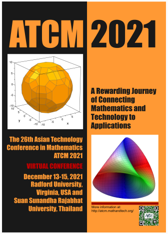
On curves with circles as their isoptics

Witold Mozgawa

Institute of Mathematics, Maria Curie-Skłodowska University, Lublin, Poland

witold.mozgawa@mail.umcs.pl

The 26th Asian Technology Conference in Mathematics, December 13-15, 2021
Hosted by Radford University, Virginia, USA, Suan Sunandha Rajabhat University, Thailand



Introduction

The main part of my lecture is based on a recent paper with prof. W. Cieślak

W. Cieślak, W. Mozgawa, On curves with circles as their isoptics,
Aequat. Math. (2021)
https://doi.org/10.1007/s00010-021-00828-4

The final part is based on

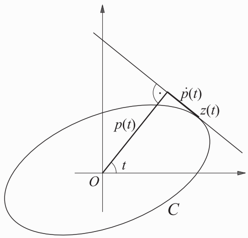
Mozgawa, W., Mellish theorem for generalized constant width curves,
Aequat. Math. 89, 1095–1105 (2015)
https://doi.org/10.1007/s00010-014-0321-3

Introduction

In this talk we consider the family $\mathcal{M}$ of all simple closed convex plane curves of class C^2 which will be called ovals. We take a coordinate system with origin O in the interior of C. Let $p(t)$, $t \in [0, 2\pi]$, be the distance from O to the support line $l(t)$ of C perpendicular to the vector $e^{it} = \cos t + i \sin t$. We call the function $p(t)$ a support function of the curve $C \in \mathcal{M}$ with respect to the origin O. It is well-known that the parametrization of C in terms of $p(t)$ is given by the formula

$$z(t) = p(t)e^{it} + p'(t)ie^{it}. \tag{1}$$

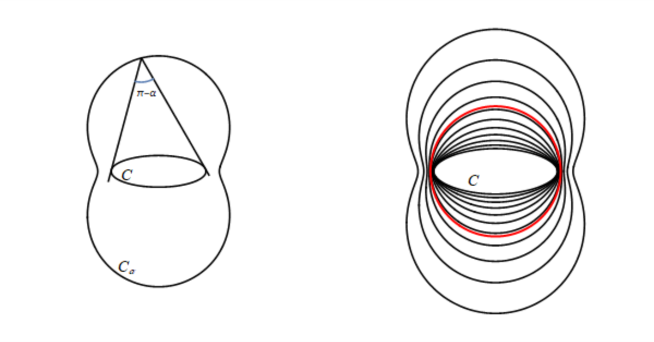
Note that the support function $p(t)$ can be extended to a periodic function on $\mathbb{R}$ with the period 2π.



Now we want to define the notion of isoptics.

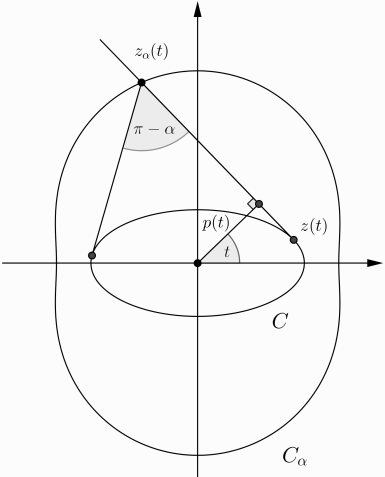
Definition 1.

Let C_α be a locus of vertices of a fixed angle $\pi - \alpha$, where $\alpha \in (0, \pi)$, formed by two support lines of the oval C. The curve C_α will be called an α-*isoptic* of C.

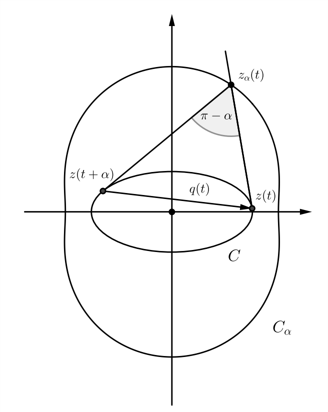


It is convenient to parametrize the α-isoptic C_α by the same angle t so that the equation of C_α takes the form

$$z_\alpha(t) = p(t)e^{it} + \left(-p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\alpha)\right) ie^{it}. \tag{2}$$



In the further part of the talk we will need the curvature k_α of the α-isoptic C_α. Thus we introduce the notation $q(t,\alpha) = z(t) - z(t+\alpha)$,



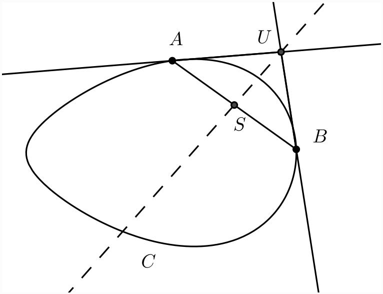
and with the aid of this vector we have that the curvature is given by

$$k_\alpha = \frac{\sin\alpha}{|q(t)|^3}\left(2|q(t)|^2 - [q(t), q'(t)]\right), \tag{3}$$

where [,] denotes the determinant of the arguments.

