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ORIGINAL PAPER

A new formula for the length of a closed curve

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Abstract

In this article we present a new formula for the length of a closed curve involving a double integral of a certain potential function. This formula is based on the construction of pedal curves, and it turns out that the integral taken over the interior of a pedal curve does not depend on the choice of a pedal point. First, using the notion of support function we derive the expression for the perimeter of oval. Next, we extend our formula to arbitrary smooth simple plane closed curves of the class C^2, but to this aim we needed to use an appropriate notion of an interior of pedal curve. Since the pedal curve of a smooth simple plane closed curve can be fairly complicated and have self-intersections and overlapping parts, we introduce and essentially use a notion of its interior based on the winding numbers and orientations of certain closed parts of the pedal curve forming its partition. At the end of the paper we propose a number of topics for further considerations, also conjecturing that our formula holds for any plane curve having its pedal curve with respect to some point.

Keywords Length · Perimeter · Pedal curve · Pedal point · Interior of a curve · Support function · Winding number · Oval · Potential function

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1 Introduction

Let $SC(n)$ denote a family of all closed hypersurfaces in $\mathbb{R}^n$, $n \geq 2$. With each $X \in SC(n)$ we associate the integral $M_n(X)$ given by the formula

$$M_n(X) = \int\limits_{\overset{m}{X}^*} \frac{1}{||y - m||^{n-1}}\, dy, \tag{1.1}$$

where m is a fixed point inside X and $\overset{m}{X}^*$ denotes a region bounded by a pedal surface $\overset{m}{X}$ of the surface X with respect to the pedal point m.

In [1] we explained the geometrical meaning of the integral $M_3(X)$. We proved that this integral $M_3(X)$ does not depend of the choice of the pedal point m, and that if the surface is an ovaloid then it reduces to its total mean curvature. Thus we could extend the notion of the total mean curvature to a wide class of strictly convex surfaces, and we gave an example that this notion coincides with the total mean curvature for convex polyhedra. Since in our way a straightforward integration technique was involved, an interesting extension of the concept of the mean curvature to non-smooth objects was possible, and intriguing resemblance of the integrand to the potential function was promising and attracting.

In this paper we explain the geometrical meaning of the integral (1.1) in the case $n = 2$. That is, we obtain the following planar counterpart:

$$M_2(X) = \int\limits_{\overset{m}{X}^*} \frac{1}{||y - m||}\, dy, \tag{1.2}$$

where X is an oval in $\mathbb{R}^2$, m lies inside the oval X, and $\overset{m}{X}^*$ denotes a region bounded by a pedal curve $\overset{m}{X}$ of the oval X with respect to the pedal point m. It turns out that we obtain the perimeter of the oval X. This was the main motivation for this paper in which we prove this formula for simple plane closed non-convex regular curves of class C^2 in $\mathbb{R}^2$.

2 Main result

Let $\alpha\colon [0, L] \to \mathbb{R}^2$, $\alpha(t) = (f(t),\ g(t))$ be a regular closed curve, and let $M(m, n)$ be a fixed point in $\mathbb{R}^2$.

Definition 2.1 The pedal curve $P(\alpha)$ of α with respect to a point M is the locus of intersection of the perpendicular from M to a tangent to α.

The pedal curve $P(\alpha)$ is given by the formula

$$t \mapsto \left(\frac{mf'^2 + (n-g)f'g' + fg'^2}{f'^2 + g'^2}, \frac{ng'^2 + (m-f)f'g' + gf'^2}{f'^2 + g'^2} \right)(t), \tag{2.1}$$

see Gray (1998).

Firstly, we consider a class of closed, smooth, strictly convex curves in $\mathbb{R}^2$. Let us consider a coordinate system with the origin O in the interior of a curve K. The function $p\colon [0, 2\pi] \to \mathbb{R}$ is called *support function* of K, if it describes the distance from the origin to the support line of K perpendicular to the vector e^{it}.

Theorem 2.1 *The perimeter L of K is given by the formula*

$$L = \iint\limits_{P^*(K)} \frac{1}{||(x, y) - (m, n)||}\, dxdy, \tag{2.2}$$

where $P^*(K)$ denote the region bounded by a pedal curve $P(K)$ of a curve K with respect to its any interior point $M(m, n)$.

