Shannon-Nyquist Sampling with a Student Exploration Application

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Abstract

The Shannon-Nyquist theorem provides a minimum number of sample points required to collect to completely determine an analog signal. First, we discuss the history following the developments of Nyquist, Shannon, and Whitakker on analyzing analog functions. Second, we outline digitizing a signal and processing the data along with aliasing and other anomalies. We finish by presenting a student application for exploring Shannon-Nyquist Sampling using Maple™.

1 Introduction and Background

The Shannon-Nyquist sampling theorem provides the fundamental connection between continuous-time signals and discrete-time samples. The theorem gives a sufficient condition for a sample rate to capture all the information from a continuous-time signal of finite bandwidth. Sampling and reconstructing a continuous signal falls under the general theory of approximation. Looking back, we see that in 1897 Borel [3] included in a theorem (under suitable conditions on the functions) the result

> If one knows the values of the function $f(z) = \int_{-\pi}^{\pi} \Psi(x)\, e^{zxı}\, dx$ at the points $z = 0, \pm1, \pm2, \ldots,$ then the function $\Psi(x)$ is completely determined.

Next, in 1915, Edmund Whittaker [17] presented a paper on interpolation with what his son J. Whittaker would later call the *cardinal series* that included the converse of the sampling theorem we'll be exploring.

Harry Nyquist published "Certain Topics in Telegraph Transmission Theory" [13] in 1928 that developed what we now call the *Nyquist rate*, twice the target function's highest frequency component, needed for sampling to be able to perfectly reconstruct a band-limited signal.

In 1933, Vladimir Kotel'nikov published the sampling theorem in "On the transmission capacity of the 'ether' and of cables in electrical communications," [8] but the paper was in Russian and remained essentially unknown outside the Soviet Union until recent times.

Claude Shannon stated that the sampling theorem was "a fact which is common knowledge in the communication art" [15, 16], citing Bennett (1941) [2], who in turn cited Raabe's 1939 PhD thesis [14]. Shannon's Theorem 1 [16, pg 11] presented sampling nicely as

> If a function $f(t)$ contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced $1/2W$ seconds apart.

While there are huge numbers of real applications, such as digital music, digital television, internet telephony, etc., we must interject an appropriate caveat: essentially no "real-world" analog signal has a maximum frequency component, i.e., is band-limited. Since 'real' signals aren't band-limited, the sampling theorem is applied in approximation.

For a comprehensive history of the sampling theorem look to A. J. Jerri's "The Shannon Sampling Theorem — Its Various Extensions and Applications: A Tutorial Review" [7].

2 The Shannon-Nyquist Sampling Theorem

Given the result’s history, it’s difficult to assign a proper name to the theorem. The most descriptive name, the *Whittaker-Nyquist-Kotel’nikov-Shannon-Kramer* Sampling Theorem is also the most cumbersome and most unlikely to be used in the literature. The theorem is also known as the *Cardinal Theorem of Interpolation*, but this name obscures the source and the result’s importance to sampling. We will call the result the *The Shannon-Nyquist Sampling Theorem*.

Let’s set the stage with a few definitions. First, we work towards a rigorous definition of band-limited.

Definition 1 (Compact Support)**.** *A function $X(\omega)$ has* compact support *when $X(\omega)$ is zero outside a finite interval* $[a, b]$.

A Fourier transform of a signal moves us to the frequency domain from the time domain, then compact support bounds the frequency components of the signal. The range of frequencies in a signal tells us its *bandwidth*.

Definition 2 (Bandwidth)**.** *The* bandwidth *of a signal is the difference between its maximum and minimum frequency components*

An audio signal heard by the human ear must lie between (approximately) 20 Hz and 20 kHz giving a bandwidth of $20{,}000 - 20 \approx 20$ kHz.