Some property of ellipses

With each curve $C \in \mathcal{M}$ we associate a certain family C^* consisting of lines constructed in the following way.
We fix a chord of the curve C such that its tangents at points $A, B \in C$ intersect. Let us denote by U the intersection point of these tangents and by S the midpoint of the segment AB. The line passing through U and S belongs to the family C^*. Moreover, given an angle $\alpha \in (0, \pi)$ denote by C^*_α the subfamily of C^* such that $\measuredangle AUB = \pi - \alpha$.



Some property of ellipses

In the first part of the talk we give the following characterization of ellipses.

Theorem 1

Let $\alpha \in (0, \pi)$ be a fixed angle such that $\frac{\alpha}{\pi}$ is a rational number and $\alpha \neq \frac{\pi}{2}$. A curve $\mathcal{C} \in \mathcal{M}$ is an ellipse if and only if all lines from $\mathcal{C}^*_\alpha$ are concurrent.

The simple part of the above theorem, namely:

If $\mathcal{C}$ is an ellipse then all lines of the family $\mathcal{C}^*$ intersect in the center of this ellipse.

follows from the properties of affine transformations since the ellipse $\mathcal{C}$ can be transformed into a circle and for the circle the mentioned property is evident.

Now, we will deal with the second part of the above theorem, namely:

If $\mathcal{C} \in \mathcal{M}$ and all lines of the family $\mathcal{C}^*_\alpha$ are concurrent then $\mathcal{C}$ is an ellipse.

The proof of this fact is divided into steps.

Step 1. Let $C \in \mathcal{M}$ and all lines of the family C^*_α be concurrent. This common point O we take as the origin of the coordinate system and the support function p in the equation (1) is determined with respect to this point. Each point $z_\alpha(t)$ of a fixed α-isoptic determines a chord of the curve C joining the points $z(t)$ and $z(t+\alpha)$. The midpoint of that chord we denote by $s(t)$. The formula (1) yields

$$\begin{aligned} 2s(t) &= z(t) + z(t+\alpha) = \\ &= (p(t) + p(t+\alpha)\cos\alpha - p'(t+\alpha)\sin\alpha)e^{it} + \\ &+ (p'(t) + p(t+\alpha)\sin\alpha + p'(t+\alpha)\cos\alpha)ie^{it}. \end{aligned} \tag{4}$$

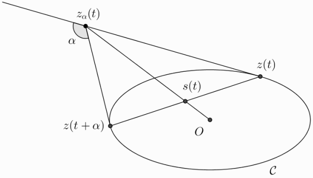


Figure: Points $z(t)$, $z(t+\alpha)$, $z_\alpha(t)$, $s(t)$

Some property of ellipses

From our assumptions the points O, $s(t)$, $z_\alpha(t)$ lie on the same line, that is we have

$$\det[s(t), z_\alpha(t)] = 0. \tag{5}$$

Thus substituting the formulae (2) and (4) into (5) we get the following equation for the support function p, namely

$$(p^2(t+\alpha) - p^2(t))\cos\alpha - (p(t+\alpha)p'(t+\alpha) + p(t)p'(t))\sin\alpha = 0. \tag{6}$$

Substituting $p = \sqrt{f}$ we get a simpler condition for f than (6)

$$2(f(t+\alpha) - f(t))\cos\alpha - (f'(t+\alpha) + f'(t))\sin\alpha = 0. \tag{7}$$

Now, we develop the function f in the Fourier series. Let

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt). \tag{8}$$

Hence we get

$$f(t+\alpha) - f(t)$$
$$= \sum_{n=1}^{\infty} \left[(a_n(\cos n\alpha - 1) + b_n \sin n\alpha) \cos nt + (-a_n \sin n\alpha + b_n(\cos n\alpha - 1)) \sin nt\right],$$

$$f'(t+\alpha) + f'(t)$$
$$= \sum_{n=1}^{\infty} \left[n(-a_n \sin n\alpha + b_n(1 + \cos n\alpha) \cos nt - n(a_n(1 + \cos n\alpha) + b_n \sin n\alpha) \sin nt\right].$$

The above relations substituted into (7) yield the following system of equations

$$\begin{cases} [2(\cos n\alpha - 1)\cos\alpha + n\sin n\alpha\sin\alpha]a_n + [2\sin n\alpha\cos\alpha - n(1+\cos n\alpha)\sin\alpha]b_n = 0 \\ [n(1+\cos n\alpha)\sin\alpha - 2\cos\alpha\sin n\alpha]a_n + [2(\cos n\alpha - 1)\cos\alpha + n\sin n\alpha\sin\alpha]b_n = 0. \end{cases}$$

Since the determinant of this system is equal to

$$[2(\cos n\alpha - 1)\cos\alpha + n\sin n\alpha\sin\alpha]^2 + [2\sin n\alpha\cos\alpha - n(1+\cos n\alpha)\sin\alpha]^2,$$

for the existence of a non-zero solution, the following system of equations must be satisfied

$$\begin{cases} 2(\cos n\alpha - 1)\cos\alpha + n\sin n\alpha\sin\alpha = 0, \\ 2\sin n\alpha\cos\alpha - n(1+\cos n\alpha)\sin\alpha = 0. \end{cases}$$

Hence

$$(9)\qquad \begin{cases} \cos n\alpha = \dfrac{4\cos^2\alpha - n^2\sin^2\alpha}{4\cos^2\alpha + n^2\sin^2\alpha}, \\ \sin n\alpha = \dfrac{4n\sin\alpha\cos\alpha}{4\cos^2\alpha + n^2\sin^2\alpha}. \end{cases}$$

Now, we prove that f has only the coefficients a_0, a_2 and b_2. From the first equation of (9) we obtain that

$$n\sin n\alpha\sin\alpha = 2(1-\cos n\alpha)\cos\alpha,$$
$$2n\sin\frac{n\alpha}{2}\cos\frac{n\alpha}{2}\sin\alpha = 4\left(\sin\frac{n\alpha}{2}\right)^2\cos\alpha,$$

which gives

$$(10)\qquad \frac{n}{2}\tan\alpha = \tan\frac{n\alpha}{2}.$$

Now, we will prove that there is no integer number $n > 2$ such that the equation (10) is fulfilled. In order to do this, we shall prove two lemmas.