Proof Let p denote the support function of K with respect to the origin. We denote by p_M the support function with respect to $M = m + ni$. We note that

$$p_M(t) = p(t) - \left\langle e^{it}, M \right\rangle,$$

and we parametrize the interior $P^*(K)$ of the pedal curve with respect to the point M using the homothety with ratio $s \in (0, 1)$. We consider a diffeomorphism

$$G\colon [0, 1] \times [0, 2\pi] \to P^*(K)$$

defined as follows:

$$G(s, t) = s\left(p(t) - \left\langle e^{it}, M \right\rangle\right) e^{it} + M.$$

The Jacobian JG of G is given by

$$JG = s\left(p(t) - \left\langle e^{it}, M \right\rangle\right)^2.$$

Therefore

$$\iint\limits_{P^*(K)} \frac{1}{||(x, y) - (m, n)||}\, dxdy = \int_0^{2\pi} \left(p(t) - \left\langle e^{it}, M \right\rangle\right) dt = \int_0^{2\pi} p_M(t)dt = L.$$

see Santalo (2004). □

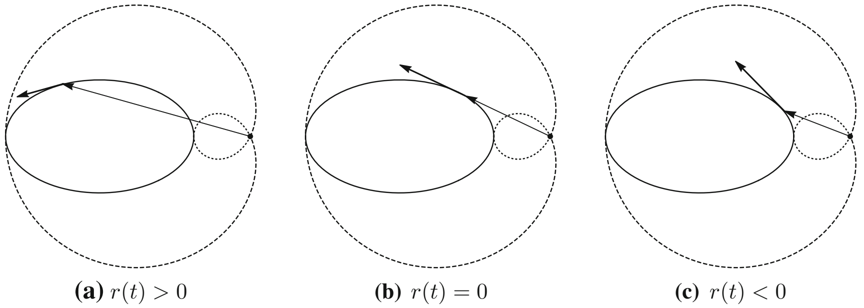

Fig. 1 The sign of $r(t)$

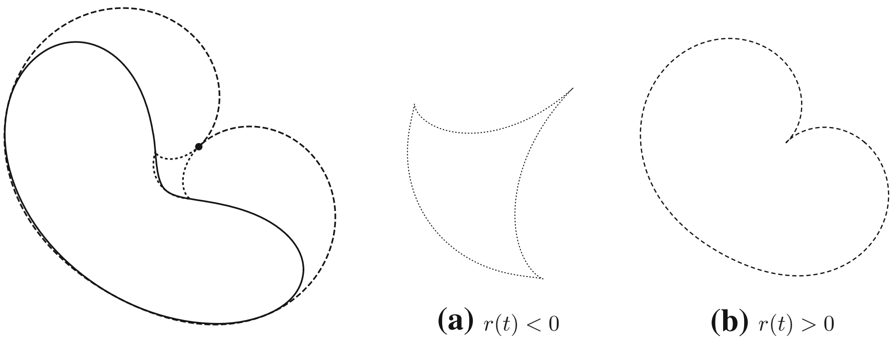

Fig. 2 A partition of a pedal curve

Theorem 1 can be extended to curves which are not strictly convex. To deal with other cases, instead of considering the pedal curve of α with respect to a point $M(m, n)$, we will consider the pedal curve of the curve $\alpha_M(t) = (f(t) - m, g(t) - n)$ with respect to the origin. α_M is the curve α translated by the vector $[-m, -n]$.

Let us denote by $\mathcal{P}$ the pedal curve of α_M with respect to the origin, which is given by the formula

$$\begin{aligned}\mathcal{P}(t) &= \left(\frac{(n-g)f'g' + (f-m)g'^2}{f'^2 + g'^2}, \ \frac{(m-f)f'g' + (g-n)f'^2}{f'^2 + g'^2}\right)(t)\\ &= -r(t)i\alpha'(t),\end{aligned} \tag{2.3}$$

where

$$r(t) = \begin{vmatrix} f(t) - m & g(t) - n \\ f'(t) & g'(t) \end{vmatrix}. \tag{2.4}$$

The sign of $r(t)$ is explained by Fig. 1.

Zeros of the function r determine a partition of a closed curve into closed curves without self-intersections and with constant sign of r. An example is the pedal curve of “a bean curve” with respect to an exterior point, see Fig. 2.

