravity. The presented materials are primarily intended for secondary school pupils and for students of the teaching of mathematics. The aim of their presence is to show how GeoGebra allows students not only to understand the meaning of the properties discussed, but also to practice their knowledge of the specific content of the school curriculum.

The activities we present are suitable for STEM (Science, Technology, Engineering, and Mathematics) education [19]. They appropriately combine the educational content of mathematics, geometry and physics, plus they reflect the history of the evolution of the knowledge of the universe and the principles on which its functioning is based. All materials created in GeoGebra are available online through the links featured in this text. Together they are available in the GeoGebra Book *Geometry of planetary motion* [7], based on the first version [8] presented at the Global GeoGebra Gathering in Linz in 2015, but which has not yet been mentioned in any publication.

The main inspiration for this study is the lecture *The Motion of Planets around the Sun* by American physicist, Nobel laureate, Richard Feynman, as recorded and further explained and illustrated in detail in the book *Feynman's Lost Lecture. The Motion of Planets Around the Sun* [4].

In 1964 Richard Feynman (1918–1988) gave a guest lecture titled *The Motion of Planets around the Sun* to the Caltech freshman class in order to introduce them to geometric proof of the elliptical motion of planets around the Sun. Inspired by Newton's geometric way of proving Kepler's laws of planetary motion, which he presented in his famous work *Philosophiæ Naturalis Principia Mathematica* [17] published in Latin in 1687 (For English translation, see e. g. [18]), Feynman created his own geometric proof using only elementary knowledge of plane geometry and the selected statements of Newton's laws of motion and gravity. As a result of the temporary disappearance of its records this lecture became known as *Feynman's Lost Lecture*. After the finding of it among other documents at the Physics Department of Caltech a number of years after the lecture was given it was professionally and literally edited by David and Judith Goodstein and published as an above mentioned book [4] in 1996.

For the sake of completeness, it should be noted that Richard Feynman addressed different approaches to the derivation of Kepler's laws of planetary motions applying Newton's laws of motion and gravity in his texts. In addition to the aforementioned purely geometrical method presented in his *Lost lecture*, which we will address further in this text, a method of numerical approximation, based on iterations recorded using a table, was published in the famous three volume book *Feynman's Lectures on Physics*, first issued in 1964, specifically in its first volume [5], Chapter 9, pages from 6 to 9 (alternatively the online [6], Chapter 9). The application of this method in GeoGebra and its use to represent the trajectory of both planets and other bodies, namely that of the peregrine falcon are presented in papers [9] and [10].

The standard method of proving the validity of Kepler's laws from Newton's laws of motion and gravity, presented in contemporary publications, employs an advanced calculus. Such proof of Kepler's Laws in a comprehensible and clear manner is presented, for example, in [20].

2 Related topics from school curriculum

As mentioned above, the activities presented in the paper cover a number of topics from the mathematics and physics curriculum. In this section we will mention the most important of them.

2.1 Area of a triangle

The area of a triangle is equal to one half its base times its altitude. Therefore, two triangles with a common base and the same altitude have the same area, regardless of the difference in their shapes, see Fig. 1.

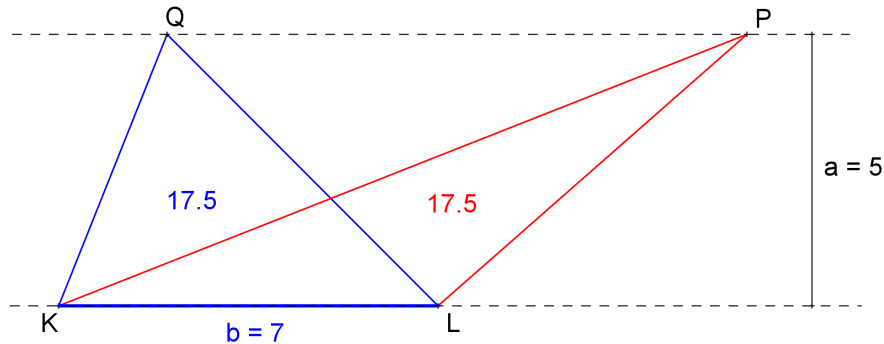


Figure 1: Triangles with a common base (b) and the same altitude (a) have the same area

2.2 Definition of an ellipse

An ellipse can be defined in various ways, see [2]. Here we use the definition of an ellipse as a locus of points, common in upper secondary school mathematics: *Given two points F and F' , called foci, and a distance $2a$ greater than the distance $|FF'|$, the ellipse is a locus of points P such that the sum of the distances $|PF|$ and $|PF'|$ is constant and equal to $2a$.*

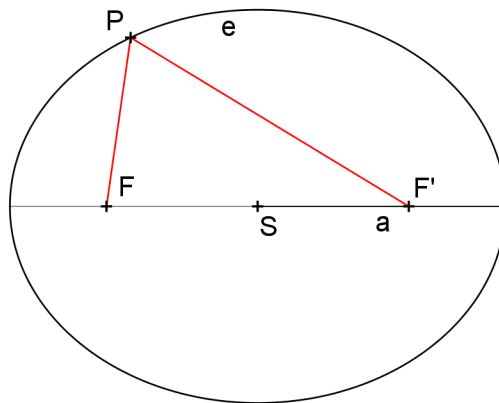


Figure 2: $e = \{P \in E_2; |PF| + |PF'| = 2a\}$

Without using the equation, this definition of an ellipse is already being introduced to lower secondary school pupils, often even to primary school pupils, in the form of the so-called “gardener’s construction of an ellipse” (i. e. elliptical bed): *At two points, drive the pins into the ground, tie a rope longer than their distance to them, fix a pole into it and then, while keeping the rope taut, you carve an ellipse on the ground with this pole.*

GeoGebra has a command `LocusEquation` that allows the given definition to be used to construct an ellipse, and will also determine its equation. Its syntax corresponding to this

purpose, specifically for the configuration shown in Figure 3, is as `LocusEquation(f + g == 8, P)`.

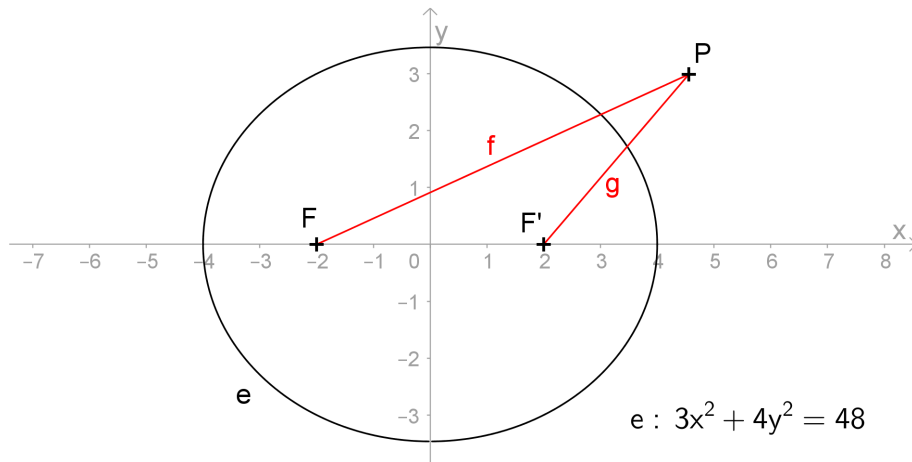


Figure 3: An ellipse determined from the definition using the `LocusEquation` command

2.3 Heron’s shortest distance problem

Given two points A and B on one side of a straight line, to find the point P on the line so that $|AP| + |PB|$ is as small as possible [1]. It is a typical task, which illustrates the use and properties of axial symmetry for lower secondary school pupils, and in addition, the use of the triangle inequality in proving the correctness of its solution.

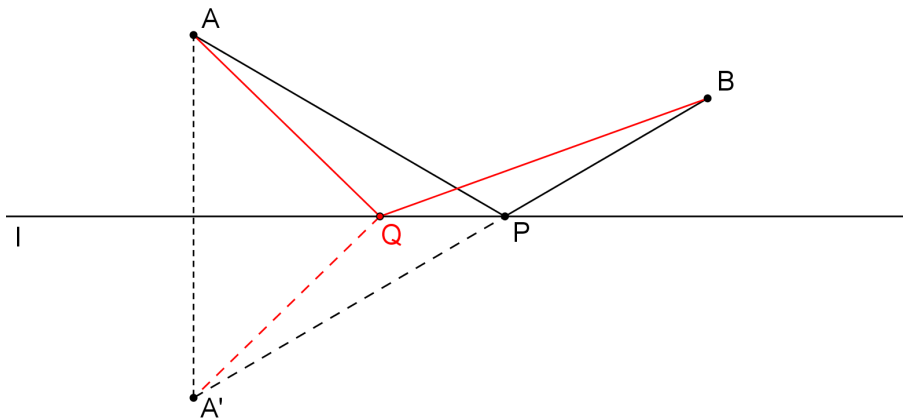


Figure 4: Heron’s shortest distance problem

Using GeoGebra we can take advantage of its dragging function. It allows us to illustratively prove the solution of Heron’s problem by contradiction, applying *the triangle inequality theorem*. See Fig. 4, where obviously $|AP| + |PB| = |A'P| + |PB| = |A'B|$; $|AQ| + |QB| = |A'Q| + |QB|$, while according to the triangle inequality is $|A'Q| + |QB| > |A'B|$. Consequently $|AQ| + |QB| > |AP| + |PB|$.

2.4 Focal properties of an ellipse

A typical feature of the ellipse explored with pupils already at lower secondary school is its reflective property: *All the light rays starting at one focus will be focused to a point at the other focus.* This property is equivalent to the fact that the tangent of the ellipse bisects the outer angle of the focal radii of its point of tangency.

Proof of it by using the triangle inequality belongs to the high school mathematics curriculum, solved by applying the same principle as in the case of solving the Heron's problem. See Fig. 5. If the line t is the angular bisector of the focal radii of P , in particular that one which

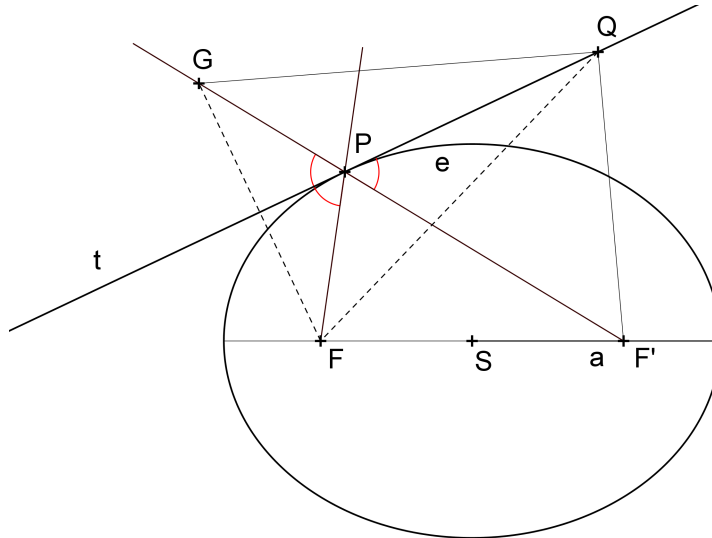


Figure 5: Tangent line t bisects the outer angle of the focal radii of its point of tangency P

does not intersect the segment FF' , then the reflection G of the focus point F in t lies on the ray $F'P$. Consequently $|PG| = |PF|$, hence

$$|F'G| = |F'P| + |PG| = |F'P| + |PF| = 2a. \quad (1)$$

It is therefore sufficient to prove that each point Q of the line t different from P lies outside the ellipse, i.e. $|F'Q| + |QF| > 2a$. However, the latter inequality always pays, thanks to the triangle inequality applied to the triangle $F'GQ$. The line t is therefore the tangent of the ellipse e .

As an advanced problem to solve, related to above mentioned properties, an upper secondary school math textbook on analytical geometry [14] presents the following task: *Show that all reflections of one focus point of an ellipse through all its tangent lines form a circle centered in the other focus point. Determine the radius of this circle.* The circle in question is known as a *circular directrix* related to the focus of an ellipse, its radius is $2a$, see Fig 6. Its existence is justified by (1).