Some property of ellipses

The first lemma below is inspired by Lemma 3 in J. Jerónimo-Castro, M. A. Rojas-Tapia, U. Velasco-García, C. Yee-Romero, An isoperimetric inequality for isoptic curves of convex bodies, *Results Math.* 75, 134 (2020), https://doi.org/10.1007/s00025-020-01261-w

Lemma 1

Suppose $\alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$. If there exists a natural number $n > 2$ such that the equation

$$\frac{n}{2} \tan \alpha = \tan\left(\frac{n\alpha}{2}\right)$$

is satisfied then

$$(n+2) \sin \frac{(n-2)\alpha}{2} = (n-2) \sin \frac{(n+2)\alpha}{2}.$$

Proof.

We know that for any complex number $z \in \mathbb{C} \setminus \left\{\left(k + \frac{1}{2}\right)\pi \colon k \in \mathbb{Z}\right\}$, it holds that

$$\tan z = i \frac{e^{-iz} - e^{iz}}{e^{iz} + e^{-iz}}.$$

By the assumptions of the lemma $\tan\alpha$ and $\tan\frac{n\alpha}{2}$ are simultaneously finite. The condition of the lemma in these terms is

$$\frac{e^{-i\frac{n}{2}\alpha}-e^{i\frac{n}{2}\alpha}}{e^{i\frac{n}{2}\alpha}+e^{-i\frac{n}{2}\alpha}}=\frac{n}{2}\cdot\frac{e^{-i\alpha}-e^{i\alpha}}{e^{i\alpha}+e^{-i\alpha}}.$$

From this equality, after some simplifications, we obtain

$$(n+2)\left(e^{i\left(\frac{n-2}{2}\right)\alpha}-e^{-i\left(\frac{n-2}{2}\right)\alpha}\right)=(n-2)\left(e^{i\left(\frac{n+2}{2}\right)\alpha}-e^{-i\left(\frac{n+2}{2}\right)\alpha}\right).$$

Since $\sin z=\frac{1}{2i}\left(e^{iz}-e^{-iz}\right)$, we have

$$(n+2)\sin\frac{(n-2)\alpha}{2}=(n-2)\sin\frac{(n+2)\alpha}{2}.$$

$\square$

Some property of ellipses

Next, we need the following lemma is due to V. Cyr, A number theoretic question arising in the geometry of plane curves and in billiard dynamics, *Proc. Amer. Math. Soc.*, 140, 2012, 3035-3040.

Lemma 2

If $\alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ is such that $\frac{\alpha}{\pi}$ is a rational number, and k and m are integer numbers such that $\sin m\alpha \neq 0$ then

$$\frac{\sin k\alpha}{\sin m\alpha}$$

is either $-1, 0, 1$ or irrational.

Using last two lemmas we prove the following lemma inspired by Lemma 5 in J. Jerónimo-Castro, M. A. Rojas-Tapia, U. Velasco-García, C. Yee-Romero, An isoperimetric inequality for isoptic curves of convex bodies, *Results Math.* 75, 134 (2020), https://doi.org/10.1007/s00025-020-01261-w

Lemma 3

If $\alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ is such that $\frac{\alpha}{\pi}$ is a rational number then there is no integer number $n > 2$ such that

$$\frac{n}{2} \tan \alpha = \tan \frac{n\alpha}{2}.$$

Proof. Suppose $\frac{\alpha}{\pi}$ is a rational number and there is an integer number $n > 2$ such that $\frac{n}{2} \tan \alpha = \tan \frac{n\alpha}{2}$. This condition implies, by Lemma 1, that

$$\frac{\sin \frac{(n-2)\alpha}{2}}{\sin \frac{(n+2)\alpha}{2}} = \frac{n-2}{n+2}.$$

From this condition we have that $\sin \frac{(n+2)\alpha}{2} \neq 0$. Moreover, since $n > 2$, we have that the fraction $\dfrac{n-2}{n+2}$ is different from $-1, 0, 1$, hence by Lemma 2 we have that

$$\frac{\sin \frac{(n-2)\alpha}{2}}{\sin \frac{(n+2)\alpha}{2}}$$

must be an irrational number, which is a contradiction since $\frac{n-2}{n+2}$ cannot be an irrational number. We conclude that there is no integer number $n > 2$ such that $\frac{n}{2}\tan\alpha = \tan\frac{n\alpha}{2}$ if $\frac{\alpha}{\pi}$ is a rational number.