A signal that has a finite bandwidth has a maximum frequency, and vice versa. Putting this observation in terms of Fourier transforms gives

Definition 3 (Band-Limited Signal)**.** *A signal $x(t)$ whose Fourier transform has compact support is* band-limited*; i.e., x has a maximum frequency component*

Harry Nyquist’ main contribution to the theorem was in determining the needed sampling rate in terms of the frequency components of a signal.

Definition 4 (Nyquist Rate)**.** *The* Nyquist rate *$\mathcal{N}$ is twice the maximum frequency component of a band-limited signal.*

All the pieces are in place, so it’s time to present the sampling theorem. We’ll use Shannon’s version as it’s the simplest and easiest for students to understand. (Note: *cps* = cycles per second = Hz.)

Theorem 1 (Shannon-Nyquist Sampling Theorem [16, pg 11])**.** *If a function $f(t)$ contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced* $1/2W$ *seconds apart.*

Shannon’s original proof showed that the Fourier coefficients of $x(t)$ were determined by a set of $2W$ uniformly spaced sample points $\{x(k/2W)\}$.

A Fourier transform version of the sampling theorem is

Theorem 2. *Let $x(t)$ be a continuous-time signal with a Fourier transform $X(\omega)$ that has compact support, and let $\Psi(S;t)$ be the trigonometric interpolation polynomial based on the set of points S. If W is the highest frequency component of x, then a set $\mathcal{S}$ of* $2W$ *uniformly-spaced samples is sufficient to have*

$$x(t) = \Psi(\mathcal{S}; t).$$

3 Maple-based Student Exploration Applications

We offer three student explorations using Maple.[1] First, we’ll explore *aliasing*, the distortion that occurs when a signal is sampled below the Nyquist rate. Then we’ll look at a second exploration that compares the sample size with the reconstructed signal in light of the spectrum graph that shows the frequency composition of the original signal. Last, we consider a small application of sampling, image resolution, using photos of the researchers. This activity also gives a “human face” to the project and helps engage students with the topic’s history.

[1]This workbook requires Maple 2020 or newer.

Exploring Aliasing

Aliasing refers to the distortions that result when an interpolated signal reconstructed from samples is different than the original continuous signal. Take the simple continuous function $y = \sin(11t)$ and investigate aliasing when the number of evenly spaced samples is less than the Nyquist rate.

In the exploration, the graph of $f(t) = \sin(11t)$ appears with the sample points and the interpolated reconstruction of f from the sample points. Change the *number of knots* to explore the relation between the number of samples and the goodness of the reconstruction. After each change, the students record their observations and conjecture as to what caused any differences that appeared.

See Figure 1.

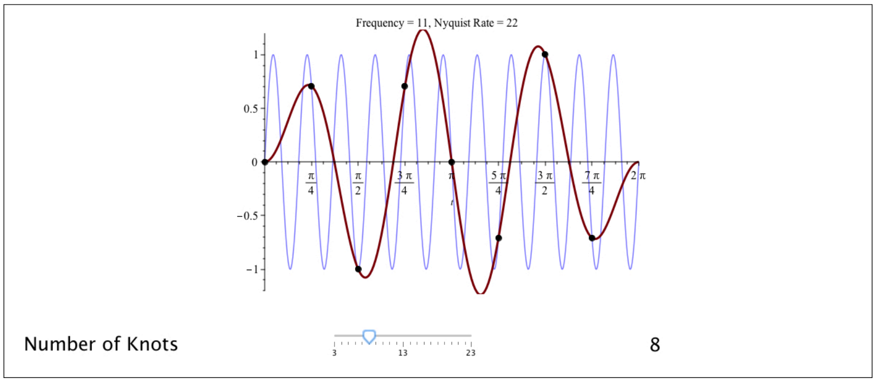


Figure 1: Explore Aliasing

Exploring Shannon-Nyquist Sampling

To explore sampling, we'll take a continuous signal built from summing sine functions. The graph appearing on the left in the exploration can show the original continuous function f, the sample points or *knots*, and the interpolated trigonometric reconstructed function. The graph on the right, the *spectrum graph* of f, shows the magnitudes of the different frequency components of f. Students are directed to change:

- the damping factor to alter the amount each frequency component contributes to the sum;
- the frequency delta to set the change in freqeuncy between components;
- the number of terms making the continuous signal function; and
- the number of knots, or nodes; that is, the number of sample points.