2.5 Kepler's laws of planetary motion

Kepler's three laws [11], which describe the motion of planets around the Sun, belong to the usual secondary school curriculum. *Johannes Kepler* (1571–1630), German mathematician

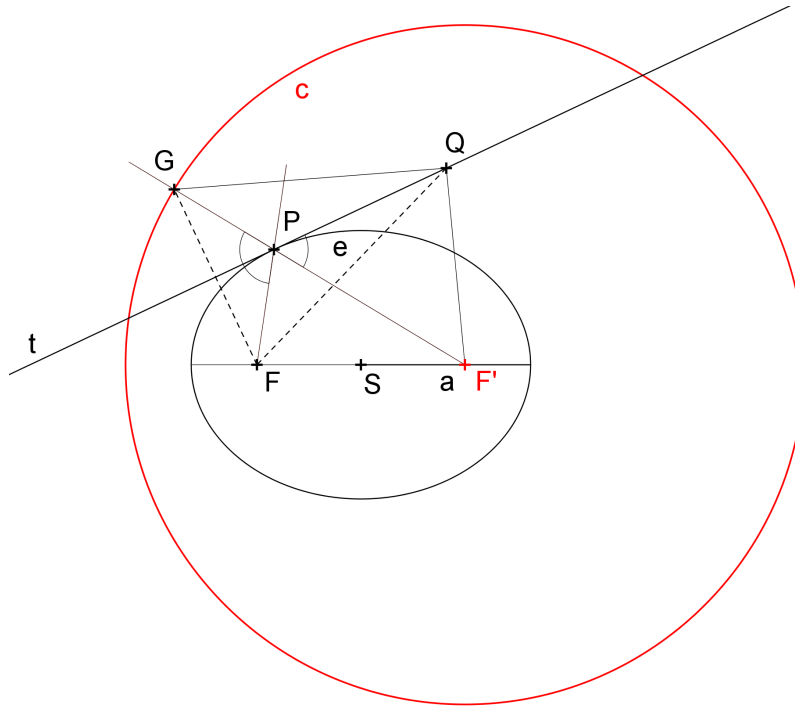


Figure 6: Circular directrix related to the focus F of the ellipse e

and astronomer who spent a significant part of his life in Graz and Linz in Austria, and in Praha in Bohemia, published the three laws over a period of time in two books. His findings were based on observations of the Danish astronomer *Tycho Brahe* (1546–1601). First, in 1609 in *Astronomia nova* [12], Kepler published two statements that are known as his 1st and 2nd laws of planetary motion, according to their focus as *the law of ellipses*, and *the law of equal areas*, respectively, then, in 1619 in *Harmonices mundi* [13], he added the 3rd law, *the law of harmonies*:

The law of ellipses: Orbits of all the planets are ellipses with the Sun at one focus.

The law of equal areas: A line segment from the Sun to a planet sweeps out equal areas in equal time.

The law of harmonies: The orbital period of a planet is proportional to the three-halves power of the size of the semi-major axis of its orbit.

2.6 Newton's laws of motion and gravity

Isaac Newton (1643–1727), English physicist and mathematician, published three laws of motion [15] and the law of gravity [16] in 1687 in his work *Philosophiæ Naturalis Principia Mathematica* [17]. Although Newton in this work also developed a mathematical method that enabled him to derive Kepler's laws as consequences of his (Newton's) laws, as we know, instead of this he provided geometric proof of Kepler's laws in the *Principia*. We are particularly interested in Newton's 1st and 2nd laws of motion, *the law of inertia* and *the law of force and*

acceleration, respectively, and in his law of gravity, often descriptively referred to as *the inverse square of the distance law of gravity*:

The law of inertia: An object keeps its state of motion, i.e. a rest or a motion at a constant speed in the same direction, unless an external force is impressed on it.

The law of force and acceleration: The change in motion is proportional to the motive force impressed; it is made in the direction of the straight line in which that force is impressed.

The inverse square of the distance law of gravity: The force of gravity diminishes as R^{-2} , where R is the distance from a planet to the Sun.

3 Proof of the 2nd Kepler's law

The geometric proof of the second Kepler's law, *the law of equal areas*, presented by Richard Feynman corresponded to the proof which was published by Isaac Newton in his *Philosophiæ Naturalis Principia Mathematica* (usually referred to as the *Principia*), 1697 [17]. Feynman used an almost identical illustration to the original one by Newton, see [17], page 32.

This geometric proof is based on the elementary properties of a triangle, particularly the determination of its area, combined with the Newton's laws of inertia and gravity. Its partial steps can be interpreted as a pair of consecutive exercises, the solution of which is the application of relevant topics from the school curriculum of mathematics and physics as mentioned in the previous section 2.1.

Exercise 1 *Given triangles SAB , SAC , SBC and SBD (D is movable along p), see Fig. 7, so that B is the midpoint of the segment AC and p is parallel to SB . Determine the relations of areas of all the given triangles. Justify your claim.*

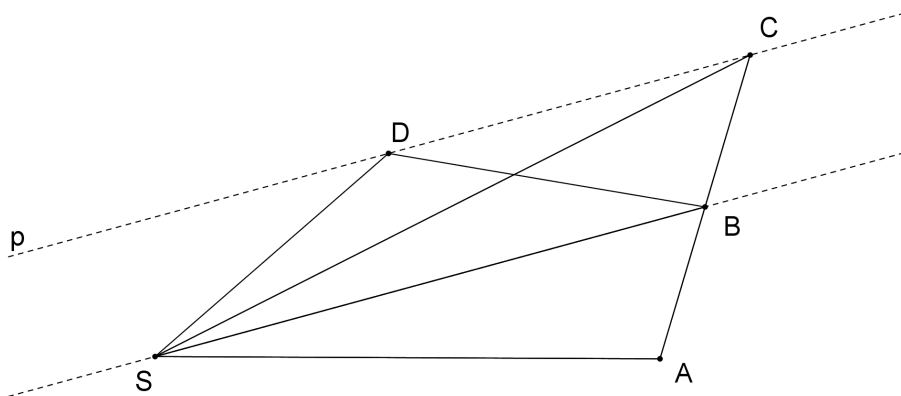


Figure 7: What are the relations between the areas of the displayed triangles?

Exercise 2 *See Fig. 8. Points A , B , C , D and E are the successive positions of a planet in its orbit around the Sun (point S) at equal intervals of time, assuming gravitational action between the Sun and the planet only at the end of each interval. The change in position due to*

gravity is represented by vectors pointing from each of these points to point S . Then, due to the law of inertia $|AB| = |Bc|$, $|BC| = |Cd|$ and $|CD| = |De|$ (Explain why!). In accordance with statement of the 2nd Kepler's law (the law of equal areas) it follows that the areas of triangles SAB , SBC , SCD and SDE are equal. Prove it! You can do it gradually, using triangles SBC , SCd and SDe . Be aware that cC , dD and eE are parallel to BS , CS and DS , respectively. Use the knowledge from the solution of the previous exercise.

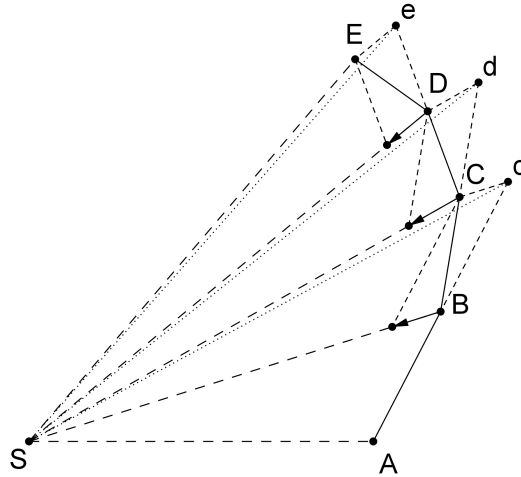


Figure 8: Newton's geometric proof of the law of equal areas

The principle of geometric proof of Kepler's second law is evident from several successive time intervals, as shown in Fig. 8, and as Newton also stated in his work [17]. With sufficient patience, however, we can draw, at least approximately, the whole ellipse of the trajectory of an imaginary planet, as shown in Fig. 9. The Kepler's *law of equal areas* thus appeared to be a consequence of the Newton's law of inertia and the fact that changes in motion of planets are caused by the gravitational force directed toward the Sun.

4 Proof of the 1st Kepler's law

The answer to the question of what causes the elliptical shape of a planet's orbit provides the proof of the Kepler's first law, *the law of ellipses*. This property arises from Kepler's 2nd and 3rd laws and from the fact that the gravitational force diminishes as R^{-2} , i. e. from *the inverse square of the distance law of gravity*.

4.1 The property of reflection of an ellipse

First of all Feynman was to prove that the property of reflection from one focus to the other of an ellipse is equivalent to the property that the sum $|F_1P| + |F_2P|$ is constant for any point P on the ellipse, using arguments analogous to those presented in section 2.4. Again, instead of proving it directly, we assign two consecutive exercises of the same nature.

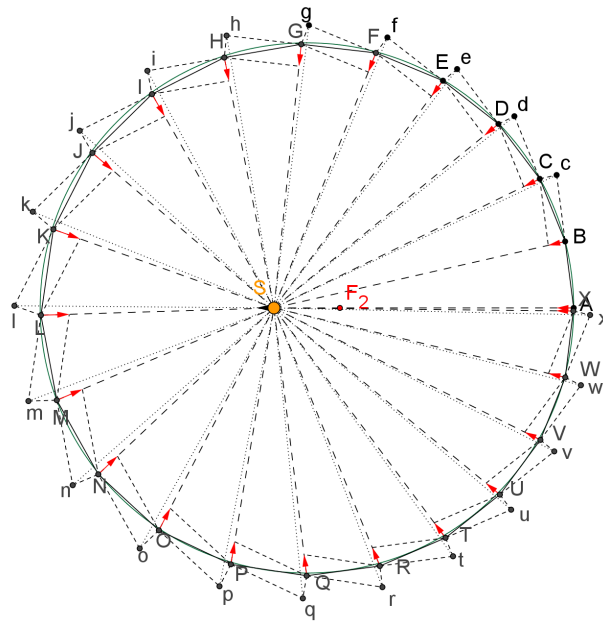


Figure 9: A line segment from the Sun to a planet sweeps out equal areas in equal time

Exercise 3 See Fig. 10. Sort by size the following lengths: $|F_1P| + |PF_2|$, $|F_1R| + |RF_2|$, $|F_2P| + |PG|$, $|F_2R| + |RG|$. Then move the line p to align point R with point P of the ellipse and make a conjecture about the relationship between the line p and the ellipse in this configuration.

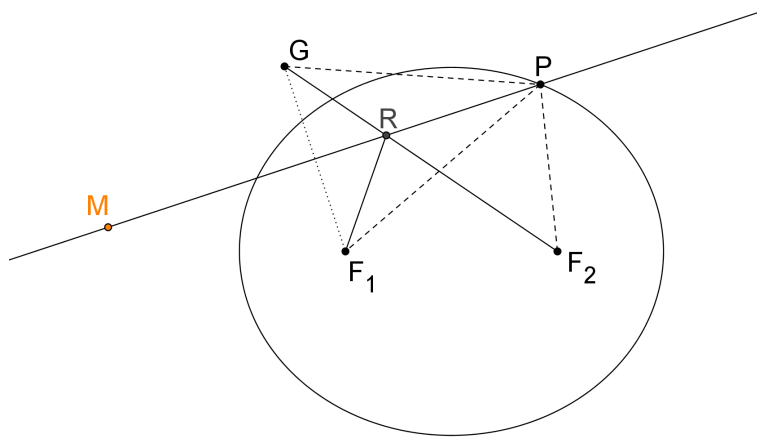


Figure 10: The property of reflection of an ellipse

Exercise 4 An ellipse is given by its two foci and by the length of its major axis, see Fig. 11. Without drawing the ellipse, using the available tools construct its movable point and the tangent of the ellipse passing through this point.