$\square$

Now, if $\frac{\alpha}{\pi}$ is a rational number (different from $\frac{1}{2}$), from Lemma 3 we have that there is no integer number $n > 2$ such that $\frac{n}{2}\tan\alpha = \tan\frac{n\alpha}{2}$. It only remains to analyze the cases $n = 1$ and $n = 2$. When $n = 1$, the only solution of the equations (9) is $\alpha = 0$, which is not a permitted value. For $n = 2$, the equations (9) become identities and so a_2 and b_2 can be chosen arbitrarily. Finally, the function f must be of the following form

$$f(t) = \frac{a_0}{2} + a_2\cos 2t + b_2\sin 2t,$$

where $a_0 > 2\sqrt{a_2^2 + b_2^2}$, since it has to have a positive value.
The case $\alpha = \frac{\pi}{2}$ will be analyzed in next theorem below.

Step 3.
We consider the function

$$p(t)=\sqrt{\frac{a_0}{2}+a_2\cos 2t+b_2\sin 2t}, \tag{11}$$

such that

$$a_0>2\sqrt{a_2^2+b_2^2}. \tag{12}$$

First we note that the condition (12) guarantees that $\frac{a_0}{2}+a_2\cos 2t+b_2\sin 2t>0$. On the other hand we have $4p^3(p+p'')=a_0^2-4a_2^2-4b_2^2>0$. The condition $p+p''>0$ guarantees that p is a support function. Next, after some straightforward calculations we show that the function p is a support function of an ellipse with its center at the origin of the coordinate system and rotated about this point.
Thus support functions given by the formula (11), where $a_0>2\sqrt{a_2^2+b_2^2}$ describe only ellipses. $\square$

Curves whose orthoptics are circles

In this part of my lecture we will consider a subfamily $\mathcal{M}\left(\frac{\pi}{2}\right)$ of the family $\mathcal{M}$ defined as follows

$$(13) \qquad \mathcal{M}\left(\frac{\pi}{2}\right) = \{C \in \mathcal{M} \colon \ C_{\frac{\pi}{2}} \text{ is a circle}\}.$$

Let a curve $C \in \mathcal{M}\left(\frac{\pi}{2}\right)$ be given by (1). We denote by $s(t)$ the midpoint of the segment with ends at $z(t)$ and $z\left(t+\frac{\pi}{2}\right)$. We present here a certain geometric characterization of the family $\mathcal{M}\left(\frac{\pi}{2}\right)$.

Theorem 2

A curve $\mathcal{C} \in \mathcal{M}$ belongs to $\mathcal{M}\left(\frac{\pi}{2}\right)$ if and only if for each fixed t the points $s(t)$, $z_{\frac{\pi}{2}}(t)$ and the origin O lie on the same line.

Proof. Note that

$$\begin{aligned} 2\det[z_{\frac{\pi}{2}}(t), s(t)] &= \det\left[z_{\frac{\pi}{2}}(t), z(t) + z\left(t+\frac{\pi}{2}\right)\right] \\ &= p(t)p'(t) + p\left(t+\frac{\pi}{2}\right)p'\left(t+\frac{\pi}{2}\right) \\ &= \frac{1}{2}\left(p^2(t) + p^2\left(t+\frac{\pi}{2}\right)\right)' = \frac{1}{2}\left(|z_{\frac{\pi}{2}}(t)|^2\right)', \end{aligned}$$

i.e.

$$4\det[z_{\frac{\pi}{2}}(t), s(t)] = \left(|z_{\frac{\pi}{2}}(t)|^2\right)'. \tag{14}$$

From (14) it follows that $s(t), z_{\frac{\pi}{2}}(t), O$ lie on the same line if and only if the orthoptic is a circle. $\square$

Curves whose orthoptics are circles

Let us consider a class $\mathcal{F}$ of all positive valued Fourier series of the form

$$(15) \qquad \frac{a_0}{2} + \sum_{k=0}^{\infty} [a_{2+4k} \cos(2+4k)t + b_{2+4k} \sin(2+4k)t].$$

We recall that the support function p of $\mathcal{C} \in \mathcal{M}\left(\frac{\pi}{2}\right)$ satisfies the following equation

$$(16) \qquad p^2(t) + p^2\left(t + \frac{\pi}{2}\right) = r^2,$$

where r is the radius of the orthoptic, see A. Miernowski, Parallelograms inscribed in a curve having a circle as $\frac{\pi}{2}$-isoptic, *Ann. Univ. Mariae Curie-Skłodowska*, 62, 2008, 105–111.

We develop the function p^2 in the Fourier series

$$p^2(t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} [a_n \cos nt + b_n \sin nt]. \tag{17}$$

Using the calculations from Step 2 we get

$$p^2(t) + p^2\left(t + \frac{\pi}{2}\right) = a_0 +$$
$$+ \sum_{n=0}^{\infty} \left[\left(a_n \left(\cos \frac{n\pi}{2} + 1 \right) + b_n \sin \frac{n\pi}{2} \right) \cos nt + \left(b_n \left(\cos \frac{n\pi}{2} + 1 \right) - a_n \sin \frac{n\pi}{2} \right) \sin nt \right].$$

Hence we have $r^2 = a_0$ and $\frac{n\pi}{2} = \pi + 2k\pi$, that is $n = 2 + 4k$ for $k = 0, 1, 2, \ldots$ Finally, the Fourier series of p^2 belongs to $\mathcal{F}$.