After each change, the students record their observations and conjecture as to what caused any differences that appeared.

See Figure 2.

Exploring an Application of Sampling

A small application is the subject of this exploration. Pictures of the researchers Nyquist, Shannon, and Whittaker are sampled at different depths and images are presented. Upper division students familiar with Maple can be asked to alter the exploration by adding separate controls for horizontal and vertical depth. After each change, the students record their observations and conjecture as to what caused any differences that appeared.

See Figure 3.

4 Conclusion

Shannon-Nyquist sampling is a fascinating, deep, but easily accessible area for student explorations that has significant application to everyday life. Exploring topics like this helps to engage and draw students in to further mathematical studies.

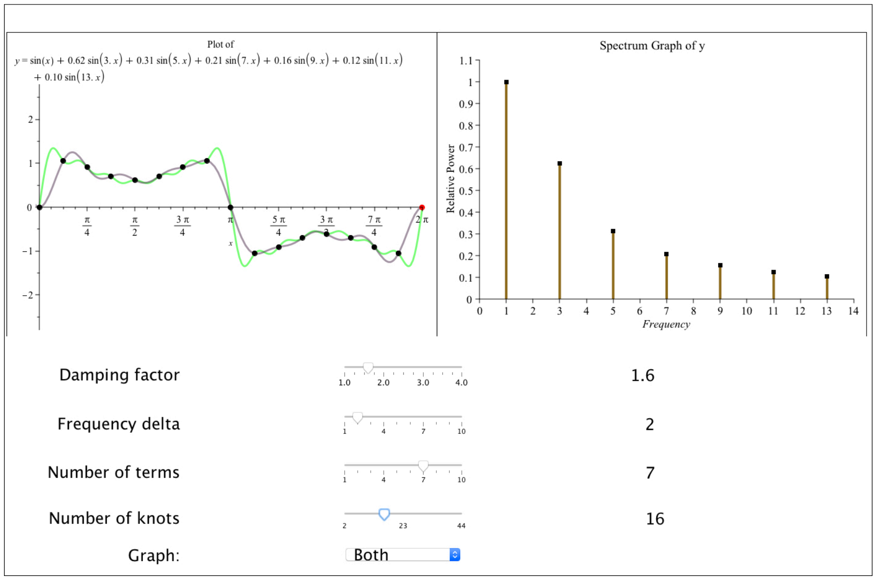
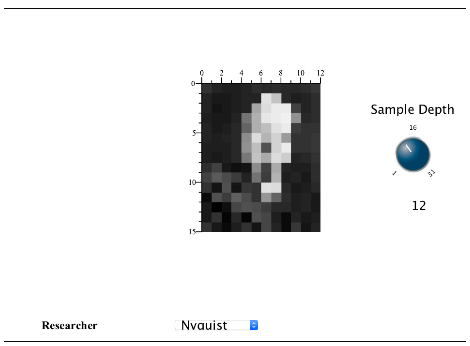
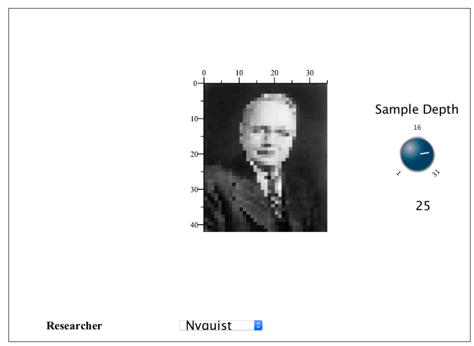


Figure 2: Explore Sampling



(a) Lower Depth Sampling



(b) Higher Depth Sampling

Figure 3: Explore an Application

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