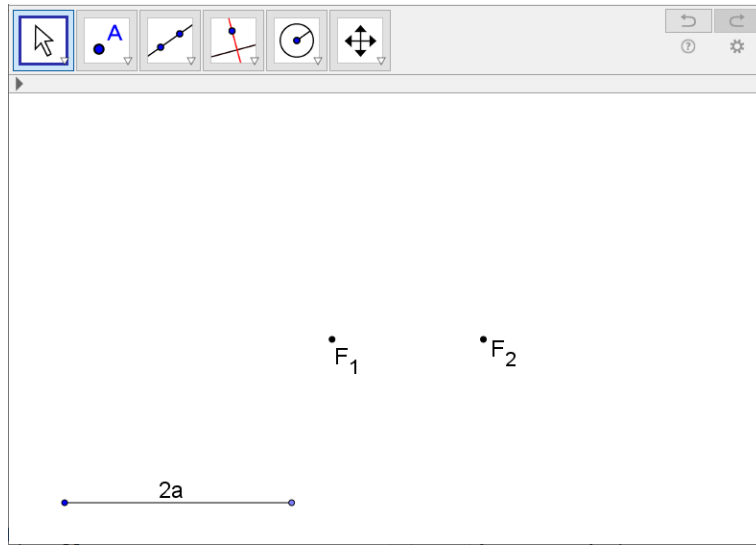


Figure 11: Using the available tools, construct an ellipse given F_1 , F_2 and a (GeoGebra with the selected tool menu)

4.2 Feynman’s proof of the law of ellipses

The ongoing steps of Feynman’s geometric proof of Kepler’s first law are quite complex and require a detailed description. For that, we refer the reader directly to [4]. Here we will only briefly describe these steps. All dynamic applets are available in [7].

First, Feynman proved that equal angles correspond to equal velocity changes. He illustrated this property by two diagrams: an orbit diagram and a velocity diagram. From the construction and from the relevant physical laws it follows that the velocity diagram is always the shape of the regular polygon, see Fig. 12.

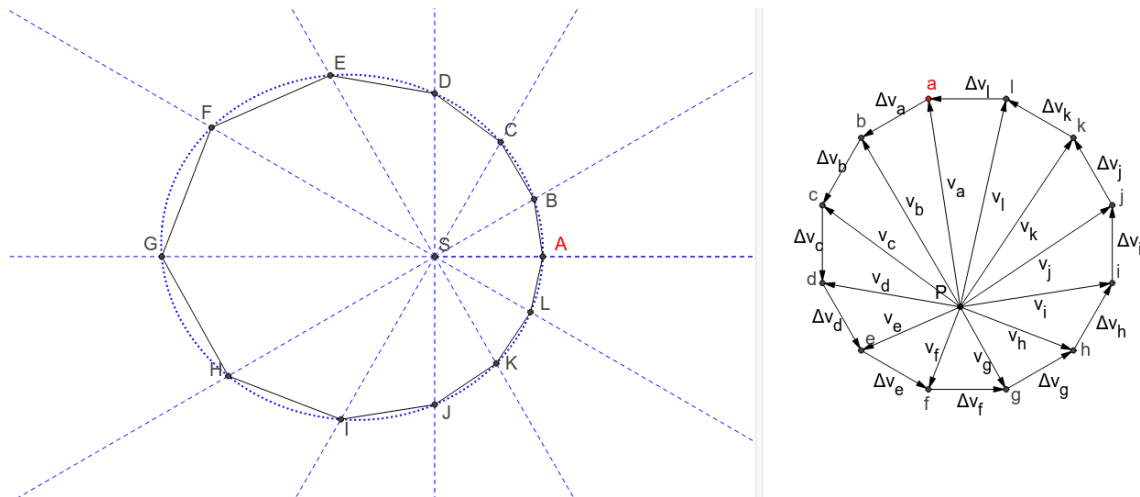


Figure 12: An orbit diagram and a velocity diagram

Then, he found the geometrical correspondence of these two diagrams based on the equality of

the angle swept by a planet with the relevant central angle in the velocity diagram, see Fig. 13.

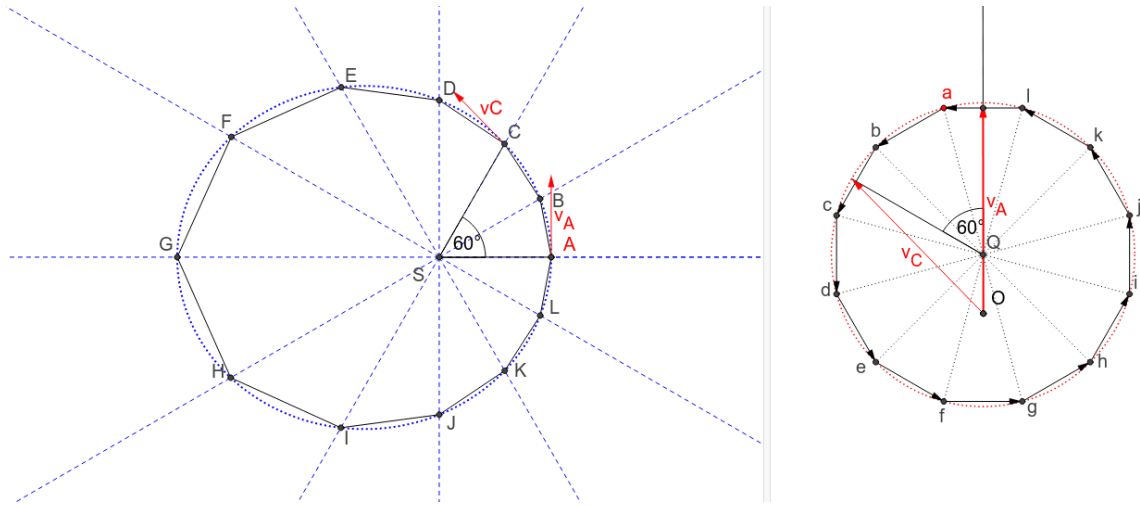


Figure 13: Geometrical correspondence of two diagrams

Rotating the velocity diagram and using the properties of ellipse that we mentioned above (section 2.4 or exercise 4) he finally proved that the shape of a planet's path is an ellipse, see Fig. 14. Of course, its size does not fit, it is only a proof of the shape of the trajectory.

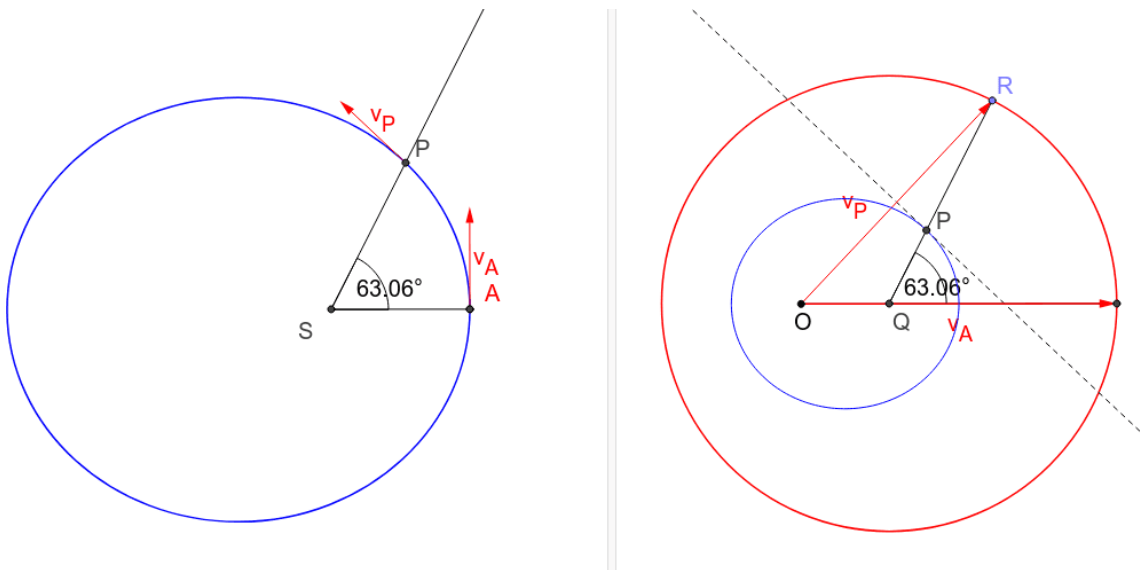


Figure 14: The shape of a planet's path is an ellipse

5 Conclusions

The aim of the paper was to show the classical secondary school geometric problems solved in the context of a physics, moreover in connection with the stories of the discovering of the

physical nature of the universe and of the life of Richard Feynman, Nobel Laureate. We will be happy if the reader tries out the presented exercises in her teaching practice and provides us with feedback.

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Delineation of the Zone of Influence of Pumping Wells using CAS and DGS

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Abstract

Groundwater pollution is a general concern in countries using pumping wells for water consumption. Also, the determination of protected zone around pumping wells is a practical concern. The delineation of this zone is frequently obtained using numerical models such as MODFLOW and or FEFLOW. In this contribution we derived the equation of the plane curve of the protected zone using a steady state solution of the groundwater flow equations and the theory of envelopes. Cassini ovals appear in some particular cases, providing new applications of these plane curves. A Dynamic Geometry System, such as GeoGebra and/or Desmos, is used to explore the protected zone. A Computer Algebra System may be used for the computations, in particular to characterise the delineation curves for the zones of influence for wells.

1 Introduction

Protection of pumping wells from pollution is one of the most important concerns in the modern management of water resources. In Israel, where almost seventy percent of the available water comes from groundwater, the protection of production wells against dissolved pollutants is of critical concern. In the past, well protection techniques have focused on the delineation of zones of influence of pumping wells using advanced modeling tools. The aim of this paper is to develop a new approach, based on analytical derivation of the sensitivity of the well locations to the zone of influence of a series of wells. Figure 1 shows a standard situation with different kinds of wells.

Inside the zone of influence of a pumping well a particle of polluted species will flow through the well. On the opposite, a particle of polluted species outside the zone of influence will not reach the pumping well. Also, in the case of recharging wells the zone of influence of these wells corresponds to the zone of depollution. For example, in the pump and treat strategy for depollution of groundwater aquifers, the zone of depollution corresponds to the zone of

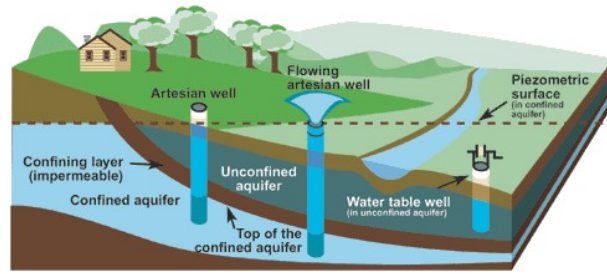


Figure 1: Acquifer and wells

influence of the recharge wells. When remediation design of a groundwater site is performed, the optimal locations of the pumping recharge wells and the pumping rates are certainly the significant parameters in the design. Therefore, a precise determination of the zone of influence with respect to the pumping rates and the distance between wells is needed for well design. A classical approach to delineation of pollution has been to use conceptual groundwater models such as MODFLOW or FEFLOW; see the USGS webpage devoted to Groundwater Modeling . These models are based on the analysis of geological, geophysical and geochemical data, without consideration of groundwater flow. Selection of an appropriate conceptual geohydrological model, based on hydrodynamics, requires the use of a calibrated three-dimensional flow and transport simulation for each conceptual scenario. This task, however, is time consuming and requires rather detailed knowledge of the model's input parameters. For this reason, a simplified and less computer-intensive model is useful in providing a good first assessment of the zone of influence of a polluted well [1].

This is precisely the situation that arises in the observation of source locations of salinization in many of Israel's deep aquifers. When many of the aquifer parameters are unknown or unavailable, the problem becomes one of identifying a conceptual model based upon the information that is available. Measurements, such as wellhead, well concentration, well pumping rate and the location of the wells must therefore be used to recover more knowledge of the contaminant transport characteristics.

We explore graphically the influence zones of the wells, using a Dynamic Geometry System, such as GeoGebra or Desmos. The curves which appear in this exploration are a parameterized family, containing Cassini ovals as a subfamily. Some envelopes and offsets of these curves have been studied by in [5]; in the present paper, these ovals are themselves the requested envelope, but with regards to another definition, as described in Section . The equations have to be transformed into polynomial equations. For this purpose, the algebraic computations are performed either with the CAS implemented into GeoGebra, or with another CAS. For this, we used Maple.

2 Flow equations

Tracing the movement of the elemental area over time, as determined by the flow field, then generates the particle paths. Because the concentration of the source, denoted by c , in the Lagrangian interpretation of advective transport, is associated with a single particle and equals

that of a particle emerging from the source, it must not change over time. It is therefore easy to see that the equation for advective transport in the form

$$\frac{DC}{DT} = \frac{q_s}{\theta}(C_s - C)$$

simplifies to become

$$\frac{DC}{DT} = 0.$$

Plainly stated, in purely advective transport the concentration associated with a fluid particle does not change with time as the particle moves along its path line. Therefore, in a series of wells, the pollution arrives to the well only when the particle flow through the well inside a zone of influence of the wells.