Curves whose orthoptics are circles

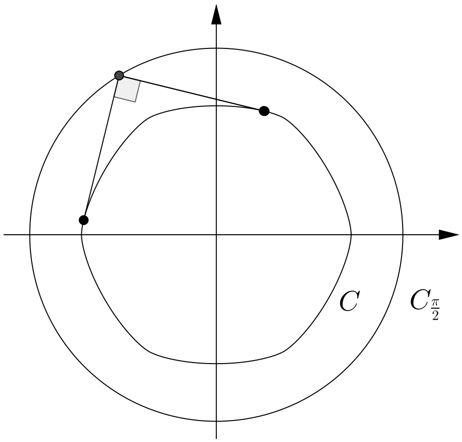


Figure: Circle of radius $\sqrt{45}$ is the orthoptic of the curve $\mathcal{C}$ with $p(t) = \sqrt{\frac{45}{2} + \cos 6t}$

Curves whose orthoptics are circles

On the other hand with respect to (16) we may take $p(t) = r\cos h(t)$ and $p\left(t+\frac{\pi}{2}\right) = r\sin h(t)$. These formulas imply that the Fourier series of the function h belongs to $\mathcal{F}$, where $a_0 = \frac{\pi}{2}$. We note here that J. W. Green in Sets subtending a constant angle on a circle, *Duke Math. J.*, 17, 1950, 263–267 introduced a curve $C \in \mathcal{M}$ with the support function $p(t) = \cos\left(\frac{\pi}{4} + k\sin 2t\right)$ where k is sufficiently small. We will develop this idea in a general setting in the next part of this talk. From the above considerations it follows that all curves of the family $\mathcal{M}\left(\frac{\pi}{2}\right)$ can be constructed using the Fourier series of the class $\mathcal{F}$. To this aim we formulate the following theorem.

Theorem 3

Let $f \in \mathcal{F}$. Each function

1. $p(t) = \sqrt{f(t)}$,
2. $p(t) = \cos f(t)$,
3. $p(t) = \sin f(t)$,

such that $p(t) > 0$ and $p(t) + p''(t) > 0$ is a support function of some curve $C \in \mathcal{M}\left(\frac{\pi}{2}\right)$, and conversely.

Curves whose isoptics are circles

In this part of my talk we extend the results from the previous section to the general case. Our goal is to describe all curves $C \in \mathcal{M}$ possessing a circle as one of its isoptics. Such curves are called curves of generalized constant width and we will talk about them at the end. Now, we will consider a subfamily $\mathcal{M}(\alpha, r)$ of $\mathcal{M}$ defined as follows

$$\mathcal{M}(\alpha, r) = \{C \in \mathcal{M} \colon C_\alpha \text{ is a circle of radius } r\}. \tag{18}$$

We fix a curve $C \in \mathcal{M}(\alpha, r)$. From Theorem 3.1 of W. Mozgawa, Mellish theorem for generalized constant width curves, *Aequationes Math.*, 89, 4, 2015, 1095–1105, we know that the Steiner centroid O of $\mathcal{C}$ and the center of the circle coincide. Thus we assume that the origin of the coordinate system is chosen at O, so the center of this circle is $(0, 0)$. Taking formula (2) into account we see that there should be

$$\begin{cases} p(t) = r \sin h(t), \\ \dfrac{p(t+\alpha) - p(t)\cos\alpha}{\sin\alpha} = r \cos h(t), \end{cases} \tag{19}$$

for some non-constant 2π-periodic function h.

Thus substituting the first formula into the second one we get

$$\sin h(t+\alpha) = \sin(h(t)+\alpha).$$

Thus either $h(t+\alpha) - h(t) = \alpha$ or $h(t+\alpha) + h(t) = \pi - \alpha$. The first case is impossible since the Fourier expansion of the left hand side has no constant term and this implies $\alpha = 0$. If we substitute the Fourier expansion of $h(t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} [a_n \cos nt + b_n \sin nt]$ into the second formula then we obtain

$$a_0 + \sum_{n=0}^{\infty} [(a_n \cos n\alpha + b_n \sin n\alpha + a_n) \cos nt + (b_n \cos n\alpha - a_n \sin n\alpha + b_n) \sin nt] = \pi - \alpha.$$

Then we have

$$a_0 = \pi - \alpha \tag{20}$$

and

$$\begin{cases} (\cos n\alpha + 1) a_n + \sin n\alpha \cdot b_n = 0, \\ -\sin n\alpha \cdot a_n + (\cos n\alpha + 1) b_n = 0. \end{cases} \tag{21}$$

Curves whose isoptics are circles

The determinant of this system of equations is equal to $2(1+\cos n\alpha)$. Thus in order to have the non-zero solutions $\cos n\alpha$ should be equal to -1, so $\alpha = \frac{1+2k}{n}\pi$, for a natural k, such that $0 < \frac{1+2k}{n} < 1$. Thus the possible angles α are rational multiples of π, and $\alpha = \frac{l}{j}\pi$, where $l, j = 1, 2, \ldots$, $l < j$ and l is odd. Then summing up our considerations we have the following theorem.

Theorem 4

Let $\alpha = \frac{l}{j}\pi$ be an angle, where l is odd, l and j are relatively prime, $l < j$ and $l, j = 1, 2, \ldots$. Then each function

(22)

$$p(t) = r\sin\left(\frac{\pi-\alpha}{2} + \sum_{k=0}^{\infty}(a_{j(1+2k)}\cos j(1+2k)t + b_{j(1+2k)}\sin j(1+2k)t)\right),$$

such that $p(t) > 0$ and $p(t) + p''(t) > 0$ is a support function of some curve $\mathcal{C} \in \mathcal{M}(\alpha, r)$, and conversely.