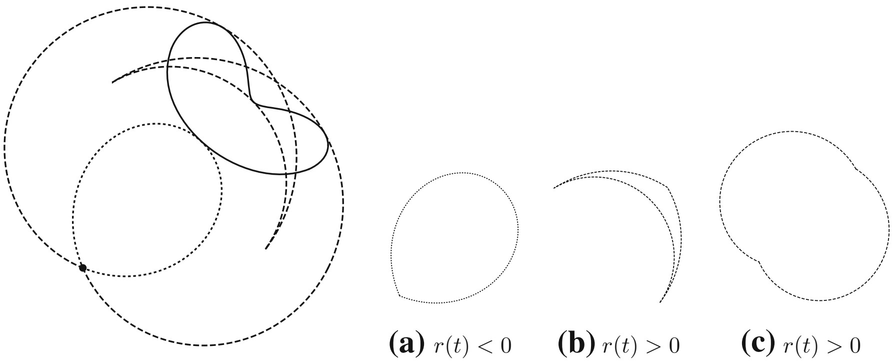

Fig. 3 A partition of a pedal curve into simple curves

The common tangents to the curve α determine double points of its pedal curve. Therefore, when obtained curves are not simple, we divide them in their self-intersection points to be simple closed curves, see Fig. 3.

Definition 2.2 *(Interior of a pedal curve)* The interior $\mathcal{P}^*$ of the pedal curve $\mathcal{P}$ is said to be the sum of interiors $\mathcal{P}_j^*$ of curves determined by this partition. We assign two numbers to each of $\mathcal{P}_j^*$: sgn $r_{\mathcal{P}_j}$, which denotes the sign of r on its boundary $\mathcal{P}_j$, and the winding number $\mathrm{Ind}_{\mathcal{P}_j}$ of a contour $\mathcal{P}_j$ around one of its interior points.

Theorem 2.2 *Let $\alpha : [0, L] \to \mathbb{R}^2$ be a regular closed curve of the class C^2. Let $M(m, n)$ be a fixed point in $\mathbb{R}^2$. The perimeter L of a curve α is given by*

$$L = \iint\limits_{P^*(\alpha)} \frac{1}{||(x, y) - (m, n)||} dxdy = \sum_j \iint\limits_{P_j^*(\alpha)} \frac{\mathrm{sgn}\, r_{P_j} \cdot \mathrm{Ind}_{P_j}}{||(x, y) - (m, n)||} dxdy, \qquad (2.5)$$

where $P^*(\alpha)$ denote the interior of the pedal curve P of a curve α with respect to a point $M(m, n)$.

Proof Let us consider the family of closed, smooth, compact curves of the class C^2 in $\mathbb{R}^2$. Let $\alpha_M : [0, L] \to \mathbb{R}^2$, $\bigcup\limits_j L_j = [0, L]$. We note that for the functions

$$P(x, y) = -\frac{1}{2} \ln \left(y + \sqrt{x^2 + y^2} \right), \quad Q(x, y) = \frac{1}{2} \ln \left(x + \sqrt{x^2 + y^2} \right)$$

we have

$$Q_x(x, y) - P_y(x, y) = \frac{1}{\sqrt{x^2 + y^2}}.$$

For the presented partition of the pedal curve $\mathcal{P}(t) = \left(g'(t)r(t), -f'(t)r(t) \right)$, where $r(t) = (f(t) - m)\, g'(t) - (g(t) - n)\, f'(t)$, we obtain regions $\mathcal{P}_j^*$ bounded by a simple closed curve $\mathcal{P}_j$. Next, by applying Green's theorem to each of $\mathcal{P}_j$ we have for $r(t) > 0$

$$\iint_{\mathcal{P}_j^*} \frac{1}{\sqrt{x^2+y^2}}\,dxdy = \frac{1}{2}\int_{L_{j-1}}^{L_j} -\ln(-f'r+r)(g'r)' + \ln(g'r+r)(-f'r)'dt$$

$$= \int_{L_{j-1}}^{L_j} (f'^2 + g'^2)dt + \frac{1}{2}\Big[\big(-\ln(-f'r+r)(g'r) + \ln(g'r+r)(-f'r)\big)$$

$$+(rg'+rf') - 2((f-m)f' + (g-n)g')\Big]\Big|_{L_{j-1}}^{L_j}$$

$$= \int_{L_{j-1}}^{L_j} 1dt + \mathcal{A} = L_j - L_{j-1} + \mathcal{A},$$

where

$$\mathcal{A} = \frac{1}{2}\Big[\big(-\ln(-f'r+r)(g'r) + \ln(g'r+r)(-f'r)\big) + (rg'+rf') - 2((f-m)f' + (g-n)g')\Big]\Big|_{L_{j-1}}^{L_j}.$$