2.1 Generation of Flow Field

We begin first with the generation of the flow field from which particle tracking will be computed. Flow fields can be generated either by exact analytical solutions or through the use of numerical approximations. In the case of two-dimensional flow in a homogeneous single aquifer of idealized geometry, the solution of steady state flow may be derived analytically using the potential flow theory as described in [12, 17]. Although analytical techniques have a greater degree of accuracy, they lack the flexibility to model irregular boundaries and complicated boundary conditions. For irregular boundaries, realistic aquifer geometries and aquifers with heterogeneous properties, numerical methods are generally used to solve the flow and transport equations. This document will focus on the combination of analytical and numerical techniques, or semi-analytical solutions, which are sometimes referred to in the literature as Analytical Element Methods (AEM).

2.2 Numerical solutions

The two most common numerical techniques for solving groundwater flow equations are the finite difference (including integrated finite difference of constant volume methods) and finite element methods [11]. These methods use a series of nodes or elements to solve the governing equations of groundwater flow and transport. The most significant restriction in numerical methods is the need to discretize the domain into a three-dimensional network of nodes or elements. This type of limitation can be lessened by solving regional Dupuit-Forchheimer flow through superposition of analytic elements, as proven in [16, 9].

In the case of finite differences, the error introduced is that identified with the Taylor series approximation and will be affected by the size of the discretization and the behavior of the function being approximated. While finite elements allow slightly more flexibility when dealing with boundaries and have a similar degree of numerical accuracy, they require more computational effort and therefore more computer time.

2.3 Analytic solutions

In analytical modeling the flow field is generated by utilizing formal mathematical closed-form solutions to generate the appropriate flow field. Analytical solutions are restricted to

simple geometries, but provide an extremely accurate way to study the behavior of groundwater flows under hypothetical conditions. The analytic element method, as developed in [17] for groundwater flow, uses the method of images and the principle of superposition to produce flows associated with various aquifer features.

2.4 Steady flow

In the analytical element method, simulation of regional flow is accomplished by using superposition to combine the equations for the fluid potential for each feature, or element. Features such as injection wells and pumping wells are simulated by using the equation describing the fluid potential in an infinite domain for a straight-line source [17] and the method of images. Derivation of the solution of a straight-line source with a given strength is obtained from the integration of a point source along a prescribed length. Aquifer heads and flow velocities may then be obtained by combining the equations of potentials for all of the prescribed elements.

In order to then transform the domain from an infinite region to one that adheres to the specified boundary conditions, the method of images is used [17]. The method of images consists of locating image conditions such that the combination of an aquifer feature and its image produces the desired equipotential. Aquifer heads and flow velocities are, consequently, obtained by combining the equations for the potentials and the images for all of the prescribed elements. For features located on the boundary, such as areas of infiltration, special analytical solutions are used. For example, the modeling of an area of infiltration can be simulated analytically using the potential for a circle derived by Haitjema [8] or through the use of the potential for a rectangle derived by Steward [15].

Closed form solutions for circular and rectangular surfaces exist in the form of specified constant discharges with varying heads. When closed form solutions are not available, one must apply the boundary element method [11]. The boundary element method combines a discretized numerical solution on the boundary, with an analytical solution inside the domain, to model more complicated boundary conditions when combined with general geometries.

3 Definition of the zone of influence

3.1 General setting

Before the pumping starts, the hydraulic head is at a level let say ϕ_0 . After the pumping starts around each pumping well, the *drawdown* due to the pumping from a groundwater level may be expressed as in [17]:

$$|Drawdown| = |\phi(x, y) - \phi_w|$$

where $|Drawdown|$ is the hydraulic head at a point $M(x, y)$. The zone of influence of a given well may be defined as the zone of very small perturbation from the initial groundwater level. Therefore, the zone of influence or and/ or the envelop of the disturbed zone may be expressed as:

$$\left\{ \begin{array}{l} |Drawdown| = |\phi(x, y) - \phi_w| < \varepsilon \\ \varepsilon \ll 1 \end{array} \right.$$

Analytical formula for the computation of the formula is difficult and we propose to use a Computer Algebra System (CAS) to derive this zone.

3.2 Conceptual Model of the Aquifer

The model developed for identifying salinization sources comprises the following steps and simplifying assumptions:

- (i) The aquifer is modeled as one homogeneous unit of constant thickness with parallel vertical planar boundaries (box like). Constant head or given flow rates on the boundaries are selected.
- (ii) The wells are represented by singularity lines of known strength while the pollutant source is represented as an area of given water flux (which may be small). To account for boundary conditions, appropriate images are added.
- (iii) A mean, effective, value of the hydraulic conductivity for the formation is determined by calibrating computed heads against measured heads at a few points, while also taking into account that the hydraulic conductivity can be expressed as the ratio between the specific discharge potential and the pressure head.
- (iv) The velocity field is determined analytically by differentiation of the potential and division of the result by the effective porosity. A simple algorithm leads to the velocity values at each selected point.

The aquifer to be modeled is considered fully-saturated, confined and incompressible. It is presumed to be at steady state with a homogeneous hydraulic conductivity. The aquifer is of constant thickness and contains two impervious boundaries formed by planar surfaces on the top and the bottom.

3.3 Specific cases

3.3.1 Two wells cases

Derivation of the flow field around a well by analytical methods originates with the solution of the potential in an infinite domain for a straight-line source [17, 12]. Wells are treated as lines with a given strength derived from the integration of a point source along a prescribed length. Consider a two dimensional infinite confined aquifer of transmissivity T and two pumping wells separated by a distance x_0 .

1. Same rates: The first well is located at the origin of the coordinates system and both wells are pumping with the same rate Q . The drawdown due to the pumping from a groundwater level ϕ_w is given by the following equation [17]:

$$\phi(r) - \phi_w = \frac{Q}{2\pi T} \ln \frac{R}{r} + \frac{Q}{2\pi T} \ln \frac{R}{r_1},$$

where R is the radius of influence of both wells, $r = \sqrt{x^2 + y^2}$ and $r_1 = \sqrt{(x - x_0)^2 + y^2}$. One may express the drawdown as:

$$\phi(r) - \phi_w = \frac{Q}{2\pi T} \ln \frac{R^2}{rr_1}.$$

The zone of influence of the wells corresponds to the domain where $\phi(r) - \phi_w \neq 0$. Therefore, the boundary of the zone of influence is given by the following equations:

$$\begin{cases} \phi(r) - \phi_w = \frac{Q}{2\pi T} \ln \frac{R^2}{rr_1} = 0 \\ R^2 = rr_1 = \sqrt{(x^2 + y^2)((x - x_0)^2 + y^2)} \end{cases} \quad (1)$$

The last equation defined a family of surfaces with a parameter :

$$F(x, y, x_0) = \sqrt{(x^2 + y^2)((x - x_0)^2 + y^2)} - R^2. \quad (2)$$

2. Different rates: The drawdown due to the pumping from a groundwater level ϕ_w is given by the equation [17]).

$$\phi(r) - \phi_w = \frac{Q_1}{2\pi T} \ln \frac{R}{r} + \frac{Q_2}{2\pi T} \ln \frac{R}{r_1},$$

One may express the drawdown as:

$$\phi(r) - \phi_w = \frac{Q_1}{2\pi T} \ln \frac{R^{q+1}}{rr_1^q}, \quad \text{where } q = \frac{Q_2}{Q_1}.$$

Therefore, the boundary of the zone of influence is given by the following equations:

$$\begin{cases} \phi(r) - \phi_w = \frac{Q_1}{2\pi T} \ln \frac{R^{q+1}}{rr_1^q} = 0 \\ R^{q+1} = rr_1^q = (x^2 + y^2)^{1/2} ((x - x_0)^2 + y^2)^{q/2}. \end{cases} \quad (3)$$

The last equation defined a family of surfaces with two parameters :

$$F(x, y, x_0, q) = R^{q+1} - (x^2 + y^2)^{1/2} ((x - x_0)^2 + y^2)^{q/2}.$$

3.4 A line of wells

Consider a two-dimensional infinite confined aquifer of transmissivity T and a line of n pumping wells separated by a distance X_0 . The first well is located at the origin of the coordinates system and both wells are pumping with the same rate Q . The drawdown due to the pumping from a groundwater level ϕ_w is given by the following equation [17]:

$$\phi(r) - \phi_w = \frac{Q}{2\pi T} \sum_{i=0}^{n-1} \ln \frac{R}{r_i},$$

where R is the common radius of influence of all the wells. We have:

$$r_i = \sqrt{(x - ix_0)^2 + y^2}, \quad i = 1 \dots n$$

One may express the drawdown as follows:

$$\phi(r) - \phi_w = \frac{Q}{2\pi T} \sum_{i=0}^{n-1} \ln \frac{R^n}{\prod_{i=0}^{n-1} r_1}$$

The boundary of the zone of influence is given by the equation:

$$\begin{cases} \phi(r) - \phi_w = 0 \\ R^n = \prod_{i=0}^{n-1} r_1 \sqrt{(x - ix_0)^2 + y^2}. \end{cases} \quad (4)$$

The last equation defined a family of surfaces with a parameter x_0 :

$$F(x, y, x_0) = \prod_{i=0}^{n-1} r_1 \sqrt{(x - ix_0)^2 + y^2} - R^n. \quad (5)$$

3.4.1 Computation of the zone of influence

With this simplified approach, the optimal design for a remediation using pump and treat system will correspond to the design that assures a total recover of the zone of influence. Mathematically speaking, it corresponds to an envelope, in the sense of Definition 4 in next Section, of the parameterized families of surfaces (either with one or two parameters) given by the functions in Equations (2) or (5).

4 Different definitions of envelopes of 1-parameter families of plane curves

Envelopes of 1-parameter families of plane curves have been studied for a long time, but there exist 4 different definitions of this kind of objects. Kock [10] gives 3 different definitions of an envelope of a 1-parameter family of plane curves:

Let $\{C_k\}$ be a family of real plane curves dependent on a real parameter k .

Definition 1 (Synthetic) *The envelope \mathcal{E}_1 is the union of the characteristic points M_k , where the characteristic point M_k is the limit point of intersections $C_k \cap C_{k+h}$ as $h \rightarrow 0$. In other words, the envelope \mathcal{E}_1 is the set of limit points of intersections of nearby curves C_k .*

Definition 2 (Impredicative) *The envelope \mathcal{E}_2 is a curve such that at each of its points, it is tangent to a unique curve from the given family. The locus of points where \mathcal{E}_2 touches C_k is called the characteristic point M_k .*

Definition 3 (Analytic) *Suppose that the family of curves is given by an equation $F(x, y, k) = 0$ (where k is a real parameter and F is differentiable with respect to k); then an envelope \mathcal{E}_3 is determined by the solution of the system of equations:*

$$\begin{cases} F(x, y, k) = 0 \\ \frac{\partial F}{\partial k}(x, y, k) = 0 \end{cases}$$

i.e., the envelope is the projection onto the (x, y) -plane of the points in the 3- dimensional (x, y, k) -space, belonging to the surface with equation $F(x, y, k) = 0$.

Simple examples are given in [7]. With other notations, Bruce and Giblin ([3], Chap. 5), show that $\mathcal{E}_1 \subset \mathcal{E}_3$ and $\mathcal{E}_2 \subset \mathcal{E}_3$, and give several examples. They add a 4th definition, different from the previous three.

Definition 4 *The envelope \mathcal{E}_4 is the boundary of the region filled by the curves C_k .*

Among the above definitions, the only one which is easily computable is Definition 3. This is the only definition given by Berger [2](sections 9.6.7 and 14.6.1). Examples for Definition 4 have been studied in [4, 5]; in this 1st paper in reference, the question was related to the determination of a safety zone around a mobile device. This is the meaning of an envelope that interests us in what follows to determine zones of influence of wells.