In 1931 a paper *Notes on differential geometry*, Ann. Math. (2) 32, 181-190 (1931), was published containing some fragments found among the papers of an early deceased Canadian mathematician, Arthur Preston Mellish (June 10, 1905 – February 7, 1930). His ideas influenced many interesting papers and are fairly far from being exhausted. The paper we are speaking about begins with some considerations about ovals, where the author gives the following fascinating theorem.

Curves of generalized constant width

Theorem 5 (Mellish)

The statements;

(i) a curve is of constant width;

(ii) a curve is of constant diameter;

(iii) all the normals of a curve (an oval) are double;

(iv) the sum of radii of curvature at opposite points of a curve (an oval) is constant;

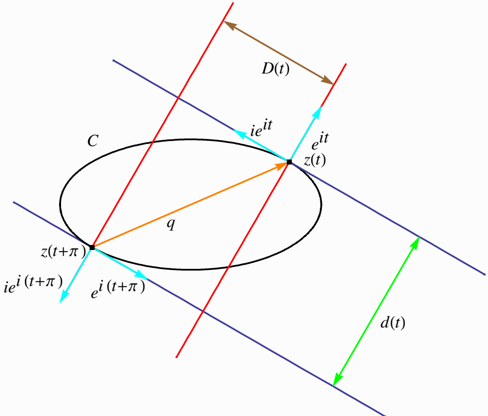
are equivalent, in the sense that whenever one of statements $(\mathrm{i} - \mathrm{iv})$ holds true, all other statements also hold.

(v) All curves of the same (constant) width a have the same length L given by

$$L = \pi a.$$

To understand this theorem we need to introduce a few definitions and notations. For $C \in \mathcal{M}$ we consider two support lines (tangents) at points $z(t)$ i $z(t+\pi)$ and note that from the definition of a support function we have that these lines are parallel. We denote by $d(t)$ the distance between these lines and by $D(t)$ the distance between the normal lines at these points. Recall that

$$q(t) = z(t) - z(t+\pi).$$



Curves of generalized constant width

Definition 2

A curve $C \in \mathcal{M}$ is said to be the curve of constant width if the distance d is constant for each direction $t \in \mathbb{R}$. The number d is called a width of the curve.
A curve $C \in \mathcal{M}$ is said to be the curve of constant diameter if the length $|q|$ is constant for each direction $t \in \mathbb{R}$. The number $|q|$ is called a diameter of the curve.

Note that $d(t) = p(t) + p(t + \pi)$, $D(t) = -d'(t)$ and $d(t) = |q(t)| = R(t) + R(t + \pi)$.

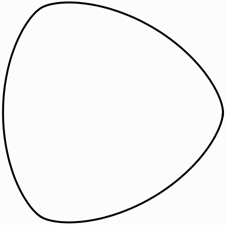


Figure: Curve (oval) of constant width for $p(t) = 10 + \cos 3t$

Curves of generalized constant width

Figure: Church of Our Lady, Bruges, Belgium

Figure: Triangle Tower, Köln, Germany

Curves of generalized constant width

Figure: manhole cover, LA, USA

Curves of generalized constant width

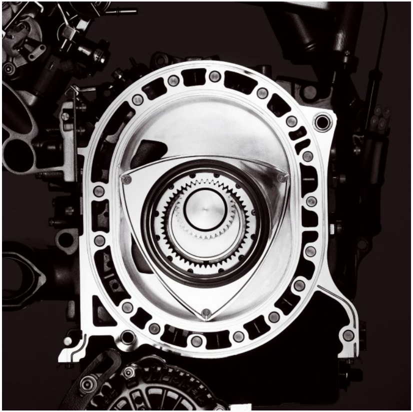
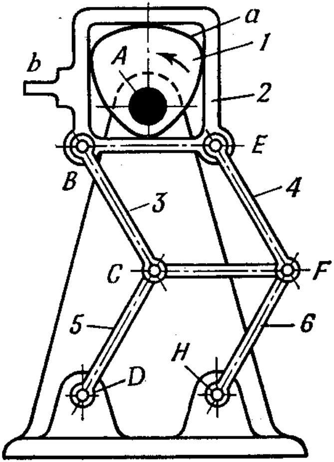


Figure: Wankel engine in Mazda RX-8

Curves of generalized constant width



Principle of operation of the film projector

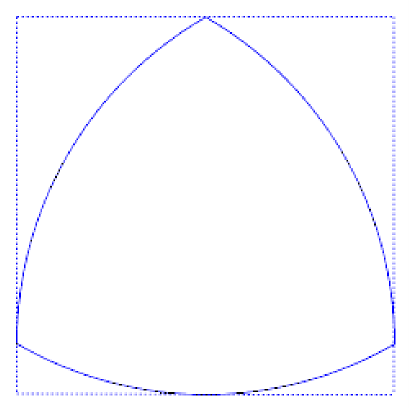
This mechanism pushes the film forward by stopping for the moment when the shutter opens.

Curves of generalized constant width

Coins 20 pence and 50 pence have a shape of curves of a constant width

Curves of generalized constant width

But most spectacular is the ability to drill square holes.