Moreover, for $r(t) < 0$ we have

$$\iint_{\mathcal{P}_j^*} \frac{1}{\sqrt{x^2+y^2}}\,dxdy = \frac{1}{2}\int_{L_{j-1}}^{L_j} \big(-\ln(-f'r-r)(g'r)' + \ln(g'r-r)(-f'r)'\big)\,dt$$

$$= -\int_{L_{j-1}}^{L_j} (f'^2 + g'^2)dt + \frac{1}{2}\Big[\big(-\ln(-f'r-r)(g'r)$$

$$+\ln(g'r-r)(-f'r)\big) - (rg'+rf')$$

$$+2((f-m)f' + (g-n)g')\Big]\Big|_{L_{j-1}}^{L_j}$$

$$= -\int_{L_{j-1}}^{L_j} 1dt + \mathcal{B} = -(L_j - L_{j-1}) + \mathcal{B},$$

where

$$\mathcal{B} = \frac{1}{2}\Big[\big(-\ln(-f'r-r)(g'r) + \ln(g'r-r)(-f'r)\big) - (rg'+rf') + 2((f-m)f' + (g-n)g')\Big]\Big|_{L_{j-1}}^{L_j}.$$

According to Definition 2.2, two signs should be considered: $\mathrm{Ind}_{\mathcal{P}_j}$ and $\operatorname{sgn} r_{\mathcal{P}_j}$. The first one was naturally reduced since Green's theorem was applied to negatively oriented curves.

Our next goal is to show that the value of (2.5) does not depend on $\mathcal{A}$ and $\mathcal{B}$. Let us include the sign of $r(t)$.

Firstly, note that logarithmic expressions for $r(t) = 0$ reduce to 0 by the formula $\lim_{x\to 0, y\to 0} \; y \; \ln x = 0$.

Moreover, we have

$$\sum_j \left[\big((rg' + rf') - 2((f-m)f' - (g-n)g')\big)\Big|_{L_{j-1}}^{L_j} \right]$$
$$= \big((rg' + rf') - 2((f-m)f' - (g-n)g')\big)\Big|_{L_N}$$
$$- \big((rg' + rf') - 2((f-m)f' - (g-n)g')\big)\Big|_{L_0} = 0$$

and

$$\sum_j (L_j - L_{j-1}) = L_N - L_0 = L.$$

Finally,

$$\sum_j \operatorname{sgn} r_{\mathcal{P}_j} \int\limits_{L_j} P dx + Q dy = L.$$

□

3 Final remarks

We proved our formula for the case of closed, simple and regular curves in the plane. The next step might be the extension of this result to closed, simple and piecewise regular curves. Such an extension should be possible for curves to which pedal curves can be defined. Thus, this can probably be done for rectifiable curves, even non-closed; but already this should be a hard problem. Another possible direction is the extension to surfaces in $\mathbb{R}^n$.

References

Charytanowicz, K., Cieślak, W., Mozgawa, W.: A generalization of the total mean curvature, arXiv:1912.10662

Gray, A.: Modern Differential Geometry of Curves and Surfaces with Mathematica, 2nd edn. CRC Press, Boca Raton (1998)

Hadwiger, H.: Altes und Neues über konvexe Körper. Birkhäuser Verlag, Basel und Stuttgart (1955)

Hopf, H.: Differential Geometry in the Large. Notes taken by Peter Lax and John Gray. With a preface by S. S. Chern, Lecture Notes in Mathematics, 1000, Berlin: Springer-Verlag (1983)

Santalo, L.A.: Integral Geometry and Geometric Probability With a foreword by Mark Kac, 2nd edn. Cambridge Mathematical Library, Cambridge University Press, Cambridge (2004)