5 Some case studies

5.1 Two wells

For two wells, one at the origin $(0, 0)$ and another one at the position $(a, 0)$. The drawdown of the head for steady state solution of the flow equation for a confined aquifer is given by:

$$\begin{cases} h(x, y) = \phi(x, y) = h_0 \\ \phi(x, y) = \frac{Q_1}{2\pi Kb} \ln \frac{r_0}{\sqrt{x^2+y^2}} + \frac{Q_2}{2\pi Kb} \ln \frac{r_0}{\sqrt{(x-a)^2+y^2}} \end{cases}$$

where h_0 denotes the initial head before pumping, K the hydraulic conductivity, b the depth of the aquifer, Q_1 the pumping rate of the 1st well and Q_2 the pumping rate of the 2nd well. A simple way of building the zone of protection is to compute the zone determined by the equation $\phi(x, y) = 0$. This equation is equivalent to the following:

$$\ln \left(\frac{r_0}{\sqrt{x^2 + y^2}} \right) \left(\frac{r_0}{\sqrt{(x - a)^2 + y^2}} \right)^q = 0,$$

where $q = \frac{Q_2}{Q_1}$.

Finally the equation

$$[x - a]^2 + y^2]^{q/2} [x^2 + y^2]^{1/2} = r_0^{q+1} \tag{6}$$

defines the curves of delineation of the zone of influence of the two wells. It depends on three parameters:

- the ratio between the pumping rates: $q = r = \frac{Q_2}{Q_1}$;
- The distance between the two wells: $a=L$;
- The radius of the well: $r_0 = p$.

It is important to understand the engineering aspects of these three parameters.

- $q = r = \frac{Q_2}{Q_1}$ described the effect of the pumping rate on the zone of influence;
- $a = L$ analyses the effect of the distance between the wells on the zone of influence;

- $r_0 = p$ analyses the effect of the well design of the singular well on the zone of influence. Depending on the internal radius of the well the the well has a " potential radius of influence" r_0 .

In the following figures, obtained with Desmos, for $Q_1 = Q_2$ we present three different types of graphs. These graphs show three cases of "zone of influence".

a. Two separate zones of influence; see Figure 2.

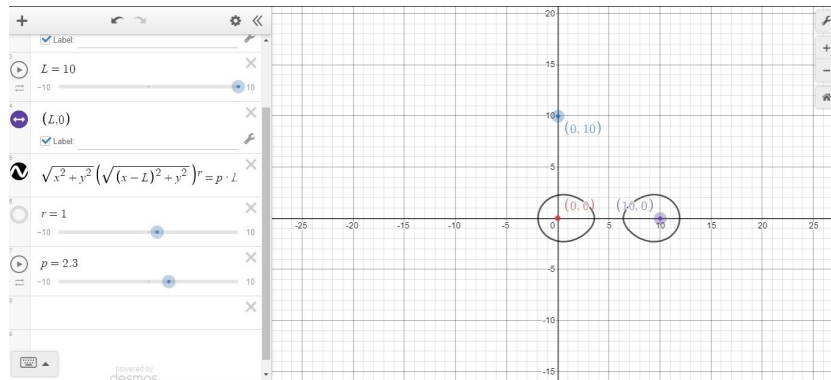


Figure 2: Equal pumping; $r = 1, L = 10, p = 2.3$

b. A narrow zone of influence, as illustrated un Figure 3.

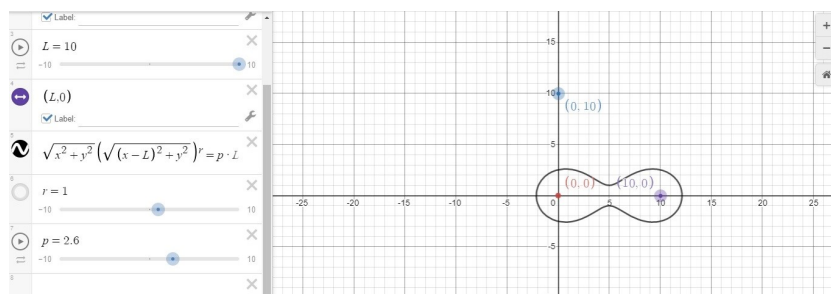


Figure 3: Equal pumping; $r = 1, L = 10, p = 2.6$

c. A large zone of influence, as shown in Figure 4.

The equations appearing in the figures have been written in a simplified form, suitable for the specific cases, namely:

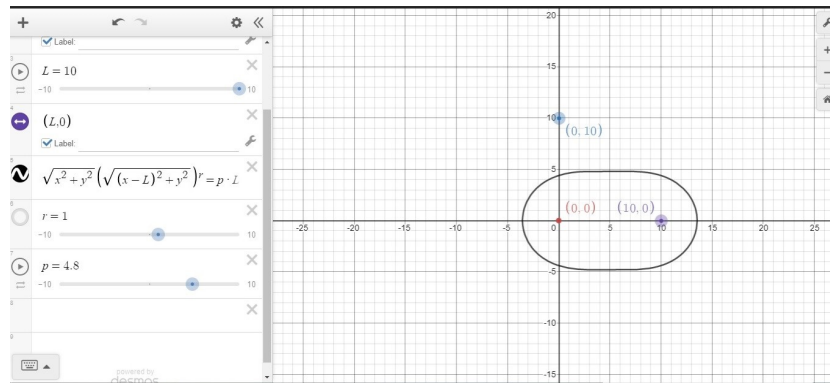
$$\sqrt{x^2 + y^2} \cdot \sqrt{(x - L)^2 + y^2} = pL. \quad (7)$$

Consider the particular case for which $r = 1$. Squaring both sides of Equation (7), we obtained a quadratic equation

$$(x^2 + y^2) \cdot ((x - L)^2 + y^2) = p^2 L^2. \quad (8)$$

The delineation curve is now a bicircular quadratic of a specific kind¹

¹Recall that a complete catalog of quadratic curves exists; as soon as a plane algebraic curve is of degree 4, it is easy to determine which kind of curve it is.


 Figure 4: Equal pumping; $r = 1, L = 10, p = 4.8$

Proposition 5 For $r = 1$, the delineation curves are Cassini ovals.

Proof. Denote $F_1(L/2, 0)$ and let $r = 1$. Now look at the equation $\sqrt{x^2 + y^2} \sqrt{(x - L)^2 + y^2} = pL$ which appears in the algebraic window of Figures 2, 3 and 4. It describes the geometric locus of points $M(x, y)$ such that $OM \cdot F_1M = pL$, i.e. the given curve is a Cassini oval with foci O and F_1 .

We can see that also by an algebraic computation. Apply the change of coordinates $(x, y) = (X + L/2, Y)$. Then we have:

$$\sqrt{x^2 + y^2} \cdot \sqrt{(x - L)^2 + y^2} = pL$$

i.e.

$$\sqrt{\left(x + \frac{L}{2}\right)^2 + y^2} \cdot \sqrt{\left(x - \frac{L}{2}\right)^2 + y^2} = pL.$$

Squaring both sides and expanding them, we have:

$$(X^2 + Y^2)^2 - \frac{1}{2}L^2X^2 + \frac{1}{2}L^2Y^2 + \frac{1}{16}L^4 - p^2L^2 = 0,$$

which is easily identified as the equation of a Cassini oval; see [5] and the references there. ■ Recall that, even if the plot shows two components, the polynomial is irreducible and the curve is irreducible. This is easy to check with the **factor** command of any CAS. The more the wells are distant, the more the curve shows points of inflexion, until it has two components, shaped as loops. This is illustrated in Figure 5.

Here are a few rows of Maple code for Figure 5.

```
restart:with(plots):setoptions(scaling = constrained):setoptions(thickness = 2);
F := (X^2 + Y^2)^2 - 1/2*L^2*X^2 + 1/2*L^2*Y^2 + 1/16*L^4 - p^2*L^2 = 0;
p := 4.8;
for k from 10 by 2 to 16 do
implicitplot(subs(L = k, F), X = -15 .. 15, Y = -15 .. 15);
end do;
```

Remark 6 Cassini ovals are defined by equations whose general form is $(x^2 + y^2)^2 + ax^2 + by^2 + c = 0$, where a, b, c are real numbers. They may have one or two components, which are

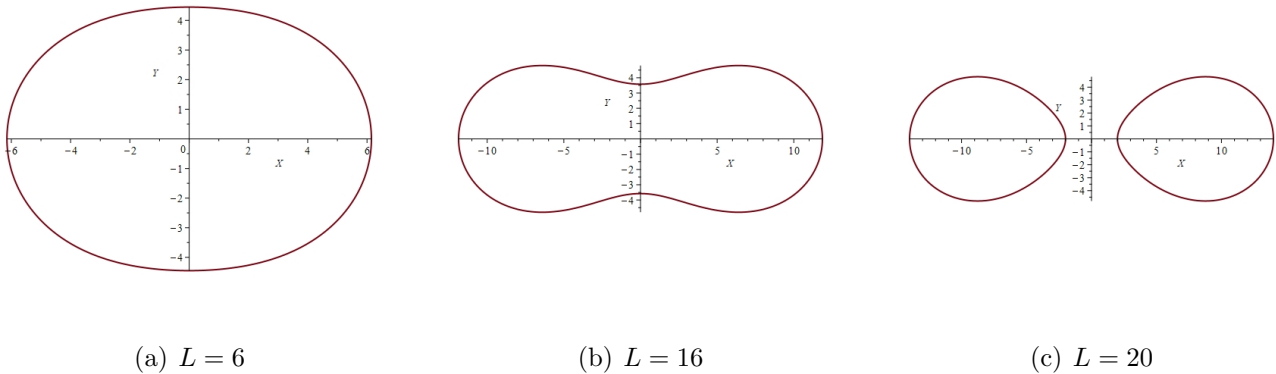


Figure 5: The influence of the distance between wells

not distinguished by factorization of the polynomial. Another description of Cassini ovals is as the intersection of a torus with a plane parallel to the torus axis. The general setting in the literature is with a regular torus. Equations as above describe sometimes the intersection of a self-intersecting torus with a plane parallel to the axis; in such a case, the intersection may have two components, one inside the other. Details are explained in see [6], where Cassini ovals are called by their other name: spiric curves. The physical meaning of the question under study here is enough to understand why such a situation does not occur here and we don't have a component in the interior of the other one.

The influence of the parameters p and L can be explored separately, using the following rows of Maple code:

- For the influence of L (note that here the value of F is fixed, but this can be easily changed, even introduced in a **for** loop):

```
plots[animate](implicitplot, [subs(p = 4.8, F) = 0, X = -20 .. 20,
                               Y = -10 .. 10], L = 0 .. 20);
```

- For the influence of p (with a similar remark as above, this time regarding the value of L):

```
plots[animate](implicitplot, [subs(L = 3, F) = 0, X = -20 .. 20,
                               Y = -10 .. 10], p = 0 .. 10);
```

For other values of the parameter r , other shapes are obtained, and have to be studied separately. Figure 6 shows an example with $r = 2$. If $r = 2$ the obtained delineation curve is as sextic². For such curves, no complete catalog exist, but an algorithm is available³, to determine the topology of a given sextic. The website Mathcurve presents rational sextics and a few non rational ones. A list of 64 cases is given. The pear-shaped curve which we obtained here does

²An algebraic curve of degree 6.

³Developed in 2021 at Max Planck Institute for Mathematics in the Sciences, and implemented in Mathematica

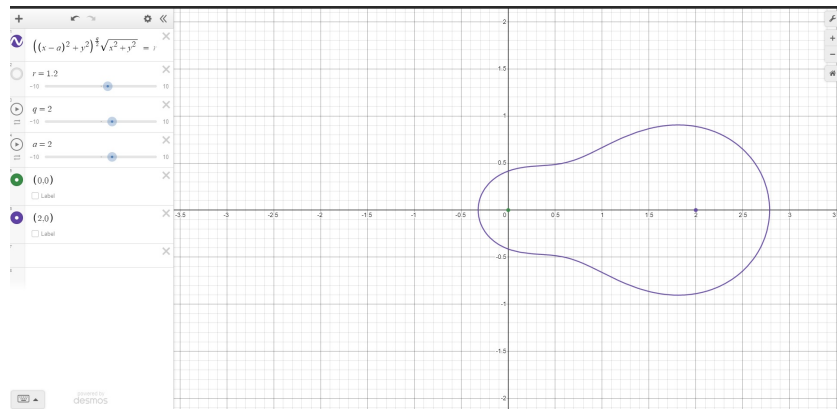


Figure 6: Inequal pumping; $r = 2$

not appear in the list, and checking whether this curve is rational or not is beyond the scope of the present work. See also [13] (the classification is performed for sextics over \mathbb{C} , which is not exactly our concern here) and the references there.