Drilling square holes 1 Play

Drilling square holes 2 Play

Drilling square holes 3 Play

Curves of generalized constant width

As we have seen in the begining of this talk an isoptic is formed as the locus of vertices of a fixed angle $\pi - \alpha$, formed by two tangents to the curve C. On the other hand, one knows that a constant width oval of width a can be freely rotated in a strip of width a. This means that parallel tangents to such oval intersect "at infinity" giving the "π-isoptic". In this talk we would like to bring together the concept of constant width and the concept of isoptics in the the framework of plane Euclidean geometry, while avoiding the projective geometry concepts. Thus the isoptic "at infinity" giving the "π-isoptic" seen as the line at the infinity is not applicable here, and in the further part of this talk we want to define and examine such isoptics.

Recall that with the aid of vector q we can express the curvature of the α-isoptic C_α by

$$k_\alpha = \frac{\sin\alpha}{|q(t)|^3}\left(2|q(t)|^2 - [q(t), q'(t)]\right), \tag{23}$$

where [,] denotes the determinant of the arguments. One of possible ways to see π-isoptic is given by the following definition.

Definition 3

For any curve $C \in \mathcal{M}$ and any its α-isoptic C_α, $0 < \alpha < \pi$, the function

$$\kappa_\alpha(t) = \frac{2|q(t)|^2 - [q(t), q'(t)]}{|q(t)|^3} \tag{24}$$

is said to be a sine-curvature of C_α.

Curves of generalized constant width

We could consider an isoptic of the curve C at infinity as a curve which is given by its curvature κ_π, but neither π-isoptic nor its equation is known yet. Moreover, computer simulations show that such curves can be fairly complicated and possibly constructed in a non-geometric way.
For us the following definition will be useful.

Definition 4

A curve with $\kappa_\alpha \equiv$ *const* will be called the curve of constant α-width.

Note, that this notion makes sense for $\alpha = \pi$. Then, we can say that the considered curve is a curve of constant width at infinity.

Theorem 6

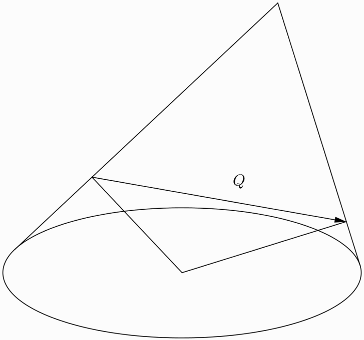
A curve C is a curve of constant width at infinity iff it is a curve of constant width.

From this theorem we immediately get

Corollary 1

For $0 < \alpha < \pi$ oval is of constant α-width iff its α-isoptic is a circle.

Before we formulate our last theorem we need three more ingredients - a generalization to our framework of the notion of width of an oval, a certain hedgehog associated to C and the sine theorem for isoptics.
In the classical projective definition the diameter is the polar line of the pole at infinity. Thus the generalization of the notion of width in this framework should rather be the section connecting the two touching points, that is the vector $q(t,\alpha)$. But $q(t,\alpha)$ does not generalize to the diameter of an oval in the direction e^{it} when $\alpha=\pi$. Moreover, as we already said, we would like to work in the framework of plane Euclidean geometry, while a avoiding the projective geometry concepts. Thus we introduce an adequate notion which we will call a α-spread of C at a point t and which has also some relevance with the diameter of the oval. For this purpose we introduce a vector $Q(t,\alpha)$ between the projections of the origin of coordinate system onto support lines of C at points $z(t)$ and $z(t+\alpha)$, as it is shown in the Figure below.



Construction of vector Q.

After some calculations one gets

$$Q(t,\alpha) = (p(t+\alpha)\cos\alpha - p(t))\, e^{it} + p(t+\alpha)\sin\alpha \; ie^{it}. \tag{25}$$

Definition 5

A number
$d_\alpha(t) = |Q(t,\alpha)| = \sqrt{p(t)^2 + p^2(t+\alpha) - 2p(t)p(t+\alpha)\cos\alpha}$ is said to be an α-spread of C at a point t.

Note that in the case $\alpha = \pi$ we get

$$d_\pi = p(t) + p(t+\pi). \tag{26}$$

By hedgehog we understand a curve given by the equation

$$z(t) = p(t)e^{it} + p'(t)ie^{it}, \tag{27}$$

where on the 2π-periodic function $p(t)$ we assume only that it is the C^1-function.

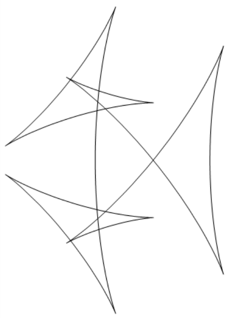
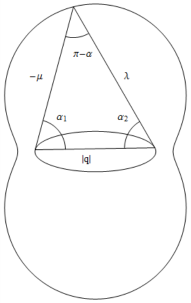


Figure: A hedgehog given by the support function $p(t) = 8 + \cos 3t + \cos 5t$

To define a necessary hedgehog we define a new support function $P(t) = \frac{|Q(t,\alpha)|}{2}$. Thus by a definition *a α-hedgehog associated* to an oval C and the angle α is a curve $H_\alpha(t) = P(t)e^{it} + P'(t)ie^{it}$. Note that the hedgehogs are intensively investigated, see papers Martinez-Maure, Y., *Hedgehogs of constant width and equichordal points*, Ann. Pol. Math. 67, No.3, 285-288 (1997) and Langevin, R., Levitt, G., Rosenberg, H., *Hérissons et multihérissons (enveloppes parametrées par leur application de Gauss)*, Singularities, Banach Cent. Publ. 20, 245-253 (1988) for references.