Figures 7 and 9 corresponds to $r = 3$ and $r = 4$ respectively. Here too, it is possible to derive from the data a polynomial equation for the delineation curve. Of course degrees are higher, making the identification with a classical curve harder, if possible. A general exploration of these

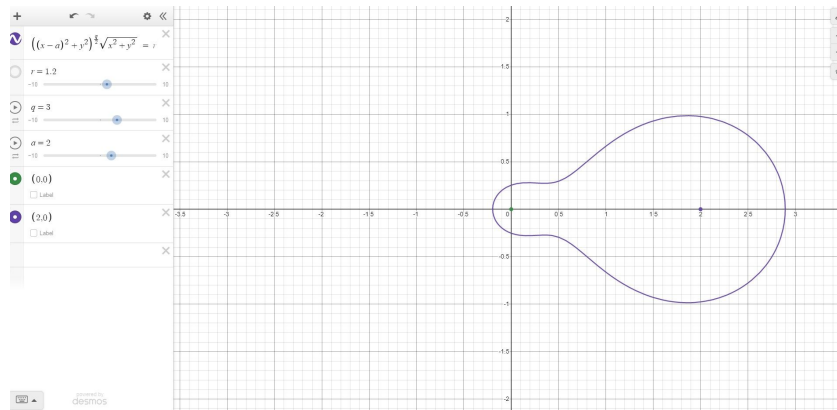


Figure 7: Inequal pumping; $r = 3, L=10, p=4.8$

sextics with similar code as above shows that for pL close to 0, the curves are convex, then have two disjoint components, then have one component with points of inflexion (shaped somehow like cougurds), and again tend to convex shapes. See Figure 8. This surprising behaviour from a single component to a single component and in-between cases with two components is not intuitive. It reinforces the need to make a computerized exploration for different values.

Similar exploration can be performed for any value of the parameter r .

6 Conclusions and directions for future work

Starting from the end, we wish to emphasize the importance of finding Cassini ovals. Jean-Dominique Cassini (8 June 1625 – 14 September 1712), an Italian and French naturalized

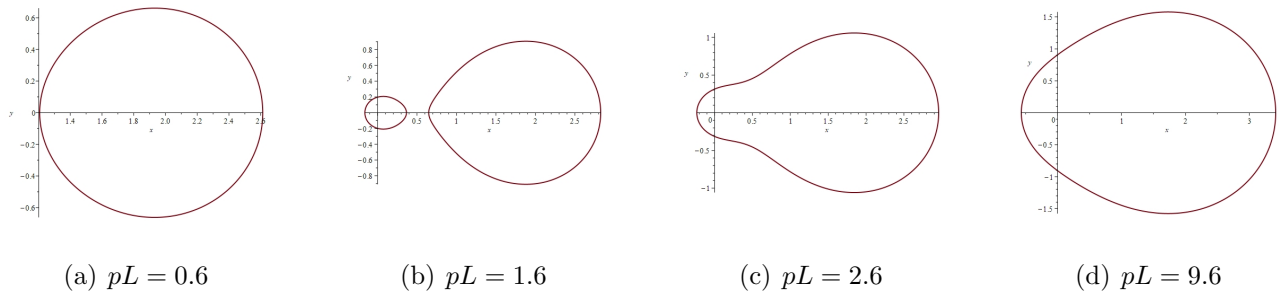


Figure 8: The influence of the distance between wells - sextics

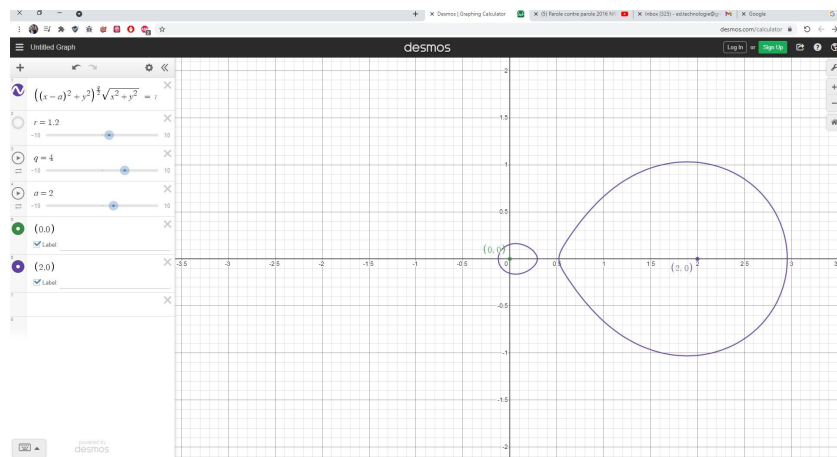


Figure 9: Inequal pumping; $r = 4, L = 10, p = 4.8$

mathematician and astronomer conjectured that the planetary orbits around the Sun were the ovals which will be later called after his name. After Kepler proved that these orbits were actually ellipses (1st Kepler law), Cassini ovals seemed to have lost of their importance and some mathematicians considered them as a nice topic in mathematics and not more. Actually, Cassini ovals appear in electrostatics and to describe some magnetic fields. A mathcurve page is devoted to numerous properties of these ovals. We describe here another application of Cassini ovals and of some of their generalizations. For a reader non familiar with Cassini ovals, a GeoGebra applet is available to check the various possible shapes, according to the choice of the foci and of the parameter..

Regarding the contents of the work above, we wish to make the following (not so) final remarks:

1. One can compile easily using a CAS the zone of influence of a *series* of wells.
2. We present the different solutions for the 2 wells problem with constant pumping wells. This can be generalized to any number of wells in a line. Subsequent work will address the issue of non aligned wells.
3. Parameters such as distances between wells are also considered. Here we did not analyse this effect available for at least three wells.

4. More work is needed for the analysis of the well design of each singular well on the zone of influence. We focused on the distance between wells. The other parameters encode the volume of pumping per time unit, whose influence deserves a separate amelioration of our model.
5. More work is needed for understanding the influence of different pumping rates and well design. In particular, we discovered through a CAS assisted dynamical exploration that for a fixed rate and different values of the distance, very different shapes for the influence zone are obtained. This can have very important consequences in the field, for actual wells connected to the aquifer.

With our approach, it will be possible to analyze the optimal location of the wells.

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Computation of voting power indices using polynomial rings and ideals

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Abstract

In many jurisdictions globally, voting is done not by individuals, but by blocks of voters. Examples are the American Electoral College, the International Monetary Fund, the European Parliament, and the houses of parliament or of congress in many national legislatures. Voting is thus done by blocks: state, country, or political party; each member of which casts the same vote. It is tempting to assume that a block with the most votes is the most powerful, and has the greatest chance of influencing the outcome of the final ballot. As with so much else in voting theory, this is quite incorrect, and it is often the case that a minor block can have an influence entirely out of keeping with its size. There are various different methods of determining the relative power of each block, and allocating a numerical measure of power; these values are called “power indices”. Some of these methods can be computationally intensive, and it took several years after the initial definition to compute power indices for the American Electoral College, with its 50 states. In this paper we explore a unified method for computing different power indices using the theory of polynomial rings. This allows not only a relatively simple computation, but one which can be adjusted to consider “coalitions” (two or more blocks which band together to increase their power), and “quarreling parties”, where two blocks refuse to agree on any vote.

1 Introduction

Although power indices had first been defined and discussed in the mid 1940’s [8], it was really a paper in 1965 which started to bring the notion into prominence. The lawyer John Banzhaf [1] was asked to mediate in a discussion of the fairness of voting allocation in Nassau County, New York state, which had been made as shown in table 1.

There were thus 30 votes in total, of which 16 (a majority) were required for any motion to pass. To determine if power was correlated with votes, Banzhaf made a list of all possible combinations which could win; that is, sum to 16 or greater. In each such combination, he then determined if a particular block was necessary for that win.

Municipality	Number of votes
Hempstead (No. 1):	9
Hempstead (No. 2):	9
North Hempstead:	7
Oyster Bay:	3
Glen Cove:	1
Long Beach:	1

Table 1: Nassau County voting allocations

For example, one winning combination would be Hempstead (No. 1), Hempstead (No. 2), Oyster Bay, and Glen Cove, for a total of 22 votes. However, both the Hempsteads are necessary for this combination, because if either withdrew their support, the votes would drop from 22 to 13. And neither Oyster Bay or Glen Cove are necessary, since if either or both withdrew their support, the total votes would drop to no lower than 18, which is still enough for a win. A necessary voting block in a winning combination is also called a “swinging” block.

Banzhaf identified 32 different winning combinations, and with them 48 different possible swings, with numbers as shown in table 2.

Municipality	Number of swings
Hempstead (No. 1):	16
Hempstead (No. 2):	16
North Hempstead:	16
Oyster Bay:	0
Glen Cove:	0
Long Beach:	0

Table 2: Nassau County necessary votes

It can be seen then that in spite of their different voting numbers, Hempstead 1, Hempstead 2 and North Hempstead are all exactly equal in their abilities to influence a vote; conversely Oyster Bay, Glen Cove and Long Beach have *no power at all*: it doesn’t matter how they vote, because in any combination which reaches a winning value, none of their votes are actually necessary.

The conclusion was that the original voting allocation was manifestly unfair, and some other allocation was required so that the smaller blocks had non-zero power. As Banzhaf said in the article: “It is hard to conceive of any theory of representative government which could justify a system under which the representatives of three of the six municipalities ”represented” are allowed to attend meetings and cast votes, but are unable to have any effect on legislative decisions. Yet this is exactly what occurs now in Nassau County.”

Although Banzhaf had in a sense reinvented a method original proposed by Lionel Penrose (and that had gone completely unnoticed), it was again reinvented in 1971 by James Coleman [3]. These indices are thus known as the Banzhaf, or the Banzhaf-Coleman, or the Penrose-Banzhaf-Coleman, power indices. In this article the term *Banzhaf power index* will be used.

Preceding Banzhaf by over 10 years, in 1954 the economists Lloyd Shapley (who would win the Nobel Prize in 2012) and Martin Shubik developed a power index now known by their

names. To compute the power index, every possible permutation of the voting blocks is listed, and reading from left to right, a block is said to be *pivotal* if it changes a hitherto losing combination to a winning one. For instance, with the Nassau County municipalities, one of the $6! = 720$ permutations is given in table 3.

Municipality	Votes	Cumulative Sum
Glen Cove:	1	1
North Hempstead:	7	8
Hempstead (No. 2):	9	17
Oyster Bay:	3	20
Long Beach:	1	21
Hempstead (No. 1):	9	30

Table 3: A permutation with its pivotal block

We see that in this permutation, Hempstead 2 is pivotal. The Shapley-Shubik index counts the number of times each block has been pivotal. (There are better ways of computing this than enumerating all permutations.) For this example they can be computed to be $(240, 240, 240, 0, 0, 0)$; so as for the Banzhaf indices: the top three municipalities have equal power, and the lower three have no power. We note that by definition the sum of these indices must be $n!$, where n is the number of voting blocks.

It is more convenient to scale indices so that their sum is one; in this instance then both the Banzhaf and Shapley-Shubik indices return $(1/3, 1/3, 1/3, 0, 0, 0)$. Although the two indices can agree, they are generally different.

2 Formal definitions

A *weighted voting game* consists of a sequence of weights w_i for $1 \leq i \leq n$ and a *quota* q . The quota represents the number of votes needed for a win, hence it must be less than or equal to the sum of all the weights. Such a game is notated as

$$[q; w_1, w_2, \dots, w_n].$$

The Nassau County issue discussed in the introduction would thus be notated as

$$[16; 9, 9, 7, 3, 1, 1].$$

A *coalition* is any non-empty set of weights, and a *winning coalition* is a coalition whose weights sum to q or more. In a winning coalition, voter i is *pivotal* or *necessary* if the removal of i from the coalition reduces the sum to below the quota.

2.1 Computation of power indices

We have seen in the introduction that the Banzhaf power indices require consideration of all subsets of voters, and the Shapley-Shubik power indices require consideration of all permutations of voters. In either case, the computation is exponential in terms of the number of voters. Efficient

computation is therefore a major consideration. For example, if we consider the American Electoral College as a weighted voting game with 51 voters (50 states plus DC) where the weights are the number of electors in each state¹, we would require $2^{51} = 2,251,799,813,685,248$ computations, and for the Shapley-Shubik indices we would need $51! \approx 1.55 \times 10^{66}$ computations.

Banzhaf power indices

We can of course compute these by simply enumerating each subset $S \subseteq \{1, 2, 3, \dots, n\}$ and if it is a winning coalition, determining which voter is necessary. But a neater polynomial method was developed by Brams and Affuso in 1976 [2], and for demonstration we shall consider the voting game

$$[39; 34, 33, 7, 1, 1].$$

This game represents the Australian Federal Senate, or upper house, in 1985. The parties are Labor (34), Liberal/National (33), Democrats (7), Nuclear Disarmament (1), Independent (1).

To determine the power of voter k , first create the formal polynomial

$$p_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^n (1 + x^{w_i})$$

so that for the first voter above,

$$p_1(x) = (1 + x^{33})(1 + x^7)(1 + x)(1 + x).$$

Then the coefficient of x^j in this polynomial is the number of ways all the other voters can combine to form a coalition with j votes. Expanding the polynomial:

$$p_1(x) = x^{42} + 2x^{41} + x^{40} + x^{35} + 2x^{34} + x^{33} + x^9 + 2x^8 + x^7 + x^2 + 2x + 1$$

shows that there are two ways, for example, of obtaining a total of 41 votes (voters 2, 3, and 4, or voters 2, 3, and 5). For the votes already above the quotient of 39, the addition of the new voter won't make any difference; the voter cannot be necessary. Voter k becomes necessary only for those votes with a sum less than the quotient q , but not less than $q - w_k$. The latter restriction is necessary because if a sum is less than $q - w_k$, then the addition of the weight w_k cannot produce a sum equal to or greater than the quotient. This means that the number we want is the sum of coefficients of all powers x^m for which $q - w_k \leq m < q$, so that the k -th Banzhaf index, b_k can be computed as

$$b_k = \sum_{j=q-w_k}^{q-1} c_j$$

where c_j is the coefficient of x_j in the polynomial.

In our example, given $p_1(x)$ above, we look at coefficients of powers x^m for which $39 - 34 \leq m < 39$:

$$x^{35} + 2x^{34} + x^{33} + x^9 + 2x^8 + x^7$$

and there are six such terms with coefficients summing to 8. Thus the Banzhaf power index for the first voter is 8.

This can be easily implemented in Python using the SymPy library as shown in Listing 1.

¹The situation is slightly muddled in actuality in that two states: Maine and Nebraska, allocate their electoral votes at least partially according to the popular vote. In all other states it's "winner take all".

```

> import sympy as sy
> x = sy.Symbols('x')
> def banzhaf(q,w):
    for k in range(n):
        qe = sy.prod([(1+x**w[i]) for i in range(n) if i != k]).expand()
        b[k] = sum(qe.coeff(x,m) for m in range(q-w[k],q))
    return(b)
> banzhaf(39, [34,33,7,1,1])
[8,8,8,0,0]

```

Listing 1: Computing the Banzhaf power indices using polynomials

The output, showing the Banzhaf indices, indicate that the three largest parties in the Senate have equivalent power, in spite of one being very much smaller than the other², and the two small parties have no power at all. As mentioned above, the “raw” indices can be normalized to sum to one.

Shapley-Shubik indices

The use of polynomials here predates the work of Brams and Affuso; in fact Shapley himself, working with Irwin Mann, produced a polynomial computation in 1962 [7]. However, before introducing the polynomials, we shall see how the Shapley-Shubik indices can be determined by a method very similar to the Banzhaf indices. Suppose that voter k is necessary in a coalition S , being a subset of $V = \{1, 2, 3, \dots, n\}$. Consider the two sets $S_k = S - \{k\}$ and $V - S$. Then k is pivotal in any permutation of the form:

[any permutation of the elements of S_k], k , [any permutation of $V - S$].

If S has r members, then the number of such permutations is $(r - 1)!(n - r)!$. This means that the Shapley-Shubik index for a voter k can be calculated as

$$\sum_S (|S| - 1)!(n - |S|)!$$

where the sum is taken over all coalitions in which k is necessary. For the Australian Senate example, the first voter is necessary to the coalitions

$\{1, 2\}$, $\{1, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 3, 4, 5\}$

and hence its power index will be

$$1!3! + 1!3! + 2!2! + 2!2! + 3!1! + 2!2! + 2!2! + 3!1! = 40.$$

²This smaller party, the “Australian Democrats” was created precisely for this purpose, to maintain a “balance of power in the Senate” and in the words of its founder: “to keep the bastards honest”. In spite of small numbers, the Democrats were a significant player in Australian politics for over a decade.

To use polynomials here, we need—as well as knowing when a voter k is necessary for a coalition—to know the size of the coalition. This is easily done by introducing a new variable y which will act as a counter:

$$p_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^n (1 + x^{w_i} y).$$

For the first voter in the Senate example,

$$\begin{aligned} p_1(x, y) &= (1 + x^{33}y)(1 + x^7y)(1 + xy)(1 + xy) \\ &= x^{42}y^4 + 2x^{41}y^3 + x^{40}y^2 + x^{35}y^3 + 2x^{34}y^2 + x^{33}y + x^9y^3 + 2x^8y^2 \\ &\quad + x^7y + x^2y^2 + 2xy + 1. \end{aligned}$$

This shows, for example, that a coalition with combined weight 41 can be obtained with 3 voters in 2 ways. For the Banzhaf example, we knew that coalitions of weight 41 could be obtained in two ways, but the polynomial did not include the information about the numbers of voters.

As with the Banzhaf indices, to obtain a coalition in which k is necessary, we add the coefficients of $x^i y^j$ for which $q - w_k \leq i < q$. For $p_1(x, y)$ above, this produces

$$x^{35}y^3 + 2x^{34}y^2 + x^{33}y + x^9y^3 + 2x^8y^2 + x^7y$$

and adding the coefficients of the powers of x produces

$$2y^3 + 4y^2 + 2y.$$

In this last polynomial in y , the powers are the sizes of the winning coalitions.

This means that the Banzhaf procedure given in Listing 1 can be used as the basis for a very similar procedure for computing the Shapley-Shubik indices; this is given in Listing 2.

This procedure can be easily adjusted to return the normalized values of $[1/3, 1/3, 1/3, 0, 0]$.

2.2 Deegan-Packel and Holler indices

More recently, some other power indices have been proposed. In 1978, Deegan and Packel [4] proposed an index based on *minimal winning coalitions*, abbreviated as MWCs, which are coalitions in which every party is critical. For example, with $w = [15, 12, 7, 4, 3, 2]$ and $q = 22$, then $(15, 4, 3)$ is a minimal winning coalition, as is $(12, 7, 3)$. However, $(15, 12, 4)$ is a winning coalition, but it is not minimal, since 4 is not needed. Let W be the set of all such minimal winning coalitions, and let $W_i \subset W$ be those that contain voter i . Then the Deegan-Packel power index is defined as

$$d_i = \sum_{S \in W_i} \frac{1}{|S|}.$$

For example, with the Nassau County example, the MWCs are

$$(9_1, 9_2), \quad (9_1, 7), \quad (9_2, 7).$$

```

> import sympy as sy
> x = sy.Symbols('x')
> def shapley_shubik(q,w):
    n = len(w)
    ss = [0]*n
    for k in range(n):
        pe = sy.prod([(1+x**w[i]*y) for i in range(n) if i != k]).expand()
        py = sum(pe.coeff(x,m) for m in range(q-w[k],q))
        ss[k] = sum(sy.factorial(m)*sy.factorial(n-1-m)*py.coeff(y,m)\
                    for m in range(n))
    return(ss)

> ss(39, [34, 33, 7, 1, 1])
[40, 40, 40, 0, 0]
    
```

Listing 2: Computing the Shapley-Shubik power indices using polynomials

where we have distinguished the two largest voters with subscripts. Since each voter is a member of exactly two of these three coalitions, each one has the index $1/2 + 1/2 = 1$. As with the Banzhaf and Shapley-Shubik indices, the top three voters have equal power; the lower three none at all.

Using MWCs only accords with *Riker's size principle*, that “parties seek to increase votes only up to the size of a minimum coalition” [6]. This makes political sense, and hence winning coalitions that include non-critical parties may be considered as irrelevant to voting power (if the size principle is assumed).

Holler's public good index [6] is obtained by normalizing the values of $|W_i|$ for each voter. Since these values are $(2, 2, 2, 0, 0, 0)$, then the public good indices are $(1/3, 1/3, 1/3, 0, 0, 0)$ as they are for all the other indices.

Although the different indices return the same values for the Nassau County example, this is not normally the case. For example, with $w = [28, 16, 5, 4, 3, 3]$ and $q = 30$, the various normalized indices are:

$$\begin{aligned}
 \text{Banzhaf} &: \left[\frac{3}{4}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20} \right] \approx [0.75, 0.05, 0.05, 0.05, 0.05] \\
 \text{Shapley-Shubik} &: \left[\frac{2}{3}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15} \right] \approx [0.67, 0.067, 0.067, 0.067, 0.067, 0.067] \\
 \text{Deegan-Packel} &: \left[\frac{5}{12}, \frac{7}{60}, \frac{7}{60}, \frac{7}{60}, \frac{7}{60}, \frac{7}{60} \right] \approx [0.42, 0.117, 0.117, 0.117, 0.117, 0.117] \\
 \text{Holler} &: \left[\frac{1}{3}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15} \right] \approx [0.33, 0.133, 0.133, 0.133, 0.133, 0.133]
 \end{aligned}$$

Although all indices give the same smaller value to the five smaller parties, the relative weightings are different.


```

> q = 30
> w = [28,16,5,4,3,3]
> n = len(w)
> xs = sy.var('x0:%d'%n)
> pr = sy.prod([1+xs[i]**w[i] for i in range(n)]).expand()
> pa = pr.args
> pw = [x for x in pa[1:] if sum(sy.degree_list(x)) >= q]
    
```

Listing 3: Using multivariate polynomials

The Deegan-Packel indices are *non-monotonic*; in that it is possible for a smaller party to have a greater index. For example (the values are approximate only):

$$[51; 35, 20, 15, 15, 15] \Rightarrow (1.5, 0.75, 0.92, 0.92, 0.92)$$

and we see that a party of size 15 is assigned a *greater* voting power than a party of size 20. Holler's public good index conforms with Riker's size principle, but is also non-monotonic. The difference between the Deegan-Packel and Holler indices is that Deegan-Packel gives higher weight to coalitions with smaller numbers: a party that can influence an MWC of two members only is seen to have greater power than one which requires three or more members before it can be critical; Holler simply counts the number of MWCs for which a party is critical.

3 Working with multivariate polynomials

In order to deal with “coalitions” (when two or more parties join together), or with “quarreling” parties (who never agree), it will be convenient to use a different symbol for each party. Listing 3 shows how this can be done using SymPy.

In this script, the fourth line creates a list (named “*xs*”) of the variables

$$x_0, x_1, \dots, x_{n-1}$$

and the next line creates the polynomial

$$\text{pr} = \prod_{i=1}^{n-1} (1 + x_i^{w_i}).$$

The next two lines first break the polynomial up into its monomials, and then selects those for which the degree sum is not less than q . In this case there are 28 of them. These monomials correspond to all the winning coalitions. We shall call this polynomial the *encoding polynomial* of the winning coalitions.

To obtain the *minimal* winning coalitions, what we need to do now is to sieve out all those monomials which are multiples of another one, and we can write a recursive function to do this.