Curves of generalized constant width

In the our generalization we will need the sine theorem proved in Benko, K., Cieślak, W. Góźdź, S., Mozgawa W., *On isoptic curves*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat. 36, No.1, 47-54 (1990).



The sine theorem

Theorem 7

Under the notations in the above Figure the following identities hold

$$\frac{|q|}{\sin\alpha} = \frac{\lambda}{\sin\alpha_1} = \frac{-\mu}{\sin\alpha_2}. \tag{28}$$

Curves of generalized constant width

Having fixed all these notions and notations we prove the following theorem

Theorem 8

For fixed $\alpha \in]0, \pi]$ the statements;

(i) an oval is of constant α-width;

(ii) an oval is of constant α-spread;

(iii) all the vectors $q(t, \alpha)$ are parallel to vectors $Q(t, \alpha)$;

(iv) with the notations from the last Figure the expression

$$\frac{1}{|q(t,\alpha)|^2}\left(2|q(t,\alpha)| - \left(\frac{\sin\alpha_1}{k(t+\alpha)} + \frac{\sin\alpha_1}{k(t)}\right)\right) \tag{29}$$

is constant;

are equivalent, in the sense that whenever one of statements $(\mathrm{i} - \mathrm{iv})$ holds true, all other statements also hold.

(v) For all curves of the same constant α-width a the associated hedgehogs H_α have the same length L given by

$$L = \pi a.$$

It is worth the effort to see why is it a generalization of Mellish theorem. Let us substitute $\alpha = \pi$ and let C be an oval of constant width d then as we have on the last Figure $\alpha_1 = \alpha_2 = \frac{\pi}{2}$ and $|q| = a$ thus we get

$$a = \frac{1}{k(t)} + \frac{1}{k(t+\pi)} \tag{30}$$

and conversely. To finish this step we have to deal with the last property (v). However, if an oval C is of constant α-width, i.e. $d_\alpha \equiv const = a$ then the length L of the associated hedgehogs H_α is constant and given by the formula

$$L = \int_0^{2\pi} P(t)dt = \int_0^{2\pi} \frac{a}{2} dt = \pi a. \tag{31}$$

In the case of $\alpha = \pi$ we get $a = p(t) + p(t+\pi)$ which gives the reason for Barbier theorem, and L is the perimeter of C in this case.
Thus we are done with this non-trivial generalization of the Mellish theorem.

At the end of my talk we mention three still interesting questions:
1). Is it possible to give a characterization of convex bodies of constant α-width analogous to that unexpected given by Makai and Martini in Makai, E. jun.; Martini, H., *A new characterization of convex plates of constant width*, Geom. Dedicata 34, No.2, 199-209 (1990).

In this paper the authors prove that a convex plate $D \subset \mathbb{R}^2$ of diameter 1 is of constant width 1 if and only if any two perpendicular intersecting chords have total length ≥ 1.

Curves of generalized constant width

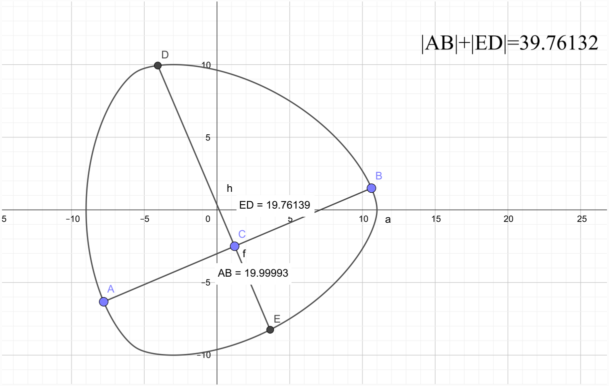


Figure: An oval given by the support function $p(t) = 10 + \cos 3t$ has the constant width equal to 20

2). Is it possible to give a counterpart of a theorem given by

A. Miernowski in *Parallelograms inscribed in a curve having a circle as $\frac{\pi}{2}$-isoptic*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A 62, 105-111 (2008)

showing that for a closed, convex curves having circles as $\frac{\pi}{2}$-isoptics, the maximal perimeter of a parallelogram inscribed in this curve can be realized by a parallelogram with one vertex at any prescribed point of the curve? Note that

Jean-Marc Richard observed in *Safe domain and elementary geometry*, Eur. J. Phys. 25 (2004), 835– 844

that maximal perimeter of a parallelogram inscribed in a given ellipse can be realized by a parallelogram with one vertex at any prescribed point of ellipse.

Alain Connes and Don Zagier gave in *A property of parallelograms inscribed in ellipses*, Amer. Math. Monthly 114 (2007), 909–914
probably the most elementary proof of this property of ellipse.

A nice proof of this fact can be found in M. Berger, Géométrie, Vol. 2, Nathan, Paris, 1990.

3). Is it possible to extend the above results to ovaloids, obtaining suitable generalization of the second part of Mellish paper *Notes on differential geometry*, Ann. Math. (2) 32, 181-190 (1931).

The end

Thank you for your attention!! Thank you for your attention!! Thank you for your attention!! Thank you for your attention!! Thank you for your attention!!