The idea is that at the beginning of the sieving routine, t is an empty list, and p consists of all the monomials. As the function works through its iterations, t picks up the “lowest” values from p , while p is reduced by multiples of the elements of t . For example:

```
def sieve(t,p):
    if len(p)==0:
        return(t)
    else:
        for x in p[1:]:
            if sy.rem(x,p[0]) == 0:
                p.remove(x)
        return(sieve(t+[p[0]],p[1:]))
```

Listing 4: A sieve function to remove polynomial multiples

```
> pw2 = pw.copy()
> pws = sieve([],pw2)
> pws
```

$$[x_0^{28}x_1^{16}, x_0^{28}x_2^5, x_0^{28}x_3^4, x_0^{28}x_4^3, x_0^{28}x_5^3, x_1^{16}x_2^5x_3^4x_4^3x_5^3]$$

The sum of these monomials will be a polynomial which, as before, can be referred to as the encoding polynomial of the MWCs. To obtain (for example) the Deegan-Packel power indices, we can work with in dictionary whose keys are the variables, and whose values will be increased by $1/k$ where k is the number of variables in each monomial:

```
> dp = {xs[i]:0 for i in range(n)}
> for p in pws:
    pv = p.free_symbols
    pm = len(pv)
    for x in pv:
        dp[x] += sy.Rational(1,pm)
> dp
```

$$\left\{ x_0 : \frac{5}{2}, x_1 : \frac{7}{10}, x_2 : \frac{7}{10}, x_3 : \frac{7}{10}, x_4 : \frac{7}{10}, x_5 : \frac{7}{10} \right\}$$

and this can be normalized to add to one:

```
> dpn = {xs[i]:0 for i in range(n)}
> s = sum(dpn.values())
> for z in dpn.keys():
    dpn[z] = dp[z]/s
> dpn
```

$$\left\{ x_0 : \frac{5}{12}, x_1 : \frac{7}{60}, x_2 : \frac{7}{60}, x_3 : \frac{7}{60}, x_4 : \frac{7}{60}, x_5 : \frac{7}{60} \right\}$$

As a proof-of-concept of the use of such polynomials, we'll consider the Banzhaf power indices for [16; 10, 9, 6, 5]. We start as above by creating four variables x_i and computing the product

```
> q, w = 16, [10,9,6,5]
> p = \prod_{i=0}^3(1+x_i^{w_i}).
```

This is done exactly as in Listing 3 and the polynomial returned is

$$x_0^{10}x_1^9x_2^6x_3^5 + x_0^{10}x_1^9x_2^6 + x_0^{10}x_1^9x_3^5 + x_0^{10}x_1^9 + x_0^{10}x_2^6x_3^5 + x_0^{10}x_2^6 + x_0^{10}x_3^5 + x_0^{10} + x_1^9x_2^6x_3^5 + x_1^9x_2^6 + x_1^9x_3^5 + x_1^9 + x_2^6x_3^5 + x_2^6 + x_3^5 + 1$$

To find the i -th power index, we reduce the polynomial modulo $x_i^{w_i}$, and add up the coefficients of those monomials whose degree sum is between $q - w_i$ and q . The reduction can be done either with a polynomial division or using Groebner bases; for this simple operation a division is adequate:

```
> i = 0
> quo, rem = sy.div(p,xs[i]**w[i])
```

and the remainder (which is what we want) is

$$x_1^9x_2^6x_3^5 + x_1^9x_2^6 + x_1^9x_3^5 + x_1^9 + x_2^6x_3^5 + x_2^6 + x_3^5 + 1.$$

We can find the coefficients and degree sums, and add the appropriate coefficients:

```
> m = list(rem.args)
> cs = [Poly(rem).coeff_monomial(x) for x in m]
> ds = [1] + [sym(sy.degree_list*x( for x in m[1:]))
> sum(x for x,y in zip(cs,ds) if (y >= q-w[i]) and (y < q))
```

These last scripts can be placed in a loop to run through all values of i .

Clearly this method is overkill for the computation of the Banzhaf indices: as we have seen earlier working with a univariate polynomial is quite sufficient. However, that polynomial cannot indicate which parties have met to form each coalition.

With the voting game [16; 10, 9, 6, 5] the products of all of $(1 + x^{w_i})$ except for each term in turn are:

$$\begin{aligned} &x^{20} + x^{15} + x^{14} + x^{11} + x^9 + x^6 + x^5 + 1 \\ &x^{21} + x^{16} + x^{15} + x^{11} + x^{10} + x^6 + x^5 + 1 \\ &x^{24} + x^{19} + x^{15} + x^{14} + x^{10} + x^9 + x^5 + 1 \\ &x^{25} + x^{19} + x^{16} + x^{15} + x^{10} + x^9 + x^6 + 1 \end{aligned}$$

In the i -th polynomial, we then have to see which of the terms when multiplied by x^{w_i} will push the degree from under to at or over the quota q .

Using different variables, we can first find the product of $(1 + x_i^{w_i})$ and eliminate all terms with a degree sum less than q . This leads to the encoding polynomial:

$$x_0^{10}x_1^9x_2^6x_3^5 + x_0^{10}x_1^9x_2^6 + x_0^{10}x_1^9x_3^5 + x_0^{10}x_1^9 + x_0^{10}x_2^6x_3^5 + x_0^{10}x_2^6 + x_1^9x_2^6x_3^5$$

and we can see immediately which parties have met to form which coalition. The encoding polynomial thus does encode all the individual winning coalitions.

An alternative approach is to start with the encoding polynomial, and for the i -th party, take the quotient when divided by $x_i^{w_i}$. The number of monomials in the quotient with degree sum less than q will be the i -th Banzhaf power index.

4 Quarreling parties

These are parties who—either temporarily or permanently—refuse to vote together. This might be a matter of principle, or of irreconcilable political differences. If the j -th and k -parties are quarreling, then the polynomial $\prod(1 + x_i^{w_i})$ must be reduced modulo $x_j x_k$. In other words, we remove all monomials which include both x_j and x_k .

As an example, we consider the 2021 composition of the Australian Federal Senate:

	Party	Support & Ideology	Numbers
Coalition	Liberal	Business & economy, right wing	31
	National	Primary Producers, right wing	5
Opposition	Labor	Workers, centre left	26
Cross-bench	Greens	Environment & sustainability, left	9
	One Nation	Anti-immigration, far right	2
	Centre Alliance	Centrist	1
	Lambie Network	Populist	1
	Patrick Team	Regional	1

Table 4: Australian Federal Senate, 2021

Confusingly, the Liberal³ and National parties are separate political entities, but for the purposes of obtaining the numbers needed to form a government, have an alliance known as “The Coalition”. And “cross-bench” simply means that the senators in those parties are not bound by coalition or opposition party lines, but can vote according to their consciences. As of late 2021, the Coalition are the governing body in Australia.

As a majority is required to pass any motion, the voting game is thus

$$[39; 31, 5, 26, 9, 2, 1, 1, 1]$$

but given the Liberal-National coalition, this is better expressed as

$$[39; 36, 26, 9, 2, 1, 1, 1]$$

The Banzhaf, Shapley-Shubik, Deegan-Packel indices with their normalizations, and the Holler index, are respectively:

$$\begin{aligned}
 & [52, 12, 12, 10, 4, 4, 4] \Rightarrow [0.531, 0.122, 0.122, 0.102, 0.041, 0.041, 0.041] \\
 & [2616, 684, 684, 516, 180, 180, 180] \Rightarrow [0.519, 0.136, 0.136, 0.102, 0.036, 0.036, 0.036] \\
 & \left[\frac{9}{4}, \frac{11}{10}, \frac{11}{10}, \frac{8}{5}, \frac{59}{60}, \frac{59}{60}, \frac{59}{60} \right] \Rightarrow [0.25, 0.122, 0.122, 0.178, 0.109, 0.109, 0.109] \\
 & \frac{1}{16} [3, 2, 2, 3, 2, 2, 2] = [0.1875, 0.125, 0.125, 0.1875, 0.125, 0.125, 0.125]
 \end{aligned}$$

Suppose that Labor and the Coalition are quarrelling; and will not vote together. We can encode this by determining the encoding polynomial modulo $x_0 x_1$. We can start by creating the polynomial $\prod(1 + x_i^{w_i})$ as shown in Listing 3, and from that obtain the encoding polynomial:

³Note that “Liberal” here has a very different meaning to that in America; Australia’s Liberal Party is a right-wing conservative party.

```

bsq = [0]*n
for i in range(n):
    Gi = sy.groebner([xs[i]**ws[i]],xs)
    quo_i, rem_i = Gi.reduce(rem)
    mn = Poly(quo_i[0],xs).monoms()
    bsq[i] = len([x for x in qi if sum(x) < qs])
    
```

Listing 5: Computing the Banzhaf indices in the case of a quarrel

```

> pmn = Poly(pr,xs).monoms()
> pmnw = [x for x in pmn if sum(x) >= qs]
> p_enc = sum([sy.prod(x**y for x,y in zip(xs,p)) for p in pmnw])
    
```

and then reduce it:

```

> Gs = sy.groebner([xs[0]*xs[1]],xs1)
> quo,rem = Gs.reduce(p_enc)
> display(rem)
    
```

$$x_0^{36} x_2^9 x_3^2 x_4 x_5 x_6 + x_0^{36} x_2^9 x_3^2 x_4 x_5 + \boxed{24 \text{ terms omitted}} + x_1^{26} x_2^9 x_3^2 x_4 x_6 + x_1^{26} x_2^9 x_3^2 x_5 x_6$$

We can now compute the Banzhaf power indices by seeing to which winning coalitions party i is necessary. We can do this by finding the quotient modulo $x_i^{w_i}$ and determining the number of monomials whose degree sum is less than q . This is shown in Listing 5 and the index list, with its normalization, is:

$$[24, 4, 12, 10, 4, 4, 4] \Rightarrow [0.387, 0.065, 0.194, 0.161, 0.065, 0.065, 0.065]$$

Comparing these values with the previous (non-quarreling results) we see the Liberal/Nationals (the first term) have decreased in power slightly, Labor (the second term) has decreased in power drastically, and the loss is distributed among the other parties. The lesson here is that it would be very unwise for Labor to adopt a permanent quarrelling stance with the Coalition.

Suppose that the Greens and One Nation are quarrelling (which is a very reasonable assumption, given their respective ideologies); this requires reducing modulo $x_2 x_3$. The resulting encoding polynomial has 40 terms, and the Banzhaf indices can be found to be

$$[40, 8, 7, 6, 2, 2, 2] \Rightarrow [0.597, 0.119, 0.104, 0.09, 0.03, 0.03, 0.03]$$

and so this particular quarrel has the effect of increasing the power of the Coalition.

A diagram of the quarrels and their effects on the normalized Banzhaf indices is given in Figure 1. One counter-intuitive result is that a quarrel may increase the power of the non-quarreling parties.

Clearly the method outlined can be used for multiple simultaneous quarrels.

5 Conclusions

Assessing the power of a voting body in weighted voting is a fundamental aspect of modern decision making; we expect and assume that voting power will be roughly proportional to the

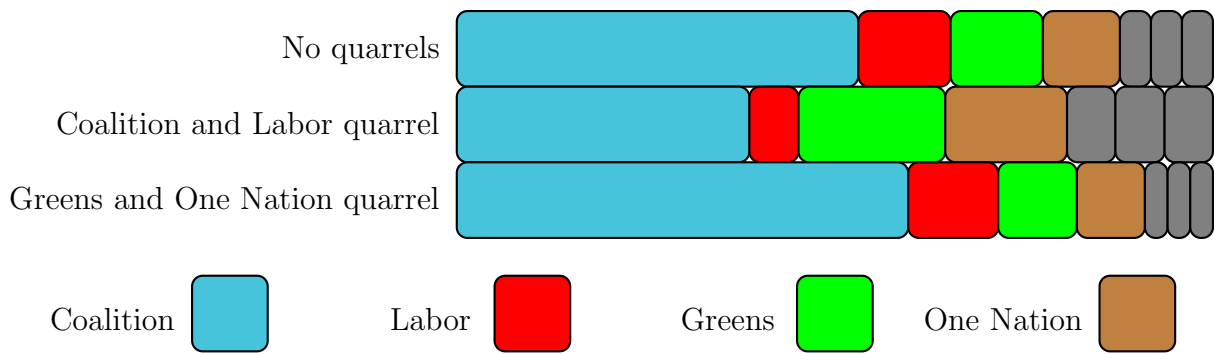


Figure 1: Effects of quarreling

voter’s weight. However, this is rarely the case. Over 60 years of investigation into power indices have resulted in many different methods of assigning power, each with particular strengths and weaknesses. And there is now a sizable literature into axiomatic power indexing: what are the basic axioms we would expect a power index to satisfy, and can they be made consistent? And if not, what properties are we prepared to forsake? As we have seen, monotonicity may be a desirable property, which is not satisfied by either the Deegan-Packel or Holler indices. Power is thus a vital topic in decision theory, and such relatively recent upheavals as Brexit may require voting weights to be reassigned so that no country increases power at the cost of another [5].

This means that the computation of voting power indices is also an important topic, and polynomials and various combinatorial algorithms are still popular. We have shown how standard polynomial methods can be enlarged and deepened, using the theory of ideals, can provide a highly general approach that can also deal with quarrels.

Acknowledgements

The author gratefully acknowledges the close reading of the reviewers, who pointed out a number of errors: factual, typographical, and mathematical, (hopefully all of) which have been amended.

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